A GAUSSIAN PROCEDURE TO TEST FOR COINTEGRATION

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TESTING COINTEGRATION

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ABSTRACT

The paper is dedicated to deriving a gaussian procedure to test for cointegration. We consider four alternative specifications, depending on the form adopted by the deterministic terms. We then define the test statistic and derive its asymptotic behaviour under both the null and the alternative hypotheses. We show that, under the null hypothesis, the test procedure follows a Standard-Normal distribution. The Monte Carlo results confirm that the performance of the proposed test procedures is quite satisfactory.

Classification Code: C12, C15, C22

Keywords: integrated process; cointegration; Gaussian procedures; Monte Carlo experiments

1.- INTRODUCTION

Following Ericson and MacKinnon (2002), we can distinguish three general approaches for testing whether or not non-stationary economic time series are cointegrated: single-equation static regressions (Engle and Granger (1987)); vector autorregressions (Johansen (1988, 1995)); and single-equation conditional error correction models (Sargan (1964), Davidson et al. (1978) and Harbo et al. (1998)).

Most of these procedures adopt a testing framework in which a null hypothesis of less cointegration, that is a simple hypothesis, is tested against the alternative composite hypothesis of more cointegration. The result is that, under the null hypothesis, the testing procedure follows a non-standard probability distribution that requires ad-hoc procedures to determine the limits of the critical region. The paper by Ericson and MacKinnon (2002) is again the reference for more details on this point.

Against this background, the aim of this paper is to develop a procedure for testing cointegration which, under the null hypothesis, follows a Standard Normal distribution, so that the usual critical points can be used to formulate the critical region of the test. We propose to test a composite null hypothesis of more cointegration, against a simple alternative hypothesis of less cointegration.

The rest of the paper is organized as follows. In Section 2 we introduce the models and some preliminaries used in the subsequent sections. The testing procedures and their asymptotic properties for the different models are described in Section 3. The Monte-Carlo results are presented and commented on in Section 4. Finally, Section 5 closes the paper with a review of the main conclusions. The proofs of the results formulated in Section 3 are given in the Appendix.

2.- MODELS AND SOME PRELIMINARIES

Let y_t be an n-dimensional vector of time series variables and let us partition it as: $y'_t = \left(y_{1t} \quad y'_{2t}\right)$, where y_{1t} has *1* element and y'_{2t} has the remaining *n-1* elements. Assume that the Data Generating Process (DGP) for y'_t is the following:

$$y_{1t} = \delta_0 + \delta_1 t + \beta' y_{2t} + u_{1t} \tag{1}$$

$$y_{2t} = y_{2t-1} + u_{2t} \tag{2}$$

where δ_0 and δ_1 are parameters and β' is an *(n-1)* vector of cointegration parameters; further, we assume that:

$$A(L)u_t = \delta_2^* + \varepsilon_t \tag{3}$$

where $\delta_2^* = (0, \delta'_2)$ $u'_t = (u_{1t}, u'_{2t})$, with u'_{2t} having *(n-1)* elements, and with L denoting the lag operator, such that $L'z_t = z_{t-1}$, $A(L) = \sum A_i L^i$ with $A_0 = I_n$ and $A_i = \begin{pmatrix} A_{11i} & A_{12i} \\ A_{21i} & A_{22i} \end{pmatrix}_{n\times n}$

$i = 1,2,\ldots p$ with $A_{12p} = A_{22p} = 0$.

We additionally assume that the roots of $|A(L)| = 0$ are outside the unit circle. A final assumption is that $\varepsilon_t = \begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{pmatrix}$ are identically and independently distributed sequences of n dimension gaussian vectors with mean zero and covariance matrix:

$$\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{10} \\ \sigma_{20} & \Sigma_{22} \end{pmatrix} \tag{4}$$

where $\sigma_{10} = \sigma'_{20}$ is the *(n-1)* vector of covariances of u_{1t} with the elements of u_{2t}.

Rewrite (1) and (2) as:

$$H(L)y_t = \delta_0^* + \delta_1^* t + u_t \tag{5}$$

where $\delta_0^{*\prime} = (\delta_0 \quad 0'_{n-1})$, $\delta_1^{*\prime} = (\delta_1 \quad 0'_{n-1})$ and:

$$H(L) = \begin{bmatrix} 1 & -\beta' \\ 0 & I_{n-1} - I_{n-1}L \end{bmatrix}$$

After premultiplying (5) by A(L) and using (3), we obtain:

$$J(L)y_t = A(1)\delta_0^* + A(L)\delta_1^* t + \delta_2^* + \varepsilon_t \tag{6}$$

with $J(L)y_t = A(L)H(L) = J_0 + J_1 L + ... + J_i L^i + ...$ and where:

$$A(1)\delta_0^* = \begin{bmatrix} a_{11}(1) \\ \vdots \\ a_{i1}(1) \\ \vdots \\ a_{n1}(1) \end{bmatrix} \delta_0 \tag{7}$$

and

$$A(L)\delta_1^* t = \begin{bmatrix} \delta_1 a_{11}(1)t \\ \vdots \\ \delta_1 a_{i1}(1)t \\ \vdots \\ \delta_1 a_{n1}(1)t \end{bmatrix} - \begin{bmatrix} \delta_1 \sum_{j=1}^{p} j a_{11j} \\ \vdots \\ \delta_1 \sum_{j=1}^{p} j a_{i1j} \\ \vdots \\ \delta_1 \sum_{j=1}^{p} j a_{n1j} \end{bmatrix} = h_1 t + h_2 \tag{8}$$

since $a_{i1}(L)\delta_1 t = \delta_1 \left[1 + a_{i11}L + ... + a_{i1p}L^p\right]t = \delta_1 \left[t + a_{i11}(t-1) + ... + a_{i1p}(t-p)\right]$

