Física Teórica, *Julio Abad*, 1–7 (2008)

The slicing problem for hyperplane projections of B_p^n

David Alonso[1*], Jesús Bastero[1†], Julio Bernués[1‡] and Pawel Wolff[2§]

[1] Departamento de Matemáticas, Facultad de Ciencias, Universidad de Zaragoza, 50009 Zaragoza, Spain

[2] Department of Mathematics, Case Western Reserve University, Cleveland, Ohio 44106, USA

Abstract. We prove that for every hyperplane $H \subset \mathbb{R}^n$ and $1 \leq p \leq \infty$, the isotropy constant of the projection $P_H(B_p^n)$ is bounded by a universal constant.

Keywords: Slicing problem, hyperplane projections.

2000 MS classification: 52A40, 46B20, 52A39.

1. Introduction

A major problem in Asymptotic Geometric Analysis (the branch of modern Functional Analysis, coming from the interaction between local theory of Banach spaces, classical convex geometry and probability) is the so called Slicing Problem or Hyperplane Conjecture.

The statement has a number of equivalent forms. A well known formulation has its roots in classical mechanics. It is based on the fact that for every convex body K (compact, convex set in $\mathbb{R}^n$ with non empty interior) there exists a unique ellipsoid $\mathcal{L}(K)$ which has the same moments of inertia as K with respect to every axis (the so called *Legendre ellipsoid* of K, see [9]). In this framework the problem can be formulated as *does there exist a universal constant $c > 0$ such that for all $n \in \mathbb{N}$ and all convex bodies $K \subset \mathbb{R}^n$ of volume 1 one has that the volume of $\mathcal{L}(K)$ is bounded from above by c?*

Nowadays, we are very far from solving the Slicing Problem. The best general upper bound is $cn^{1/2}$ (due to Klartag, see [6]) improving an earlier bound by Bourgain $cn^{1/2} \log^2 n$ ([2], [13]). On the other hand, the Slicing Problem is known to have a positive answer for many classes of convex bodies (see for instance [9]).

For K of volume 1, the volume of $\mathcal{L}(K)$ is equivalent, up to an absolute constant, to L_K^2, the isotropy constant associated to K (see definition below). So it is more common to present the Slicing Problem as *does there exist a universal constant $c > 0$ such that for all $n \in \mathbb{N}$ and all convex bodies $K \subset \mathbb{R}^n$ one has that L_K is bounded from above by c?* (The name *Slicing Problem* arises from geometry and is due to another equivalent version *is there an absolute*

* daalonso@unizar.es

† bastero@unizar.es

‡ bernues@unizar.es

§ pwolff@mimuw.edu.pl

constant $c > 0$ such that for every dimension n, every convex body K in $\mathbb{R}^n$ of volume 1 has a hyperplane section (slice) with volume greater than c?)

Consider the unit ball of the ℓ_p^n space, $1 \leq p \leq \infty$, denoted by B_p^n. For p finite this is $B_p^n = \{x \in \mathbb{R}^n \mid \sum_{i=1}^n |x_i|^p \leq 1\}$. M. Junge [4] proved that the isotropy constants of all orthogonal projections (of any dimension) of B_p^n, $1 < p \leq \infty$ are bounded by $cp\prime$, (p' is the conjugate exponent of p, i.e. $\frac{1}{p} + \frac{1}{p'} = 1$). See also [8] for a different proof of this fact and [5] for further generalizations. Notice that, as p approaches 1, the constant $cp\prime$ explodes.

Our aim is to improve this bound, for hyperplane projections of B_p^n, to a numerical constant independent of p (Theorem 2.1) and moreover extend it to the limiting case $p = 1$ (Theorem 2.2).

We will use the following notation: m_n (resp. m_{n-1}) will denote the n dimensional (resp. $n-1$ dimensional) Hausdorff measure on R^n. We choose the normalization so that m_n coincides with the Lebesgue measure on R^n. The measure of a set will be denoted by $|\cdot|_n$ (resp. $|\cdot|_{n-1}$). For a hyperplane $H \subset \mathbb{R}^n$, P_H is the orthogonal projection onto H. The letters C, c, c'... will denote numerical constants whose value may change from line to line. The rest of the notation is standard as it appears in [12].

