

# ASYMPTOTIC BEHAVIOUR OF AVERAGES OF $k$ -DIMENSIONAL MARGINALS OF MEASURES ON $\mathbb{R}^n$

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ABSTRACT. We study the asymptotic behaviour, as  $n \rightarrow \infty$ , of the Lebesgue measure of the set  $\{x \in K ; |P_E(x)| \leq t\}$  for a random  $k$ -dimensional subspace  $E \subset \mathbb{R}^n$  and  $K \subset \mathbb{R}^n$  in a certain class of isotropic bodies. For  $k$  growing slowly to infinity, we prove it to be close to the suitably normalised Gaussian measure in  $\mathbb{R}^k$  of a  $t$ -dilute of the Euclidean unit ball. Some of the results hold for a wider class of probabilities on  $\mathbb{R}^n$ .

## 1. PRELIMINARIES AND NOTATION

Let  $E$  be a  $k$ -dimensional subspace of  $\mathbb{R}^n$ ,  $1 \leq k \leq n$  and denote by  $P_E$  the orthogonal projection onto  $E$ . For any Borel probability  $\mathbb{P}$  on  $\mathbb{R}^n$ , its marginal probability on  $E$  is defined as  $\mathbb{P}_E(A) := \mathbb{P}(A + E^\perp) = \mathbb{P}(x \in \mathbb{R}^n \mid P_E(x) \in A)$ ,  $A \subseteq E$ . A Borel probability  $\mathbb{P}$  is isotropic if  $\int_{\mathbb{R}^n} x \, d\mathbb{P}(x) = 0$  and its covariance matrix is a multiple of the identity. A convex body  $K$  of volume 1 is isotropic if the uniform measure on  $K$  is. In this case, such multiple of the identity is denoted by  $L_K^2$ .

In [KL2] the author solved the so called *Central limit problem for convex bodies* (posed in [ABP], [BV] for  $k = 1$  and considered in [BK], [BHV], [KL], [MM], [Mi], [Wo]). He showed that every isotropic convex body  $K$  (and more generally, every isotropic log-concave probability measure) has the property that *most* of its  $k$ -dimensional marginal distributions are *approximately* Gaussian, with respect to the total variation metric, provided that  $k \ll \frac{\log n}{\log \log n}$ .

In a more general probabilistic setting, the  $k$ -dimensional version of the problem goes back to [W] (see also [DF], [Bo], [Su]). In [NR], the authors studied proximity of  $k$ -marginals to the Gaussian measure with respect to the (weaker) T-distance, for a class of isotropic probabilities satisfying some concentration hypothesis. In [M], Gaussian approximation of  $k$ -marginals with respect to the Wasserstein distance is studied for isotropic probabilities with geometric symmetries.

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A key tool in all those results is the use of some kind of concentration property of the Euclidean norm with respect to the probability  $\mathbb{P}$ .

Let  $K$  be an isotropic convex body and consider the distribution function

$$(1.1) \quad F_K(t, E) := |\{x \in K : |P_E(x)| \leq t\}|, \quad t \geq 0$$

where  $|\cdot|$  denotes both the Euclidean norm and the Lebesgue measure on  $\mathbb{R}^n$ . The function  $F_K(t, E)$  is the marginal measure (of the uniform measure on  $K$ ) on  $E$  of a  $t$ -dilate of the Euclidean unit ball. Denote by  $\Gamma_K^k(t)$  the  $k$ -dimensional Gaussian measure (centred with variance  $L_K^2$ ) of  $\{s \in \mathbb{R}^k : |s| \leq t\}$ .

We are interested in studying the closeness between  $F_K(t, E)$  and  $\Gamma_K^k(t)$ . Estimates  $|F_K(t, E) - \Gamma_K^k(t)|$  are particular cases of the results in [K12]. It was pointed out to the authors by V. Milman the interest of the (stronger) comparison  $|\frac{F_K(t, E)}{\Gamma_K^k(t)} - 1|$  in the spirit of [So] and we will address this question. With a concentration assumption on  $K$  (see (3.3) below) we will show,

**Theorem 3.11.** *Let  $K \subset \mathbb{R}^n$  in the family of isotropic convex bodies satisfying  $L_K \leq c$  and condition (3.3). Then for every  $0 < \varepsilon < 1$  and  $1 \leq k \leq \frac{c_1 \varepsilon \log n}{(\log \log n)^2}$  we have*

$$\nu \left\{ E \in G_{n,k}; \sup_{t \geq \varepsilon} \left| \frac{F_K(t, E)}{\Gamma_K^k(t)} - 1 \right| \leq \varepsilon \right\} \geq 1 - \exp(-c_2 n^{0.9})$$

where  $c_1, c_2$  depend only on  $c$  and on the constants in (3.3).

We follow a somewhat standard procedure: first we show that the average of  $F_K(t, E)$  on the grassmannian  $G_{n,k}$  is close to the Gaussian measure. Then, by the concentration of measure phenomena on  $G_{n,k}$ , we show that for most subspaces  $E$ ,  $F_K(t, E)$  is close to its average.

It turns out that the average of  $F_K(t, E)$  can be written in a way that admits generalisation to any probability  $\mathbb{P}$ . In the second and in the last section of the paper we study properties of this averaging, including proximity to the Gaussian measure in the uniform distance.

The paper is organised as follows:

In section 2 we introduce an average of  $k$ -dimensional marginals for any probability  $\mathbb{P}$  on  $\mathbb{R}^n$ , compute the (radial) density  $\varphi_{\mathbb{P}}^k(s)$ ,  $s \in \mathbb{R}^k$  of its absolutely continuous part (Proposition 2.1) and explain its geometrical meaning (Proposition 2.3). For  $\mathbb{P}$  the uniform measure on  $K$ , the relation with our problem is given by the formula:

$$(1.2) \quad F_K^k(t) := \int_{G_{n,k}} F_K(t, E) d\nu(E) = \int_{\{|s| \leq t\}} \varphi_K^k(s) ds$$

where  $\varphi_K^k(s) = \int_{O(n)} |(s_1 \xi_1 + \dots + s_k \xi_k + \{\xi_1, \dots, \xi_k\}^\perp) \cap K|_{n-k} dU$ , integration is with respect to the Haar probability on the orthogonal group  $O(n)$  and  $U = (\xi_1, \dots, \xi_n) \in O(n)$ . Moreover, each  $F_K(t, E)$  is a certain average on  $O(E)$  of marginal densities, see Remark 2.4.

In section 3.1, we investigate the closeness of the average density  $\varphi_K^k(s)$  to a suitably normalised Gaussian density  $\gamma_K^k(s)$  and obtain estimates for

$$\left| \frac{\varphi_K^k(s)}{\gamma_K^k(s)} - 1 \right|$$

(Theorem 3.5.1)). At this stage, it is still possible to state the result for general probabilities  $\mathbb{P}$  verifying (3.3) with no extra effort and we do so (Theorem 3.1). We extend the ideas in [So] ( $k = 1$  in that reference) for the estimates when  $s$  is far from the origin. The study of estimates for  $s$  near the origin lead us to consider the parameter  $M_{\mathbb{P}}$ , see definition below.

A simple integration yields to relations between the *average* of  $F_K(t, E)$  and  $\Gamma_K^k(t)$ , that is

$$\left| \frac{F_K^k(t)}{\Gamma_K^k(t)} - 1 \right|$$

(Theorem 3.5.2)). In section 3.2, the concentration of measure phenomena on  $G_{n,k}$  will be the key ingredient to show that for “most” subspaces  $E$ ,  $F_K(t, E)$  is close to its average  $F_K^k(t)$ . For that matter we estimate the modulus of continuity of  $F_K(t, E)$ .

All the results in this section, valid for the uniform probability on isotropic convex bodies, can be stated and hold true for log-concave probabilities  $\mathbb{P}$ .

Finally, in section 4 we return to the study of the average density  $\varphi_{\mathbb{P}}^k(s)$ . For class of probability measures  $\mathbb{P}$ , we estimate

$$\sup_{s \in \mathbb{R}^k} \left| \varphi_{\mathbb{P}}^k(s) - \gamma_{\mathbb{P}}^k(s) \right| \quad \text{and} \quad \sup_{t \geq 0} \left| F_K^k(t) - \Gamma_{\mathbb{P}}^k(t) \right|$$

(Theorem 4.2) and show that such difference tends to 0 (as  $n \rightarrow \infty$ ) provided that  $k = O\left(\frac{\sqrt{\log n}}{(\log \log n)^{\frac{1}{2} + \delta}}\right)$ ,  $\delta > 0$ . We extend the ideas in [BK] and solve the difficulties appearing in that paper for  $s = 0$ .

When  $k$  increases very fast to infinity,  $k = n - \ell$ ,  $\ell$  fixed and  $k = (1 - \lambda)n$ ,  $0 < \lambda < 1$  we cannot expect a Gaussian behaviour. We obtain upper bounds for the average marginal density (Proposition 4.7) which, for some cases, are shown to be sharp. Such upper bounds are also needed in the first part of the section (Lemma 4.5).

Next we shall introduce some notation and definitions. We denote by  $D_n$  the Euclidean ball in  $\mathbb{R}^n$  and by  $\omega_n$  its Lebesgue measure. The area measure of the unit sphere  $S^{n-1}$  is  $|S^{n-1}| = n \omega_n$ . The letters  $c, C, c_1 \dots$  will denote absolute numerical constants whose value may change from line to line.

The elements of the orthogonal group  $O(n)$  are denoted by  $U = (\xi_1 \dots \xi_n)$  so the columns  $(\xi_i)$  form an orthonormal basis in  $\mathbb{R}^n$  and  $dU$  is the Haar probability on  $O(n)$ . The Haar probability on  $S^{n-1}$  is denoted by  $\sigma_{n-1}$ .

Let  $\mathbb{P}$  be a Borel probability on  $\mathbb{R}^n$ . We introduce the following parameters

$$M_2^2 = M_2^2(\mathbb{P}) := \frac{1}{n} \int_{\mathbb{R}^n} |x|^2 d\mathbb{P}(x) \quad M_{\mathbb{P}} := \sup_{t > 0} \frac{\mathbb{P}\{tD_n\}}{|tD_n|}$$

and

$$\sigma_{\mathbb{P}}^2 := n \left( \frac{\int_{\mathbb{R}^n} |x|^4 d\mathbb{P}(x)}{(\int_{\mathbb{R}^n} |x|^2 d\mathbb{P}(x))^2} - 1 \right) = \frac{\text{Var}(|x|^2)}{nM_2^4(\mathbb{P})}$$

When  $\mathbb{P}$  is the uniform measure on  $K$  we change the notation accordingly, that is,  $\sigma_{\mathbb{P}}$  to  $\sigma_K$  and so on.

*Remark 1.1.*  $\sigma_{\mathbb{P}}$  is a concentration parameter. Chebyshev's inequality implies (see [ABP]),

$$\mathbb{P}\{x \in \mathbb{R}^n ; | |x|^2 - nM_2^2(\mathbb{P}) | > \varepsilon nM_2^2(\mathbb{P})\} \leq \frac{\sigma_{\mathbb{P}}^2}{n^2\varepsilon^2}$$

For  $\mathbb{P}$  the uniform measure on an isotropic convex body  $K$ , the parameter  $\sigma_K$  is conjectured to be bounded by an absolute constant (the Variance Hypothesis).

When  $\mathbb{P}$  has density  $f$ ,  $M_{\mathbb{P}}$  is the Hardy-Littlewood maximal function of  $f$  at the origin. It is finite, for instance, when this the origin is a regular Lebesgue point of  $f$  (Lebesgue differentiation theorem holds) or when  $f$  is bounded and in such case  $M_{\mathbb{P}} \leq \|f\|_{\infty}$  (the supremum norm of  $f$ ). Also observe that  $M_{\mathbb{P}} < \infty$  implies  $\mathbb{P}(\{0\}) = 0$ .

