A Note on $L(\infty, q)$ Spaces and Sobolev Embeddings

JESÚS BASTERO, MARIO MILMAN &
FRANCISCO J. RUIZ BLASCO

ABSTRACT. We prove a sharp version of the Sobolev embedding theorem using $L(\infty, n)$ spaces and we compare our result with embeddings due to Hansson, Brézis-Wainger and Malý-Pick.

1. INTRODUCTION

Sobolev embedding theorems play a fundamental role in PDEs and have been intensively studied in the literature. In particular, the limiting cases of the Sobolev embedding theorem have been considered by many authors. Among the extensive list of contributions we mention here [22], [20], [11], [7], [16], [1], [21], [8], [15], as well as the references therein.

While the limiting results obtained in [11] and [7] are optimal[1] (cf. [11], [8]), it turns out to be possible to improve on these results by means of replacing the limiting spaces with limiting inequalities ([15]). This idea is, of course, a familiar one in analysis and, in particular, in interpolation theory (cf. [6], [4], [19]) as well as extrapolation theory (cf. [13]).

In this paper we introduce a new scale of spaces (conditions) that interpolate between L^∞ and the space weak-L^∞ of Bennett-DeVore-Sharpley [4]. We show that the $L(\infty, q)$ spaces are natural target spaces for sharp endpoint Sobolev embedding theorems, even for domains with infinite measure. Moreover, we show that when $|\Omega| < \infty$, $L(\infty, n)(\Omega)$ is contained in the Hansson-Brézis-Wainger space. We also show that $L(\infty, n)$ coincides with the space introduced in [15], also in the context of extreme Sobolev embeddings.

In what follows given an open domain $\Omega \subset \mathbb{R}^n$ and a r.i. space $X(\Omega)$, we denote by $W_0^{1,X}(\Omega)$ the closure of $C_0^\infty(\Omega)$ with the seminorm $\|\nabla f\|_X$. We started our work on sharp limiting Sobolev theorems in [2][2] where we observed that (cf.

[1] At least within the class of rearrangement invariant spaces.

[2] However, for editorial reasons, the final version of the paper [3] did not include Sobolev embedding theorems.

Indiana University Mathematics Journal ©, Vol. 52, No. 5 (2003)

[2, Example 13.12, page 94])

$$W_0^{1,L(n,\infty)}(\mathbb{R}^n) \subset L(\infty,\infty)(\mathbb{R}^n). \tag{1.1}$$

Here $L(n,\infty)(\mathbb{R}^n)$ is the classical Lorentz space weak-$L^n(\mathbb{R}^n)$, and $L(\infty,\infty)(\mathbb{R}^n)$ is the Bennett-DeVore-Sharpley space weak-$L^\infty(\mathbb{R}^n)$[3] defined by (cf. [4])

$$L(\infty,\infty)(\mathbb{R}^n) = \{f : \|f\|_{L(\infty,\infty)} = \sup_t\{f^{**}(t) - f^*(t)\} < \infty\}.$$

More generally, (1.1) also holds if we replace $\mathbb{R}^n$ by an open domain $\Omega \subset \mathbb{R}^n$:

$$W_0^{1,L(n,\infty)}(\Omega) \subset L(\infty,\infty)(\Omega). \tag{1.2}$$

The proof of (1.1)-(1.2) is an easy consequence of the following lemma and well known properties of symmetric spherical decreasing rearrangements.

Lemma 1.1. (cf. [1]) *Suppose that f is a smooth function equal to its symmetric spherical decreasing rearrangement, and zero at infinity. Then*

$$f^{**}(t) - f^*(t) \le \beta_n(|\nabla f|)^{**}(t)t^{1/n}, \tag{1.3}$$

where β_n[4]*> 0 is independent of f.*

Proof. See Appendix. □

In this note we show that similar considerations lead to an optimal embedding theorem for the usual Sobolev space $W_0^{1,n}(\Omega) = W_0^{1,L^n}(\Omega)$.

2. $L(\infty, q)$ SPACES

According to the usual definition of the scale of Lorentz spaces (cf. [6]) we have

$$L(\infty,q) = \begin{cases} \{0\} & \text{if } q \neq \infty\,, \\ L^\infty & \text{if } q = \infty\,. \end{cases}$$

In fact, if $q < \infty$, we readily see that

$$|E| > 0 \Rightarrow \|\chi_E\|_{\infty,q} = \infty.$$

We are thus led to use an idea introduced in [4]: we replace f^{**} (or f^*) by the quantity $f^{**} - f^*$ in the definition of the norm of $L(\infty,q)$.

---

[3] $L(\infty,\infty)$ is not a linear space and $\|\cdot\|_{L(\infty,\infty)}$ is not a norm.

[4] In fact $\beta_n = (n\gamma_n^{1/n})^{-1}$, where γ_n =measure of the unit ball in $\mathbb{R}^n$.

