The theorems of Caratheodory and Gluskin for $0 < p < 1$

by

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Abstract. *In this note we prove the p-convex analogue of both Caratheodory's convexity theorem and Gluskin's theorem concerning the diameter of Minkowski compactum.*

Throughout this note X will denote a real vector space and p will be a real number, $0 < p < 1$. A set $A \subseteq X$ is called p-convex if $\lambda x + \mu y \in A$, whenever $x, y \in A$, and $\lambda, \mu \geq 0$, with $\lambda^p + \mu^p = 1$. Given $A \subseteq X$, the p-convex hull of A is defined as the intersection of all p-convex sets that contain A. Such set is denoted by p-conv (A). A (real) p-normed space $(X, \|\cdot\|)$ is a (real) vector space equipped with a quasi-norm such that $\|x+y\|^p \leq \|x\|^p + \|y\|^p, \forall\, x, y \in X$. The unit ball of a p-normed space is a p-convex set and will be denoted by B_X.

We denote by $\mathcal{M}_n^p$ the class of all n-dimensional p-normed spaces. If $X, Y \in \mathcal{M}_n^p$ the Banach-Mazur distance $d(X,Y)$ is the infimun of the products $\|T\| \cdot \|T^{-1}\|$, where the infimun is taken over all the isomorphisms T from X onto Y. We shall use the notation and terminology commonly used in Banach space theory as it appears in [Tmcz].

The problem we are concerned about is an aspect of the local structure of finite dimensional p-Banach spaces. The well known theorem of Gluskin gives a sharp lower bound of the diameter of the Minkowski compactum. In [Gl] it is proved that $\operatorname{diam}(\mathcal{M}_n^1) \geq cn$ for some absolute constant c. Our purpose is to study this problem in the p-convex setting. In [Pe], T. Peck gave an upper bound of the diameter of $\mathcal{M}_n^p$ namely, $\operatorname{diam}(\mathcal{M}_n^p) \leq n^{2/p-1}$. We will show that such bound is optimum (Theorem 2). When proving it, in order to compute some volumetric estimates, it will be necessary to have the corresponding version for $p < 1$ of Caratheodory's convexity theorem (Theorem 1).

The results of this note are the following:

Theorem 1. *Let $A \subseteq \mathbb{R}^n$ and $0 < p < 1$. For every $x \in p\text{-conv}\,(A), x \neq 0$ there exist linearly independent vectors $\{P_1 \dots P_k\} \subseteq A$ with $k \leq n$, such that $x \in p\text{-conv}\,\{P_1 \dots P_k\}$. Moreover, if $0 \in p\text{-conv}\,(A)$, there exits $\{P_1 \dots P_k\} \subseteq A$ with $k \leq n+1$ such that $0 \in p\text{-conv}\,\{P_1 \dots P_k\}$.*

and

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Theorem 2. *Let $0 < p < 1$. There exits a constant $C_p > 0$ such that for every $n \in I\!N$*

$$C_p n^{2/p-1} \leq \mathrm{diam}(\mathcal{M}_n^p) \leq n^{2/p-1}.$$

Observe that Theorem 1 looks stronger than Caratheodory's one in the sense that we get $k \leq n$ and only $k \leq n+1$ can be assured for $p = 1$ (see [Eg], pg 35). It will be clear that this is not such since vector 0 plays a particularly special role.

We begin by recalling the main property of p-convex hulls. It is probably known but since we have not found it in any reference we sketch its proof.

Lemma 1. *Let $A \subset X$. The p-convex hull of A coincides with the set of all finite sums $\sum \lambda_i x_i$ where x_i are taken from A (possibly with repetition), $\lambda_i \geq 0$ and $0 < \sum \lambda_i^p \leq 1$.*

Proof. Straighforward arguments show that p-conv (A) coincides with the set of all finite sums $\sum \lambda_i x_i$, $x_i \in A$, $\lambda_i \geq 0$ and $\sum \lambda_i^p = 1$. Now, we only have to prove that every non zero element x of the form $x = \sum_{i=1}^n \lambda_i x_i$, $x_i \in A$, $\sum_{i=1}^n \lambda_i^p < 1$ can be written as $x = \sum_{i=1}^m \mu_i y_i$, $y_i \in A$, $\sum_1^m \mu_i^p = 1$. Suppose $\lambda_1 \neq 0$. Write $\lambda_1 = \sum_{i=1}^k \beta_i$, with $\beta_i \geq 0$. We have $\sum_{i=1}^n \lambda_i^p \leq \sum_{i=1}^k \beta_i^p + \sum_{i=2}^n \lambda_i^p \leq k^{1-p}\lambda_1^p + \sum_{i=2}^n \lambda_i^p$. It is now clear, by a continuity argument, that we can find k and $\beta_i \geq 0$, $1 \leq i \leq k$, such that $\lambda_1 = \sum_{i=1}^k \beta_i$ and $\sum_{i=1}^k \beta_i^p + \sum_{i=2}^n \lambda_i^p = 1$. The representation $x = \sum_{i=1}^k \beta_i x_i + \sum_{i=2}^n \lambda_i x_i$ does the job. ///

Remark. Observe in particular says that for every $0 \neq x \in X$, p-conv $\{x\} = (0, x] = \{\lambda x; 0 < \lambda \leq 1\}$. This situation is rather different from the case when $p = 1$.