*Remark 1.2.* For  $M_{\mathbb{P}}$  and  $M_2(\mathbb{P})$  finite the parameter  $M_2(\mathbb{P})M_{\mathbb{P}}^{1/n}$  plays an important role. In the particular case of  $\mathbb{P}$  being the uniform measure of an isotropic convex body  $K$ , such constant is  $L_K (= M_2(\mathbb{P})$  and  $M_{\mathbb{P}} = 1)$ . If  $\mathbb{P}$  has density  $f$  an even log-concave function, the constant  $M_2(\mathbb{P})M_{\mathbb{P}}^{1/n}$  is the isotropy constant of the function since  $M_{\mathbb{P}} = f(0)$ , (see [B]).

The following fact due to Hensley [H], whose proof follows from the results in [B], Lemma 6, will be extensively used along the paper:

**Lemma 1.3.** *There exists an absolute constant  $c > 0$  so that for any probability  $\mathbb{P}$  on  $\mathbb{R}^n$ ,  $M_2(\mathbb{P})M_{\mathbb{P}}^{1/n} \geq c$ .*

We finish this section with some

**1.1. Technical preliminaries.** Let  $\mathbb{P}$  be a Borel probability on  $\mathbb{R}^n$  with  $M_{\mathbb{P}} < \infty$ ,  $M_2 = M_2(\mathbb{P}) < \infty$ . Denote

$$\gamma_{\mathbb{P}}^k(s) = \frac{1}{(2\pi)^{k/2}M_2^k} e^{-\frac{|s|^2}{2M_2^2}}, \quad s \in \mathbb{R}^k \quad \Gamma_{\mathbb{P}}^k(t) = \int_{|s| \leq t} \gamma_{\mathbb{P}}^k(s) ds, \quad t \geq 0$$

In the next three lemmas we state some inequalities that will be useful in the sequel.

Given  $g, h: [0, \infty) \rightarrow \mathbb{R}$ , write  $g \sim h$  for  $c_1 h(x) \leq g(x) \leq c_2 h(x), \forall x \geq 0$ .

**Lemma 1.4.** *The following estimates are well known,*

$$\text{i) } \Gamma(x+1) = x^x e^{-x} \sqrt{2\pi x} (1 + \frac{1}{12x} + O(\frac{1}{x^2}))$$

$$\text{ii) } |S^{n-1}| = n \omega_n = \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})}, \quad \omega_n \leq \frac{c^n}{n^{n/2}}, \quad \omega_n^{1/n} \sim \frac{\sqrt{2\pi e}}{\sqrt{n}} \text{ and for } k = o(n),$$

$$\frac{|S^{n-k-1}|}{|S^{n-1}|} \leq C \frac{n^{k/2}}{(2\pi)^{k/2}}.$$

**Lemma 1.5.**

$$\text{i) } \frac{t^k \omega_k}{(\sqrt{2\pi} M_2)^k} e^{-\frac{t^2}{2M_2^2}} \leq \Gamma_{\mathbb{P}}^k(t) \leq \frac{t^k \omega_k}{(\sqrt{2\pi} M_2)^k}, \quad \forall t \geq 0$$

$$\text{ii) } \Gamma_{\mathbb{P}}^k(t) \geq 1 - 2^{k/2} e^{-\frac{t^2}{4M_2^2}}, \quad \forall t \geq 0$$

$$\text{iii) } \Gamma_{\mathbb{P}}^k(t + \delta) \leq \left(1 + \frac{\delta}{t}\right)^k \Gamma_{\mathbb{P}}^k(t), \quad \forall \delta, t > 0$$

*Proof.* i) is straightforward as for ii),

$$1 - \Gamma_{\mathbb{P}}^k(t) = \int_{|s| \geq t} \gamma_{\mathbb{P}}^k(s) ds \leq \frac{e^{-\frac{t^2}{4M_2^2}}}{(\sqrt{2\pi} M_2)^k} \int_{|s| \geq t} e^{-\frac{|s|^2}{4M_2^2}} ds \leq 2^{k/2} e^{-\frac{t^2}{4M_2^2}}$$

iii) follows from

$$\frac{\Gamma_{\mathbb{P}}^k(t + \delta)}{\Gamma_{\mathbb{P}}^k(t)} = 1 + \frac{\int_{t < |s| \leq t + \delta} \gamma_{\mathbb{P}}^k(s) ds}{\int_{|s| \leq t} \gamma_{\mathbb{P}}^k(s) ds}$$

$$\leq 1 + \omega_k \frac{((t + \delta)^k - t^k) e^{-\frac{t^2}{2M_2^2}}}{\int_{|s| \leq t} \gamma_{\mathbb{P}}^k(s) ds} \leq 1 + \frac{(t + \delta)^k - t^k}{t^k}.$$

□

**Lemma 1.6.** *Let  $n \geq k + 3$ . There exists an absolute  $C > 0$  such that*

$$\text{i) } \left| \left(1 - \frac{2u}{n}\right)^{\frac{n-k-2}{2}} - e^{-u} \right| \leq C \frac{k}{n-k}, \quad \forall u \in [0, \frac{n}{2}].$$

$$\text{ii) } \left| \left( \frac{|S^{n-k-1}|}{|S^{n-1}|} \frac{(2\pi)^{k/2}}{n^{k/2}} \right) - 1 \right| \leq C \frac{k^2}{n}$$

$$\text{iii) } \left| e^u \left(1 - \frac{2u}{n}\right)^{\frac{n-k-2}{2}} - 1 \right| \leq 8 \left( \frac{ku}{n} + \frac{u^2}{n} \right), \quad \forall u \in [0, \sqrt{n}/4], \text{ provided}$$

that  $\frac{ku}{n} + \frac{u^2}{n} \leq 1/8$ .

*Proof.* The proof of i) is the same as in [BK]. As for ii), it is a consequence of the formula  $|S^{n-1}| = \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})}$  and the asymptotic formula for the Gamma function in Lemma 1.4. Let us show the proof of iii). Write  $y = u + \frac{n-k-2}{2} \log\left(1 - \frac{2u}{n}\right)$ . We use the inequality  $|e^y - 1| \leq 2|y|$ , provided  $|y| \leq 1$  and Taylor's formula with Lagrange's error term  $\log(1-x) = -x - \frac{x^2}{2(1-\xi)^2}$ ,  $0 < \xi < x \leq 1$  with  $x = 2u/n$ . Thus,

$$\begin{aligned}
& \left| e^u \left(1 - \frac{2u}{n}\right)^{\frac{n-k-2}{2}} - 1 \right| \leq 2 \left| u + \frac{n-k-2}{2} \log \left(1 - \frac{2u}{n}\right) \right| \\
& \leq 2 \left| u - \frac{n-k-2}{2} \left( \frac{2u}{n} + \frac{4u^2}{2n^2(1-\xi)^2} \right) \right| \leq 2 \left( \frac{k+2}{n} u + \frac{(n-k-2)u^2}{(n-2u)^2} \right) \\
& \leq 2 \left( \frac{3ku}{n} + \frac{n u^2}{(n - \frac{\sqrt{n}}{2})^2} \right) \leq 8 \left( \frac{ku}{n} + \frac{u^2}{n} \right)
\end{aligned}$$

□

## 2. AVERAGE OF $k$ -DIMENSIONAL MARGINALS

Let  $\mathbb{P}$  be a Borel probability on  $\mathbb{R}^n$ . For every  $k \in \mathbb{N}$ ,  $1 \leq k \leq n$  we define the following average of  $k$ -marginals

$$A_k(\mathbb{P})(B) = \int_{O(n)} \mathbb{P}(U(B + \mathbb{R}^{n-k})) dU, \quad B \subset \mathbb{R}^k$$

$A_k(\mathbb{P})$  is a Borel probability on  $\mathbb{R}^k$  invariant under the action of the orthogonal group in  $\mathbb{R}^k$ . Clearly,  $A_k(A_n(\mathbb{P})) = A_k(\mathbb{P})$ .

The following proposition was considered in [BV], [BK] and [So] for the case  $k = 1$ .

**Proposition 2.1.** *Let  $\mathbb{P}$  be a Borel probability on  $\mathbb{R}^n$ . Then, for all  $1 \leq k < n$  and any Borel set  $B \subset \mathbb{R}^k$  we have,*

$$A_k(\mathbb{P})(B) = \mathbb{P}(\{0\})\delta_0(B) + \int_B \frac{|S^{n-k-1}|}{|S^{n-1}|} \int_{\{|x| \geq |s|\}} \left(1 - \frac{|s|^2}{|x|^2}\right)^{\frac{n-k-2}{2}} \frac{d\mathbb{P}(x)}{|x|^k} ds$$

where  $\delta_0$  is the Dirac measure at 0. The density function of the absolutely continuous part is denoted by  $s \in \mathbb{R}^k \rightarrow \varphi_{\mathbb{P}}^k(s)$ .

*Proof.* Since  $A_k(A_n(\mathbb{P})) = A_k(\mathbb{P})$  and the function in the inner integral is radial it is enough to prove it for probabilities  $\mathbb{P}$  that are invariant under orthogonal transformations. First we consider the case  $\mathbb{P} = \sigma_{n-1}$ . It is enough to prove the equality for dilates of the Euclidean ball, that is, to show that

$$\begin{aligned}
A_k(\sigma_{n-1})(rD_k) &= \int_{rD_k} \frac{|S^{n-k-1}|}{|S^{n-1}|} \int_{\{|x| \geq |s|\}} \left(1 - \frac{|s|^2}{|x|^2}\right)^{\frac{n-k-2}{2}} \frac{d\sigma_{n-1}(x)}{|x|^k} ds \\
&= \frac{|S^{n-k-1}|}{|S^{n-1}|} \int_{rD_k} (1 - |s|^2)^{\frac{n-k-2}{2}} \mathcal{X}_{D_k}(s) ds
\end{aligned}$$

$$\text{If } r \geq 1, \text{ then } A_k(\sigma_{n-1})(rD_k) = 1 = \frac{|S^{n-k-1}|}{|S^{n-1}|} \int_{D_k} (1 - |s|^2)^{\frac{n-k-2}{2}} ds.$$

If  $r < 1$ , after passing to polar coordinates, the right hand side equals to

$$\frac{|S^{n-k-1}||S^{k-1}|}{|S^{n-1}|} \int_0^r (1-t^2)^{\frac{n-k-2}{2}} t^{k-1} dt$$

On the other hand,  $A_k(\sigma_{n-1})(rD_k) = \sigma_{n-1}(rD_k \times \mathbb{R}^{n-k})$  and

$$\sigma_{n-1}(rD_k \times \mathbb{R}^{n-k}) = \frac{\omega_k |S^{n-k-1}|}{\omega_n} \left( \frac{r^k}{n} (1-r^2)^{\frac{n-k}{2}} + \int_{\sqrt{1-r^2}}^1 t^{n-k-1} (1-t^2)^{\frac{k}{2}} dt \right)$$

Now, the derivative of the two expressions are equal and we have the result. Observe that, by re-scaling, the formula also holds for the Haar probabilities on  $\lambda S^{n-1}$ ,  $\lambda > 0$ .

In the general case, we use the fact that any probability  $\mathbb{P}$  invariant under orthogonal transformations is, up to  $\mathbb{P}(\{0\})$ , the product measure of a positive measure on  $(0, \infty)$  and the Haar measure on  $S^{n-1}$  and so, it can be approximated by convex combinations of Haar probabilities on  $\lambda S^{n-1}$ ,  $\lambda > 0$ . For  $\lambda = 0$ , the associated probability is  $\delta_0$ .  $\square$

*Remark 2.2.* If  $\mathbb{P}$  is a probability with density  $f$ , that is  $\mathbb{P}(C) = \int_C f(x) dx$ , then  $A_k(\mathbb{P})(B) = \int_B \varphi_{\mathbb{P}}^k(s) ds$  where

$$\varphi_{\mathbb{P}}^k(s) = \int_{O(n)} \int_{\mathbb{R}^{n-k}} f(s_1 \xi_1 + \dots + s_n \xi_n) ds_{k+1} \dots ds_n dU$$

and  $s = (s_1, \dots, s_k)$ .

In the particular case  $\mathbb{P}(C) = |K \cap C|$  for  $K \subset \mathbb{R}^n$  a Borel set of volume 1 we have  $A_k(\mathbb{P})(B) = \int_B \varphi_K^k(s) ds$  where

$$\varphi_K^k(s) = \int_{O(n)} |(s_1 \xi_1 + \dots + s_k \xi_k + \{\xi_1, \dots, \xi_k\}^\perp) \cap K|_{n-k} dU$$

This integral, an average of sections by  $n - k$  dimensional subspaces at distance  $|s|$  from the origin is the density function of a certain average of  $k$ -dimensional marginals of  $K$  (further applications of this formula appear in [BBR]).