Definition 2.1. Let (Ω, μ) be a sigma finite measure space and let $q > 0$. We define

$$||f||_{L(\infty,q)(\Omega)} = \left\{ \int_0^{|\Omega|} (f^{**}(t) - f^*(t))^q \frac{dt}{t} \right\}^{1/q},$$

$$L(\infty, q)(\Omega) = \{f : ||f||_{L(\infty,q)(\Omega)} < \infty\}.$$

Due to the cancellation afforded by $f^{**} - f^*$, the $L(\infty, q)$ spaces defined in this fashion are nontrivial. For example, if $f = \chi_E$, then $f^{**}(t) - f^*(t) = (|E|/t)\chi_{(|E|,\infty)}(t)$, and therefore

$$\|\chi_E\|_{L(\infty,q)} = \left(\frac{1}{q}\right)^{1/q}.$$

Once again we remind the reader that, in general, the $L(\infty, q)$ spaces we have just defined are not linear spaces, and that $\|.\|_{L(\infty,q)}$ is not a norm.

The $L(\infty, q)$ spaces form a scale of conditions that interpolate between $L^\infty = L(\infty, 1)$ (see (5.1) below) and weak-$L^\infty = L(\infty, \infty)$.

Our embedding theorem can be now stated as follows

Theorem 2.2. *Let Ω be an open domain in $\mathbb{R}^n$. Then*

$$W_0^{1,n}(\Omega) \subset L(\infty, n)(\Omega).$$

Proof. Suppose first that $|\Omega| < \infty$. Let f° denote the symmetric spherical decreasing rearrangement of f. Recall that f° is defined by

$$f^\circ(x) = f^*(\gamma_n |x|^n), \tag{2.1}$$

where γ_n = measure of the unit ball in $\mathbb{R}^n$, and f^* is the usual non-decreasing rearrangement of f[5]. Suppose first that $\Omega = B$ is a ball with measure $|\Omega|$, and let $f \in W_0^{1,n}(\Omega)$ be such that $f = f^\circ$. Rewrite (1.3) as

$$\frac{f^{**}(t) - f^*(t)}{t^{1/n}} \leq \beta_n(|\nabla f|)^{**}(t).$$

Raising to the power n, and then integrating, it follows that

$$||f||_{L(\infty,n)(\Omega)} \leq c\, ||\nabla f||_{L^n(\Omega)}. \tag{2.2}$$

Now we dispense with the extra assumptions and prove (2.2) in general. Let $\tilde{\Omega}$ be the ball centered at the origin with $\left|\tilde{\Omega}\right| = |\Omega|$. Let $f \in W_0^{1,n}(\Omega)$. Since

[5]For the properties of symmetric spherical rearrangements we refer to [17].

the operation $f \to |f|$ does not increase the Sobolev norm (cf. [16]) we may assume without loss of generality that $f \geq 0$. By the Pólya -Szegö symmetrization theorem (cf. [17, page 12]) we have that $f^\circ \in W_0^{1,n}(\tilde{\Omega})$, and

$$\|\nabla f^\circ\|_{L^n(\tilde{\Omega})} \leq \|\nabla f\|_{L^n(\Omega)}. \tag{2.3}$$

Moreover, it is well known, and plain from (2.1), that f° is equimeasurable with f and therefore

$$(f^\circ)^* = f^*. \tag{2.4}$$

It follows from the first part of the proof (applied to f°) that

$$\|f^\circ\|_{L(\infty,n)(\tilde{\Omega})} \leq c\,\|\nabla f^\circ\|_{L^n(\tilde{\Omega})}. \tag{2.5}$$

But inserting (2.3) and (2.4) in (2.5) we obtain

$$\|f\|_{L(\infty,n)(\Omega)} = \|f^\circ\|_{L(\infty,n)(\tilde{\Omega})} \leq c\,\|\nabla f^\circ\|_{L^n(\tilde{\Omega})} \leq c\,\|\nabla f\|_{L^n(\Omega)}.$$

The argument if $|\Omega| = \infty$ is the same; the only change needed is that in this case $\tilde{\Omega} = \mathbb{R}^n$. Thus (2.2) holds in general concluding the proof. □

Remark 2.3. It is of interest for future applications (cf. Remark 5.5) to streamline the method of proof of Theorem 2.2 as follows. We note that the proof of Lemma 1.1 in the Appendix together with the argument of Theorem 2.2 yields the following variant of (1.3): if $f \in C_0^\infty(\Omega)$,

$$f^{**}(t) - f^*(t) \leq \beta_n (\nabla f^\circ)^{**}(t) t^{1/n}.$$

Morever, using the version of the Pólya -Szegö principle one can find in [9, (4.4), page 66], we get

$$\frac{f^{**}(t) - f^*(t)}{t^{1/n}} \leq \beta_n (\nabla f)^{**}(t).$$

Therefore, raising the previous inequality to the power n, integrating, and then applying Hardy's inequality, we get

$$\|f\|_{L(\infty,n)(\Omega)} \leq \frac{n}{n-1}\beta_n \|\nabla f\|_{L^n(\Omega)}.$$