Proof of Theorem 1. Let $x \in p$-conv (A), $x \neq 0$. Let N be the smallest integer so that x in the p-convex hull of a subset $\{P_1, ..., P_N\}$ of A. Consider the set of all $(\alpha_i) \geq 0$ with $x = \sum_{i=1}^N \alpha_i P_i$, $0 < \sum_{i=1}^N \alpha_i^p \leq 1$. Minimize $\sum_{i=1}^N \alpha_i^p$ on this set and denote the optimum by (λ_i). Clearly $\lambda_i > 0$, for all $i = 1, ..., N$. Suppose $\{P_1, ..., P_N\}$ are linearly dependent; then there exists nontrivial coeficients (μ_i) so that $\sum_{i=1}^N \mu_i P_i = 0$. If $\delta > 0$ is small enough all the coefficients $\lambda_i + t\mu_i > 0$ and the function $\phi(t) = \sum_{i=1}^N (\lambda_i + t\mu_i)^p$ defined for $t \in (-\delta, \delta)$ has a minimum in $t = 0$, which contradicts the fact that the second derivative of $\phi(t)$ is negative.

If $0 \in p$-conv (A) then $0 = \sum_{i=1}^N \lambda_i P_i$, $P_i \in A$, $\lambda_i > 0$, $\forall\, i$ and $\sum_{i=1}^N \lambda_i^p = 1$. We can suppose $P_1 \dots P_m$ linearly independent with $m \leq n$. We consider $\sum_{i=1}^{m+1} \lambda_i P_i = -\sum_{i=m+2}^N \lambda_i P_i$. If we apply the first part of the proof to $\tilde{x} = \sum_{i=1}^{m+1} \lambda_i s^{-1} P_i$, $s^p = \sum_{i=1}^{m+1} \lambda_i^p$ we obtain $\sum_{i=1}^m \beta_i P_i = -\sum_{i=m+2}^N \lambda_i P_i$, with $\sum_{i=1}^m \beta_i^p \leq 1$. Hence $0 \in p$-convex envelope of $N-1$ points. Repeat the argument until reaching a representation of length $\leq n+1$. ///

Next we are going to prove Theorem 2. The proof follows Gluskin's original ideas. We first introduce some notation. S^{n-1} will denote the euclidean sphere in $I\!R^n$ with its normalized Haar measure μ_{n-1} and Ω will be the product space $S^{n-1}\times \overset{n)}{\cdots} \times S^{n-1}$ endowed with the product probability $I\!P$. If $K \subseteq I\!R^n$, $|K|$ is the Lebesgue measure of K. If $A = (P_1, \dots, P_n) \subset \Omega$, we write $Q_p(A) = p\text{-conv}\,\{\pm e_i, \pm P_i \mid 1 \le i \le n\}$, being $\{e_i\}_{i=1}^n$ the canonical basis of $I\!R^n$. We denote by $\|\cdot\|_{Q_p(A)}$ the p-norm in $I\!R^n$ whose unit ball is $Q_p(A)$.

We only need to prove that for some absolute constant $C_p > 0$, there exist $A, A' \in \Omega$ such that simultaneously both $\|T\|_{Q_p(A)\to Q_p(A')} \ge C_p n^{1/p-1/2}$ and $\|T^{-1}\|_{Q_p(A')\to Q_p(A)} \ge C_p n^{1/p-1/2}$ hold for any $T \in \mathrm{SL}(n)$ (that is, any linear isomorphism in $I\!R^n$ with det $T = 1$).

Straightforward argument shows that it is enough to see that for any $A' \in \Omega$, $I\!P\{\, A \in \Omega \mid \; \|T\|_{Q_p(A)\to Q_p(A')} < C_p n^{1/p-1/2}$ for some $T \in \mathrm{SL}(n)\,\} < \frac{1}{2}$.

Fix $A' \in \Omega$, $t > 0$, and write $\Omega(A', t) = \{A \in \Omega \mid \|T\|_{Q_p(A)\to Q_p(A')} < t$ for some $T \in \mathrm{SL}(n)\}$.

The proof of the following lemma is analogous to the one in the case $p = 1$ (see [Tmcz], §38).