Let J_0 be defined as:

$$J_0 = \begin{bmatrix} 1 & -\beta' \\ 0 & I_{n-1} \end{bmatrix}$$

If we premultiply (6) by J_0^{-1}, we obtain:

$$y_t = \delta^* + \lambda^* t + \Phi(L)y_{t-1} + v_t \tag{9}$$

where:

$$\delta^{*} = J_0^{-1}\left(A(1)\delta_0^{*} + \delta_2^{*} + h_2\right) \tag{10}$$

$$\lambda^{*} = J_0^{-1}h_1 \tag{11}$$

$$\Phi(L) = \sum_{i=1}^{p} \Phi_i L^{i-1}$$

with $\Phi_i = -J_0^{-1}J_i$ and $v_t = J_0^{-1}\varepsilon_t$. The error correction form (ECM) of the model can be written as:

$$\Delta y_t = \delta + \lambda t + \alpha(1 \quad -\beta')y_{t-1}^{+} + \sum \Gamma_i \Delta y_{t-i} + v_t \tag{12}$$

where:

$$y_t^{+} = \left(y_{1t} - \delta_0 - \delta_1 t,\; y_{2t}'\right) \tag{13}$$

with:

$$\alpha\left(1 \quad -\beta'\right) = I_n - \Phi(1) = J_0^{-1}A(1)H(1) = \begin{bmatrix} 1 & \beta' \\ 0 & I_{n-1} \end{bmatrix}\begin{bmatrix} a_{11}(1) & a_{1.}(1) \\ a_{2.}(1) & a_{22}(1) \end{bmatrix}\begin{bmatrix} 1 & -\beta' \\ 0 & 0 \end{bmatrix} =$$
$$\begin{bmatrix} a_{11}(1) + \beta' a_{2.}(1) & -a_{1.}(1)\beta' - \beta' a_{2.}(1)\beta' \\ a_{2.}(1) & -a_{2.}(1)\beta' \end{bmatrix} = \begin{bmatrix} a_{11}(1) + \beta' a_{2.}(1) \\ a_{2.}(1) \end{bmatrix}\begin{bmatrix} 1 & -\beta' \end{bmatrix} \tag{14}$$

and $\Gamma_i = \sum \Phi_j$ and where δ and λ are adjusted so that (9) follows from (8). In particular, we have:

$$\delta = \delta^{*} + \alpha\delta_0 \tag{15}$$

$$\lambda = \lambda^{*} + \alpha\delta_1 \tag{16}$$

Using this framework, we are going to distinguish four different testing cases with the term in the parenthesis being the notation used by Johansen (1995, Chapter 5) to denote them.

CASE 1: $\delta_0 = \delta_1 = \delta_2 = 0$. Variables around a constant mean; no constant and no linear trend in the cointegration relation (H_2(r)).

In this case, the model is that written in (12), with:

$$\delta = 0, \lambda = 0 \text{ and } y_t^+ = y_t$$

CASE 2: $\delta_0 \neq 0; \delta_1 = \delta_2 = 0$. Variables around a constant mean; a non-zero constant and no linear trend in the cointegration relation (H_1^*(r)).

$$\begin{aligned}
&\delta = J_0^{-1} A(1)\delta_0^* + \alpha\delta_0 \\
&\lambda = 0 \\
&y_t^{+\prime} = (y_{1t} - \delta_0 , y'_{2t})
\end{aligned}$$

CASE 3: $\delta_0 \neq 0; \delta_1 = 0 , \delta_2 \neq 0$. Variables around a cointegrated linear trend; a non-zero constant and no linear trend in the cointegration relation (H_1(r)).

In this case,

$$\begin{aligned}
&\delta = J_0^{-1}(A(1)\delta_0^* + \delta_2^*) + \alpha\delta_0 \\
&\lambda = 0 \\
&y_t^{+\prime} = (y_{1t} - \delta_0 , y'_{2t})
\end{aligned}$$

CASE 4: $\delta_0 \neq 0; \delta_1 \neq 0 , \delta_2 \neq 0$. Variables around a non-cointegrated linear trend; a non-zero constant and a linear trend in the cointegration relation (H^*(r)).

In this case, the model is the general model derived previously with y_t^+ as defined in (12) and δ and λ as defined in (15) and (16), respectively.

3.- TESTING PROCEDURES

In this section, we propose testing procedures for each of the four cases contemplated in the previous Section 2. The proof of the asymptotic properties of the proposed tests-statistics will be derived in the first case and then extended, without proof, to the other three cases.

CASE 1: $\delta_0 = \delta_1 = \delta_2 = 0$

Using the restrictions corresponding to this case, we rewrite (1) and the first relation of (12), respectively, as:

$$y_{1t} = \beta' y_{2t} + u_{1t} \tag{17}$$

$$\Delta y_{1t} = \alpha u_{1t-1} + \sum_{i=1}^{n} \sum_{j=1}^{p-1} \gamma_{1ij} \Delta y_{it-j} + v_{1t} \tag{18}$$

Let M_1 and M_2 be two models defined by the following sets of regressors:

$$\begin{array}{l} M_1 : \{u_{1t-1}, \Delta y_{it-j}\} \quad i = 1,...,n,\ j = 1,2,...,p-1 \\ M_2 : \{\Delta y_{it-j}\} \end{array}$$

Note that under the null hypothesis, u_{1t-1} is an I(0) process while, under the alternative, u_{1t-1} is an I(1) process. Thus, when we compare M_1 and M_2 , the first model has an I(0) variable as the first regressor when the data are generated by the null hypothesis while M_1 has an I(1) process as the first regressor when the data are generated by the alternative hypothesis.