The following proposition gives a more geometrical interpretation of such function,

**Proposition 2.3.** *Let  $K \subset \mathbb{R}^n$  be a Borel set of volume 1. Then if  $1 \leq k < n$  and  $s \in \mathbb{R}^k$  we have*

$$\varphi_K^k(s) = \int_{S^{n-1}} \left( \int_{G(\theta^\perp, n-k)} |(|s|\theta + E) \cap K|_{n-k} d\nu(E) \right) d\sigma_{n-1}(\theta)$$

where  $G(\theta^\perp, n - k)$  is the Grassman manifold of the  $n - k$ -dimensional subspaces of the hyperplane  $\theta^\perp$  and  $d\nu(E)$  its Haar measure.

That is, consider the sphere  $|s|S^{n-1}$ ; for any  $\theta \in S^{n-1}$  we first average over all the  $(n - k)$ -dimensional sections of  $K$  at distance  $|s|$  from the origin

in the direction  $\theta$ , that is inside  $|s|\theta + \theta^\perp$  and then we average over the sphere.

*Proof.* Since  $\varphi_K^k(s)$  is radial,

$$\varphi_K^k(s) = \int_{O(n)} \left| (|s|\xi_1 + \{\xi_1, \dots, \xi_k\}^\perp) \cap K \right|_{n-k} dU.$$

Next we consider the following consequence of the conditional expectation theorem as it appears in [Ko] Lemma 1: for any (say) continuous function  $F$  on  $O(n)$ ,

$$\int_{O(n)} F(U) dU = \int_{G(n,k)} \int_{\xi_{k+1}, \dots, \xi_n \in E^\perp} dU_{n-k} \int_{\xi_1, \dots, \xi_k \in E} F(U) dU_k d\nu(E)$$

where  $dU_{n-k}$  and  $dU_k$  are the Haar measures on  $O(n-k)$  and  $O(k)$ . We apply this formula for  $k=1$  and any continuous function and we have in particular

$$\int_{O(n)} F(\xi_1, \dots, \xi_n) dU = \int_{S^{n-1}} \left( \int_{O(\xi_1^\perp)} F(\xi_1, \xi_2, \dots, \xi_n) dU_1 \right) d\sigma_{n-1}(\theta)$$

where  $O(\xi_1^\perp)$  is the orthogonal group in the hyperplane  $\xi_1^\perp$ ,  $dU_1$  its Haar measure (this formula can also be proved for any (say) continuous function  $F$  directly, by using the uniqueness of the Haar measure on  $O(n)$ ). Applying again Koldobsky's formula in the whole space  $\xi_1^\perp$  and  $n-k$  for the function  $F(\xi_1, \xi_2, \dots, \xi_n) = \left| (|s|\xi_1 + \{\xi_1, \dots, \xi_k\}^\perp) \cap K \right|_{n-k}$  we eventually get the result.  $\square$

*Remark 2.4.* Let  $E$  be a  $k$ -dimensional subspace of  $\mathbb{R}^n$ . We show some relations between the function  $F_K(t, E) := |\{x \in K : |P_E(x)| \leq t\}|$  (formula (1.1)) and the average marginal density  $\varphi_K^k(s)$ .

Fix  $\{\xi_1, \dots, \xi_k\} \subset \mathbb{R}^n$  an orthonormal basis of  $E$ . By Fubini's theorem we have

$$F_K(t, E) = \int_{|s| \leq t} \left| \left( \sum_{i=1}^k s_i \xi_i + E^\perp \right) \cap K \right|_{n-k} ds_1 \dots ds_k.$$

We now integrate when  $U = (\xi_1, \dots, \xi_k)$  runs over the orthogonal group  $O(E)$  which allows us to express  $F_K(t, E)$  as a convenient average of marginal densities.



$$\begin{aligned}
 F_K(t, E) &= \int_{O(E)} \left( \int_{|s| \leq t} \left| \left( \sum_{i=1}^k s_i \xi_i + E^\perp \right) \cap K \right|_{n-k} ds_1 \dots ds_k \right) dU \\
 &= \int_{|s| \leq t} \left( \int_{O(E)} \left| \left( \sum_{i=1}^k s_i \xi_i + E^\perp \right) \cap K \right|_{n-k} dU \right) ds_1 \dots ds_k \\
 &\quad \text{(by the invariance under the orthogonal group)} \\
 &= \int_{|s| \leq t} \left( \int_{O(E)} \left| (|s| \xi_1 + E^\perp) \cap K \right|_{n-k} dU \right) ds_1 \dots ds_k \\
 &\quad \text{(by using the Lemma 1 in [Ko])} \\
 &= \int_{|s| \leq t} \left( \int_{S_E} \left| (|s| \theta + E^\perp) \cap K \right|_{n-k} d\sigma_E(\theta) \right) ds_1 \dots ds_k \\
 &\quad \text{(by passing to polar coordinates in } E \text{)} \\
 &= |S^{k-1}| \int_0^t r^{k-1} f_K(r, E) dr
 \end{aligned}$$

where  $S_E = S^{n-1} \cap E$ ,  $\sigma_E$  its Haar probability and

$$f_K(r, E) = \int_{S_E} \left| (r\theta + E^\perp) \cap K \right|_{n-k} d\sigma_E(\theta), \quad r \geq 0.$$

Finally, observe that we also obtain formula (1.2)

$$F_K^k(t) = \int_{G_{n,k}} F_K(t, E) d\nu(E) = \int_{\{|s| \leq t\}} \varphi_K^k(s) ds$$

Our last lemma provides bounds for  $f_K(r, E)$  and  $F_K(t, E)$  that will be useful in the next section.

**Lemma 2.5.** *Let  $K \subset \mathbb{R}^n$  be an isotropic convex body and  $E \in G_{n,k}$ . Denote  $\mathcal{L}_k = \sup\{L_M \mid M \subset \mathbb{R}^k, \text{ isotropic}\}$ . There exist an absolute constant  $c_1 > 0$  such that*

$$\begin{aligned}
 i) \quad & f_K(r, E) \leq e^k f_K(r', E) \leq \frac{(c_1 \mathcal{L}_k)^k}{L_K^k}, \quad \forall r \geq r' \geq 0 \\
 ii) \quad & F_K(t, E) \geq 1 - \exp\left(-\frac{c_1 t}{L_K \sqrt{k}}\right) \quad \forall t \geq 0.
 \end{aligned}$$

*Proof.* *i)* A result by Fradelizi, see [F], states that

$$\left| (r\theta + E^\perp) \cap K \right|_{n-k} \leq e^k \left| (r'\theta + E^\perp) \cap K \right|_{n-k}, \quad \forall r \geq r' \geq 0$$

and the first inequality follows. As for the second inequality, it is a consequence of the previous one (for  $r' = 0$ ) and a result by Ball and Milman and Pajor, see [B], [MP], which states that  $|E^\perp \cap K|_{n-k} \leq \frac{(c_2 \mathcal{L}_k)^k}{L_K^k}$ .

ii) It is a consequence of a more general result: Let  $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a linear map such that  $\dim T(\mathbb{R}^n) = k$ , then

$$|\{x \in K; |T(x)| \leq t\}| \geq 1 - \exp\left(-\frac{c_1 t}{L_K \|T\|_{HS}}\right) \quad \forall t \geq 0$$

where  $\|T\|_{HS}$  denotes the Hilbert-Schmidt norm. Indeed, Borell's inequality (see [MS]) states

$$\left|\{x \in K; \frac{|T(x)|}{(\int_K |Tx|^2 dx)^{1/2}} > t\}\right| \leq \exp(-c_1 t) \quad \forall t \geq 0$$

We can suppose that  $T = \sum_{j=1}^k u_j \otimes e_j$ , where  $\{u_j\}_{j=1}^k$  are  $k$  vectors in  $\mathbb{R}^n$  and  $\{e_j\}_{j=1}^k$  is an orthonormal basis in the subspace  $T(\mathbb{R}^n)$ . Then

$$\int_K |Tx|^2 dx = \int_K \sum_{j=1}^k |\langle u_j, x \rangle|^2 dx = \sum_{j=1}^k |u_j|^2 \int_K |\langle \frac{u_j}{|u_j|}, x \rangle|^2 dx = L_K^2 \|T\|_{HS}^2$$

In our case simply take  $T = P_E$ . □

### 3. ESTIMATING THE QUOTIENT.

Our aim is to estimate  $|\frac{F_K(t, E)}{\Gamma_K^k(t)} - 1|$  for a random  $k$ -dimensional subspace  $E \subset \mathbb{R}^n$ . Some of the steps hold true for more general probabilities  $\mathbb{P}$  and we will state them in full generality. The following hypothesis will be imposed on  $\mathbb{P}$  throughout the section.

**Concentration hypothesis ([So]):**

$$(3.3) \quad \mathbb{P}\{x \in \mathbb{R}^n : ||x| - \sqrt{n}M_2| > t\sqrt{n}M_2\} \leq A \exp(-Bn^\alpha t^\beta)$$

for all  $0 \leq t \leq 1$  and for some constants  $\alpha, \beta, A, B > 0$ .

#### 3.1. Gaussian approximation of the average density and distribution.

We first consider Gaussian approximation of the average density  $\varphi_{\mathbb{P}}^k(s)$ .

**Theorem 3.1.** *Let  $\mathbb{P}$  a probability on  $\mathbb{R}^n$  satisfying (3.3) and  $M_2, M_{\mathbb{P}} < \infty$ . Denote  $h(n) = n^{\min\{\alpha, \frac{\alpha}{\beta}, \frac{1}{2}\}}$  and let  $\tilde{h}(n)$  such that  $\tilde{h}(n) < c(B, \beta)h(n)$  with  $c = c(B, \beta) = \min\{\frac{1}{8}, (\frac{B}{2})^{\min\{1, \frac{1}{\beta}\}}\}$ . Then, if  $k \leq \frac{c \tilde{h}(n)}{\log(1 + M_2 M_{\mathbb{P}}^{1/n})}$  we*

have

$$\sup_{|s| \leq \sqrt{\tilde{h}(n)} M_2} \left| \frac{\varphi_{\mathbb{P}}^k(s)}{\gamma_{\mathbb{P}}^k(s)} - 1 \right| \leq c_1 \frac{\tilde{h}(n)}{h(n)}$$

for some constant  $c_1 = c_1(\alpha, A, \beta, B) > 0$

*Proof.* Recall that, by Proposition 2.1,  $\varphi_{\mathbb{P}}^k(s) = \int_{\{|x| \geq |s|\}} g_{|s|}(|x|) d\mathbb{P}(x)$ , where

$$g_t(r) = \frac{|S^{n-k-1}|}{|S^{n-1}|} \frac{1}{r^k} \left(1 - \frac{t^2}{r^2}\right)^{\frac{n-k-2}{2}}, r \geq t > 0.$$

Consider the image probability of  $\mathbb{P}$  under the map  $x \rightarrow |x|$ , that is, the probability on  $[0, \infty)$  also denoted by  $\mathbb{P}$  with distribution function  $\mathbb{P}\{x \in \mathbb{R}^n \mid |x| \leq r\}$ . With this notation,

$$\varphi_{\mathbb{P}}^k(s) = \int_{[|s|, \infty)} g_{|s|}(r) d\mathbb{P}(r)$$

In order to estimate asymptotic behaviour of  $\varphi_{\mathbb{P}}^k(s)$  as  $n \rightarrow \infty$  we write

$$(3.4) \quad \varphi_{\mathbb{P}}^k(s) = g_{|s|}(\sqrt{n}M_2) \mathbb{P}\{|x| \geq |s|\} + \int_{[|s|, \infty)} (g_{|s|}(r) - g_{|s|}(\sqrt{n}M_2)) d\mathbb{P}(r)$$

Write  $g_{|s|}(r) - g_{|s|}(\sqrt{n}M_2) = \int_{\sqrt{n}M_2}^r g'_{|s|}(u) du$ . By using Fubini's theorem in (3.4), it is easy to see that

$$\begin{aligned} \varphi_{\mathbb{P}}^k(s) - g_{|s|}(\sqrt{n}M_2) &= -g_{|s|}(2\sqrt{n}M_2) \mathbb{P}\{2\sqrt{n}M_2 \leq |x|\} \\ &\quad - \int_{|s|}^{\sqrt{n}M_2} g'_{|s|}(r) \mathbb{P}\{|x| \leq r\} dr \\ &\quad + \int_{\sqrt{n}M_2}^{2\sqrt{n}M_2} g'_{|s|}(r) \mathbb{P}\{|x| > r\} dr \\ &\quad + \int_{[2\sqrt{n}M_2, \infty)} g_{|s|}(r) d\mathbb{P}(r) \end{aligned}$$

The summands above are estimated with the help of the following three technical lemmas which extend the ideas in [So] to a general  $k$  for  $|s|$  far from the origin. The behaviour at the origin (included in Lemma 3.4) is estimated via the parameter  $M_{\mathbb{P}}$ .