2. The results

The starting point in the proof of the theorems is the following formula, which serves us as a definition for the isotropy constant of a convex body K, see [9]:

$$nL_K^2 = \inf\left\{\frac{1}{|K|_n^{2/n}} \cdot \frac{1}{|K|_n} \int_{a+TK} |x|^2 \, dm_n(x);\ a \in \mathbb{R}^n, T \in GL(n), \det(T) = 1\right\} \tag{1}$$

Theorem 2.1. *There exists a universal constant $c > 0$ such that for every hyperplane H of $\mathbb{R}^n$ and $1 < p \leq \infty$,*

$$L_{P_H(B_p^n)} \leq c$$

Proof. The case $2 \leq p \leq \infty$ follows by the result of M. Junge [4] since $p\prime \leq 2$. We will assume $1 < p \leq 2$. Let $H = \{x \in \mathbb{R}^n; \langle x, \theta\rangle = 0\}$ with $\theta \in S^{n-1}$.

In view of the representation (1) we have

$$(n-1)L_{P_H(B_p^n)}^2 \leq \frac{1}{|P_H(B_p^n)|_{n-1}^{2/(n-1)}} \frac{1}{|P_H(B_p^n)|_{n-1}} \int_{P_H(B_p^n)} |x|^2 dm_{n-1}(x)$$

Thus, we need to estimate $|P_H(B_p^n)|_{n-1}$ from below and the remaining integral from above. The following inequality is well known $|B_p^n|_n \leq |P_H(B_p^n)|_{n-1}\ |\langle\theta\rangle \cap B_p^n|_1$ (see Lemma 8.8 in [12]) where $|\langle\theta\rangle \cap B_p^n|_1$ is a segment length. Since $p \leq 2$, clearly $|\langle\theta\rangle \cap B_p^n|_1 \leq 1$ and so

$$|B_p^n|_n \leq |P_H(B_p^n)|_{n-1}$$

Thus, by the well known formula $|B_p^n|_n = \dfrac{2^n(\Gamma(1+1/p))^n}{\Gamma(1+n/p)}$ and Stirling's estimate we have

$$|P_H(B_p^n)|_{n-1}^{2/(n-1)} \geq |B_p^n|_n^{2/(n-1)} \geq \frac{c}{n^{2/p}}$$

It remains to bound $\frac{1}{|P_H(B_p^n)|_{n-1}}\int_{P_H(B_p^n)}|x|^2 dm_{n-1}(x)$ from above. For that matter, we will use the method developed in [3]. We will denote by σ_p^n the normalized area measure on ∂B_p^n and by μ_p^n the cone probability measure on ∂B_p^n, defined by

$$\mu_p^n(A)=\frac{|\{ta; a\in A, 0\le t\le 1\}|_n}{|B_p^n|_n}$$

whenever $A\subseteq\partial B_p^n$. It was proved in [11] that

$$\frac{d\sigma_p^n}{d\mu_p^n}(x)=\frac{n|B_p^n|_n}{|\partial B_p^n|_{n-1}}\left|\triangledown(\|\cdot\|_p)(x)\right|$$

for almost every point $x\in\partial B_p^n$, where $\triangledown$ denotes the gradient (of the norm). For any (say) bounded measurable function f defined on $P_H(B_p^n)$ we have, by Cauchy's formula

$$\begin{aligned}\int_{P_H(B_p^n)} f(x)dm_{n-1}(x) &= \frac{1}{2}|\partial B_p^n|_{n-1}\int_{\partial B_p^n} f(P_H(y))|\langle N(y),\theta\rangle|d\sigma_p^n(y)\\ &\quad (N(y)\text{ is the unit normal vector to }\partial B_p^n)\\ &= \frac{n}{2}|B_p^n|_n\int_{\partial B_p^n} f(P_H(y))|\langle\triangledown(\|\cdot\|_p)(y),\theta\rangle|d\mu_p^n(y)\\ &= \frac{n}{2}|B_p^n|_n\int_{\partial B_p^n} f(P_H(y))\left|\sum_{i=1}^n |y_i|^{p-1}\operatorname{sgn} y_i\theta_i\right| d\mu_p^n(y)\end{aligned}$$