In this paper, we develop a procedure to compare M_1 to M_2 such that the first regressor of M_1 is always an I(0) process no matter which hypothesis generates the data. To that end, the first regressor of M_1 is defined as a sum of two terms in such a way that when the data are generated by the null hypothesis the first regressor is dominated by the first term while when the data are generated by the alternative hypothesis, this first regressor is dominated by the second term.

Let $\hat{u}_1$ be the vector of OLS residuals from the estimation of (17). Then, we estimate (18) after substituting u_{1t-1} by $\hat{u}_{1t-1}$. Let $\hat{v}_1$ be the vector of OLS residuals from this model. Finally, consider the following model:

$$\Delta\hat{u}_{1t} = \phi\hat{u}_{1t-1} + \sum_{j=1}^{p-1}\lambda_j \Delta\hat{u}_{1t-j} + \varepsilon_t^* \tag{19}$$

Let $\hat{\phi}$ be the OLS estimation of ϕ in (19). Assume that we define $\hat{\phi}^* = g(\hat{\phi})$, in such a way that when the data are generated by the model under the null hypothesis, $g(\hat{\phi})$ behaves asymptotically as $\sqrt{T}$, while when the data are generated by the model under the alternative hypothesis, $g(\hat{\phi})$ converges to zero. This can be achieved because, as it is well known, under the null hypothesis, $\hat{\phi}$ converges to ϕ at a rate equal to $\sqrt{T}$, while under the alternative hypothesis, $\hat{\phi}$ converges to zero at a rate equal to T. After carrying out different simulation exercises, by using models similar to those employed in the Monte Carlo study presented in Section 4, we find that for sample sizes habitual in applied work –say T = 100 or T = 200- the estimates of $\hat{\phi}$ obtained are always negative and smaller than one in absolute value. Further, we find that, under the null hypothesis, the first decimal digit is different from zero while, under the alternative, most of the estimates are smaller than 0.05, in absolute value. Of course, there is a border-line area for values of the composite null hypothesis close to the value under the alternative hypothesis, for which this rule is not so clear. However this is always the case when one is testing a null hypothesis that is composite, against an alternative hypothesis that is simple.

On the basis of these results, we propose the following expression for g($\bullet$): $\hat{\phi}^* = (c\hat{\phi})^{2b}$, where, for c, we recommend a value around 20 and *b* is defined as: $b = \left|\dfrac{log(T)}{2\,log(c\hat{\phi})^2}\right|$ where | |

denotes absolute value. For these values of *b* and *c*, given the values taken by $\hat{\phi}$ it is clear that, under the null hypothesis, we obtain $\left|c\hat{\phi}\right| > 1$, while under the alternative hypothesis, we have that $\left|c\hat{\phi}\right| < 1$. Hence, defining $\hat{\phi}^*$ as $(c\hat{\phi})^{2b}$, $\hat{\phi}^*$ satisfies the limiting behaviour assumed for g($\bullet$).

Consider the two following sets of regressors:

$$x_t' = (\hat{\phi}^* \hat{u}_{1t-1} + u_{1t}^*, \Delta y_{1t-1}, ..., \Delta y_{1t-p+1}, \Delta y_{2t-1}', ..., \Delta y_{2t-p+1}') \tag{20}$$

$$z_t = (\hat{\phi}^* u_t^* + \Delta y_{1t}) \tag{21}$$

where u_t^* and u_{1t}^* are artificial generated variables with mean zero and variance σ^{*2}. Note that x_t has $n(p\text{-}1)+1$ elements and that z_t has only one. It can be seen that under the null hypothesis, the first regressor of (20) behaves asymptotically as the first term of the sum, whilst when the data are generated by the alternative, the first regressor behaves asymptotically following the pattern of the second term. The same can be said about the regressor written in (21).

If the null hypothesis holds, then note that when we project Δy_{1t} on the first set of regressors written in (20), the projection is on the same space spanned by the regressors under that hypothesis. On the other hand, when we project this increase on the second set of regressors in (21), if the data are generated by the null hypothesis, we are projecting this increase on a process without any structure, whilst if the data are generated by the alternative hypothesis we are projecting Δy_{1t} on itself.

Let x_1^+ be the same vector defined in (20) after dividing the first term of it by $\sqrt{T}$.

$$\Delta y_{1t} = x_t^+ \delta_1 (1-\lambda) + z_t \lambda + (1-\lambda) v_{1t} \tag{22}$$

where δ_1 is the vector of the parameters of the model in (18). Notice that, under the alternative hypothesis, for $\lambda = 0$, this model asymptotically becomes the model written in (18). The Ordinary Least Square estimator of λ can be written as:

$$\hat{\lambda} = \left(z'M_X z\right)^{-1} z'M_X \Delta y_1 \tag{23}$$

where z is the $T \times 1$ vector of observations of the variable defined in (21), $M_X = I - X(X'X)^{-1}X'$ with X being the $T \times [n(p\text{-}1)+1]$ matrix of observations of the $n(p\text{-}1)+1$ elements of the vector defined in (20) and Δy_1 is the $T \times 1$ vector of observations of Δy_{1t}. In the following theorem we derive the asymptotic properties of this estimator.