3. COMPARISON WITH CLASSICAL EMBEDDINGS

Let Ω be an open domain in $\mathbb{R}^n$ with $|\Omega| < \infty$. In [11], and independently, and by different methods in [7], it was shown that

$$W_0^{1,n}(\Omega) \subset H_n(\Omega), \tag{3.1}$$

where[6]

$$H_n(\Omega) = \{f : \|f\|_{H_n(\Omega)} = \left[\int_0^{|\Omega|} \left(\frac{f^{**}(s)}{1+\log\frac{|\Omega|}{s}}\right)^n \frac{ds}{s}\right]^{1/n} < \infty\}.$$

Moreover, Hansson [11] (cf. also [8]) has shown that $H_n(\Omega)$ is the optimal target space in the class of rearrangement invariant spaces.

We now show that if we give up on the requirement that the target space be a linear space, and frame the limiting Sobolev embeddings in terms of rearrangement invariant inequalities, then the $L(\infty, q)$ classes give sharper results[7].

Theorem 3.1. $L(\infty, n)(\Omega) \subset H_n(\Omega)$, $n > 1$.

Proof. We shall first obtain an *a priori* estimate and prove that, if f is bounded (which in particular implies that $f \in H_n(\Omega)$),

$$\|f\|_{H_n(\Omega)} \leq \frac{n}{n-1} \|f\|_{L(\infty,n)(\Omega)} + c(|\Omega|, n) \|f\|_{L^1(\Omega)}, \tag{3.2}$$

where $c(|\Omega|, n) > 0$ is a constant independent of f. We shall then extend (3.2) by a limiting argument.

We will integrate by parts. To this end we observe that since f is bounded, $f^{**}(t) \to \|f\|_{L^\infty(\Omega)}$ as $t \to 0$. Therefore

$$\lim_{t\to 0} f^{**}(t)^n \frac{1}{\left(1+\log\frac{|\Omega|}{t}\right)^{n-1}} = 0. \tag{3.3}$$

[6]The weight $w(s) = (1+\log(|\Omega|/s))^{-n}s^{-1}$ satisfies the Ariño-Muckenhoupt condition: for all $0 < t < |\Omega|$,

$$\int_t^{|\Omega|} \left(\frac{t}{s}\right)^n w(s)\,ds \leq c \int_0^t \left(\frac{t}{s}\right)^n w(s)\,ds.$$

It follows that (cf. [3])

$$\|f\|_{H_n(\Omega)} \approx \left\{\int_0^{|\Omega|} \left(\frac{f^*(s)}{1+\log\frac{|\Omega|}{s}}\right)^n \frac{ds}{s}\right\}^{1/n}.$$

[7]In Remark 5.3 we give a different proof connected with the use of Hardy operators (cf. [3]).

Now,

$$\|f\|_{H_n(\Omega)}^n = \int_0^{|\Omega|} \left(\frac{f^{**}(s)}{1+\log\frac{|\Omega|}{s}}\right)^n \frac{ds}{s} \tag{3.4}$$
$$= \frac{-1}{1-n}\int_0^{|\Omega|} [f^{**}(s)]^n d\left(\left(1+\log\frac{|\Omega|}{s}\right)^{1-n}\right)$$
$$= \frac{-1}{1-n}[f^{**}(s)]^n \left(1+\log\frac{|\Omega|}{s}\right)^{1-n}\Bigg|_0^{|\Omega|}$$
$$+\frac{1}{1-n}\int_0^{|\Omega|}\left(1+\log\frac{|\Omega|}{s}\right)^{1-n} d\left(f^{**}(s)^n\right) .$$

We estimate the integrated term first. It follows from (3.3) that

$$\frac{-1}{1-n}[f^{**}(s)]^n \left(1+\log\frac{|\Omega|}{s}\right)^{1-n}\Bigg|_0^{|\Omega|}$$
$$= \lim_{s\to|\Omega|}\frac{-1}{1-n}[f^{**}(s)]^n \left(1+\log\frac{|\Omega|}{s}\right)^{1-n}$$
$$+\lim_{s\to 0}\frac{1}{1-n}[f^{**}(s)]^n \left[1+\log\frac{|\Omega|}{s}\right]^{1-n}$$
$$= \frac{1}{n-1}[f^{**}(|\Omega|)]^n = \frac{1}{n-1}\left(\frac{1}{|\Omega|}\int_0^{|\Omega|} f^*(s)\,ds\right)^n .$$

Now, since we obviously have

$$\int_0^{|\Omega|} f^*(s)\,ds \le c(|\Omega|,n)\,\|f\|_{H_n(\Omega)} ,$$

it follows that, (with c representing a different constant at each appearance)

$$\lim_{s\to|\Omega|}\frac{-1}{1-n}[f^{**}(s)]^n\left(1+\log\frac{|\Omega|}{s}\right)^{1-n} \le c(|\Omega|,n)\,\|f\|_{H_n(\Omega)}^{n-1}\,\|f\|_{L^1(\Omega)} . \tag{3.5}$$