And in particular for $f(y)=|y|^2$,

$$\int_{P_H(B_p^n)}|x|^2dm_{n-1}(x)\le\frac{n}{2}|B_p^n|_n\int_{\partial B_p^n}|y|^2\left|\sum_{i=1}^n|y_i|^{p-1}\operatorname{sgn} y_i\theta_i\right|d\mu_p^n(y)$$

In order to compute this integral we use a concrete probabilistic description of μ_p^n (see, for instance, [15], [3], [11], [10]).

Let g be a random variable with density $e^{-|t|^p}/(2\Gamma(1+1/p))$, $t\in\mathbb{R}$. For $g_1,\dots,g_n$ i.i.d. copies of g we define $S=\left(\sum_{i=1}^n|g_i|^p\right)^{1/p}$.

Then the random vector $(g_1/S,\dots,g_n/S)\in\partial B_p^n$ is independent of S and is distributed on ∂B_p^n according to the cone measure μ_p^n (see [14] and [15]).

Hence, by applying Cauchy's formula with $f(y)=1$ and this representation

$$\begin{aligned}\frac{1}{|P_H(B_p^n)|_{n-1}}\int_{P_H(B_p^n)}|x|^2dm_{n-1}(x) &\le \frac{\mathbb{E}\sum_{i=1}^n\frac{|g_i|^2}{S^2}\left|\sum_{i=1}^n\frac{|g_i|^{p-1}}{S^{p-1}}\operatorname{sgn}(g_i)\theta_i\right|}{\mathbb{E}\left|\sum_{i=1}^n\frac{|g_i|^{p-1}}{S^{p-1}}\operatorname{sgn}(g_i)\theta_i\right|}\\ &= \text{(by independence)}\\ &= \frac{\mathbb{E}S^{p-1}}{\mathbb{E}S^{p+1}}\sum_{i=1}^n\frac{\mathbb{E}|g_i|^2\left|\sum_{i=1}^n|g_i|^{p-1}\operatorname{sgn}(g_i)\theta_i\right|}{\mathbb{E}\left|\sum_{i=1}^n|g_i|^{p-1}\operatorname{sgn}(g_i)\theta_i\right|}\end{aligned}$$

We estimate the first fraction,

$$\begin{aligned}\mathbb{E}S^{p-1} &= \mathbb{E}\left(\sum_{i=1}^n |g_i|^p\right)^{(p-1)/p} \leq \quad \text{(by Hölder's inequality)}\\ &\leq \left(\mathbb{E}\sum_{i=1}^n |g_i|^p\right)^{(p-1)/p} = \left(\frac{n}{p}\right)^{(p-1)/p}\end{aligned}$$

since $\mathbb{E}|g|^p = 1/p$ and

$$\begin{aligned}\mathbb{E}S^{p+1} &= \mathbb{E}\left(\sum_{i=1}^n |g_i|^p\right)^{(p+1)/p} \geq \quad \text{(by Hölder's inequality)}\\ &\geq \left(\mathbb{E}\sum_{i=1}^n |g_i|^p\right)^{(p+1)/p} = \left(\frac{n}{p}\right)^{(p+1)/p}\end{aligned}$$

therefore

$$\frac{\mathbb{E}S^{p-1}}{\mathbb{E}S^{p+1}} \leq \frac{p^{2/p}}{n^{2/p}} \leq \frac{c}{n^{2/p}}$$

