Comment on ''Magnetic levitation force and penetration depth in type-II superconductors''

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A new approach for examining the various superconducting pairing states was proposed by Xu, Miller, and Ting [Phys. Rev. B **51**, 424 (1995)]. The method consists of investigating the levitation force measured by a magnetic force microscope tip above a superconducting sample. Nevertheless, the quantitative analysis of experimental results needs to take into account important geometrical effects. In this Comment I show that, related to several physical arguments, the geometrical coefficients given by the authors admit much more simple expressions, allowing straightforward calculations for practical purposes. Furthermore, the method is presented for extending their linear theory to higher-order approximations. [S0163-1829(97)00918-1]

Xu, Miller, and Ting[1] have emphasized that the magnetic force microscope (MFM) provides a powerful technique in the study of superconducting materials. In particular, the analysis of the temperature dependence of the London penetration depth $\lambda(T)$ is shown to give useful information about the pairing state. The relation between the measured force F and λ can be obtained by means of Maxwell and London equations.

In order to obtain force values within the resolution of a MFM, the experimental range for the distance between the tip and the sample should be below $\approx 1\ \mu$m.[1] Then, considering a typical size for the tip to be around 0.5 μm and sample surface dimensions of the order of millimeters it is obvious that a realistic model should deal with a finite-size magnet above a semi-infinite superconductor. In what follows, the superconductor will be assumed to lie on the XY plane. Calling $\mathbf{B}_2(\mathbf{r})$ the induced magnetic induction due to the superconductor, the force acting on the magnetic tip can be obtained by integration of the relation $F_i = (\frac{1}{2})\partial_i(\mathbf{m}\cdot\mathbf{B}_2)$ for a magnetic dipole. This idea is applied in Ref. 1, splitting the magnet into elementary dipoles $d\mathbf{m} = \mathbf{M}dv$, computing the total induced field $\mathbf{B}_2(\mathbf{r})$, and integrating again over the magnet dimensions to obtain the total force. This approach leads to

$$F = \frac{\mu_0}{4\pi}\int_0^\infty dk \int_V d^3r \int_V d^3r' \left\{ k^3 \frac{\sqrt{1+\lambda^2k^2}-\lambda k}{\sqrt{1+\lambda^2k^2}+\lambda k} e^{-2ak}\mathbf{M}(\mathbf{r})\cdot\mathbf{M}(\mathbf{r}')e^{-k(z+z')}J_0[k\sqrt{(x-x')^2+(y-y')^2}]\right\}, \tag{1}$$

where a is the distance between the lower end of the magnet and the superconductor (measured along the z axis) and V the volume of the magnet. The integrals in Eq. (1) become intricate even when F is linearized with respect to the λ dependence. In particular, the authors consider the limit $\lambda/a \ll 1$ in which $F \approx \alpha - \beta(\lambda/a)$ and calculate the geometrical coefficients α and β for uniformly magnetized cylindrical and spherical tips. For the cylindrical magnetic tip they obtain

$$\alpha = \frac{\mu_0 m^2}{4\pi h^2 R^2}\sum_{i=0}^{\infty}\frac{(-1)^i(2i+1)!(2i+2)!}{i![(i+1)!]^2(i+2)!}\left[\frac{R}{4a+4h}\right]^{2i+2} \times\left[\left(1+\frac{h}{a+h}\right)^{-2i-2}+\left(1-\frac{h}{a+h}\right)^{-2i-2}-2\right], \tag{2}$$

$$\beta = \frac{\mu_0 m^2}{4\pi h^2 R^2}\sum_{i=0}^{\infty}\frac{(-1)^i[(2i+2)!]^2}{i![(i+1)!]^2(i+2)!}\left[\frac{R}{4a+4h}\right]^{2i+2} \times\left[\left(1+\frac{h}{a+h}\right)^{-2i-3}+\left(1-\frac{h}{a+h}\right)^{-2i-3}-2\right]. \tag{3}$$

On the other hand, the following expressions are given for the case of a spherical tip:

$$\alpha = \frac{9\mu_0 m^2}{4\pi R^4}\sum_{i,j=1}^{\infty}\left[\frac{R}{2a+R}\right]^{2i+2j} \times\sum_{k=0}^{\infty}(-1)^k\left[\frac{R}{8a+8R}\right]^{2k} C_{ijk}^{-1}, \tag{4}$$

$$\beta = \frac{9\mu_0 m^2}{4\pi R^4} \sum_{i,j=1}^{\infty} \left[\frac{R}{2a+R}\right]^{2i+2j} \sum_{k=0}^{\infty} (-1)^k \left[\frac{R}{8a+8R}\right]^{2k} C^0_{ijk}, \tag{5}$$

where

$$C^{\mu}_{ijk} \equiv \frac{(2i+2j+2k+\mu)!(i+j+2k+1)!}{k!(i+j+k+1)!(2i+2k+1)!!(2j+2k+1)!!}. \tag{6}$$

