Asymptotic theory for the inverse problem in magnetic force microscopy of superconductors

A. Badía*
Departamento de Física de la Materia Condensada, CPSUZ-ICMA, María de Luna 3, E-50 015 Zaragoza, Spain
(Received 12 April 1999)

An asymptotic theory is formulated, which allows us to recover the London penetration depth λ in superconductors from magnetic force microscopy measurements. An *ad hoc* interpretation of the forward problem allows us to accomplish a complex variable based power series reversion scheme. The asymptotic series expansion of λ can be generated in terms of measurable quantities. Simulations for analytically tractable situations have been performed which confirm the stability of our approach, even for cases where noise corrupted data is considered. The method does not restrict to any particular symmetry and is suited for covering the whole temperature range $\lambda(T)$, with obvious implications on the knowledge of the superconducting pairing state. By comparing with the analytic model of Coffey [Phys. Rev. B **57**, 11 648 (1998)] we discuss the limitations and possible extensions of the existing theory. [S0163-1829(99)05637-4]

I. INTRODUCTION

Asymptotic expansions have been widely used in fundamental physical theories for the computation of a variety of functions. In particular, a celebrated approximation of quantum mechanics, the WKB method is based on an asymptotic series in terms of $\hbar$. Recall that the representation of a function $f(x)$ by partial sums of an asymptotic infinite series is an arbitrarily good approximation for large values of x. On the contrary, for a finite value of x such series can even be divergent and one has a corresponding number of terms which provide an optimum approximation. In this article, we propose an asymptotic series method to solve an inverse problem in the magnetic force microscopy[1] (MFM) of superconductors.

Specifically, we will deal with a theory which allows us to recover the London penetration depth λ from the measured force between a magnetic tip and a Meissner state superconductor. This statement belongs to a class of problems to which both theoretical physics and applied mathematics devote special consideration: one wants to reconstruct a system's properties from indirect observations. Ideally, a *tabular* approach, which consists of building as complete as possible a catalog of direct transformations and then looking up the desired inverse should be avoided. Unless one possesses a solid previous knowledge of the system, this can lead to an ambiguous fitting procedure with questionable free parameters. Additionally, fundamental inversion topics as uniqueness and robustness can be considered if a more general procedure is afforded. This will be our aim for the mentioned MFM problem.

MFM has become a high performance technique for the investigation of superconductors[2,3] and other magnetic materials. It is noteworthy to mention that MFM instrumentation is under continuous development,[3–5] while the interpretation of measurements is still a challenge[6–8] in some aspects. Thus, accurate MFM calibration has been described[4] by using micronscale current rings biased by a precision current source, spatial resolution of less than 10 nm has been achieved[5] based on a force gradient measurement, and low-temperature inherent problems have been overcome with a reported pico-Newton resolution at 4.3 K.[3]

For a given magnetic tip and a superconducting sample in the Meissner state, the repulsion force F is determined by the temperature dependence $\lambda(T)$.[9] This quantity has been selected as a sensitive probe distinguishing between different coupling states for the superconducting carriers. In fact, for a given s-wave BCS gap Δ, $\lambda(T)-\lambda(0)$ experimentally approaches zero as $\exp(-\Delta/kT)$. On the other hand, other pairing states as the $d_{x^2-y^2}$ type give rise to a linear dependence $1-aT$ at low temperature.[10] Recently, the topic has been addressed for high-T_c superconductors in which the pairing state is still a controversy. See Ref. 10 for a very comprehensive review on this subject. The main conclusion is that there is increasing experimental evidence of an unconventional quasi-particle excitation spectrum as would correspond to a d-wave pairing state, instead of the BCS s wave. Nevertheless, several inconsistencies remain to be solved. One of the potential reasons for these discrepancies stems from the fact that for *macroscopic* measurements, $\lambda(T)$ holds nothing but an averaged permeability. Recall that, in high-T_c compounds, even the presumed high quality crystals are not absent of grain boundaries and other inhomogeneities to which these materials are highly sensitive, owing to their short coherence length. Nonetheless, MFM offers a local probe by means of which very small (submicrometric scale) regions of the sample can be scanned.

In the magnetostatic case, the relation between the measured force F and λ can be established by combining Ampère law and the London equations (the London theory is preferred to the Ginzburg-Landau approach because the range of interest is especially at low temperatures). Nevertheless, when boundary conditions are included, even for the simplest configurations (i.e., axisymmetry) the arising integral form of Helmholtz equation is cumbersome enough. Thus, in general, extracting λ from the measured force turns out to be a formidable task. As a consequence, the procurement of a practical scheme for obtaining λ as a function of F is still under development. The availability of a consistent and robust theory should stimulate novel experiments on the basic physics of superconductors.