THEOREM 1: Assume that the DGP is given by (1) and (2) with $\delta_0 = \delta_1 = \delta_2 = 0$ and that the disturbances follow the same stochastic properties commented on in the previous section. Then:

$$T\hat{\lambda} \xrightarrow{d} N\left(\frac{p\lim \dfrac{v_1' v_1}{T}}{\sigma^{*2}}, \frac{\sigma^2}{\sigma^{*2}} \right) \tag{24}$$

PROOF: See the Appendix.

Thus, we propose the following statistic to test for cointegration:

$$J^* = \frac{z'M_X \Delta y_1 - \hat{v}_1'\hat{v}_1}{s(z'M_X z)^{1/2}} \tag{25}$$

where s^2 is the OLS estimated residual variance of the regression of Δy_{1t} on x_t^+ and z_t in (22).

The asymptotic properties of this statistic under both hypotheses are as follows:

THEOREM 2: under the null hypothesis that there is one cointegration relation, we have:

$$J^{*} \xrightarrow{d} N(0,1)$$

Proof: See the Appendix

THEOREM 3: Under he alternative hypothesis that there is no cointegration relation, it holds that:

$$J^{*} \longrightarrow \infty$$

Proof: See the Appendix

CASE 2: $\delta_0 \neq 0; \delta_1 = \delta_2 = 0$.

Here, we follow using the same testing procedure, but adopting it to the new restrictions. In particular, a constant is included in (17) and (18), while in (18) and (19) we use $\left(1 \quad -\hat{\beta}'\right)\hat{y}_{t-1}^{+}$ instead of $\hat{u}_{1t-1}$. This same change is introduced in (20), where we define x_t. Taking into account these modifications, we redefine the J-test written in (25) and the null hypothesis is rejected when the value of this statistic is over the critical point corresponding to a Standard Normal distribution once the nominal size is chosen.

CASE 3: $\delta_0 \neq 0; \delta_1 = 0, \delta_2 \neq 0$.

Since, in this case, the linear trends of the two variables are cointegrated, the testing procedure is the same as that commented on for Case 2.

CASE 4: $\delta_0 \neq 0; \delta_1 \neq 0, \delta_2 \neq 0$.

The modifications to be introduced in this case are as follows: a constant and a linear trend should be included in (17) and (18), while in (18) and (19) we use (13) to define u_{1t-1}. Introducing these changes in (25), the new J-test is obtained and the corresponding critical region is derived from the Standard Normal distribution.

By using the triangular form of the system of Phillips (1991) and Phillips and Loretan (1991), the extension of the testing procedure just commented to situations whith more than one cointegration relation is straightforward.

4. MONTE CARLO STUDY

This section is dedicated to presenting the results from a Monte Carlo simulation study. Considering a model with two variables, we analyze the behaviour of the test proposed in this paper for the four cases introduced in the previous sections.

Two broad approaches have been followed in the literature in order to evaluate the performance of different cointegration testing procedures using Monte Carlo analysis. The first, based on a transformation of the model into a “canonical form”, can be seen, for example, in Toda (1994,1995) and in Hubrich et al. (2001); the second, with a DGP close to what can be regarded as a “structural form”, has been used in Haug (1996), who follows the same framework used in Gonzalo (1994), among others. In this section, we assume a model close to this second approach.

In the study, we test for the null hypothesis that the rank of the cointegration space is one, as against the alternative hypothesis that the rank is zero, i. e., H_0: $r = 1$ against H_1: $r = 0$. We assume that the data are generated by the following model:

$$y_{1t} = \delta_0 + \delta_1 t + \beta y_{2t} + u_{1t} \tag{26}$$

$$\Delta y_{2t} = u_{2t} \tag{27}$$

with

$$u_{1t} = \rho_{11} u_{1t-1} + \rho_{12} u_{1t-2} + \varepsilon_{1t} \tag{28}$$

$$u_{2t} = \delta_2 + \rho_2 u_{2t-1} + \varepsilon_{2t} \tag{29}$$

and $\varepsilon_t = \begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{pmatrix}$ iid ~N(0, Σ), with $\Sigma = \begin{pmatrix} \sigma_1^2 & \rho_0\sigma_1\sigma_2 \\ & \sigma_2^2 \end{pmatrix}$

Note that if we define $w_{1t} = u_{1t}$ and $w_{2t} = u_{2t}$, this model is the same as that in Toda (1994, 1995) and Hubrich et al. (2001).

The parameter values are as follows:

$$
\begin{aligned}
&\delta_0 = 0, 1, 5 \\
&\delta_1 = 0, 1, 5 \\
&\delta_2 = 0, 1, 5 \\
&\beta = 1, 5 \\
&\rho_{11} = 0.5 \\
&\rho_{12} = 0, 0.2, 0.5 \\
&\rho_2 = 0.1, 0.3, 0.5 \\
&\sigma_1^2 = 1, 5 \\
&\sigma_2^2 = 1, 5 \\
&\rho_0 = -0.9, -0.5, 0, 0.5, 0.9 \\
&T = 100, 200, 500
\end{aligned}
$$

All programmes have been written in GAUSS and all simulations are performed on a PENTIUM-III PC. For each Monte Carlo simulation we generate 10,000 series of length T+20. As in Haug, we start at $y_{1,0} = 0$, $y_{2,0} = 0$, $u_{1,0} = 0$ and $u_{2,0} = 0$, and then discard the first 20 observations to eliminate start-up effects. The RNDN function in GAUSS with a fixed seed has been used in order to generate the pseudo-normal variates $\varepsilon_{1,t}$ and $\varepsilon_{2,t}$ and u_{1t}^* and u_t^*. We have generated u_{1t}^* as an $N(0, \sigma_1^{*2})$ with $\sigma_1^{*2} = 1, 5$, with the results being robust with respect to those two values.