To estimate the second term of (3.4) we note that

$$d\left(f^{**}(s)^n\right) = n(f^{**}(s))^{n-1}(f^{**}(s))'\,ds$$
$$= n(f^{**}(s))^{n-1}\frac{[f^*(s)-f^{**}(s)]}{s}\,ds.$$

Thus,

$$
\begin{aligned}
(3.6)\quad & \frac{1}{1-n}\int_0^{|\Omega|}\left(1+\log\frac{|\Omega|}{s}\right)^{1-n} d(f^{**}(s)^n) \\
& \leq \frac{n}{1-n}\int_0^{|\Omega|}\left(1+\log\frac{|\Omega|}{s}\right)^{1-n}(f^{**}(s))^{n-1}\frac{[f^*(s)-f^{**}(s)]}{s}\,ds \\
& \leq \frac{n}{n-1}\left(\int_0^{|\Omega|}\left(\frac{f^{**}(s)}{1+\log\frac{|\Omega|}{s}}\right)^n\frac{ds}{s}\right)^{(n-1)/n}\left(\int_0^{|\Omega|}[f^{**}(s)-f^*(s)]^n\frac{ds}{s}\right)^{1/n},
\end{aligned}
$$

where in the last step we used Hölder's inequality with exponents $n/(n-1), n$. Inserting (3.5) and (3.6) in (3.4) we find

$$
||f||^n_{H_n(\Omega)} \leq \frac{n}{n-1}||f||^{n-1}_{H_n(\Omega)}\,||f||_{L(\infty,n)(\Omega)} + \frac{1}{n-1}c(|\Omega|, n)\,||f||^{n-1}_{H_n(\Omega)}\,||f||_{L^1(\Omega)}.
$$

Thus we have proved that for all bounded functions

$$
(3.7)\qquad ||f||_{H_n(\Omega)} \leq \frac{n}{n-1}||f||_{L(\infty,n)(\Omega)} + \frac{1}{n-1}c(|\Omega|, n)\,||f||_{L^1(\Omega)}.
$$

To prove the inequality (3.7) in general we use an approximation argument. We may assume without loss of generality that $f \geq 0, f \in L(\infty, n)(\Omega)$. Let

$$
f_k(x) = f^*\left(\frac{1}{k}\right)\chi_{\{f>f^*(1/k)\}} + f(x)\chi_{\{f\leq f^*(1/k)\}},
$$

then $f_k \uparrow f$ a.e.. Moreover, since $f_k^*(t) = \min\{f^*(t), f^*(1/k)\}$ we readily see that

(i) $f_k^*(t) \uparrow f^*(t)$ a.e.,
(ii) $f_k^{**}(t) - f_k^*(t) \leq f^{**}(t) - f^*(t)$, and
(iii) $f_k^*(t) \leq f^*(t)$.

By (ii),

$$
||f_k||_{L(\infty,n)(\Omega)} \leq ||f||_{L(\infty,n)(\Omega)},
$$

and by (iii)

$$
||f_k||_{L^1(\Omega)} \leq ||f||_{L^1(\Omega)}.
$$

Thus, by the monotone convergence theorem, we get

$$\begin{aligned} \|f\|_{H_n(\Omega)} &= \lim_{k\to\infty} \|f_k\|_{H_n(\Omega)} \\ &\le \lim_{k\to\infty} \sup\{c\,\|f_k\|_{L(\infty,n)(\Omega)} + c(|\Omega|, n)\,\|f_k\|_{L^1(\Omega)}\} \\ &\le c\,\|f\|_{L(\infty,n)(\Omega)} + c(|\Omega|, n)\,\|f\|_{L^1(\Omega)}. \end{aligned}$$

Therefore (3.7) holds in full generality, which concludes the proof of the theorem. □

4. COMPARISON WITH THE MALÝ-PICK CONSTRUCTION

In [15] Malý and Pick consider (nonlinear) spaces defined as follows

$$MP_n(\Omega) = \{f : f^*\left(\frac{t}{2}\right) - f^*(t) \in L^n\left(\frac{dt}{t}\right)(0, |\Omega|)\},$$

and show that if $|\Omega| < \infty$, then[8]

$$W_0^{1,n}(\Omega) \subset MP_n(\Omega).$$

The connection with the $L(\infty, q)$ spaces is given by the following result:[9]

Theorem 4.1. $L(\infty, n)(\Omega) = MP_n(\Omega)$.