A significant advance in the inversion theory is due to Coffey[7,8] who has recently solved the inversion problem for a depth-dependent penetration $\lambda(z)$. However, as recognized by this author, inverse problems are typically ill posed and stable algorithms must be sought. On the other hand, he as-

0163-1829/99/60(14)/10436(6)/$15.00

sumes a point dipole magnet, whereas non-negligible finite-size effects are present for many experimental situations.

Finally, we want to mention that the crudest approximation to the inverse problem consists of using λ as a small parameter (respect to the other relevant length scales) and linearizing the expressions which involve this quantity. Along this line, the direct(inverse) problem was posed in Refs. 9,11. There, a linear dependence of the levitation force on λ was assumed. In principle, this approach is only valid at very low temperatures, i.e., well apart from the divergence of λ at $T=T_c$.

In this report, we hope to shed some more light on the inverse problem in the MFM of superconductors. The work is focused on the very general idea of getting a useful power series expansion of λ in terms of the measured force F. We will show that this is feasible, obtaining an asymptotic series and giving a practical scheme for recovering λ in a realistic situation. Finally, we will establish the comparison with the previously existing theory[7,8] showing the complementarity of both approaches and discussing possible further extensions.

II. ASYMPTOTIC EXPANSION

The starting point builds on previous work[12] in which the direct problem was treated by the author. Considering the typical length scales and the arguments developed there one can show that the customary MFM experiment may be modeled as a finite size magnet over a semi-infinite superconductor. In order to bring out the essential physics without unnecessary mathematical difficulties, at first, we will assume a problem with rotational symmetry (eventually, this condition will be relaxed in Sec. IV). Now, under the assumption of rotational symmetry, the vertical force can be expressed as[9]

$$F=\frac{\mu_0}{4\pi}\int_0^\infty dk\int_V d^3r\int_V d^3r' \times\left\{k^3\frac{\sqrt{1+\lambda^2k^2}-\lambda k}{\sqrt{1+\lambda^2k^2}+\lambda k}\times e^{-2ak}\mathbf{M}(\mathbf{r})\cdot\mathbf{M}(\mathbf{r}') \times e^{-k(z+z')}J_0[k\sqrt{(x-x')^2+(y-y')^2}]\right\}. \quad (1)$$

Here, the magnet is assumed to rest at a distance a above the superconductor, which is set to lie parallel to the XY plane. $\mathbf{M}(\mathbf{r})$ stands for the magnetization of the magnet, V its volume, and J_0 is the Bessel function of the first kind of order 0. The origin of coordinates has been taken at the lower end of the magnet. Along this section $\mathbf{M}(\mathbf{r})$ will be considered uniform and parallel to the Z axis.

Formally, one can integrate Eq. (1) by power series expansion in terms of the dimensionless combination λk. In practice, this will provide a method for computing F for large values of a because the high wave numbers are attenuated by the exponential function. Then λk becomes a small parameter and one gets a convergent series.

Observe that the expansion of the pre-exponential factor leads to

$$F(a)=F_0(a)+\sum_{n=1}^{\infty} f_n\left(\frac{\lambda}{a}\right)^n, \quad (2)$$

where F_0 can be identified as the force between the magnet and the completely shielded superconductor $(\lambda\to 0)$. On the other hand, the coefficients f_n can be evaluated from the relation

$$f_n=\alpha_n\frac{(-1)^n}{2^n}\frac{d^nF_0}{da^n}a^n, \quad (3)$$

with

$$\alpha_1=-\alpha_2=-2,\alpha_3=-1,\alpha_{2m+1}=\frac{(-1)^m(2m-3)!!}{2^{m-1}m!}. \quad (4)$$

Thus, Eqs. (1) and (2) allow to anticipate $F[a,\lambda(T)]$ for any foregone dependence $\lambda(T)$ and this quantity can be compared to the experimental results. On the contrary, if one attempts to recover the material property $\lambda(T)$ from the observable $F(a,T)$, an inversion scheme is required. Below, an inversion program based on the infinite power series reversion is presented. We want to stress that, hereafter, Eq. (2) will be used at a formal level. In fact, a useful asymptotic inverse series is obtained even for situations in which the forward one does not converge.