**Lemma 3.2.** *If  $|s| \leq \sqrt{\frac{n}{2}} M_2$ , then*

$$\int_{[2\sqrt{n}M_2, \infty)} \frac{g_{|s|}(r)}{g_{|s|}(\sqrt{n}M_2)} d\mathbb{P}(r) \leq \frac{A}{2^k} \exp\left(\frac{|s|^2}{M_2^2} - Bn^\alpha\right)$$

and

$$\frac{g_{|s|}(2\sqrt{n}M_2)}{g_{|s|}(\sqrt{n}M_2)} \mathbb{P}\{2\sqrt{n}M_2 \leq |x|\} \leq \frac{A}{2^k} \exp\left(\frac{|s|^2}{M_2^2} - Bn^\alpha\right).$$

*Proof.* Use the elementary inequalities  $(1-x)^{-1} \leq e^{2x}$ ,  $0 \leq x \leq 1/2$  and  $1-x \leq e^{-x}$ ,  $x \geq 0$ .  $\square$

**Lemma 3.3.** *If  $\left(\frac{|s|^2}{M_2^2}\right)^{\max\{\beta,1\}} < \frac{Bn^\alpha}{2}$ , we have*

$$\int_{\sqrt{n}M_2}^{2\sqrt{n}M_2} \frac{|g'_{|s|}(r)|}{g_{|s|}(\sqrt{n}M_2)} \mathbb{P}\{|x| > r\} dr \leq \max\left\{\frac{|s|^2}{M_2^2}, k\right\} \frac{A c(\beta)}{(Bn^\alpha)^{1/\beta}}.$$

*Proof.* By straightforward computations and the inequalities  $(1-x)^{-1} \leq e^{2x}$ ,  $0 \leq x \leq 1/2$  and  $1-x \leq e^{-x}$ ,  $x \geq 0$ ,

$$\frac{|g'_{|s|}(r)|}{g_{|s|}(\sqrt{n}M_2)} \leq \frac{|(n-2)|s|^2 - kr^2|}{r^3} e^{-\frac{(n-k-4)|s|^2}{2r^2}} e^{\frac{(n-k-2)|s|^2}{2nM_2^2}}$$

For  $r \in [\sqrt{n}M_2, 2\sqrt{n}M_2]$ ,

$$\frac{|(n-2)|s|^2 - kr^2|}{r^3} \leq \frac{c}{r} \max\left\{\frac{|s|^2}{M_2^2}, k\right\}.$$

On the other hand,  $\frac{(n-k-2)|s|^2}{2nM_2^2} - \frac{(n-k-4)|s|^2}{2r^2} \leq 1 + \frac{(n-k-4)|s|^2}{2nM_2^2} \left(1 - \frac{nM_2^2}{r^2}\right) \leq 1 + \frac{|s|^2}{2M_2^2} \left(1 - \frac{nM_2^2}{r^2}\right)$ .

After using such bounds, the change of variables  $r = (1+u)\sqrt{n}M_2$  and the inequality  $1 - \frac{1}{(1+u)^2} \leq 2u$ ,  $u \geq 0$  yield

$$\int_{\sqrt{n}M_2}^{2\sqrt{n}M_2} \frac{|g'_{|s|}(r)|}{g_{|s|}(\sqrt{n}M_2)} \mathbb{P}\{|x| > r\} dr \leq c \int_0^1 e^{\frac{|s|^2}{M_2^2}u} \mathbb{P}\{|x| > (1+u)\sqrt{n}M_2\} du$$

Now use the concentration hypothesis (3.3). The proof finishes by estimating the remaining integral with the aid of the following *Claim* (with  $K = \frac{|s|^2}{M_2^2}$  and  $L = Bn^\alpha$ ), see [So] Lemma 9.

*Claim.* Let  $K, L > 0$  such that  $K^{\max\{\beta,1\}} < L/2$ . Then,

$$\int_0^1 \exp(Ku - Lu^\beta) du \leq \frac{c(\beta)}{L^{1/\beta}}.$$

□

**Lemma 3.4.** *There exists  $c > 0$  such that whenever  $|s| \leq \sqrt{\frac{n}{2}} M_2$ ,  $k < n/2$  and  $\left(8k \log(cM_2M_{\mathbb{P}}^{1/n})\right)^{\max\{1,\beta\}} < \frac{B}{2} n^\alpha$ ,*

$$\int_{|s|}^{\sqrt{n}M_2} \frac{|g'_{|s|}(r)|}{g_{|s|}(\sqrt{n}M_2)} \mathbb{P}\{|x| \leq r\} dr \leq \max\left\{\frac{|s|^2}{M_2^2}, k\right\} \frac{A c(\beta)}{(Bn^\alpha)^{1/\beta}} + \frac{1}{2^n}$$

*Proof.* Denote  $\lambda := (cM_2M_{\mathbb{P}}^{1/n})^{-2}$ , with  $c > 0$  to be chosen below and split the integral into two parts

$$\int_{\max\{|s|, \lambda\sqrt{n}M_2\}}^{\sqrt{n}M_2} + \int_{|s|}^{\max\{|s|, \lambda\sqrt{n}M_2\}} = I_1 + I_2$$

By Hensley's Lemma 1.3 we choose  $c$  so that  $\lambda < 1$ . It is easy to see that

$$\frac{|g'_{|s|}(r)|}{g_{|s|}(\sqrt{n}M_2)} \leq 2 \frac{|(n-2)|s|^2 - kr^2|}{r^{k+3}} (\sqrt{n}M_2)^k$$

The change of variables  $r = \sqrt{n}M_2u$  and the inequality  $\frac{|(n-2)|s|^2 - knM_2^2u^2|}{nM_2^2} \leq \max\left\{\frac{|s|^2}{M_2^2}, k\right\}$ ,  $0 \leq u \leq 1$  yield to

$$I_1 \leq 2 \max\left\{\frac{|s|^2}{M_2^2}, k\right\} \int_{\frac{\max\{|s|, \lambda\sqrt{n}M_2\}}{\sqrt{n}M_2}}^1 \frac{\mathbb{P}\{|x| \leq \sqrt{n}M_2u\}}{u^{k+3}} du$$

Set  $a = \frac{\max\{|s|, \lambda\sqrt{n}M_2\}}{\sqrt{n}M_2}$ . We have  $0 \leq a \leq 1$ . By the change of variables  $u = 1 - v$  and the concentration hypothesis (3.3),

$$I_1 \leq 2A \max\left\{\frac{|s|^2}{M_2^2}, k\right\} \int_0^{1-a} \exp(-(k+3)\log(1-v) - Bn^\alpha v^\beta) dv$$

Finally, use the inequalities

$$-\log(1-v) \leq \frac{-\log a}{1-a}v \leq \log\left(\frac{3}{a}\right)v \leq 2\log(cL_{\mathbb{P}})v, \quad v \in [0, 1-a]$$

and the *Claim* above.

For the second integral  $I_2$  we can suppose  $|s| \leq \lambda\sqrt{n}M_2$ . Proceeding as before, we have

$$I_2 \leq 2 \int_{|s|}^{\lambda\sqrt{n}M_2} \frac{|(n-2)|s|^2 - kr^2|}{r^{k+3}} (\sqrt{n}M_2)^k \mathbb{P}\{|x| \leq r\} dr$$

By the inequality  $|(n-2)|s|^2 - kr^2| \leq nr^2$ , the definition of  $M_{\mathbb{P}}$  and  $k < n/2$  we have

$$I_2 \leq 2n(\sqrt{n}M_2)^k M_{\mathbb{P}} \omega_n \int_{|s|}^{\lambda\sqrt{n}M_2} r^{n-k-1} dr \leq 4 (L_{\mathbb{P}} \omega_n^{1/n} \sqrt{n} \lambda^{1/2})^n$$

since  $\lambda^{n-k} < \lambda^{n/2}$ . Finally, the sequence  $\omega_n^{1/n} \sqrt{n}$  is bounded by an absolute constant and we can choose  $c > 0$  in the definition of  $\lambda$  so that  $I_2 \leq 2^{-n}$ .  $\square$

*End of proof of Theorem 3.1:*

Notice that the hypothesis of the Lemmas are satisfied and therefore

$$\begin{aligned} \left| \frac{\varphi_{\mathbb{P}}^k(s)}{g_{|s|}(\sqrt{n}M_2)} - 1 \right| &\leq \frac{A}{2^{k-1}} \exp\left(\frac{|s|^2}{M_2^2} - Bn^\alpha\right) \\ &\quad + \frac{c(A, B, \beta)}{n^{\alpha/\beta}} \max\left\{\frac{|s|^2}{M_2^2}, k\right\} + \frac{1}{2^n} \end{aligned}$$

Finally, use the inequality

$$\left| \frac{\varphi_{\mathbb{P}}^k(s)}{\gamma_{\mathbb{P}}^k(s)} - 1 \right| \leq \frac{g_{|s|}(\sqrt{n}M_2)}{\gamma_{\mathbb{P}}^k(s)} \left| \frac{\varphi_{\mathbb{P}}^k(s)}{g_{|s|}(\sqrt{n}M_2)} - 1 \right| + \left| \frac{g_{|s|}(\sqrt{n}M_2)}{\gamma_{\mathbb{P}}^k(s)} - 1 \right|$$

By Lemma 1.6 ii), iii) for  $u = \frac{|s|^2}{2M_2^2}$ , we have  $\frac{g_{|s|}(\sqrt{n}M_2)}{\gamma_{\mathbb{P}}^k(s)} \leq c_1$  and

$$\begin{aligned} \left| \frac{\varphi_{\mathbb{P}}^k(s)}{\gamma_{\mathbb{P}}^k(s)} - 1 \right| &\leq c_1 \left( \exp(\tilde{h}(n) - Bn^\alpha) + \frac{\tilde{h}(n)}{n^{\alpha/\beta}} + \frac{1}{2^n} \right) + \\ &+ \left[ \frac{8}{n} \left( \frac{k|s|^2}{2M_2^2} + \frac{|s|^4}{4M_2^4} \right) \right] \leq c_1 \left( \frac{\tilde{h}(n)}{n^{\alpha/\beta}} + \frac{\tilde{h}^2(n)}{n} \right) \end{aligned}$$

which finishes the proof.  $\square$

For  $\mathbb{P}$  the uniform measure on an isotropic convex body  $K$  we obtain as a corollary,

**Theorem 3.5.** *Let  $K \subset \mathbb{R}^n$  be an isotropic convex body satisfying the concentration hypothesis (3.3). For some  $c, c_1$  depending on the constants in (3.3),*

$$1) \text{ If } k \leq c \frac{\tilde{h}(n)}{\log(1+L_K)},$$

$$\sup_{|s| \in I} \left| \frac{\varphi_K^k(s)}{\gamma_K^k(s)} - 1 \right| \leq c_1 \frac{\tilde{h}(n)}{h(n)}$$

where  $I = [0, L_K \sqrt{\tilde{h}(n)}]$ .

$$2) \text{ If } k \leq c \frac{\tilde{h}(n)}{\log^2 n},$$

$$\sup_{t \geq 0} \left| \frac{F_K^k(t)}{\Gamma_K^k(t)} - 1 \right| \leq c_1 \frac{\tilde{h}(n)}{h(n)}$$

*Proof.* The statement 1) is a consequence Theorem 3.1, since in our case  $M_{\mathbb{P}} = 1$ ,  $M_2 = L_K$ .