Some of the results from CASE 1 are presented in Table I, corresponding to three sample sizes, (T = 100, 200 and 500). For each sample size and a combination of values of the parameters on the left hand side, the values in this table give the probability of rejection of the null hypothesis that the cointegrating rank is 1. Notice that for values of $(\rho_{11} + \rho_{12})$

smaller than one, the values in the table give the empirical size, while for $(\rho_{11} + \rho_{12}) = 1$, these values give account of the power. A 5% nominal size has been used in all cases.

TABLE I

From an examination of the values, we can appreciate that the empirical size is close to the nominal size, even for small sample sizes. The power, although not very high in small samples, increases as expected, as the sample size increases. Moreover, the results are robust with respect to changes of ρ_0, ρ_2 and σ_i^2, i = 1, 2.

The results for CASE 2 are presented in Table II. As in the previous case, they make clear that the empirical size is close to the nominal 5% size in almost every case. Further, they appear to cofirm that when $(\rho_{11} + \rho_{12}) = 1$, the power is rather low for small sample sizes. However, the power increases as the sample size grows and approach unity when T = 500.

TABLE II

The same conclusions are derived from Table III, CASE 3, and Table IV, CASE 4: empirical size close to the 5% nominal size and low power for samples around T = 100; however, this power increases as the sample size grows, although at a slower rate than in the two previous cases:

TABLE III

TABLE IV

5.- CONCLUSIONS

This paper has been dedicated to deriving a test procedure for testing cointegration that, under the null hypothesis, follows a Standard Normal distribution. To that end, we have considered four alternative specifications, depending on the form adopted by the deterministic terms.

The test procedure has been based on the comparison of the form adopted by a relation of the Error Correction Form, under the null hypothesis that there is one coitegration relation, with respect to the form adopted by that relation when it is assumed that there is no cointegraion. Since the form corresponding to the case with cointegration has one regressor that behaves as an I(0) process under the null hypothesis, and as an I(1) process under the alternative hypothesis, we have defined a new set of regressors in which one of them is defined in terms of the sum of the two elements in such a way that the I(0) character of the regressor is maintained no matter which hypothesis generates the data.

By using these new set of regressors, we have derived the cointegration testing procedure in Section 3 and it has been shown that the procedure, under the null hypothesis, follows a standard Normal distribution.

In the closing section of the paper, we have considered the results from some Monte-Carlo simulations. For a wide range of values of the parameters of the model, it has been shown that the proposed test has a good performance, both in terms of how close the empirical size is to the nominal size, and in terms of the high power values.

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APPENDIX

First, some preliminary results, useful when proving the three theorems, are collected in the following lemma.

LEMMA1: Let the DGP be (1)-(2) with $\delta_0 = 0, \delta_1 = 0, \delta_2 = 0$. Then:

(i) $\hat{u}_{1t} \xrightarrow{p} u_{1t}$

(ii) $H_1^{-1} X'X H_2^{-1} \xrightarrow{p} Q = \begin{pmatrix} q_{00} & q_{01} \\ q_{10} & Q_{11} \end{pmatrix}$

(iii) $H_2^{-1} X'X_1^* \xrightarrow{p} Q$

(iv) $H_2^{-1} X'z$ is $O_p(1)$

(v) $M_x z \xrightarrow{p} z + O_p(1)$

(vi) $\frac{1}{T^2} z' M_X z \xrightarrow{p} \sigma^{*2}$

where H_1 and H_2 are, respectively:

$$H_1 = \begin{pmatrix} T^{1/2} & 0 \\ 0 & I_{n(p-1)} \end{pmatrix} \text{ and } H_2 = \begin{pmatrix} T^{3/2} & 0 \\ 0 & TI_{n(p-1)} \end{pmatrix}$$

and where Q is a square matrix of constants of order $n(p\text{-}1)+1$, X_1^* is the matrix of observations of the regressors in (18) and δ_1 is the vector of $n(p\text{-}1)+1$ parameters of these regressors.

PROOF:

(i) Write $\hat{u}_{1t}$ as:

$$\hat{u}_{1t} = y_{1t} - \hat{\beta} y_{2t} = u_{1t} - (\hat{\beta} - \beta) y_{2t}$$

where $\hat{\beta}$ is the Ordinary Least Square estimator of β: $\hat{\beta} = \frac{\sum y_{1t} y_{2t}}{\sum y_{2t}}$. Since, when the variables are cointegrated, $(\hat{\beta}-\beta)$ is $O_p(T^{-1})$, then the result follows.