Assume that $F_0(a)$ is known. In any case, one can calculate it by using the image technique or other methods in the potential theory.[13] Define $\bar{F}\equiv F-F_0$ as the difference between the actual force and the limit F_0. The problem can be posed as follows. (i) We have the direct power series expansion

$$\bar{F}(a)=\sum_{n=1}^{\infty} b_n\lambda^n, \quad (5)$$

(ii) b_n being known functions of a

$$b_n=\alpha_n\left(-\frac{1}{2}\right)^n\frac{d^nF_0}{da^n}, \quad (6)$$

(iii) and we want to obtain an explicit expression of the kind

$$\lambda=\sum_{n=1}^{\infty} c_n\bar{F}^n, \quad (7)$$

(iv) so that we have to solve for the unknowns c_n.

A quite general approach for solving the problem can be made by the use of complex variables. Starting with Cauchy's integral formula[14]

$$\lambda(\bar{F})=\frac{1}{2\pi i}\oint_C\frac{\lambda(\bar{F}')d\bar{F}'}{\bar{F}'-\bar{F}} \quad (8)$$

which is valid for λ analytic on the complex plane contour C and within its interior (at least, this will be valid for some circle of convergence), and using the convergent expansion for the infinite geometric series $\Sigma_n(\bar{F}/\bar{F}')^n=(1-\bar{F}/\bar{F}')^{-1}$ one can show that

$$c_n=\frac{1}{n!}\lim_{\lambda\to 0}\left[\frac{d^{n-1}}{d\lambda^{n-1}}\frac{\lambda^n}{\bar{F}(\lambda)^n}\right], \quad (9)$$

which can be identified with the so-called *residue* of the function $1/n\bar{F}^n$ at the point $\lambda=0$

TABLE I. Several inverse series coefficients for the penetration depth expansion in terms of the measured force: $\lambda=\Sigma_n c_n[F_{\mathrm{MFM}}(a)-F_0(a)]^n$. F_{MFM} stands for the MFM data and F_0 is the theoretical force between the magnetic tip and a completely shielded ($\lambda=0$) superconductor.

c_1	$1/F_0'$
c_2	$-(1/2)F_0''/(F_0')^3$
c_3	$(1/8)[4(F_0'')^2-F_0'F_0^{(3)}]/(F_0')^5$
c_4	$(5/8)[(1/2)F_0'F_0''F_0^{(3)}-(F_0'')^3]/(F_0')^7$
c_5	$(1/8)\{7(F_0'')^4-(21/4)F_0'(F_0'')^2F_0^{(3)}+(F_0')^2[(3/8)(F_0^{(3)})^2+(1/16)F_0'F_0^{(5)}]\}/(F_0')^9$

$$c_n=\mathrm{Res}_{\lambda=0}\left[\frac{1}{n\bar{F}(\lambda)^n}\right]. \tag{10}$$

This follows from the fact that $1/\bar{F}^n$ possesses a pole of order n at $\lambda=0$.

Therefore, the inversion program can be summarized as follows. (a) Evaluate the residues of the function $1/n\bar{F}^n$ for increasing values of n. Here, it is convenient to calculate $\bar{F}$ starting with Eq. (2). This will provide the coefficients c_n in terms of F_0 and its derivatives. (b) Calculate the difference between the MFM data F_{MFM} and the zero penetration depth limit F_0: $\bar{F}_{\mathrm{MFM}}=F_{\mathrm{MFM}}-F_0$ as a function of the recording distance a (occasionally, just for a given distance). (c) Reconstruct the inverse series by means of Eq. (7) and $\bar{F}_{\mathrm{MFM}}$.

We should mention that the evaluation of residues is currently standardized, for instance, by using the algebraic manipulation package MATHEMATICA.[15] However, for the readers' benefit we display several coefficients in Table I. The number of listed coefficients has been chosen for providing a very nice inversion in simulated experiments [by using Eq. (1)] for analytically solvable models. Closed form expressions for $F_0(a)$ in the case of spherical and cylindrical tips are given in Ref. 11. Additionally, we include here the formulas for semispherical and conical tips in Table II. They are given in terms of definite integrals of the Legendre function $Q_{-1/2}$. For computation purposes, recall that $Q_{-1/2}(z)=\sqrt{2/(z+1)}K[\sqrt{2/(z+1)}]$, where K stands for the complete elliptic integral of the first kind. Notice that alternative expressions in terms of improper integrals have been obtained for all these cases in Ref. 16.

As a practical example on how the method works we include Fig. 1, in which the reconstruction of λ for a simulated experiment is illustrated. Synthetic data $F(a,\lambda)$ were generated for a cylindrical tip, whose dimensions $L=R\equiv D$ established the length units of the problem. The recovered λ is plotted as a function of the distance a for an increasing number of terms in the inverse series. We want to stress that the asymptotic character displayed in Fig. 1 has also been observed for $\lambda/D>1$. One gets a very good approximation with a small number of terms (eventually one) for $a\gg\lambda$, whereas an optimum number appears for small distances ($n=3$ in the above example).