Part 2) follows from 1). Indeed, by the Lemmas 2.5 ii) and 1.5 ii),

$$\left| F_K(t, E) - \Gamma_K^k(t) \right| = \left| (1 - F_K(t, E)) - (1 - \Gamma_K^k(t)) \right| \leq e^{-\frac{c_2 t}{L_K \sqrt{k}}} + 2^{k/2} e^{-\frac{t^2}{4L_K^2}}$$

Therefore, in the range  $t \geq C \log n \sqrt{k} L_K$  (for suitable  $C > 0$ ) we trivially have  $\left| F_K(t, E) - \Gamma_K^k(t) \right| \leq \frac{2}{n}$  for every  $k$ -dimensional subspace  $E$ . For that range of  $t$ , Lemma 1.5 ii) gives  $\Gamma_K^k(t) \geq c_0 > 0$  and so  $\left| \frac{F_K(t, E)}{\Gamma_K^k(t)} - 1 \right| \leq \frac{c_1}{n}$ .

Finally, observe that  $t \leq C \log n \sqrt{k} L_K$  implies  $t \leq L_K \sqrt{\tilde{h}(n)}$  and so by integrating 1) and formula (1.2) we have the result.  $\square$

*Example.* It is proved in [So] that the uniform probability on the unit ball of  $\ell_p^n$ ,  $p \geq 1$  verifies the concentration hypothesis (3.3) for  $\alpha = \frac{1}{2} \min\{p, 2\}$  and  $\beta = \min\{p, 2\}$ . So,  $h(n) = \sqrt{n}$  and by taking  $\tilde{h}(n) = o(h(n))$ , Theorem 3.5.1) implies that  $\sup_{|s| \in I} \left| \frac{\varphi_K^k(s)}{\gamma_K^k(s)} - 1 \right| \rightarrow 0$  as  $n \rightarrow \infty$  for  $I = [0, o(n^{1/4})]$  and  $k = o(n^{1/2})$  (since in this case  $L_K$  is uniformly bounded by a constant depending only on  $p$ ).

If we study the behaviour at  $t = 0$  of Theorem 3.1 2), we obtain the following strong form of reverse Hölder's inequality in the spirit of [V].

**Corollary 3.6.** *Let  $K \subset \mathbb{R}^n$  in the family of isotropic convex bodies satisfying (3.3). If  $k = o(\frac{h(n)}{\log^2 n})$ ,*

$$\left( \int_K |x|^2 \right)^{1/2} \left( \int_K \frac{dx}{|x|^k} \right)^{1/k} \rightarrow 1 \quad \text{as } n \rightarrow \infty$$

*Proof.* By Remark 2.4. and L'Hopital's rule

$$\lim_{t \rightarrow 0^+} \frac{F_K(t, E)}{\Gamma_K^k(t)} = \lim_{t \rightarrow 0^+} \frac{f_K(t, E)}{(\sqrt{2\pi}L_K)^{-k} e^{-\frac{t^2}{2L_K^2}}} = (\sqrt{2\pi}L_K)^k |E^\perp \cap K|_{n-k}$$

Therefore,

$$\lim_{t \rightarrow 0^+} \frac{F_K^k(t)}{\Gamma_K^k(t)} = (\sqrt{2\pi}L_K)^k \int_{G_{n, n-k}} |E \cap K|_{n-k} dv$$

But this is equal to

$$(\sqrt{2\pi}L_K)^k \frac{\omega_{n-k}}{\omega_n} \tilde{W}_k(K) = (\sqrt{2\pi}L_K)^k \frac{|S^{n-k-1}|}{|S^{n-1}|} \int_K \frac{dx}{|x|^k}$$

by the dual Kubota formula, where  $\tilde{W}_k(K)$  denotes the  $k$ -th dual mixed volume of  $K$  (see [BBR]). Since  $L_K^2 = \frac{1}{n} \int_K |x|^2 dx$ , we have

$$\lim_{t \rightarrow 0^+} \frac{F_K^k(t)}{\Gamma_K^k(t)} = \frac{(2\pi)^{k/2}}{n^{k/2}} \frac{|S^{n-k-1}|}{|S^{n-1}|} \left( \int_K |x|^2 \right)^{k/2} \left( \int_K \frac{dx}{|x|^k} \right)$$

By Lemma 1.6 ii) and Theorem 3.5.2), the result follows.  $\square$

### 3.2. Gaussian behaviour of a typical subspace.

The main tool of this subsection is the concentration of measure phenomena in the space  $G_{n,k}$  equipped with its Haar probability and the distance  $\|P_{E_1} - P_{E_2}\|_{HS}$ ,  $E_1, E_2 \in G_{n,k}$ , where  $P_E$  is the orthogonal projection onto  $E$ . Recall that the modulus of continuity of a continuous  $f: G_{n,k} \rightarrow \mathbb{R}$  is

$$\omega(a) = \sup_{\|P_{E_1} - P_{E_2}\|_{HS} \leq a} |f(E_1) - f(E_2)|, \quad a > 0$$

**Theorem 3.7** (Concentration of measure). *Denote by  $\nu$  the Haar probability on  $G_{n,k}$ . Let  $f: G_{n,k} \rightarrow \mathbb{R}$  continuous. There exist absolute constants  $c_1, c_2 > 0$  such that for every  $a > 0$ ,*

$$\nu \{E \in G_{n,k}; |f(E) - \mathbb{E}(f(E))| > \omega(a)\} \leq c_1 \exp(-c_2 n a^2)$$

*Proof.* The inequality above can be stated with  $G_{n,k}$  equipped with the distance

$$d(E_1, E_2) = \min \left\{ \left( \sum_{j=1}^k |u_j - v_j|^2 \right)^{1/2} \mid (u_j), (v_j) \text{ orthonormal basis of } E_1, E_2 \right\}$$

for  $E_1, E_2 \in G_{n,k}$  (see [MS]).

In order to finish the proof we show  $\|P_{E_1} - P_{E_2}\|_{HS} \leq \sqrt{2} d(E_1, E_2)$ . Indeed, for any  $(u_j), (v_j)$  orthonormal basis of  $E_1, E_2$  we write  $P_{E_1} = \sum_{j=1}^k u_j \otimes u_j$  and  $P_{E_2} = \sum_{i=1}^k v_i \otimes v_i$  and by definition

$$\|P_{E_1} - P_{E_2}\|_{HS}^2 = 2k - 2 \sum_{i,j=1}^k \langle u_j, v_i \rangle^2 \leq 2 \sum_{j=1}^k (1 - \langle u_j, v_j \rangle^2) \leq 2 \sum_{j=1}^k |u_j - v_j|^2$$

since  $1 - \langle u_j, v_j \rangle^2 \leq 2(1 - \langle u_j, v_j \rangle) = |u_j - v_j|^2$ . □

We will compute the modulus of continuity of  $E \rightarrow \frac{F_K(t, E)}{\Gamma_K^k(t)}$ :

**Lemma 3.8.** *Let  $0 < \varepsilon < 1$ ,  $t > 0$  and  $K \subset \mathbb{R}^n$  an isotropic convex body. Then for every  $E_1, E_2 \in G_{n,k}$  and some universal constant  $c > 0$  we have*

$$\left| \frac{F_K(t, E_1)}{\Gamma_K^k(t)} - \frac{F_K(t, E_2)}{\Gamma_K^k(t)} \right| \leq \varepsilon$$

provided that  $\|P_{E_1} - P_{E_2}\|_{HS} \leq a$  where

$$a = \begin{cases} \frac{c^k \varepsilon^2 t^2}{L_K^2 \mathcal{L}_k^k}, & \text{if } t \leq 2\sqrt{k} L_K; \\ \frac{c \varepsilon^2}{t^{k-1}}, & \text{otherwise.} \end{cases}$$

*Proof.* Let  $0 < \delta (< t)$  to be fixed later. By the triangle inequality,

$$F_K(t, E_2) - F_K(t, E_1) \leq F_K(t + \delta, E_1) - F_K(t, E_1) + |\{x \in K, |(P_{E_1} - P_{E_2})(x)| \geq \delta\}|$$

Let us estimate each summand. By Remark 2.4,

$$F_K(t + \delta, E_1) - F_K(t, E_1) = |S^{k-1}| \int_t^{t+\delta} r^{k-1} f_K(r, E_1) dr$$

we can apply Lemma 2.5 i) and

$$F_K(t + \delta, E_1) - F_K(t, E_1) \leq |S^{k-1}| c^k \frac{\mathcal{L}_k^k}{L_K^k} \frac{1}{k} ((t + \delta)^k - t^k)$$



By the mean value theorem,  $(t + \delta)^k - t^k \leq k(t + \delta)^{k-1}\delta \leq k2^{k-1}t^{k-1}\delta$  so,

$$F_K(t + \delta, E_1) - F_K(t, E_1) \leq |S^{k-1}| c^k \frac{\mathcal{L}_k^k}{L_K^k} t^{k-1} \delta$$

Now we compute the second summand. Repeat the arguments in Lemma 2.5 ii) with  $T = P_{E_1} - P_{E_2}$  and we have,

$$|\{x \in K, |(P_{E_1} - P_{E_2})(x)| \geq \delta\}| \leq \exp\left(-\frac{c_1 \delta}{L_K \|P_{E_1} - P_{E_2}\|_{HS}}\right)$$

Put the estimates together, exchange  $E_1$  and  $E_2$  and conclude that

$$(3.5) \quad \left| \frac{F_K(t, E_1)}{\Gamma_K^k(t)} - \frac{F_K(t, E_2)}{\Gamma_K^k(t)} \right| \leq \frac{|S^{k-1}| c^k \mathcal{L}_k^k}{L_K^k \Gamma_K^k(t)} t^{k-1} \delta + \frac{\exp\left(-\frac{c_1 \delta}{L_K a}\right)}{\Gamma_K^k(t)}$$

If  $t \geq 2\sqrt{k}L_K$ , Lemma 1.5 ii) gives  $\Gamma_K^k(t) \geq c_0 > 0$ . Take  $\delta = \frac{c \varepsilon L_K}{t^{k-1}}$  ( $c > 0$  small enough). Substituting in formula (3.5) together with  $|S^{k-1}| \leq \frac{c_0^k}{k^{k/2}}$  (Lemma 1.4.ii),  $\mathcal{L}_k \leq c_1 k^{1/4}$  ([K12]) and  $L_K \geq c_2$ , we have

$$\left| \frac{F_K(t, E_1)}{\Gamma_K^k(t)} - \frac{F_K(t, E_2)}{\Gamma_K^k(t)} \right| \leq \frac{\varepsilon}{2} + \exp\left(-\frac{c_3 \varepsilon}{t^{k-1} a}\right) \quad \text{if } \|P_{E_1} - P_{E_2}\|_{HS} \leq a$$

Finally set  $a = \frac{c_3 \varepsilon^2}{t^{k-1}}$  so the second summand reads  $\exp(-\frac{1}{\varepsilon}) \leq \frac{\varepsilon}{2}$ .

If  $t \leq 2\sqrt{k}L_K$ , Lemma 1.5 i) implies  $\Gamma_K^k(t) \geq \frac{e^{-2k} t^k \omega_k}{(\sqrt{2\pi}L_K)^k}$ . We substitute this estimate in formula (3.5) and so,

$$\left| \frac{F_K(t, E_1)}{\Gamma_K^k(t)} - \frac{F_K(t, E_2)}{\Gamma_K^k(t)} \right| \leq c^k \mathcal{L}_k^k \frac{\delta}{t} + \frac{c^k L_K^k}{\omega_k t^k} \exp\left(-\frac{c\delta}{L_K a}\right)$$

We take  $\delta = \frac{\varepsilon t}{2c^k \mathcal{L}_k^k}$ . Thus the first summand is equal to  $\frac{\varepsilon}{2}$ . With this choice of  $\delta$ , if we also write  $u = \frac{t}{2L_K} \in [0, \sqrt{k}]$  the second summand becomes

$$\frac{c_1^k}{\omega_k u^k} \exp\left(-\frac{\varepsilon u}{\mathcal{L}_k^k c_2^k a}\right)$$

Finally set  $a = \frac{c \varepsilon^2 u^2}{c_2^k k^{3/2} \mathcal{L}_k^k}$  for some appropriately chosen  $c > 1$  and substitute in the previous formula,

$$\frac{c_1^k}{\omega_k u^k} \exp\left(-\frac{k^{3/2} c}{\varepsilon u}\right) := h(u)$$

The maximum value of  $h$  is obtained at  $u_0$  so that  $h'(u_0) = 0$ , that is  $u_0 = \frac{c \sqrt{k}}{\varepsilon}$  and  $h(u_0) = \frac{c_1^k \varepsilon^k}{\omega_k c^k k^{k/2}} e^{-k} \leq \frac{\varepsilon}{2}$ .