Proof. The result will follow from the following pointwise estimates

$$f^*\left(\frac{t}{2}\right) - f^*(t) \le 2(f^{**}(t) - f^*(t)), \tag{4.1}$$

for all $t \in (0, |\Omega|)$ and

$$f^{**}(t) - f^*(t) \le \frac{1}{t}\int_0^t \left(f^*\left(\frac{s}{2}\right) - f^*(s)\right) ds + \left(f^*\left(\frac{t}{2}\right) - f^*(t)\right) \tag{4.2}$$

for all $t \in (0, |\Omega|)$. Indeed, from (4.1) it immediately follows that

$$\|f\|_{MP_n(\Omega)} \le 2\,\|f\|_{L(\infty,n)(\Omega)}.$$

To prove the opposite inequality note that the Hardy operator

$$Pf(t) = \frac{1}{t}\int_0^t f(s)\,ds$$

[8] Note that our embedding theorem does not require the measure to be finite.
[9] For the case $q = \infty$ see [19].

is bounded on $L^n(dt/t)$ (cf. [12, page 246])[10]. In terms of the operator P the inequality (4.2) can be rewritten as

$$f^{**}(t) - f^*(t) \leq P\left(f^*\left(\frac{s}{2}\right) - f^*(s)\right)(t) + \left(f^*\left(\frac{t}{2}\right) - f^*(t)\right).$$

Thus

$$\|f\|_{L(\infty,n)(\Omega)} \leq \left(\|P\|_{L^n(dt/t)\to L^n(dt/t)} + 1\right)\|f\|_{MP_n(\Omega)}.$$

It remains to prove (4.1) and (4.2). To prove (4.1) we rewrite it as

$$f^*\left(\frac{t}{2}\right) + f^*(t) \leq 2f^{**}(t). \tag{4.3}$$

Now

$$\begin{aligned} 2f^{**}(t) &= \frac{2}{t}\int_0^t f^*(s)\,ds \\ &\geq 2\left[\frac{\frac{1}{2}}{t/2}\int_0^{t/2} f^*(s)\,ds + \frac{\frac{1}{2}}{t/2}\int_{t/2}^t f^*(s)\,ds\right] \\ &\geq f^*(t/2) + f^*(t), \end{aligned}$$

and (4.3) follows.

To show (4.2) note that

$$\begin{aligned} f^{**}(t) &= \frac{1}{t}\int_0^{t/2} f^*(s)\,ds + \frac{1}{t}\int_{t/2}^t f^*(s)\,ds \\ &\leq \frac{1}{t}\int_0^{t/2} f^*(s)\,ds + \frac{1}{2}f^*(t/2) \\ &= \frac{1}{2t}\int_0^t f^*(s/2)\,ds + \frac{1}{2}f^*(t/2) \end{aligned}$$

gives

$$2f^{**}(t) \leq \frac{1}{t}\int_0^t f^*(s/2)\,ds + f^*(t/2).$$

Therefore,

$$\begin{aligned} f^{**}(t) - f^*(t) &\leq \frac{1}{t}\int_0^t f^*(s/2)\,ds - f^{**}(t) + f^*(t/2) - f^*(t) \\ &= \frac{1}{t}\int_0^t (f^*(s/2) - f^*(s))\,ds + f^*(t/2) - f^*(t), \end{aligned}$$

as we wished to show. $\square$

[10] Alternatively note that the weight $1/t$ satisfies the M_n condition of Muckenhoupt (cf. [3]).

5. FINAL REMARKS

As far as we are aware, the $L(\infty, q)$ spaces studied in this paper have not been considered explicitly or systematically in the literature except in the case $q = \infty$, cf. [4], and [19] and its references, and the case $q = 1$, which was considered in [10]. In this respect we note that in [10] it is claimed[11] (cf. Theorem 4.1) that

$$\int_0^1 (f^{**}(t) - f^*(t))\frac{dt}{t} < \infty \Rightarrow f \in \mathrm{VMO}.$$

However, since (see the proof of Theorem 3.1)

$$-(f^{**}(t))' = \frac{f^{**}(t) - f^*(t)}{t}$$

we see that as sets

$$L(\infty, 1)(0,1) = L^\infty(0,1). \tag{5.1}$$

Remark 5.1. Although our proof of the Sobolev embedding theorem breaks down at $n = 1$, the $L(\infty, 1)$ spaces still give the correct embedding spaces in dimension one. This breakdown occurs because, starting from

$$\frac{f^{**}(t) - f^*(t)}{t} \le f'^{**}(t)$$

and integrating, we only get

$$\|f\|_{L(\infty,1)} \le \left\|f'^{**}\right\|_{L^1} = \left\|f'\right\|_{L(LogL)}.$$

On the other hand an elementary application of the fundamental theorem of calculus gives

$$W_0^{1,1}(0,1) \subset L^\infty(0,1),$$

which agrees with (5.1). It should be noted that the equivalence with the H_n spaces also breaks down at $n = 1$.

---

[11] We note in passing that another condition in [10]

$$\int_0^1 \frac{f^{**}(t) - f^*(t)}{f^{**}(t)}\frac{dt}{t} < \infty$$

can be also be recognized as

$$\int_0^1 d[-(\ln f^{**}(t))] < \infty$$

which ends up being

$$\ln\left(\frac{\|f\|_{L^\infty}}{\|f\|_{L^1(dt)}}\right) < \infty.$$

This form could be useful in interpolation theory.

Remark 5.2. Our proof of Theorem 3.1 follows closely the proof of (3.1) given by one of us many years ago. Since that proof was never published it seems of interest to present it here since it may also suggest a method to prove Gagliardo-Nirenberg inequalities in the extreme cases.