On the other hand, let $(\varepsilon_1, \ldots, \varepsilon_n)$ a vector sequence of i.i.d random ± 1 signs, independent of $(g_1, \ldots, g_n)$. It is clear that for any value of $(\varepsilon_1, \ldots, \varepsilon_n)$

$$\mathbb{E}_g\left|\sum_{i=1}^n |\varepsilon_i g_i|^{p-1}\,\mathrm{sgn}(g_i\varepsilon_i)\theta_i\right| = \mathbb{E}\left|\sum_{i=1}^n |g_i|^{p-1}\,\mathrm{sgn}(g_i)\theta_i\right|,$$

so

$$\begin{aligned}\mathbb{E}\left|\sum_{i=1}^n |g_i|^{p-1}\,\mathrm{sgn}(g_i)\theta_i\right| &= \mathbb{E}_\varepsilon\mathbb{E}_g\left|\sum_{i=1}^n |\varepsilon_i g_i|^{p-1}\,\mathrm{sgn}(g_i\varepsilon_i)\theta_i\right|\\ &\geq \quad \text{(by Khinchine's inequality, see[12])}\\ &\geq C\,\mathbb{E}_g\left(\sum_{i=1}^n |g_i|^{2p-2}\theta_i^2\right)^{1/2}\\ &\geq \quad \text{(by Jensen's inequality)}\\ &\geq C\,\mathbb{E}\sum_{i=1}^n |g_i|^{p-1}\theta_i^2 = C\,\mathbb{E}|g|^{p-1} \geq C > 0\end{aligned}$$

whenever $1 \leq p \leq 2$ since $\sum_{i=1}^n \theta_i^2 = 1$.

With an analogous argument we have

$$\begin{aligned}
\mathbb{E}|g_1|^2\left|\sum_{i=1}^n |g_i|^{p-1}\operatorname{sgn}(g_i)\theta_i\right| &= \mathbb{E}_\varepsilon\mathbb{E}_g|g_1\varepsilon_1|^2\left|\sum_{i=1}^n |g_i\varepsilon_i|^{p-1}\operatorname{sgn}(g_i\varepsilon_i)\theta_i\right|\\
&= \mathbb{E}_g|g_1|^2\mathbb{E}_\varepsilon\left|\sum_{i=1}^n |g_i\varepsilon_i|^{p-1}\operatorname{sgn}(g_i\varepsilon_i)\theta_i\right|\\
&\geq \quad \text{(by Khinchine's inequality)}\\
&\leq C\ \mathbb{E}_g|g_1|^2\left(\sum_{i=1}^n |g_i|^{2p-2}\theta_i^2\right)^{1/2}\\
&\leq \quad \text{(by Jensen's inequality)}\\
&\leq C\left(\mathbb{E}|g_1|^4\sum_{i=1}^n |g_i|^{2p-2}\theta_i^2\right)^{1/2}\\
&= C\left(\sum_{i=1}^n \mathbb{E}|g_1|^4|g_i|^{2p-2}\theta_i^2\right)^{1/2} \geq C > 0
\end{aligned}$$

Therefore, $\displaystyle\sum_{i=1}^n \frac{\mathbb{E}|g_i|^2\left|\sum_{i=1}^n |g_i|^{p-1}\operatorname{sgn}(g_i)\theta_i\right|}{\mathbb{E}\left|\sum_{i=1}^n |g_i|^{p-1}\operatorname{sgn}(g_i)\theta_i\right|} \leq Cn.$

Collecting the estimates together we have thus proved

$$(n-1)L^2_{P_H(B_p^n)} \leq cn^{2/p}\frac{1}{|P_H(B_p^n)|_{n-1}}\int_{P_H(B_p^n)}|x|^2dm_{n-1}(x) \leq \frac{cn^{2/p}n}{c'n^{2/p}} = Cn$$

and the result follows. □

Theorem 2.2. *There exists a universal constant $c > 0$ such that for every hyperplane $H \subset \mathbb{R}^n$,*

$$L_{P_H(B_1^n)} \leq c$$

Proof. Let $H = \{x \in \mathbb{R}^n; \langle x, \theta\rangle = 0\}$ with $\theta \in S^{n-1}$. Let $\{F_i, i \in I\}$ be the faces of B_1^n and ν_i the normal vector of $F_i,\ i \in I$. First, suppose $\langle\theta, \nu_i\rangle \neq 0, \forall\ i \in I$.