These cumbersome series expansions can be greatly simplified on the basis of several physical arguments. Instead of using Eq. (1) for calculating α and β, I will start with the physical meaning of α. This coefficient corresponds to the magnetic force on the magnet above a semi-infinite superconductor with complete flux expulsion $(\lambda=0)$. Then, α can be straightforwardly obtained by means of the image technique in the potential theory. On the other hand, observing Eq. (1), it can be seen that $\beta=-a\, d\alpha/da$, which allows calculating this coefficient just by taking a derivative. Notice also that higher-order approximations can be easily obtained using this iterative technique [for instance, the coefficient of $(\lambda/a)^2$ is $\gamma=(a^2/2)d^2\alpha/da^2$]. It should be emphasized that the terms beyond the linear approximation can be of utter importance for the analysis of $\lambda(T)$. Recall that λ diverges as T approaches the critical temperature.

Let us concentrate on the calculation of α for several geometries. The starting point is that, according to the image technique, $\mathbf{B}_2(\mathbf{r})$ is just the magnetic induction created by the corresponding image magnet. For a cylindrical magnet (radius R, height $2h$, magnetic moment m) uniformly magnetized along its axis, which coincides with the z direction, the image is an identical magnet, symmetrically located with respect to the surface of the superconductor and with antiparallel magnetization. Now, α can be evaluated (see Ref. 2, p. 207) using the force equation on a magnetic body due to an externally applied magnetic induction:

$$\alpha = -\int_V (\boldsymbol{\nabla}\cdot\mathbf{M}) B_{2,z} d^3r + \int_S (\mathbf{M}\cdot\hat{n}) B_{2,z} ds. \tag{7}$$

In our case (uniform magnetization) the volume integral reduces to zero. Then, inserting the field created by a cylindrical magnet in $\mathbf{B}_2(\mathbf{r})$ and evaluating the surface integral we obtain

$$\alpha = \frac{2\mu_0 m^2}{\pi^2 R^3 h^2} \left\{ 2a_2 \frac{K(k_2)-E(k_2)}{k_2} - a_1 \frac{K(k_1)-E(k_1)}{k_1} - a_3 \frac{K(k_3)-E(k_3)}{k_3} \right\}, \tag{8}$$

where $k_i \equiv R/\sqrt{a_i^2+R^2}$; $a_1 \equiv a$; $a_2 \equiv a+h$; $a_3 \equiv a+2h$. $K(k)$ and $E(k)$ are the complete elliptic integrals of the first and second kind, respectively, in Legendre's notation.

Now, β is straightforwardly obtained, just considering the recurrence relations between the complete elliptic integrals and their derivatives (see Ref. 3, p. 326). The result is

$$\beta = \frac{4\mu_0 m^2 a}{\pi^2 R^3 h^2} \left\{ -2K(k_2)\left(\frac{2}{k_2}-k_2\right) + K(k_1)\left(\frac{2}{k_1}-k_1\right) + K(k_3)\left(\frac{2}{k_3}-k_3\right) + 4\frac{E(k_2)}{k_2} - 2\frac{E(k_1)}{k_1} - 2\frac{E(k_3)}{k_3} \right\}. \tag{9}$$

For the case of a sphere (radius R) uniformly magnetized along the z axis, the situation simplifies even more because for this geometry the induction outside the magnet is exactly a dipole field (see Ref. 2, p. 194). One can, therefore, calculate α starting with the repulsion force between a magnetic dipole and a superconducting sheet and substituting $a \rightarrow R + a$. This leads to

$$\alpha = \frac{3\mu_0 m^2}{32\pi(R+a)^4} \tag{10}$$

and

$$\beta = \frac{3\mu_0 m^2 a}{8\pi(R+a)^5}. \tag{11}$$

It is apparent that Eqs. (8)–(11) allow straightforward calculation of the geometrical coefficients involved in the levitation force experiments as they are exact expressions in terms of well-known mathematical functions. These results, as well as the underlying physical arguments, are intended to facilitate the development of MFM in the research of superconducting materials, following the ideas presented in the work by Xu, Miller, and Ting.

[1] J. H. Xu, J. H. Miller, Jr., and C. S. Ting, Phys. Rev. B **51**, 424 (1995); **55**, 11 877(E) (1997).

[2] J. D. Jackson, *Classical Electrodynamics*, 2nd ed. (Wiley, New York, 1975).

[3] G. Arfken, *Mathematical Methods for Physicists*, 3rd ed. (Academic, Boston, 1985).