Proof of (3.1), $n > 1$. The arguments given during the course of the proof of Theorem 2.2 show that it is enough to consider the case where Ω is a ball centered at the origin. Moreover, we may assume that $f = f^\circ$, and f is bounded. By using arguments similar to those of theorem 3.1 combined with Hölder's inequality we obtain

$$\int_0^{|\Omega|} \left(\frac{f^*(s)}{1 + \log \frac{|\Omega|}{s}} \right)^n \frac{ds}{s}$$

$$\leq \frac{n}{1-n} \int_0^{|\Omega|} \left(\frac{f^*(s)}{1 + \log \frac{|\Omega|}{s}} \right)^{n-1} \left(\frac{d}{ds} f^*(s) \right) ds$$

$$= \frac{n}{1-n} \int_0^{|\Omega|} \left(\frac{f^*(s)}{1 + \log \frac{|\Omega|}{s}} \right)^{n-1} s^{1/n-1} \left(\frac{d}{ds} f^*(s) \right) s^{1-1/n} ds$$

$$\leq \frac{n}{n-1} \left(\int_0^{|\Omega|} \left(\left(\frac{f^*(s)}{1 + \log \frac{|\Omega|}{s}} \right)^{n-1} s^{1/n-1} \right)^{n/(n-1} ds \right)^{(n-1)/n}$$

$$\times \left(\int_0^{|\Omega|} \left(\left| \frac{d}{ds} f^*(s) \right| s^{1-1/n} \right)^n ds \right)^{1/n}.$$

Now observe that $f = f^\circ$ implies that we can write

$$\|\nabla f\|_{L^n(\Omega)} = c_n \left(\int_0^{|\Omega|} \left(-\frac{d}{ds} f^*(s) s^{1-1/n} \right)^n ds \right)^{1/n}.$$

Inserting this into our chain of estimates we see that we have shown

$$\int_0^{|\Omega|} \left(\frac{f^*(s)}{1 + \log \frac{|\Omega|}{s}} \right)^n \frac{ds}{s} \leq \frac{n}{n-1} c_n \left(\int_0^{|\Omega|} \left(\frac{f^*(s)}{1 + \log \frac{|\Omega|}{s}} \right)^n \frac{ds}{s} \right)^{(n-1)/n} \|\nabla f\|_{L^n(\Omega)},$$

and the result follows. $\square$

Remark 5.3. Using the theory of weights it is possible to give a more conceptual proof of Theorem 3.1 that could be of interest in generalizations involving weighted spaces. The proof that follows uses the theory of weighted norm inequalities for the Hardy operator $Pf(t) = (1/t)\int_0^t f(s)\,ds$ and its adjoint $Qf(t) = \int_t^\infty f(s)\,ds/s$. It will be convenient to deal with weighted norm inequalities in a separate lemma.

Lemma 5.4. *Let* $1 \le p < \infty$*, and suppose that* (w, v) *is a pair of weights satisfying the following condition: there exists* $C > 0$ *such that for all* $0 < t < 1$,

$$\left(\int_0^t w(x)\,dx\right)^{1/p} \left(\int_t^1 \frac{v(x)^{-p'/p}}{x^{p'}}\,dx\right)^{1/p'} \le C. \tag{5.2}$$

Then

$$\|f^{**}\|_{L^n((0,1),w(s)\,ds)} \le c\,\|f^{**} - f^*\|_{L^n((0,1),v(s)\,ds)} + \left(\int_0^1 w\right)^{1/n} \int_0^1 f^*.$$

Proof. Consider the operator $\overline{Q}$ defined by

$$\overline{Q}f(t) = \int_t^1 \frac{f(s)}{s}\,ds.$$

From (5.2) it follows that (cf. [16, page 45]),

$$\overline{Q} : L^p((0,1), v(x)\,dx) \to L^p((0,1), w(x)\,dx).$$

The connection between $\overline{Q}$ and $f^{**} - f^*$ can be seen from the following easily verified identity (cf. [2, Lemma 13.9, page 90])

$$(P + \overline{Q})f(t) = \overline{Q}Pf(t) + \int_0^1 f(s)\,ds,$$

which implies

$$Pf(t) = \overline{Q}(P - I)f(t) + \int_0^1 f(s)\,ds. \tag{5.3}$$

If we apply (5.3) to $f = f^*$, we get

$$f^{**}(t) = Pf^*(t) = \overline{Q}(f^{**} - f^*)(t) + \int_0^1 f^*(s)\,ds.$$

Thus,

$$\|f^{**}\|_{L^n((0,1),w(s)\,ds)} \le \left\|\overline{Q}(f^{**} - f^*)\right\|_{L^n((0,1),w(s)\,ds)} + \left(\int_0^1 w\right)^{1/n} \int_0^1 f^*(s)\,ds.$$