The effectiveness of the method has been also checked for noise corrupted data as would be the case for experimental situations. As an example, Fig. 1 incorporates the result of inverting force data to which a random noise, corresponding to 0.1% resolution of the measured force has been added. Raw data with no preprocessing have been directly fed into the inversion algorithm, which has been implemented for $n=3$. The statistical properties of the derived λ are a mean value $\langle\lambda\rangle=0.0999D$ and a standard deviation $\sigma=0.0009D$. However, notice that scattering increases with the distance a. This fact can be explained by a simple error propagation argument. If one assumes an uncertainty δF_{MFM} in the measured force it is simple to show that $\delta\lambda=[1/F_0'+\mathcal{O}(F_{\mathrm{MFM}}-F_0)]\delta F_{\mathrm{MFM}}$ and $1/F_0'$ is an increasing function of a while $F_{\mathrm{MFM}}-F_0$ tends to zero. As regards the extent of the previous evaluation, we should mention that pico-Newton resolution corresponds to 0.1% of the measured signal for a cylindrical tip with $D=1$ μm, $\mu_0M=1$ T, a recording distance $a\simeq1$ μm and a superconductor penetration depth $\lambda=0.1$ μm as can be tested by means of Eq. (1).

III. EXTENDED LAPLACE INVERSION METHOD

As mentioned in the Introduction, a comparison of our work with other models should enrich the state of the art in

TABLE II. Collected expressions for the repulsion force F_0 between semispherical and conical tips and a completely shielded superconducting plane. The magnets are assumed to be uniformly magnetized along the symmetry axis, which is perpendicular to the plane and oriented with the sharp end towards the superconductor. The distance between the lower end of the magnet and the superconductor is denoted by a while R and h stand for the radius and height of the tip. m stands for the magnetic moment and $Q_{-1/2}$ represents a conical Legendre function. $\alpha_1\equiv[R^2+2(a+R)(a+R-\sqrt{R^2-r^2}-\sqrt{R^2-r'^2})+\sqrt{R^2-r^2}\sqrt{R^2-r'^2}]/rr'$, $\alpha_2\equiv[R^2+4(a+R)(a+R-\sqrt{R^2-r^2})+r'^2]/2rr'$, $\alpha_3\equiv[4(a+R)^2+r^2+r'^2]/2rr'$, $\beta_1\equiv\{[2a+h(r+r')/R]^2+r^2+r'^2\}/2rr'$, $\beta_2\equiv[(2a+rh/R+h)^2+r^2+r'^2]/2rr'$, $\beta_3\equiv[(2a+2h)^2+r^2+r'^2]/2rr'$.

Semisphere	$-\frac{9\mu_0m^2}{8\pi^2R^6}\frac{\partial}{\partial a}\int_0^R dr\int_0^R dr'\{\sqrt{rr'}[Q_{-1/2}(\alpha_1)-2Q_{-1/2}(\alpha_2)+Q_{-1/2}(\alpha_3)]\}$
Cone	$-\frac{9\mu_0m^2}{2\pi^2R^4h^2}\frac{\partial}{\partial a}\int_0^R dr\int_0^R dr'\{\sqrt{rr'}[Q_{-1/2}(\beta_1)-2Q_{-1/2}(\beta_2)+Q_{-1/2}(\beta_3)]\}$

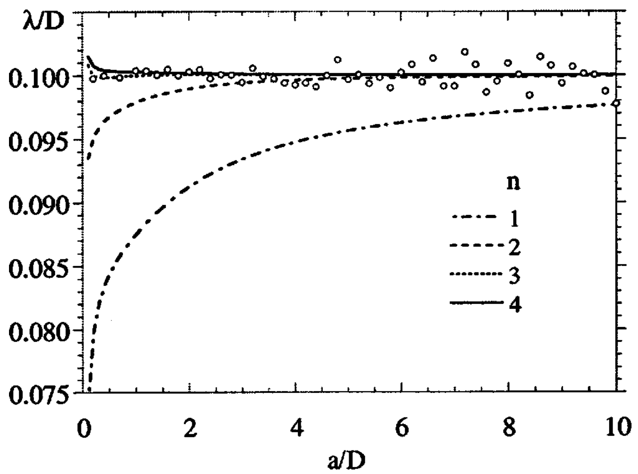

FIG. 1. Reconstruction of the penetration depth by means of Eq. (7) for a simulated force vs distance experiment in the case of a cylindrical magnetic tip (radius R, length L, and $R=L\equiv D$) and $\lambda=0.1D$. The different lines correspond, as labeled, to the truncation of the series by means of a 1, 2, 3, and 4 term sequence. Both the penetration depth and the distance between the tip and the superconductor are expressed in units of D. Points correspond to the reconstruction of λ with a superimposed 0.1% noise on F and by means of a 3 term sequence.