From the representation (1) we have

$$(n-1)L^2_{P_H(B_1^n)} \leq \frac{1}{|P_H(B_1^n)|_{n-1}^{\frac{2}{n-1}}}\cdot\frac{1}{|P_H(B_1^n)|_{n-1}}\int_{P_H(B_1^n)}|x|^2\,dm_{n-1}(x)$$

As in the case $1 < p \leq 2$, we first need to estimate $|P_H(B_1^n)|_{n-1}$ from below. Lemma 8.8 in [12] states

$$|B_1^n|_n \leq |P_H(B_1^n)|_{n-1}\ |\langle\theta\rangle \cap B_1^n|_1 \leq |P_H(B_1^n)|_{n-1}$$

The equality $|B_1^n|_n = \frac{2^n}{n!}$ and Stirling's formula yield $|P_H(B_1^n)|_{n-1}^{\frac{1}{n-1}} \geq c/n$.

On the other hand, let us denote by I^+ (resp. I^-) the set of i's for which $\langle\theta, \nu_i\rangle > 0$ (resp. < 0). The following disjoint decomposition, up to a set of $(n-1)$-dimensional Hausdorff measure zero, holds

$$P_H(B_1^n) = \bigcup_{i\in I^+} P_H(F_i) = \bigcup_{i\in I^-} P_H(F_i)$$

and moreover, for every (say) bounded measurable function $f\colon P_H(B_1^n) \to \mathbb{R}$

$$\int_{P_H(B_1^n)} f(x)\, dm_{n-1}(x) = \frac{1}{2}\sum_{i\in I}\int_{P_H(F_i)} f(x)\, dm_{n-1}(x)$$

In particular,

$$2|P_H(B_1^n)|_{n-1} = \sum_{i\in I} |P_H(F_i)|_{n-1}$$

and

$$\frac{2}{|P_H(B_1^n)|_{n-1}}\int_{P_H(B_1^n)} |x|^2 dm_{n-1}(x) = \sum_{i\in I}\frac{1}{|P_H(B_1^n)|_{n-1}}\int_{P_H(F_i)} |x|^2 dm_{n-1}(x)$$

If we write $\alpha_i = \frac{|P_H(F_i)|_{n-1}}{2|P_H(B_1^n)|_{n-1}}$ we have $\alpha_i \geq 0$, $\sum_{i\in I}\alpha_i = 1$ and

$$\frac{1}{|P_H(B_1^n)|_{n-1}}\int_{P_H(B_1^n)} |x|^2 dm_{n-1}(x) = \sum_{i\in I}\frac{\alpha_i}{|P_H(F_i)|_{n-1}}\int_{P_H(F_i)} |x|^2 dm_{n-1}(x)$$

Since the latter formula is a convex combination, we have

$$\frac{1}{|P_H(B_1^n)|_{n-1}}\int_{P_H(B_1^n)} |x|^2 dm_{n-1}(x) \leq \sup_{i\in I}\frac{1}{|P_H(F_i)|_{n-1}}\int_{P_H(F_i)} |x|^2 dm_{n-1}(x)$$

Each face F_i is of the form $\mathrm{conv}\{\pm e_1, \dots, \pm e_n\}$ for some choice of signs $\pm$ so all we need is to have a bound from above of

$$\frac{1}{|P_H(\Delta^n)|_{n-1}}\int_{P_H(\Delta_n)} |x|^2 dm_{n-1}(x).$$

where Δ_n is the canonical simplex in $\mathbb{R}^n$, i.e. $\Delta^n = \mathrm{conv}\{e_1, \dots, e_n\}$

Write $\nu = (1/\sqrt{n}, \dots, 1/\sqrt{n})$ the normal vector to Δ^n. In order to change variables in the integral above we need to show that $P_H : \Delta_n \to P_H(\Delta_n)$ is a diffeomorphism. Indeed, if $x, \bar{x} \in \Delta^n$ such that $P_H(x) = P_H(\bar{x})$, then $x - \bar{x} = \langle x - \bar{x}, \theta\rangle\,\theta$. Since $x, \bar{x} \in \Delta^n$ we have $0 = \langle x - \bar{x}, \theta\rangle\,\langle\theta, \nu\rangle$ and, since $\langle\nu, \theta\rangle \neq 0$, $\langle x, \theta\rangle = \langle\bar{x}, \theta\rangle$ which implies $x = \bar{x}$.