(ii) Let X be partitioned as $X = (x_0 \quad X_1)$, where x_0 is the $T \times 1$ vector of observations of the first element of x_t and X_1 is the $T \times n(p-1)$ matrix of observations of the rest of elements of x_t. The term on the left hand side of (ii) can be written as:

$$H_1^{-1} X'XH_2^{-1} = \begin{pmatrix} \frac{x_0' x_0}{T^2} & \frac{x_0' X_1}{T^{3/2}} \\ & \frac{X_1' X_1}{T} \end{pmatrix} \tag{A.1}$$

The convergence of the (1,1) element of this matrix is:

$$\frac{x_0' x_0}{T^2} = \frac{1}{T^2} \hat{\phi}^{*2} \sum \hat{u}_{1t-1}^2 + \frac{1}{T^2} \sum u_{1t}^{*2} + \frac{1}{T^2} 2\hat{\phi}^* \sum \hat{u}_{1t-1} u_{1t}^*$$

and, since, as we have already stated, $\hat{\phi}^{*2} \approx T$ and $\hat{\phi}^* \approx \sqrt{T}$ (asymptotically), by using (i) we have that:

$$\frac{x_0' x_0}{T^2} \xrightarrow{p} \gamma_1(0) = q_{00} \tag{A.2}$$

which is the variance of u_{1t}.

The asymptotic convergence of the i-th generic element of $\frac{x_0' X_1}{T^{3/2}}$ is, for $i = 1, 2, ..., n$ and $j = 1,...,p-1$:

$$\frac{1}{T^{3/2}}\sum x_{0t}\Delta y_{it-j} = \frac{1}{T^{3/2}}\hat{\phi}^*\sum \hat{u}_{1t-1}\Delta y_{it-j} + \frac{1}{T^{3/2}}\sum u_t^*\Delta y_{it-j} \xrightarrow{p} \frac{\sum u_{1t-1}\Delta y_{it-j}}{T} \xrightarrow{p} \gamma_{u_1,\Delta y_i}(j-1) \tag{A.3}$$

With respect to the convergence of the ij-th element of $\frac{X_1' X_1}{T}$, we have that:

$$\frac{\sum \Delta y_{it-j}\Delta y_{lt-k}}{T} \xrightarrow{p} \gamma_{\Delta y_i,\Delta y_l}(k-j) \tag{A.4}$$

(iii) The left hand side term of this expression can be written as:

$$H_2^{-1}X'X_1^* = \begin{bmatrix} \frac{x_0' u_{1,-1}}{T^{3/2}} & \frac{x_0'\Delta Y}{T^{3/2}} \\ .... & \frac{X_1'\Delta Y}{T} \end{bmatrix} \tag{A.5}$$

The convergence of $\frac{x_0' u_{1,-1}}{T^{3/2}}$ is as follows:

$$\frac{1}{T^{3/2}}\left(\hat{\phi}^*\sum \hat{u}_{1t-1}u_{1t-1} + \frac{1}{T^{3/2}}\sum u_{1t}^* u_{1t-1}\right) \xrightarrow{p} q_{00} \tag{A.6}$$

using (i) and the fact that $\hat{\phi}^* \approx \sqrt{T}$.

The convergence of $\frac{x_0'\Delta Y}{T^{3/2}}$ and $\frac{X_1'\Delta Y}{T}$ is straightforward, because $X_1' = \Delta Y$.

(iv) We have that:

$$H_2^{-1}X'z = H_2^{-1}\hat{\phi}^* X'u^* + H_2^{-1}X'\Delta y_1 \tag{A.7}$$

Note that

$$H_2^{-1}X'\Delta y_1 = H_2^{-1}X'(X_1^*\delta_1 + v_1) = H_2^{-1}X'X_1^*\delta_1 + H_2^{-1}X'v_1 \tag{A.8}$$

Using (iii), the first term on the right hand side converges to $Q\delta_1$. The second term can be written as:

$$H_2^{-1}X'v_1 = \begin{pmatrix} T^{-3/2}\hat{\phi}^* \sum \hat{u}_{1t-1}v_{1t} \\ T^{-1}X_1'V_1 \end{pmatrix} \xrightarrow{p} 0 \qquad \text{(A.9)}$$

With respect to the first term of the right hand side of (A.7), we have:

$$H_2^{-1}\hat{\phi}^* X'u^* = \begin{pmatrix} T^{-3/2}\hat{\phi}^{*2} \sum \hat{u}_{1t-1}u_t^* \\ T^{-1}\hat{\phi}^* \Delta Y u^* \end{pmatrix} \xrightarrow{p} 0 \qquad \text{(A.10)}$$

Since, by (i), $\hat{u}_{1t-1} \xrightarrow{p} u_{1t-1}$ and $\hat{\phi}^{*2}$ asymptotically behave as T, the first term of (A.10) converges to the same limit as $\frac{1}{\sqrt{T}}\sum u_{1t-1}u_t^*$. Further, since this expression is an scalar martingale difference sequence, by using Proposition 7.8 in Hamilton (1994) we obtain:

$$T^{-3/2}\hat{\phi}^{*2}\sum \hat{u}_{1t-1}u_t^* \xrightarrow{d} N\left(0, E(u_{1t-1}u_t^*)^2\right) \qquad \text{(A.11)}$$

The same line of reasoning can be used for the asymptotic distribution of the lower *n(p-1)* elements of (A.9).

Using the results about the convergence of (A.8) and (A.10), the proof follows.

(v) We can write:

$$M_X z = z - X(X'X)^{-1}X'z = z - XH_1^{-1}\left(H_1^{-1}X'XH_2^{-1}\right)^{-1}H_2^{-1}X'z$$

Using (ii) and (iv) and the fact that:

$$XH_1^{-1} \xrightarrow{p} X_1^* \qquad \text{(A.12)}$$

the result again follows.