the inversion theory for MFM. Below, we will concentrate on the inversion method developed in Ref. 8, which can be conveniently generalized by incorporating $F_0(a)$ to the analysis. Under the assumptions of axisymmetry and point dipole magnetic source, that contribution solves the problem of a depth dependent penetration $\lambda(z)$. The author employs a wave number-dependent kernel function $K(k)$ which is obtained via Laplace transform inversion of the MFM data. Hereafter, we shall refer to this approach as a Laplace inversion method (LIM). In order to show in what sense the LIM can be complemented by our ideas, we will develop a particular extension of the method below.

Consider the case of a cylindrical magnetic tip uniformly magnetized along its axis, which is directed perpendicular to the semi-infinite superconductor (thus, fulfilling rotational symmetry). If one assumes λ to be a constant, a suitable expression for $K(k)$ can be derived starting with Eq. (1). Calling L,R to the length and radius of the magnet and after some algebra one gets

$$F=\pi\mu_0 M^2R^2\int_0^\infty dk \times\left\{k^3 e^{-2ak}\frac{K(k)-1}{K(k)+1}\left[\frac{J_1(kR)(1-e^{-kL})}{k^2}\right]^2\right\}, \quad (11)$$

where the kernel function $K(k)$ is defined as $K(k)\equiv\sqrt{k^2+\lambda^{-2}}/k$.

Next, we introduce the *form factor* f

$$f(k,R,L)\equiv\left[\frac{RLk^2}{2J_1(kR)(1-e^{-kL})}\right]^2.$$

Identifying Eq. (11) as a Laplace transform, K can be expressed in terms of the inverse Laplace transform of the vertical force

$$K\left(\frac{k}{2}\right)=\frac{1+(c_m/k^3)f(k/2,R,L)\mathcal{L}^{-1}[F(a)](k)}{1-(c_m/k^3)f(k/2,R,L)\mathcal{L}^{-1}[F(a)](k)}, \quad (12)$$

where we have used $c_m\equiv 64\pi/\mu_0 m^2$, m standing for the magnetic moment. Notice the analogy to Eq. (16) of Ref. 8. Both expressions differ in the multiplying *form factor* $f(k/2,R,L)$. On the other hand, it is apparent that they coincide when one takes the limit $R,L\to 0$, as in this case f tends to unity.

Observe that f may be expressed in terms of the zero penetration depth limit $F_0(a)$. In fact, starting with Eq. (11) for $\lambda=0$ and inverting again, one has

$$1/f(k/2,R,L)=\frac{c_m}{k^3}\mathcal{L}^{-1}[F_0(a)](k),$$

which leads the straightforward generalization for axisymmetric situations

$$K\left(\frac{k}{2}\right)=\frac{1+\mathcal{L}^{-1}[F(a)](k)/\mathcal{L}^{-1}[F_0(a)](k)}{1-\mathcal{L}^{-1}[F(a)](k)/\mathcal{L}^{-1}[F_0(a)](k)}. \quad (13)$$

The relevance of accounting for the *form factor* when inverting MFM data has been checked for simulated experiments in the cylindrical system. Assuming concrete values for M,R,L,λ we have generated a set of synthetic force vs distance measurements. Then, we get the kernel function by means of either Eq. (12) or the uncorrected version ($f=1$). Laplace transform inversion has been implemented by analytic continuation of $F(a)$ onto the complex plane and then using the Bromwich integral.[14] Then, the numerically computed inverse function shows very stable behavior. Nevertheless, stability strongly depends on the smoothness of $F(a)$. In particular, we want to mention that general numerical inversion procedures display severe convergence problems for noise corrupted data.

The reconstruction of $K(k)$ for a particular case in which faked data have been produced by means of Eq. (1) is shown in Fig. 2. For completion $\mathcal{L}^{-1}[F(a)](k)$ is also displayed as an inset of the figure. For comparison purposes we plot the relation $K(k)=\sqrt{k^2+\lambda^{-2}}/k$, which should act as a basis for extracting λ from the experimental data. It is apparent that neglecting the correction factor may introduce large errors if one attempts a fit of the uncorrected points to this dependence. The best fit of $K(k)$ for our simulated experiment with $\lambda/D=0.1$ outputs a value $\lambda/D=0.0998(8)$ for the redressed data when the cutoff $k=20$ is used. One can compare the achieved accuracy with the asymptotic method by means of Fig. 1. For instance, $n=2$ would provide the same accuracy provided $a/D>6$, and $n=3$ if $a/D>0.15$.