The modulus of the jacobian is $|\langle\theta, \nu\rangle|$, so $|P_H(\Delta^n)|_{n-1} = |\langle\theta, \nu\rangle||\Delta^n|_{n-1}$ and

$$\begin{aligned}\frac{1}{|P_H(\Delta^n)|_{n-1}}\int_{P_H(\Delta^n)} |x|^2 dm_{n-1}(x) &= \frac{1}{|\Delta^n|_{n-1}}\int_{\Delta_n} |P_H(y)|^2 dm_{n-1}(y)\\ &\leq \frac{1}{|\Delta^n|_{n-1}}\int_{\Delta^n} |y|^2 dm_{n-1}(y)\end{aligned}$$

Now, it is well known (see for instance [7]) that

$$\frac{1}{|\Delta^n|_{n-1}}\int_{\Delta^n} |y|^2 dm_{n-1}(y) = \frac{2}{n+1}$$

Therefore, we have seen $(n-1)L^2_{P_H(B_1^n)} \leq cn^2\frac{2}{n+1}$ and so $L_{P_H(B_1^n)} \leq c$ for some absolute constant $c > 0$. Now, by a continuity argument, we can eliminate the restriction on θ which finishes the proof.

□

3. Conclusion

In this paper we added all hyperplane projections of B_p^n $1 \leq p \leq \infty$, to the list of convex bodies for which the Slicing Problem has a positive solution (Theorems 2.1 and 2.2).

In a forthcoming paper [1], the authors extend this approach to lower dimensional projections in the case $p = 1$. Indeed, for any k-dimensional subspace $E \subset \mathbb{R}$, it is possible to show that $L_{P_E(B_1^n)} \leq c\sqrt{\frac{n}{k}}$. We believe that the probabilistic tools we introduced here for $p > 1$ will produce a similar result.

Acknowledgements

We aknowledge the financial support by the European Union through the project Marie Curie RTN CT-2004-511953. The first three authors were also supported by MCYT Grant MTM2007-61446 and DGA E-64 (Spain).

References

[1] D. Alonso, J. Bastero, J. Bernués and P. Wolff, in preparation.

[2] J. Bourgain Lecture Notes in Math. Springer **1469** (1991) 127.

[3] F. Barthe and A. Naor, Discrete Comput. Geom. **27** (2002) 215.

[4] M. Junge, Forum Math. **6** (1994) 617.

[5] M. Junge, GAFA, Oper. Theory Adv. Appl., **77** (1995) 121.

[6] B. Klartag, Geom. and Funct. Anal. (GAFA) **16** (2006) 1274.

[7] B. Klartag and G. Kozma, to appear in Israel J. Mathematics.

[8] E. Milman, Advances in Math. **207** (2006) 566.

[9] V. Milman and A. Pajor, GAFA Seminar Lecture Notes in Math. **1376** (1989) 64.

[10] A. Naor, Trans. Amer. Math. Soc. **357** (2007) 1045.

[11] A. Naor and D. Romik, Ann. Inst. H. Poincaré Probab. Statist. **39** (2003) 241.

[12] G. Pisier, Cambridge Univ. Press, Cambridge (1989).

[13] G. Paouris Lecture Notes in Math. Springer **1745** (2000) 239.

[14] S.T. Rachev and L. Rüschendorf, Ann. Probab. **19** (1991) 1311.

[15] G. Schechtmann, J. Zinn, Proc.Amer. math. Soc **110** (1990) 217.