It follows that for any pair of weights that satisfies (5.2) for $p = n$, we have

$$(5.4)\quad \|f^{**}\|_{L^n((0,1),w(s)\,ds)} \le c\,\|f^{**} - f^*\|_{L^n((0,1),\nu(s)\,ds)} + \left(\int_0^1 w\right)^{1/n} \int_0^1 f^*.$$

∎

Proof of Theorem 3.1. Without loss of generality we suppose that $|\Omega| = 1$. By computation it is readily verified that the pair of weights

$$w(s) = \left(1 + \log\frac{1}{s}\right)^{-n} s^{-1}, \quad \nu(s) = s^{-1}$$

satisfies (5.2)[12]. Therefore we may apply the previous lemma. In our case (5.4) takes the form

$$\|f\|_{H_n(\Omega)} \le c\left(\|f\|_{L(\infty,n)(\Omega)} + \int_\Omega |f|\right),$$

concluding the proof. ∎

Remark 5.5. It is interesting to point out the role that the full scale of $L(\infty, q)$ spaces plays in the borderline case of the Sobolev embedding theorem. Using the method of proof Theorem 2.2 (see also Remark 2.3) we can easily prove[13]

$$(5.5)\qquad W_0^{1,L(n,q)}(\Omega) \subset L(\infty, q)(\Omega), \quad 1 \le q \le \infty.$$

The case $q = n$ corresponds to Theorem 2.2, while $q = \infty$ corresponds to (1.2). We also note that if $q = 1$, (5.5) gives

$$W_0^{1,L(n,1)}(\Omega) \subset L(\infty, 1)(\Omega) = L^\infty(\Omega),$$

another important borderline case of the Sobolev embedding theorem obtained in [7] by a different method.

Proof. The only modification to the proof of Theorem 2.2 that is needed is to observe that the Pólya -Szegö principle holds for Lorentz spaces (this is a consequence [9, (4.4), page 66]). Alternatively we can use the proof given in Remark 2.3 without changes. ∎

Remark 5.6. Suppose that $|\Omega| < \infty$, and let $1 \le p \le q \le \infty$. Then,

$$L(\infty, 1)(\Omega) \subset L(\infty, p)(\Omega) \subset L(\infty, q)(\Omega) \subset L(\infty, \infty)(\Omega),$$

and the inclusions are strict.

---

[12] note that $\int_0^1 w < +\infty$.

[13] We are grateful to Jie Xiao who asked about the validity of (5.5) after the first version of our paper was circulated.

Proof. Both end point inclusions are trivial (for example recall that $L(\infty,1)(\Omega) = L^\infty(\Omega)$). The corresponding inclusion for $1 < p < q < \infty$ can be seen by observing that, in view of Theorem 4.1 and [15], we have

$$\int_0^{|\Omega|}\left(f^*\left(\frac{t}{2}\right)-f^*(t)\right)^q\frac{dt}{t}\sim\sum_{k=1}^\infty\left(f^*\left(\frac{|\Omega|}{2^k}\right)-f^*\left(\frac{|\Omega|}{2^{k-1}}\right)\right)^q.$$

The fact that the inclusions are strict can be seen using the following family of functions kindly suggested to us by the referee. Let $f_\alpha(x) = (x\log^\alpha\frac{1}{x})', x \in (0,e^{-1}), 0 < \alpha \le 1$. Then it is readily seen that $f_\alpha > 0, f_\alpha^* = f_\alpha$, and $f_\alpha^{**}(t) - f_\alpha^*(t) = \alpha\log^{\alpha-1}(1/t)$. Therefore, for $1 < q < \infty$, $f_\alpha \in L(\infty,q)$ iff $\alpha < 1/q'$, and $f_1 \in L(\infty,\infty)$. □

6. APPENDIX

See [1] and also [14]. Since $f = f^\circ$ we can write (using polar coordinates, i.e., $\nabla f(s) = \nabla f(|s|)$)

$$f^*(t) = f^\circ\left(\left(\frac{t}{\gamma_n}\right)^{1/n}\right) = \int_{(t/\gamma_n)^{1/n}}^\infty |\nabla f(s)|\,ds$$

so that

$$\begin{aligned}
f^{**}(t) &= \frac{1}{t}\int_0^t\int_{(r/\gamma_n)^{1/n}}^\infty |\nabla f(s)|\,ds\,dr\\
&= f^*(t) + \frac{\gamma_n^{-1/n}}{nt}\int_0^t s^{1/n}\left|\nabla f\left(\left(\frac{s}{\gamma_n}\right)^{1/n}\right)\right|\,ds\\
&= f^*(t) + \frac{\gamma_n}{t}\int_0^{(t/\gamma_n)^{1/n}} s^n\,|\nabla f(s)|\,ds\\
&= f^*(t) + \frac{1}{nt}\int_{|y|\le(t/\gamma_n)^{1/n}} |y|\,|\nabla f(y)|\,dy\\
&\le f^*(t) + \frac{1}{nt}\left(\frac{t}{\gamma_n}\right)^{1/n}\int_{|y|\le(t/\gamma_n)^{1/n}} |\nabla f(y)|\,dy\\
&\le f^*(t) + \frac{1}{n}\left(\frac{t}{\gamma_n}\right)^{1/n}\left(\frac{1}{t}\int_0^t(|\nabla f|)^*(s)\,ds\right).
\end{aligned}$$

Therefore

$$f^{**}(t) - f^*(t) \le \frac{t^{1/n}}{n\gamma_n^{1/n}}(|\nabla f|)^{**}(t),$$

as we wished to show.