IV. NONSYMMETRICAL PROBLEMS

In this section we will show that none of the previous results is restricted to rotationally symmetric situations. In fact, they completely generalize for any shape of the tip and any magnetization pattern. Specifically, we will consider a

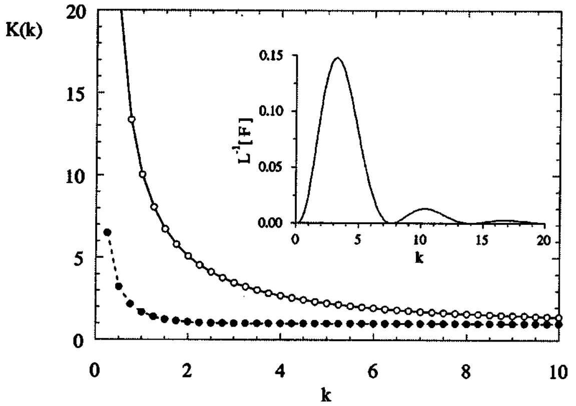

FIG. 2. Recovered kernel function $K(k)$ for a simulated force vs distance experiment in the case of a cylindrical magnetic tip (radius R, length L and $R=L\equiv D$) and $\lambda=0.1D$. Open symbols stand for the modified Laplace inversion method [Eq. (12)] while full symbols correspond to the uncorrected theory [$f=1$ in Eq. (12)]. The solid line is a plot of $K(k)=\sqrt{k^2+\lambda^{-2}}/k$, whereas the dashed line is only a guide for the eye. The inset shows the inverse Laplace transform of the simulated force, obtained with a complex variable based algorithm. In order to avoid distracting powers of 10, the normalization $\mu_0 M^2=1$ has been used.

magnet of arbitrary shape and a position dependent magnetization $[M_x(\mathbf{r}),M_y(\mathbf{r}),M_z(\mathbf{r})]$ above the semi-infinite superconductor.

Using the customary form for the magnetic field induction[9]

$$\mathbf{B}(\mathbf{r})=\begin{cases}\mathbf{B}_1(\mathbf{r})+\mathbf{B}_2(\mathbf{r}), & z\geqslant 0,\\ \mathbf{B}_3(\mathbf{r}), & z\leqslant 0,\end{cases}\tag{14}$$

where $\mathbf{B}_1(\mathbf{r})$ is the direct contribution from the magnetic source, $\mathbf{B}_2(\mathbf{r})$ is the induced field, and $\mathbf{B}_3(\mathbf{r})$ the penetrating field inside the superconductor (with surface $z=0$), one can solve the problem by imposing

(i) $\nabla^2\mathbf{B}_2=0$ and $\nabla^2\mathbf{B}_3=(1/\lambda^2)\mathbf{B}_3$,

(ii) $\mathbf{B}_1+\mathbf{B}_2=\mathbf{B}_3$ at $z=0$,

(iii) $\nabla\cdot\mathbf{B}_2=0$ and $\nabla\cdot\mathbf{B}_3=0$,

(iv) $\mathbf{B}\rightarrow 0$ as $z\rightarrow\pm\infty$.

Then, particular solutions for Laplace's and Helmholtz's equations arise for $\mathbf{B}_2$ and $\mathbf{B}_3$ after the source field $\mathbf{B}_1$ is fixed. For instance, if the source is a magnetic dipole at the point $(0,0,a)$ above the superconductor and components (m_x,m_y,m_z) one can show that $\mathbf{B}_2$ is given by

$$B_{2i}=\frac{\mu_0}{4\pi}\int_0^\infty dk\left\{k^2\frac{\lambda k-\sqrt{1+\lambda^2k^2}}{\sqrt{1+\lambda^2k^2}+\lambda k}e^{-k(a+z)}\left[\sum_j g_{ij}m_j\right]\right\},\tag{15}$$

where Latin indices are used for indicating the three cartesian components and the matrix elements g_{ij} are defined as

$$g_{11}=\frac{1}{2}J_0(k\sqrt{x^2+y^2})-\frac{x^2-y^2}{2(x^2+y^2)}J_2(k\sqrt{x^2+y^2}),$$