(vi) This last result follows in a straightforward manner from the previous result. By using (v) we have that:

$$\frac{1}{T^2} z' M_X z \xrightarrow{p} \frac{1}{T^2} z'z$$

and, by the form adopted by z, that:

$$\frac{1}{T^2} z'z = \frac{1}{T^2} \hat{\phi}^{*2} \sum u_t^{*2} + \frac{2}{T^2} \hat{\phi}^{*} \sum u_t^{*} \Delta y_{1t} + \frac{1}{T^2} \sum \Delta y_{1t}^2 \xrightarrow{p} \frac{\sum u_t^{*2}}{T} \xrightarrow{p} \sigma^{*2} \qquad \text{(A.13)}$$

PROOF OF THEOREM 1

The estimator of λ in (23) can be written as:

$$\hat{\lambda} = \left(\frac{1}{T^2} z' M_X z \right)^{-1} \frac{1}{T^2} z' M_X \Delta y_1$$

Note that, by using (ii) and the results in (iv) and (A.12), we obtain:

$$M_X \Delta y_1 = \Delta y_1 - X H_1^{-1} \left(H_1^{-1} X' X H_2^{-1} \right)^{-1} H_2^{-1} X' \Delta y_1 \xrightarrow{p} \Delta y_1 - X_1^{*} Q^{-1} Q \delta_1 = v_1 \qquad \text{(A.14)}$$

Hence, $\frac{1}{T} z' M_X \Delta y_1$ converges to the limit of:

$$\frac{1}{T} z' v_1 = \frac{\hat{\phi}^{*} \sum u_t^{*} v_{1t}}{T} + \frac{\sum \Delta y_{1t} v_{1t}}{T} \longrightarrow \frac{\sum u_t^{*} v_{1t}}{\sqrt{T}} + \frac{\sum v_{1t}^2}{T} \qquad \text{(A.15)}$$

Further, note that $\frac{1}{T} z' v_1$ converges to the limiting distribution of $\frac{\sum u_t^{*} v_{1t}}{\sqrt{T}}$ plus σ^2.

Since u_t^{*} and v_{1t} are independent with variances σ^{*2} and σ^2, respectively, the application of the central limit theorem permits us to conclude that:

$$\frac{\sum u_t^{*} v_{1t}}{\sqrt{T}} \xrightarrow{d} N\left(0, \sigma^2 \sigma^{*2}\right) \qquad \text{(A.16)}$$

Combining all these results, we obtain:

$$\frac{1}{T} z' M_X \Delta y_1 \xrightarrow{d} N\left(0, \sigma^2 \sigma^{*2}\right) + p\lim \frac{v_1' v_1}{T} \qquad \text{(A.17)}$$

By using (vi), the result is:

$$T\hat{\lambda} = \left(\frac{1}{T^2} z'M_X z \right)^{-1} \frac{1}{T} z'M_X \Delta y_1 \xrightarrow{d} N\left(\frac{p\lim \frac{v_1'v_1}{T}}{\sigma^{*2}}, \frac{\sigma^2}{\sigma^{*2}} \right) \qquad \text{(A.18)}$$

PROOF OF THEOREM 2:

We have:

$$J = \frac{z'M_X \Delta y_1 - \hat{v}_1'\hat{v}_1}{s(z'M_X z)^{1/2}} = \frac{\left(\frac{z'M_X z}{T^2} \right)^{-1} \frac{1}{T} z'M_x \Delta y_1 - \left(\frac{z'M_X z}{T^2} \right)^{-1} \frac{\hat{v}_1'\hat{v}_1}{T}}{\left(\frac{z'M_X z}{T^2} \right)^{-1} s\left(\frac{z'M_X z}{T^2} \right)^{1/2}} =$$

$$= \frac{T\hat{\lambda} - \left(\frac{z'M_X z}{T^2} \right)^{-1} \frac{\hat{v}_1'\hat{v}_1}{T}}{\frac{s}{\left(\frac{z'M_X z}{T^2} \right)^{1/2}}} \xrightarrow{d} N(0,1)$$

By using (vi) and the fact that, under the null hypothesis, $\frac{\hat{v}_1'\hat{v}_1}{T} \xrightarrow{p} \frac{v_1'v_1}{T} \xrightarrow{p} \sigma^2$ and $s^2 \xrightarrow{p} \sigma^2$, the proof follows.

PROOF OF THEOREM 3

Here, the proof follows because, under the alternative hypothesis, $\frac{1}{T} z'M_X \Delta y_1$ and $\frac{\hat{v}_1'\hat{v}_1}{T}$ converge to different limits, and furthermore, $s^2 \xrightarrow{p} 0$ because Δy_{1t} is projected on itself.