□

Next, we apply Theorem 3.7. Recall that  $c_1 \frac{\tilde{h}(n)}{h(n)}$  is the error term in Theorem 3.5.

**Lemma 3.9.** *Let  $0 < \varepsilon < 1$ ,  $t > 0$ ,  $K \subset \mathbb{R}^n$  an isotropic convex body satisfying the concentration hypothesis (3.3) and  $k \leq c \frac{\tilde{h}(n)}{\log^2 n}$ . Then,*

$$\nu \left\{ E \in G_{n,k}; \left| \frac{F_K(t, E)}{\Gamma_K^k(t)} - 1 \right| > \varepsilon + c_1 \frac{\tilde{h}(n)}{h(n)} \right\} \leq c_1 \exp(-c_2 a^2 n)$$

where

$$a = \begin{cases} \frac{c^k \varepsilon^2 t^2}{L_K^2 C_k^k}, & \text{if } t \leq 2\sqrt{k} L_K ; \\ \frac{c \varepsilon^2}{t^{k-1}}, & \text{otherwise .} \end{cases}$$

*Proof.* Theorem 3.5 states

$$\left| \frac{F_K^k(t)}{\Gamma_K^k(t)} - 1 \right| \leq c_1 \frac{\tilde{h}(n)}{h(n)}$$

Hence,

$$\begin{aligned} \nu \left\{ E; \left| \frac{F_K(t, E)}{\Gamma_K^k(t)} - 1 \right| > \varepsilon + c_1 \frac{\tilde{h}(n)}{h(n)} \right\} &\leq \\ &\leq \nu \left\{ E; \left| \frac{F_K(t, E)}{\Gamma_K^k(t)} - \frac{F_K^k(t)}{\Gamma_K^k(t)} \right| > \varepsilon \right\} \leq c_1 \exp(-c_2 n a^2) \end{aligned}$$

since Lemma 3.8 reads  $\omega(a) \leq \varepsilon$  □

In our last result of the section we pass from Lemma 3.9, valid for any fixed  $t$ , to a statement that holds for every  $t$  simultaneously.

**Lemma 3.10.** *Let  $0 < \varepsilon < \frac{1}{2}$ ,  $t_0 > 0$ ,  $K \subset \mathbb{R}^n$  an isotropic convex body satisfying (3.3) and  $k \leq c \frac{\tilde{h}(n)}{\log^2 n}$ . Suppose  $c_1 \frac{\tilde{h}(n)}{h(n)} \leq \frac{1}{2}$ . Then*

$$\nu \left\{ E \in G_{n,k}; \left| \frac{F_K(t, E)}{\Gamma_K^k(t)} - 1 \right| \leq 2\varepsilon + 2c_1 \frac{\tilde{h}(n)}{h(n)}, \forall t \geq t_0 \right\} \geq 1 - N \exp(-c A^2 n)$$

where

$$N \sim \left( \frac{c \log n \sqrt{k} L_K}{t_0} \right)^{\frac{c_1 k}{\varepsilon}} \quad \text{and} \quad A = \frac{c_2^k}{k^{k/2}} \varepsilon^2 \min \left\{ \frac{t_0^2}{L_K^2}, \frac{1}{L_K^{k-1} (\log n)^{k-1}} \right\}$$

*Proof.* By the arguments in the proof of Theorem 3.5 2), we only need to compute the probability  $\forall t \in [t_0, T]$  with  $T = C \log n \sqrt{k} L_K$ .

Pick  $0 < t_0 < t_1 \leq t_2 \leq \dots \leq t_N = T$  in the following way

$$t_i = t_0 \prod_{j=1}^i \left( 1 + \frac{\varepsilon}{8kj} \right) \sim t_0 i^{\frac{c_1 \varepsilon}{k}} \quad i = 1, \dots, N$$

Write  $\eta = 2\varepsilon + 2c_1 \frac{\tilde{h}(n)}{h(n)}$ ,  $0 < \eta < 2$ . By Lemma 3.9,

$$\nu \left\{ E; \left| \frac{F_K(t_i, E)}{\Gamma_K^k(t_i)} - 1 \right| > \frac{\eta}{2}, \text{ for some } i \right\} \leq c_1 \sum_{i=0}^N \exp(-c_2 n a_i^2)$$

where

$$a_i = \begin{cases} \frac{c^k \varepsilon^2 t_i^2}{L_K^2 \mathcal{L}_k^k}, & \text{if } t_i \leq 2\sqrt{k}L_K; \\ \frac{c\varepsilon^2}{t_i^{k-1}}, & \text{otherwise.} \end{cases}$$

If  $t \in [t_i, t_{i+1}]$ , the fact that

$$\left| \frac{F_K(t, E)}{\Gamma_K^k(t)} - 1 \right| > \eta$$

implies that either

$$F_K(t_{i+1}, E) > (1 + \eta)\Gamma_K^k(t_i) \quad \text{or} \quad F_K(t_i, E) < (1 - \eta)\Gamma_K^k(t_{i+1}).$$

Taking into account the choice of  $t_i$ , Lemma 1.5 iii) (with  $t = t_i$ ,  $\delta = t_{i+1} - t_i$ ) reads

$$\frac{\Gamma_K^k(t_{i+1})}{\Gamma_K^k(t_i)} \leq \left( \frac{t_{i+1}}{t_i} \right)^k \leq \left( 1 + \frac{\varepsilon}{8k(j+1)} \right)^k \leq e^{\varepsilon/8} \leq 1 + \frac{\eta}{4}$$

and so, by the elementary inequalities  $(1 + \eta)(1 + \frac{\eta}{4})^{-1} \geq (1 + \frac{\eta}{2})$  and  $(1 - \eta)(1 + \frac{\eta}{4}) < 1 - \frac{\eta}{2}$  we have that either

$$F_K(t_{i+1}, E) > (1 + \frac{\eta}{2})\Gamma_K^k(t_{i+1}) \quad \text{or} \quad F_K(t_i, E) < (1 - \frac{\eta}{2})\Gamma_K^k(t_i)$$

thus,

$$\begin{aligned} & \nu \left\{ E \in G_{n,k} \mid \left| \frac{F_K(t, E)}{\Gamma_K^k(t)} - 1 \right| > \eta, \text{ for some } t \in [t_0, T] \right\} \leq \\ & \leq 2 \nu \left\{ E \in G_{n,k} \mid \left| \frac{F_K(t_i, E)}{\Gamma_K^k(t_i)} - 1 \right| > \frac{\eta}{2}, \text{ for some } i \right\} \leq c_1 N \exp(-c_2 n A^2) \end{aligned}$$

where  $A = \min_{1 \leq i \leq N} a_i$ . By definition,

$$c \log n \sqrt{k} L_K = T = t_N \sim t_0 N^{\frac{c_1 \varepsilon}{k}}$$

That is,

$$N \sim \left( \frac{c \log n \sqrt{k} L_K}{t_0} \right)^{\frac{c_1 k}{\varepsilon}} \quad \text{and} \quad A \geq \frac{c_2^k}{k^{k/2}} \varepsilon^2 \min \left\{ \frac{t_0^2}{L_K^2}, \frac{1}{L_K^{k-1} (\log n)^{k-1}} \right\}$$

□

**Theorem 3.11.** *Let  $K \subset \mathbb{R}^n$  in the family of isotropic convex bodies satisfying  $L_K \leq c$  and condition (3.3). Then, for every  $0 < \varepsilon < 1$  and  $1 \leq k \leq \frac{c_1 \varepsilon \log n}{(\log \log n)^2}$  we have*

$$\nu \left\{ E \in G_{n,k}; \sup_{t \geq \varepsilon} \left| \frac{F_K(t, E)}{\Gamma_K^k(t)} - 1 \right| \leq \varepsilon \right\} \geq 1 - \exp(-c_2 n^{0.9})$$

where  $c_1, c_2$  depend only on  $c$  and on the constants in (3.3).

*Proof.* By hypothesis,  $\frac{k}{\varepsilon} \leq \frac{c_1 \log n}{(\log \log n)^2}$  and  $\varepsilon \geq \frac{(\log \log n)^2}{c_1 \log n}$ . We can clearly choose  $\tilde{h}(n)$  to fulfill the hypothesis of Lemma 3.10 and moreover  $c_1 \frac{\tilde{h}(n)}{h(n)} \leq \varepsilon$ . Now, direct computation  $N \leq n^{\frac{c_3}{\log \log n}}$  and  $A^2 \geq \frac{\log \log^4 n}{\log^3 n} n^{-\frac{c_4}{\log \log n}}$  and the result follows.  $\square$

#### 4. ASYMPTOTIC RESULTS ON THE AVERAGE DENSITY AND DISTRIBUTION

**4.1. Gaussian approximation of the average density and distribution.** In this section we show that, for a range of  $k$  and a class of probabilities  $\mathbb{P}$ , the average density is uniformly close to the Gaussian density. Furthermore, if  $\mathbb{P}$  has exponential tails on half spaces (see definition below), we can also approximate the average distribution. Recall that  $F_{\mathbb{P}}(t, E) = \mathbb{P}\{x \in \mathbb{R}^n : |P_E(x)| \leq t\}$ .

**Definition 4.1.** Let  $c > 0$ . Denote by  $\mathcal{P}_{c,n}$  the set of Borel probabilities such that  $\sigma_{\mathbb{P}}, M_2, M_{\mathbb{P}}^{1/n} \leq c$

**Theorem 4.2.** *Let  $k \leq \frac{c \sqrt{\log n}}{(\log \log n)^{\frac{1}{2} + \delta}}, \delta > 0$ . Then there exist  $c_1 > 0$  (depending only on  $c$  and  $\delta$ ) such that  $\forall \mathbb{P} \in \mathcal{P}_{c,n}$ ,*

1)

$$\sup_{s \in \mathbb{R}^k} \left| \varphi_{\mathbb{P}}^k(s) - \gamma_{\mathbb{P}}^k(s) \right| \leq \frac{c_1^k k^{k/2}}{n^{1/(k+3)}}$$

Furthermore, if  $\mathbb{P}$  satisfies  $\mathbb{P}\{x \in \mathbb{R}^n : |\langle \theta, x \rangle| > t\} \leq c_2 \exp -\frac{c_3 t}{M_2}$ , for some  $c_2, c_3 > 0$  and all  $t > 0, \theta \in S^{n-1}$  then

2)

$$\sup_{t \geq 0} \left| F_K^k(t) - \Gamma_{\mathbb{P}}^k(t) \right| \leq \frac{c_4^k k^{k/2}}{n^{1/(k+3)}} (\log n)^k$$

for some  $c_4 > 0$  depending only on the constants.

*Proof.* Observe, by straightforward computation, that the bound on  $k$  insures that the error terms in parts 1) and 2) tend to 0 as  $n \rightarrow \infty$ .

The proof of 1) will be done in 3 steps. Step 3 takes care of very large values of  $|s|$ , Step 2 of values of  $|s|$  near, and including, the origin and Step 1 of the remaining case. Fix  $c_0 > 0$  small enough that will be chosen below.  $c_0$  is used to separate these three steps.