$$g_{12}=-\frac{xy}{x^2+y^2}J_2(k\sqrt{x^2+y^2}),$$

$$g_{13}=\frac{x}{\sqrt{x^2+y^2}}J_1(k\sqrt{x^2+y^2}),$$

$$g_{21}=g_{12},$$

$$g_{22}=\frac{1}{2}J_0(k\sqrt{x^2+y^2})-\frac{y^2-x^2}{2(x^2+y^2)}J_2(k\sqrt{x^2+y^2}),$$

$$g_{23}=\frac{y}{\sqrt{x^2+y^2}}J_1(k\sqrt{x^2+y^2}),$$

$$g_{31}=-g_{13},$$

$$g_{32}=-g_{23},$$

$$g_{33}=J_0(k\sqrt{x^2+y^2}).\tag{16}$$

Now, the self-interaction energy $U=-(1/2)\mathbf{m}\cdot\mathbf{B}_2$ and superposition allow us to calculate the levitation force on an arbitrary shape magnet at a distance a above the superconductor $(F=-\partial U/\partial a)$. Using $\mathbf{M}(\mathbf{r})$ for the magnetization pattern and V for its volume one gets the generalization of Eq. (1)

$$F=\frac{\mu_0}{4\pi}\int_0^\infty dk\int_V d^3r\int_V d^3r'\times\left\{k^3\frac{\sqrt{1+\lambda^2k^2}-\lambda k}{\sqrt{1+\lambda^2k^2}+\lambda k}e^{-2ak}e^{-k(z+z')}\times\left[\sum_{i,j}M_i(\mathbf{r})g_{ij}(\mathbf{r}-\mathbf{r}')M_j(\mathbf{r}')\right]\right\},\tag{17}$$

where $g_{ij}(\mathbf{r}-\mathbf{r}')$ indicates that the matrix elements must be evaluated according to Eq. (16) by replacing $x\rightarrow x-x'$ and $y\rightarrow y-y'$. Notice that volume integration must be performed with the origin of coordinates at the lower end of the magnet.

By comparing Eqs. (1) and (17) one can see that the expansion given by Eq. (2) is still valid, $F_0(a)$ being the force between the arbitrary magnet and the completely shielded superconductor. In fact, such expansion directly depends on the pre-exponential factor, which is determined by the relation between Laplace's and Helmholtz's equations solutions in conjoined half-spaces. This result extends to the achievements of Secs. II and III to problems in which no convenient coordinate system is allowed by symmetry. In general, the inversion procedure can be performed in terms of the zero penetration depth limit $F_0(a)$.

V. DISCUSSION AND CONCLUSIONS

An asymptotic method for solving the inverse problem in the magnetic force microscopy of superconductors has been proposed. The theory developed forms an alternative to other models in the search of stable inversion algorithms.

The London penetration depth λ is obtained as the asymptotic value of a power series expansion in terms of the difference between the experimental data $F_{\mathrm{MFM}}(a)$ and the zero penetration depth limit $F_0(a)$. Although this is a zero-dimensional method (one could work at a given distance a_0), it is useful to observe the recovered λ as a function of the distance between the tip and the sample a. This provides a consistency check, as λ should tend to the correct value as a increases. Notice that the appearance of an asymptotic series[14] is guaranteed by the factor $\exp(-2ak)$ in the integral representation of F [see Eq. (1)].

It should be noted that the asymptotic expansion comes from a standard power series inversion technique, allowed by the knowledge of $F_0(a)$ regardless the symmetry of the problem. In fact, the theory has been extended to situations in which both the shape of the magnet and magnetization pattern are completely arbitrary. $F_0(a)$ can be computed if the magnetization pattern of the tip is known and we give the results for several customary tips under axisymmetry assumption. Nevertheless, an assessment of the computation can be made by experimental means. It can be either interpreted as the repulsion force between the tip and a perfectly diamagnetic plane or the attraction force between the tip and an infinitely permeable plane. Thus, one could use a superconducting film in a setup in which λ can be neglected or a high permeability material for calibration purposes. Additionally, if the zero penetration depth limit is known one can use it as a test against irregularities in surface topography or inhomogeneous penetration depth.

The validity of our theory has been checked for a wide range of values of λ, which was successfully recovered with a few terms for simulated experiments in analytically tractable situations. Moreover, the asymptotic theory provides a practical method for real data with the concomitant noise. This point has been checked for artificially corrupted data which were directly fed into the algorithm. This seems an advantage respect to LIM methods in which preprocessing smoothing steps can be seldom eluded.