TABLE I: CASE 1: $\delta_0 = \delta_1 = \delta_2 = 0$

SIZE AND POWER OF THE TEST WHEN THE COINTEGRATION RANK IS $r = 1$ ($\rho_{12} = 0, 0.2$) OR $r = 0$ ($\rho_{12} = 0.5$); ($\rho_{11} = 0.5, \beta = 1, \sigma_1^2 = 5$ AND $\sigma_2^2 = 1$)

ρ_{12}	ρ_2	ρ_0	**T = 100**	**T = 200**	**T = 500**
0	0.1	-0.9	0.117	0.052	0.052
		0	0.080	0.053	0.054
		0.9	0.065	0.049	0.053
	0.3	-0.9	0.112	0.053	0.053
		0	0.080	0.053	0.054
		0.9	0.064	0.049	0.053
0.2	0.1	-0.9	0.056	0.051	0.052
		0	0.055	0.050	0.055
		0.9	0.057	0.048	0.053
	0.3	-0.9	0.056	0.051	0.053
		0	0.055	0.050	0.054
		0.9	0.056	0.049	0.053
0.5	0.1	-0.9	**0.289**	**0.489**	**0.804**
		0	**0.438**	**0.693**	**0.886**
		0.9	**0.282**	**0.494**	**0.806**
	0.3	-0.9	**0.245**	**0.420**	**0.738**
		0	**0.455**	**0.708**	**0.881**
		0.9	**0.245**	**0.426**	**0.734**

TABLE II: CASE 2: $\delta_0 \neq 0$**;** $\delta_1 = \delta_2 = 0$.

SIZE AND POWER OF THE TEST WHEN THE COINTEGRATION RANK IS $r = 1$ ($\rho_{12} = 0.$ 0.2) OR $r = 0$ ($\rho_{12} = 0.5$); ($\rho_{11} = 0.5, \beta = 1, \sigma_1^2 = 5$ AND $\sigma_2^2 = 1$) ($\delta_0 = 1$)

ρ_{12}	ρ_2	ρ_0	**T = 100**	**T = 200**	**T = 500**
0	0.1	-0.9	0.081	0.052	0.052
		0	0.075	0.050	0.048
		0.9	0.058	0.050	0.049
	0.3	-0.9	0.061	0.052	0.052
		0	0.061	0.050	0.047
		0.9	0.058	0.050	0.048
0.2	0.1	-0.9	0.054	0.050	0.052
		0	0.058	0.050	0.048
		0.9	0.056	0.049	0.048
	0.3	-0.9	0.054	0.051	0.052
		0	0.056	0.050	0.048
		0.9	0.055	0.048	0.049
0.5	0.1	-0.9	**0.205**	**0.469**	**0.897**
		0	**0.195**	**0.465**	**0.901**
		0.9	**0.198**	**0.459**	**0.899**
	0.3	-0.9	**0.193**	**0.468**	**0.895**
		0	**0.197**	**0.466**	**0.900**
		0.9	**0.195**	**0.457**	**0.902**

TABLE III: CASE 3: $\delta_0 = 0; \delta_1 \neq 0 , \delta_2 \neq 0$

SIZE AND POWER OF THE TEST WHEN THE COINTEGRATION RANK IS

$r = 1$ ($\rho_{12} =0.\ 0.2$) OR $r = 0$ ($\rho_{12} = 0.5$); ($\rho_{11} = 0.5, \beta = 1, \sigma_1^2 = 5$ AND $\sigma_2^2 = 1$) ($\delta_1 = 1, \delta_2 = 1$)

ρ_{12}	ρ_2	ρ_0	**T = 100**	**T = 200**	**T = 500**
0	0.1	-0.9	0.085	0.054	0.054
		0	0.080	0.053	0.048
		0.9	0.062	0.050	0.048
	0.3	-0.9	0.061	0.053	0.053
		0	0.063	0.052	0.048
		0.9	0.061	0.049	0.049
0.2	0.1	-0.9	0.055	0.054	0.054
		0	0.059	0.052	0.048
		0.9	0.059	0.049	0.049
	0.3	-0.9	0.055	0.051	0.052
		0	0.057	0.051	0.048
		0.9	0.058	0.050	0.049
0.5	0.1	-0.9	**0.219**	**0.472**	**0.883**
		0	**0.194**	**0.453**	**0.886**
		0.9	**0.208**	**0.462**	**0.882**
	0.3	-0.9	**0.198**	**0.466**	**0.884**
		0	**0.202**	**0.458**	**0.887**
		0.9	**0.198**	**0.448**	**0.881**

TABLE IV: CASE 4: $\delta_0 \neq 0; \delta_1 \neq 0 \ , \delta_2 \neq 0$

SIZE AND POWER OF THE TEST WHEN THE COINTEGRATION RANK IS

$r = 1$ ($\rho_{12} = 0.\ 0.2$) OR $r = 0$ ($\rho_{12} = 0.5$); ($\rho_{11} = 0.5, \beta = 1, \sigma_1^2 = 5$ AND $\sigma_2^2 = 1$)

$(\delta_0 = 1, \delta_1 = 1, \delta_2 = 1)$

ρ_{12}	ρ_2	ρ_0	**T = 100**	**T = 200**	**T = 500**
0	0.1	-0.9	0.069	0.056	0.054
		0	0.082	0.053	0.048
		0.9	0.059	0.049	0.048
	0.3	-0.9	0.060	0.053	0.053
		0	0.066	0.052	0.048
		0.9	0.060	0.049	0.048
0.2	0.1	-0.9	0.056	0.056	0.054
		0	0.060	0.053	0.048
		0.9	0.057	0.049	0.048
	0.3	-0.9	0.055	0.052	0.053
		0	0.057	0.052	0.048
		0.9	0.057	0.049	0.049
0.5	0.1	-0.9	**0.123**	**0.270**	**0.788**
		0	**0.089**	**0.242**	**0.793**
		0.9	**0.091**	**0.247**	**0.790**
	0.3	-0.9	**0.095**	**0.254**	**0.787**
		0	**0.090**	**0.242**	**0.793**
		0.9	**0.090**	**0.241**	**0.786**