**Step 1.** Let  $k = o(n)$ . There exists a constant  $C > 0$  such that for  $0 < |s| \leq \frac{c_0 \sqrt{n}}{M_{\mathbb{P}}^{1/n}}$  and every Borel probability  $\mathbb{P}$  we have

$$\left| \varphi_{\mathbb{P}}^k(s) - \gamma_{\mathbb{P}}^k(s) \right| \leq C k^{k/2} \left( \frac{\sigma_{\mathbb{P}} M_2}{\sqrt{n} |s|^{k+1}} + \frac{1}{n M_{\mathbb{P}}^{k/n}} \right).$$

*Proof of Step 1.* By formula (3.4),

$$\begin{aligned} \varphi_{\mathbb{P}}^k(s) - \gamma_{\mathbb{P}}^k(s) &= (g_{|s|}(\sqrt{n}M_2) - \gamma_{\mathbb{P}}^k(s)) + g_{|s|}(\sqrt{n}M_2) \mathbb{P}\{|x| < |s|\} + \\ &+ \int_{[|s|, \infty)} \left( g_{|s|}(r) - g_{|s|}(\sqrt{n}M_2) \right) d\mathbb{P}(r) \end{aligned}$$

We compute the second and third summand with the aid of the following lemmas,

**Lemma 4.3.** Let  $k = o(n)$ . There exists an absolute constant  $C > 0$  such that

$$\begin{aligned} \text{i) } \sup_{r \geq t} g_t(r) &\leq \frac{1}{t^k} \frac{C k^{k/2}}{(2\pi e)^{k/2}}, \\ \text{ii) } \sup_{r \geq t} |g'_t(r)| &\leq \frac{1}{\sqrt{n} t^{k+1}} \frac{C k^{(k+3)/2}}{(2\pi e)^{k/2}}. \end{aligned}$$

*Proof.* By Lemma 1.4,  $\frac{|S^{n-k-1}|}{|S^{n-1}|} \leq C \frac{n^{k/2}}{(2\pi)^{k/2}}$ . Proceed as in [BK].  $\square$

**Lemma 4.4.** Let  $k = o(n)$ . There exists  $C > 0$  such that  $\forall s \in \mathbb{R}^k$  with  $0 < |s| \leq \sqrt{n}M_2$

$$\begin{aligned} \text{i) } g_{|s|}(\sqrt{n}M_2) \mathbb{P}\{|x| < |s|\} &\leq \frac{C k^{k/2}}{(2\pi e)^{k/2}} M_{\mathbb{P}} \omega_n |s|^{n-k}. \\ \text{ii) } \int_{\{r \geq |s|\}} \left( g_{|s|}(r) - g_{|s|}(\sqrt{n}M_2) \right) d\mathbb{P}(r) &\leq \frac{C k^{k/2} \sigma_{\mathbb{P}} M_2}{\sqrt{n} |s|^{k+1}} \end{aligned}$$

*Proof.* i) By Lemma 4.3,  $g_{|s|}(\sqrt{n}M_2) \leq \frac{C}{|s|^k} \frac{k^{k/2}}{(2\pi e)^{k/2}}$  and, by definition of  $M_{\mathbb{P}}$ ,  $\mathbb{P}\{|x| \leq |s|\} \leq M_{\mathbb{P}} \omega_n |s|^n$

ii) By the mean value theorem

$$\int_{\{|x| \geq |s|\}} |g_{|s|}(|x|) - g_{|s|}(\sqrt{n}M_2)| d\mathbb{P}(x) \leq \sup_{r \geq |s|} |g'_{|s|}(r)| \int_{\mathbb{R}^n} \left| |x| - \sqrt{n}M_2 \right| d\mathbb{P}(x)$$

Now use Lemma 4.3 and the inequality  $\int_{\mathbb{R}^n} \left| |x| - \sqrt{n}M_2 \right| d\mathbb{P}(x) \leq \sigma_{\mathbb{P}} M_2$  (see [BK]).  $\square$

Observe that, by suitably choosing  $c_0$  we have: a)  $|s| \leq \frac{c_0\sqrt{n}}{M_{\mathbb{P}}^{1/n}}$  implies  $|s| \leq \sqrt{n}M_2$ , by Hensley's Lemma 1.3 and b) the second error term in the previous Lemma absorbs the first one.

It remains to estimate the first summand,  $|\gamma_{\mathbb{P}}^k(s) - g_{|s|}(\sqrt{n}M_2)|$  where

$$g_{|s|}(\sqrt{n}M_2) = \frac{|S^{n-k-1}|}{|S^{n-1}|} \frac{1}{n^{k/2}M_2^k} \left(1 - \frac{|s|^2}{nM_2^2}\right)^{\frac{n-k-2}{2}}.$$

Write  $|s|^2 = 2M_2^2u$ . Then,  $0 < |s| \leq \sqrt{n}M_2$  is equivalent to  $0 < u \leq n/2$  and so, for such a values of  $u$  we need to estimate

$$\frac{1}{(2\pi)^{k/2}M_2^k} \left| \frac{|S^{n-k-1}|}{|S^{n-1}|} \frac{(2\pi)^{k/2}}{n^{k/2}} \left(1 - \frac{2u}{n}\right)^{\frac{n-k-2}{2}} - e^{-u} \right|$$

By Lemma 1.3 we have  $\frac{1}{(2\pi)^{k/2}M_2^k} \leq C^k M_{\mathbb{P}}^{k/n}$ . Finally add the value  $\pm \frac{|S^{n-k-1}|}{|S^{n-1}|} \frac{(2\pi)^{k/2}}{n^{k/2}} e^{-u}$  and use Lemma 1.6 to conclude the proof of Step 1.

**Step 2.** Let  $\mathbb{P} \in \mathcal{P}_{c,n}$  and  $k = o(n)$ . Then

$$\left| \varphi_{\mathbb{P}}^k(s) - \gamma_{\mathbb{P}}^k(s) \right| \leq \frac{c_1^k k^{k/2}}{n^{1/(k+3)}}$$

for all  $|s| \leq \frac{c_0\sqrt{n}}{M_{\mathbb{P}}^{1/n}}$  ( $c_1$  depending only on  $c$ ).

*Proof of Step 2.* By Lemma 1.3 we also have  $M_2, M_{\mathbb{P}}^{1/n} \geq c_2 > 0$ .

Let  $(s_n)$  be a sequence such that  $\sqrt{n}|s_n|^{k+1} = n^{\frac{1}{k+3}}$ , or equivalently,  $|s_n| = n^{-\frac{1}{2(k+3)}}$ . For  $|s| \geq |s_n|$  we have

$$\left| \varphi_{\mathbb{P}}^k(s) - \gamma_{\mathbb{P}}^k(s) \right| \leq C k^{k/2} \left( \frac{\sigma_{\mathbb{P}}M_2}{\sqrt{n}|s|^{k+1}} + \frac{1}{n M_{\mathbb{P}}^{k/n}} \right) \leq c_1^k k^{k/2} n^{-\frac{1}{k+3}}$$

If  $0 \leq |s| \leq |s_n|$ , write

$$\left| \varphi_{\mathbb{P}}^k(s) - \gamma_{\mathbb{P}}^k(s) \right| \leq \left| \varphi_{\mathbb{P}}^k(s) - \varphi_{\mathbb{P}}^k(s_n) \right| + \left| \varphi_{\mathbb{P}}^k(s_n) - \gamma_{\mathbb{P}}^k(s_n) \right| + \left| \gamma_{\mathbb{P}}^k(s_n) - \gamma_{\mathbb{P}}^k(s) \right|.$$

The second summand was estimated above. As for the third one, the inequality  $|e^{-x} - e^{-y}| \leq |x|, x \geq y > 0$  implies

$$\left| \gamma_{\mathbb{P}}^k(s) - \gamma_{\mathbb{P}}^k(s_n) \right| \leq \frac{|s_n|^2}{2(2\pi)^{k/2}M_2^{k+2}} \leq \frac{c_1^k}{n^{\frac{1}{k+3}}}$$

For the first summand, we use the following lemma

**Lemma 4.5.** Let  $n \geq 2k$ . There exists  $c_0, c_1 > 0$  such that for all  $s \in \mathbb{R}^k$  with  $|s| \leq \frac{c_0\sqrt{n}}{M_{\mathbb{P}}^{1/n}}$ ,

$$\left| \varphi_{\mathbb{P}}^k(s) - \varphi_{\mathbb{P}}^k(0) \right| \leq c_1^k \left( n^{k/2} M_{\mathbb{P}} \omega_n |s|^{n-k} + M_{\mathbb{P}}^{\frac{k+2}{n}} |s|^2 \right).$$

This finishes the proof of Step 2 since the estimate of the remaining first summand readily follows from

$$\left| \varphi_{\mathbb{P}}^k(s) - \varphi_{\mathbb{P}}^k(s_n) \right| \leq \left| \varphi_{\mathbb{P}}^k(s_n) - \varphi_{\mathbb{P}}^k(0) \right| + \left| \varphi_{\mathbb{P}}^k(s) - \varphi_{\mathbb{P}}^k(0) \right|$$

*Proof of the Lemma.* By definition,  $|\varphi_{\mathbb{P}}^k(s) - \varphi_{\mathbb{P}}^k(0)|$  equals

$$\frac{|S^{n-k-1}|}{|S^{n-1}|} \int_{\{|x| \leq |s|\}} \frac{d\mathbb{P}(x)}{|x|^k} + \frac{|S^{n-k-1}|}{|S^{n-1}|} \int_{\{|x| \geq |s|\}} \frac{1}{|x|^k} \left( \left(1 - \frac{|s|^2}{|x|^2}\right)^{\frac{n-k-2}{2}} - 1 \right) d\mathbb{P}(x)$$

We estimate the first summand. By Fubini's theorem,

$$\int_{\{|x| \leq |s|\}} \frac{d\mathbb{P}(x)}{|x|^k} = \int_0^{\frac{1}{|s|^k}} \mathbb{P}\{|x| \leq |s|, \frac{1}{|x|^k} > t\} dt = \int_0^{\frac{1}{|s|^k}} + \int_{\frac{1}{|s|^k}}^{\infty}$$

The first integral is equal to  $\int_0^{\frac{1}{|s|^k}} \mathbb{P}\{|x| \leq |s|\} dt$  and by definition of  $M_{\mathbb{P}}$ , it follows that this integral is bounded by  $|s|^{n-k} M_{\mathbb{P}} \omega_n$ .

The second integral is equal to

$$\int_{\frac{1}{|s|^k}}^{\infty} \mathbb{P}\left\{\frac{1}{|x|^k} > t\right\} dt \leq \int_{\frac{1}{|s|^k}}^{\infty} M_{\mathbb{P}} \omega_n t^{-n/k} dt = M_{\mathbb{P}} \omega_n \frac{k}{n-k} |s|^{n-k}$$

Therefore, by Lemma 1.4

$$\frac{|S^{n-k-1}|}{|S^{n-1}|} \int_{\{|x| \leq |s|\}} \frac{d\mathbb{P}(x)}{|x|^k} \leq \frac{c n^{k/2}}{(2\pi)^{k/2}} M_{\mathbb{P}} \omega_n \frac{n}{n-k} |s|^{n-k} \leq c^k n^{k/2} M_{\mathbb{P}} \omega_n |s|^{n-k}$$

Next we compute the second summand. Use in the integrand the elementary inequality  $|a^p - b^p| \leq p|a - b|$ ,  $a, b \in [0, 1]$  with  $p = \frac{n-k-2}{2}$  and conclude that the second summand is bounded by

$$\frac{|S^{n-k-1}|}{|S^{n-1}|} \frac{n-k-2}{2} \int_{\{|x| \geq |s|\}} \frac{|s|^2}{|x|^{k+2}} d\mathbb{P}(x) = \frac{(n-k-2)|S^{n-k-1}|}{2 |S^{n-k-3}|} |s|^2 \varphi_{\mathbb{P}}^{k+2}(0)$$

By Proposition 4.7. below we have, for  $|s| \leq c(\omega_n M_{\mathbb{P}})^{-1/n} \sim \frac{c_0 \sqrt{n}}{M_{\mathbb{P}}^{1/n}}$ ,

$$\varphi_{\mathbb{P}}^{k+2}(0) \leq c_1 M_{\mathbb{P}}^{(k+2)/n} \frac{\omega_{n-k-2}}{\omega_n^{\frac{1-k+2}{n}}} \leq c_1^k M_{\mathbb{P}}^{(k+2)/n}$$

Finally,  $(n-k-2) \frac{|S^{n-k-1}|}{|S^{n-k-3}|} = \pi(n-k) \frac{\Gamma(\frac{n-k-2}{2})}{\Gamma(\frac{n-k}{2})} \leq c_2$  by Lemma 1.4

and putting the estimates together, the second summand is bounded by  $c_1^k |s|^2 M_{\mathbb{P}}^{(k+2)/n}$ , which finishes the proof of the lemma.  $\square$

**Step 3.** For every probability  $\mathbb{P}$  with  $M_{\mathbb{P}}^{1/n} \leq c$ ,  $|s| \geq \frac{c_0\sqrt{n}}{M_{\mathbb{P}}^{1/n}}$  and  $k \leq \frac{n}{\log n}$ ,

$$\left| \varphi_{\mathbb{P}}^k(s) - \gamma_{\mathbb{P}}^k(s) \right| \leq \frac{c_1^k k^{k/2}}{n^{k/2}}$$

where  $c_1 > 0$  depends only on  $c$ .