Lemma 2. *Let $A' \in \Omega$ and $t > 0$.*

i) There exists a t^p-net $N(A', t)$ in $\{T \in \mathrm{SL}(n) \mid \|T\|_{\ell_p^n\to Q_p(A')} \le t\}$ with respect to the metric induced by $\|\cdot\|^p_{\ell_2^n\to Q_p(A')}$ of cardinality

$$|\,N(A', t)\,| \le (3^{1/p} n^{1/p-1/2})^{n^2} \frac{|Q_p(A')|^n}{|\,\{T \in \mathrm{SL}(n) \mid \|T\|_{\ell_2^n\to\ell_2^n} \le 1\}\,|}$$

ii)

$$\Omega(A', t) \subseteq \bigcup_{T\in N(A',t)} \{\, A \in \Omega \mid \|T(P_i)\|_{Q_p(A')} \le 2^{1/p} t, \forall P_i \in A\}$$

iii) Given $T \in \mathrm{SL}(n)$,

$$I\!P\{\, A \in \Omega \mid \|T(P_i)\|_{Q_p(A')} \le 2^{1/p} t, \forall P_i \in A\} \le (2^{1/p} t)^{n^2} \left(\frac{|\,Q_p(A')\,|}{|B_{\ell_2^n}|}\right)^n$$

Proof of Theorem 2: Numerical constants are always denoted by the same letters C (or C_p, if it depends only on p) although they may have different value from line to line. Using consecutively the three preceding lemmas we have for every $A' \in \Omega$ and $t > 0$,

$$I\!P\big(\Omega(A', t)\big) \le (C_p t n^{1/p-1/2})^{n^2} \frac{|Q_p(A')|^{2n}}{|B_{\ell_2^n}|^n \cdot |\,\{T \in \mathrm{SL}(n) \mid \|T\|_{\ell_2^n\to\ell_2^n} \le 1\}\,|}$$

It is well known that for some absolute constant $C > 0$, (see [Tmcz]), we have $|\,\{T \in \mathrm{SL}(n) \mid \|T\|_{\ell_2^n\to\ell_2^n} \le 1\}\,| \ge C^{n^2} |B_{\ell_2^n}|^n$.

Let $A' = \{P_1, \dots P_n\}$. By Theorem 1, $Q_p(A') \subseteq \bigcup p\text{-conv}\,\{P_{k_1}, \dots, P_{k_n}\}$ where the union runs over the $\binom{4n}{n}$ choices of $\{P_{k_i}\}_{i=1}^n \subseteq \{\pm e_i, \pm P_i, 1 \le i \le n\}$. Since $\|P_i\|_2 = 1$ and $|p\text{-conv}\,\{P_{k_1}, \dots, P_{k_n}\}|$ is equal to $|\det\,[P_{k_1}, \dots, P_{k_n}]\,| \cdot$

$|p\text{-conv}\,\{e_1,\dots,e_n\}|$, we get $|Q_p(A')| \le \binom{4n}{n}\frac{|B_{\ell_p^n}|}{2^n} \le C_p^n n^{-n/p}2^{-n}$ for some constant C_p (see [Pi], pg 11). Hence, $I\!P\big(\Omega(A',t)\big) \le (C_p t n^{1/2-1/p})^{n^2}$. If we take a suitable $t>0$, we can assure $I\!P\big(\Omega(A',t)\big) < \dfrac{1}{2}$ and the result follows. ///

Remark. With straighforward variations in the proof we can state the following result: *Given $0<p\le 1$ and $0<\alpha<1$, there exists a constant $0<C(p,\alpha)<1$ such that for any natural number N we can find two αN-dimensional quotients of ℓ_p^N having Banach-Mazur distance greater than or equal to $C(p,\alpha)N^{2/p-1}$.*

Remark. Given a p-normed space X and $p<q\le 1$, we define the q-Banach envelope of X as the q-normed space, X^q, whose unit ball es the q-convex envelope of the unit ball of X. It is easy to see that $d(X,X^q)\le d(X,Y)$ for any n-dimensional q-normed space Y (see [Pe],[G-K]). Theorem 1 shows that $d(X,X^q)\le n^{1/p-1/q}$. Indeed, for every $x\in B_{X^q}, \|x\|_{X^q}=1$ there exist $P_1,\dots,P_n\in B_X$ such that $x=\sum_{i=1}^n \lambda_i P_i$ with $\lambda_i\ge 0, \sum_{i=1}^n \lambda_i^q \le 1$ and $1\le \|x\|_X \le \sum_{i=1}^n \lambda_i^p \|P_i\|_X^p \le \sum_{i=1}^n \lambda_i^p \le n^{1/p-1/q}$; by homogeneity we achieve the result. Now it is easy to see that if X,Y are the spaces appearing in Theorem 2, then $d(X,X^q)\ge C_p n^{1/p-1/q}, d(Y,Y^q)\ge C_p n^{1/p-1/q}$ and $d(X^q,Y^q)\ge C_p n^{2/q-1}$. In particular, for $q=1$, $d(X,X^1)\ge C_p n^{1/p-1}$, $d(Y,Y^1)\ge C_p n^{1/p-1}$ and $d(X^1,Y^1)\ge C_p n$.

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