Acknowledgment. We are very grateful to the referee for his/her careful review and for many suggestions to improve our paper.

References

[1] A. Alvino, G. Trombetti, and P-L. Lions, *On optimization problems with prescribed rearrangements*, Nonlinear Anal. **13** (1989), 185–220.

[2] J. Bastero, M. Milman, and F. Ruiz Blasco, *On the connection between weighted norm inequalities, commutators and real interpolation*, Seminario Garcia de Galdeano **18** (1996).

[3] ______, *On the Connection between Weighted Norm Inequalities, Commutators and Real Interpolation*, Mem. Amer. Math. Soc., vol. 154, American Mathematical Society, Providence, Rhode Island, 2001.

[4] C. Bennett, R. De Vore, and R. Sharpley, *Weak-L^∞ and BMO*, Annals of Math. **113** (1981), 601-611.

[5] C. Bennett and R. Sharpley, *Interpolation of Operators*, Pure and Applied Mathematics, vol. 129, Academic Press, Boston, 1988.

[6] J. Bergh and J. Löfstrom, *Interpolation Spaces. An Introduction*, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1976.

[7] H. Brézis and S. Wainger, *A note on limiting cases of Sobolev embeddings and convolution inequalities*, Comm. Partial Diff. Eq. **5** (1980), 773-789.

[8] M. Cwikel and E. Pustylnik, *Sobolev type embeddings in the limiting case*, J. Fourier Anal. Appl. **4** (1998), 433–446.

[9] J. J. F. Fournier, *Mixed norms and rearrangements: Sobolev's inequality and Littlewood's inequality*, Ann. Mat. Pura Appl. **148** (1987), 51–76.

[10] M. Franciosi, *A condition implying boundedness and VMO for a function*, Studia Math. **123** (1997), 109–116.

[11] K. Hansson, *Imbedding theorems of Sobolev type in potential theory*, Math. Scand. **45** (1979), 77-102.

[12] G. H. Hardy, G. E. Littlewood, and G. Pólya, *Inequalities*, Cambridge Univ. Press, Oxford, 1952.

[13] B. Jawerth and M. Milman, *Extrapolation theory with applications*, Mem. Amer. Math. Soc., vol. 89, American Mathematical Society, Providence, Rhode Island, 1991.

[14] V.I. Kolyada, *Rearrangements of functions, and embedding theorems*, Russian Math. Surveys **44** (1989), 73–117.

[15] J. Malý and L. Pick, *An elementary proof of sharp Sobolev embeddings*, Proc. Amer. Math. Soc. **130** (2002), 555–563.

[16] V.G. Maz'ja, *Sobolev Spaces*, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1985.

[17] J. Mossino, *Inégalités Isopérimétriques et Applications en Physique, Travaux en Cours*, Hermann, Paris, 1984.

[18] J. Peetre, *Espaces d'interpolation et théorème de Soboleff*, Ann. Inst. Fourier **16** (1966), 279-317.

[19] Y. Sagher and P. Shvartsman, *The approximation functional and interpolation with perturbed continuity*, J. Approx. Theory **110** (2001), 236–260.

[20] R. Strichartz, *A note on Trudinger's extension of Sobolev's inequalities*, Indiana Univ. Math. J. **21** (1971/72), 841–842.

[21] L. TARTAR, *Imbedding theorems of Sobolev spaces into Lorentz spaces*, Boll. Un. Mat. Ital. **1-B** (1998), 479-500.

[22] N.S. TRUDINGER, *On imbeddings into Orlicz spaces and some applications*, J. Math. Mech. **17** (1967), 473-483.

JESÚS BASTERO:
Department of Mathematics
University of Zaragoza
Zaragoza, Spain
E-MAIL: bastero@posta.unizar.es
ACKNOWLEDGMENT: The first author was partially supported by DGES.

MARIO MILMAN:
Department of Mathematics
Florida Atlantic University
Boca Raton, FL 33431, U.S.A.
E-MAIL: extrapol@bellsouth.net
URL: http://www.math.fau.edu/milman

FRANCISCO J. RUIZ BLASCO:
Department of Mathematics
University of Zaragoza
Zaragoza, Spain
E-MAIL: fjruiz@posta.unizar.es
ACKNOWLEDGMENT: The third author was partially supported by DGES and U. Rioja.

KEY WORDS AND PHRASES: Lorentz spaces; BMO; Sobolev spaces.
2000 MATHEMATICS SUBJECT CLASSIFICATION: Primary 46E30, 26D10.
Received: *October 9th, 2002; revised: May 8th, 2003.*