The zero penetration depth limit $F_0(a)$ has also permitted an extension of the Laplace inversion method to situations in which the dipole approximation is not valid. Thus, we propose a modified model readily applicable when λ is a constant and which incorporates $\mathcal{L}^{-1}[F_0(a)]$. We want to mention here that complex variable inversion of the Laplace transform by means of Bromwich integral provides a robust numerical method. This method has been preferred for simulations, as the analytical continuation of $F(a)$ is straightforward. At least, this is a convenient tool at the theoretical level, because of the difficulties inherent to real variable based inversion methods.

We have shown the advantages of incorporating $F_0(a)$ to the inversion problem: (i) this function allows an asymptotic method which gives an alternative to the delicate task of Laplace inversion and (ii) it is also a key for correcting Laplace inversion methods when λ is a constant. As forthcoming research, it would be advisable to extend this concept to situations in which λ is depth dependent and one is concerned with the unavoidable range of small distances between the tip and the superconductor. This region cannot be neglected as it provides the higher experimental resolution. In particular, recall that Laplace transform inversion is intrinsically unstable and tiny uncertainties in $F(a)$ may result in an important degradation of $\mathcal{L}^{-1}[F(a)](k)$.

The results of this work can be expeditiously applied to the determination of the temperature dependence of the penetration depth in superconductors by means of low temperature MFM. λ is assumed to be constant along the thickness of the sample s, which has been considered infinite. In fact, this is a very good approximation, excepting very thin films in which $s<\lambda$.[17] On the other hand, lateral variations can be detected over distances below 1 μm. The infinite superconductor assumption of our calculations relies thus on having homogeneous areas well over 1 μm^2.

Finally, we want to remark that the recovery of $\lambda(T)$ is of great interest as it allows us to get insight into the basic physics of high-T_c superconductors. This quantity provides useful information on the pairing wave function. Consensus has not yet developed about the nature of the superconducting state in these compounds. Allegedly, an inversion theory such as the one presented here, which allows unambiguous recovery of λ from physical observables should be at our disposal.

Note added in proof. We recently became aware of a paper[18] that should be mentioned here for completeness.

ACKNOWLEDGMENT

The author wants to acknowledge partial financial support from CICYT under Project No. MAT95-0921-C02-02.

*Electronic address: anabadia@posta.unizar.es

[1] Y. Martin and H.K. Wickramasinghe, Appl. Phys. Lett. **50**, 1455 (1987).
[2] H.J. Hug *et al.*, Physica C **175**, 357 (1991); H.J. Reittu and R. Laiho, Supercond. Sci. Technol. **5**, 448 (1992); A. Wadas *et al.*, Z. Phys. B **88**, 317 (1992); H.J. Hug *et al.*, Physica C **235-240**, 2695 (1994); A. Moser *et al.*, J. Vac. Sci. Technol. B **12**, 1586 (1994); A. Moser *et al.*, Phys. Rev. Lett. **74**, 1847 (1995).
[3] A. Volodin *et al.*, Appl. Phys. Lett. **73**, 1134 (1998).
[4] L. Kong and S.Y. Chou, Appl. Phys. Lett. **70**, 2043 (1998).
[5] S. Hosaka *et al.*, Appl. Phys. Lett. **65**, 3407 (1994).
[6] C.D. Wright and E.W. Hill, Appl. Phys. Lett. **67**, 433 (1995); **68**, 1726 (1996).
[7] M.W. Coffey, J. Supercond. **10**, 567 (1997).
[8] M.W. Coffey, Phys. Rev. B **57**, 11 648 (1998).
[9] J.H. Xu, J.H. Miller, Jr., and C.S. Ting, Phys. Rev. B **51**, 424 (1995); **55**, 11 877(E) (1997).
[10] M. Tinkham, *Introduction to Superconductivity*, 2nd ed. (McGraw-Hill, New York, 1996).
[11] A. Badía, Phys. Rev. B **55**, 11 875 (1997).
[12] A. Badía and H.C. Freyhardt, J. Appl. Phys. **83**, 2681 (1998).
[13] J.D. Jackson, *Classical Electrodynamics*, 2nd ed. (Wiley, New York, 1975).
[14] G. Arfken, *Mathematical Methods for Physicists*, 3rd ed. (Academic, Boston, 1985).
[15] S. Wolfram, *The Mathematica book*, 3rd ed. (Cambridge University Press, Cambridge, 1996).
[16] Z.J. Yang, J. Supercond. **5**, 259 (1992).
[17] Z.J. Yang, Jpn. J. Appl. Phys., Part 2 **31**, L938 (1992).
[18] M.W. Coffey, Phys. Rev. Lett. **83**, 1648 (1999).