*Proof of Step 3.* By Lemma 1.3,  $M_2 \geq c_2 > 0$  (depending on  $c$ ) and trivially

$$\gamma_{\mathbb{P}}^k(s) \leq c_1^k \exp(-c_3 n) \leq \frac{c_1^k}{n^{k/2}}$$

On the other hand, by Lemma 4.3

$$\varphi_{\mathbb{P}}^k(s) \leq \max_{|s| \leq |x|} g_{|s|}(x) \mathbb{P}\{x; |x| \geq |s|\} \leq \frac{k^{k/2}}{(2\pi e)^{k/2} |s|^k} \leq \frac{c_1^k k^{k/2}}{n^{k/2}}$$

This finishes the proof of 1).

Now we prove 2). Let  $t \leq C \log n \sqrt{k} M_2$  (for suitable  $C > 0$ ). By integrating  $\int_{|s| \leq t} ds$  the result in 1) and using the identity (1.2) and Lemma 1.4 we have

$$\left| F_K^k(t) - \Gamma_{\mathbb{P}}^k(t) \right| \leq \frac{c_1^k}{n^{1/(k+3)}} t^k \leq \frac{c_2^k k^{k/2}}{n^{1/(k+3)}} (\log n)^k$$

In the range  $t \geq C \log n \sqrt{k} M_2$ , we proceed as in Theorem 3.5 2). Observe that if we write  $P_E(x) = \sum_{i=1}^k \langle x, u_i \rangle u_i$  for some  $(u_i)$  orthonormal basis of  $E$  then

$$1 - F_{\mathbb{P}}(t, E) = \mathbb{P}\left\{ \sum_{i=1}^k |\langle x, u_i \rangle|^2 > t^2 \right\} \leq \mathbb{P}\left\{ \sqrt{k} \max_{1 \leq i \leq k} |\langle x, u_i \rangle| > t \right\}$$

And so, by the hypothesis,  $1 - F_{\mathbb{P}}(t, E) \leq c_2 k \exp - \frac{c_3 t}{\sqrt{k} M_2}$

By this estimate and Lemma 1.5 ii)

$$\left| F_{\mathbb{P}}(t, E) - \Gamma_{\mathbb{P}}^k(t) \right| = \left| (1 - F_{\mathbb{P}}(t, E)) - (1 - \Gamma_{\mathbb{P}}^k(t)) \right| \leq c_2 k e^{-\frac{c_3 t}{M_2 \sqrt{k}}} + 2^{k/2} e^{-\frac{t^2}{4M_2^2}}$$

and we conclude, as in Theorem 3.5 2), that  $\left| F_{\mathbb{P}}(t, E) - \Gamma_{\mathbb{P}}^k(t) \right| \leq \frac{2}{n}$  for every  $k$ -dimensional subspace  $E$ . □

*Remark 4.6.* The hypothesis on  $M_{\mathbb{P}}$ ,  $M_2$  and  $\sigma_{\mathbb{P}}$  are necessary due to the behaviour at  $s = 0$ . Indeed, consider the probability given by  $\mathbb{P} = \frac{1}{2} \sigma_{n-1} + \frac{1}{2} \tilde{\sigma}_{n-1}$  where  $\tilde{\sigma}_{n-1}$  is the Haar probability on  $2S^{n-1}$ . Straightforward computations show that  $M_2 \sim cn^{-1/2}$ ,  $M_{\mathbb{P}}^{1/n} \sim cn^{1/2}$  and  $\sigma_{\mathbb{P}} \sim c\sqrt{n}$  and  $|\varphi_{\mathbb{P}}^k(0) - \gamma_{\mathbb{P}}^k(0)| \sim c n^{k/2}$  and so, it tends to  $+\infty$  as  $n \rightarrow \infty$ .

*Examples.* We give some examples with  $\sigma_{\mathbb{P}}$ ,  $M_2$ ,  $M_{\mathbb{P}}^{1/n}$  uniformly bounded.



1. Let  $\mathbb{P}$  be uniform measure on  $K$ , the normalised unit ball of the space  $\ell_p^n, p > 0$ . Clearly  $M_{\mathbb{P}} = 1$ . The parameters  $M_2(= L_K)$  and  $\sigma_K$  are uniformly bounded on  $n$  as it appears in [ABP]. (In that paper for  $p \geq 1$ , but by similar arguments also for  $0 < p < 1$ ).

2. Let  $\mathbb{P}$  be a Borel probability on  $\mathbb{R}$  with finite fourth moment. Consider the product measure  $\tilde{\mathbb{P}} = \mathbb{P} \otimes \cdots \otimes \mathbb{P}$  on  $\mathbb{R}^n$  and suppose  $M_{\tilde{\mathbb{P}}} = 1$ . A simple computation show that  $M_2(\tilde{\mathbb{P}}) = M_2(\mathbb{P})$  and  $\sigma_{\tilde{\mathbb{P}}} = \sigma_{\mathbb{P}}$ .

3. Consider the density function on  $\mathbb{R}^n$  given by  $f(|x|)$  where  $f: \mathbb{R} \rightarrow [0, \infty)$  is an even log-concave function. Then, we have that  $M_{\mathbb{P}} = f(0)$  and, by Lemma 2.6 in [Kl],  $\sigma_{\mathbb{P}}, M_2$  are bounded by an absolute constant.

4. Let  $f(x) = \exp -a^p|x|^p, 0 < p < 1$  be a density function on  $\mathbb{R}^n$ . Then,  $M_{\mathbb{P}} = 1$  and  $\sigma_{\mathbb{P}}, M_2$  are bounded by constants depending only on  $p$ .

**4.2. Upper bounds for a fast growth of  $k$ .** A Gaussian behaviour for large  $k$  is not expected: Consider the case  $K = \omega_n^{-\frac{1}{n}} D_n$ . We have

$$\varphi_K^k(s) = \begin{cases} \omega_{n-k} \omega_n^{\frac{k-n}{n}} \left(1 - |s|^2 \omega_n^{\frac{2}{n}}\right)^{\frac{n-k}{2}} & \text{for } |s| \leq \omega_n^{-\frac{1}{n}} \\ 0 & \text{otherwise} \end{cases}$$

For  $k = n - \ell, \ell$  fixed or  $k = (1 - \lambda)n, 0 < \lambda < 1$  the asymptotic behaviour of  $\varphi_K^k(s)$  is

- If  $k = n - \ell, \ell$  fixed, then the equivalence  $\omega_{n-k} \omega_n^{\frac{k}{n}-1} \sim n^{\ell/2} (2\pi e)^{-\ell/2}$  implies  $\varphi_K^{n-\ell}(s) n^{-\ell/2} \rightarrow \omega_{\ell} (2\pi e)^{-\ell/2}$ .

- If  $k = (1 - \lambda)n, 0 < \lambda < 1$ , we have  $\omega_{n-k} \omega_n^{\frac{k}{n}-1} \sim \frac{\lambda(2\pi)^{\lambda/2}}{\lambda^{\lambda n/2} n^{(1-\lambda)/2}}$  which implies  $\varphi_K^{(1-\lambda)n}(s) \lambda^{\lambda n/2} n^{(1-\lambda)/2} \rightarrow \lambda(2\pi)^{\lambda/2} e^{-\pi e \lambda |s|^2}$ .

For general probabilities we find the following upper bounds of  $\varphi_{\mathbb{P}}^k(s)$ .

**Proposition 4.7.** *Let  $\mathbb{P}$  be a probability measure on  $\mathbb{R}^n$  with  $M_{\mathbb{P}} < \infty$ . Then there exist numerical constants  $c, C > 0$  so that*

i) *If  $1 \leq k \leq n - 2$ ,*

$$\varphi_{\mathbb{P}}^k(s) \leq C M_{\mathbb{P}}^{k/n} \frac{\omega_{n-k}}{\omega_n^{1-\frac{k}{n}}} \left(1 - \frac{k}{n} \frac{|s|^{n-k}}{(\omega_n M_{\mathbb{P}})^{\frac{k}{n}-1}}\right)$$

*whenever  $|s| \leq \left(\frac{k}{n}\right)^{1/(n-k)} (\omega_n M_{\mathbb{P}})^{-1/n}$ .*

ii) *If  $k = n - 1$  and  $\mathbb{P}$  has bounded density  $f$ ,*

$$\varphi_{\mathbb{P}}^{n-1}(s) \leq C \|f\|_{\infty}$$

*whenever  $|s| \leq c\sqrt{n} \|f\|_{\infty}^{-1/n}$ .*

*Proof.* i) Case  $1 \leq k \leq n - 2$ . Recall

$$\begin{aligned}\varphi_{\mathbb{P}}^k(s) &= \frac{|S^{n-k-1}|}{|S^{n-1}|} \int_{\{|s| \leq |x|\}} \left(1 - \frac{|s|^2}{|x|^2}\right)^{\frac{n-k-2}{2}} \frac{d\mathbb{P}(x)}{|x|^k} \\ &\leq \frac{|S^{n-k-1}|}{|S^{n-1}|} \int_{\{|s| \leq |x|\}} \frac{d\mathbb{P}(x)}{|x|^k}\end{aligned}$$

Let  $A \geq |s|$  to be chosen later.

$$\begin{aligned}\varphi_{\mathbb{P}}^k(s) &\leq \frac{|S^{n-k-1}|}{|S^{n-1}|} \left( \int_{|s| \leq |x| \leq A} \frac{d\mathbb{P}(x)}{|x|^k} + \int_{A \leq |x|} \frac{d\mathbb{P}(x)}{|x|^k} \right) \\ &\leq \frac{|S^{n-k-1}|}{|S^{n-1}|} \left( \int_{|s| \leq |x| \leq A} \frac{d\mathbb{P}(x)}{|x|^k} + \frac{1}{A^k} \right)\end{aligned}$$

Fix  $I > 1$  and let  $N_s$  be a natural number such that  $\frac{A}{I^{N_s+1}} \leq |s| < \frac{A}{I^{N_s}}$ .

Since  $\int_{tD_n} d\mathbb{P}(x) \leq M_{\mathbb{P}} t^n \omega_n$  for all  $t > 0$ , we have

$$\begin{aligned}\int_{|s| \leq |x| \leq A} \frac{d\mathbb{P}(x)}{|x|^k} &\leq \sum_{m=0}^{N_s} \int_{\frac{A}{I^{m+1}} \leq |x| < \frac{A}{I^m}} \frac{d\mathbb{P}(x)}{|x|^k} \leq \sum_{m=0}^{N_s} \left(\frac{I^{m+1}}{A}\right)^k \int_{|x| \leq \frac{A}{I^m}} d\mathbb{P}(x) \\ &\leq \frac{I^k}{A^k} \sum_{m=0}^{N_s} I^{mk} \left(\frac{A}{I^m}\right)^n \omega_n M_{\mathbb{P}} = I^k A^{n-k} \omega_n M_{\mathbb{P}} \sum_{m=0}^{N_s} \left(\frac{1}{I^{n-k}}\right)^m \\ &\leq I^k A^{n-k} \omega_n M_{\mathbb{P}} \left(1 - \left(\frac{1}{I^{n-k}}\right)^{N_s+1}\right) \left(1 - \frac{1}{I^{n-k}}\right)^{-1} \\ &\leq I^k \left(1 - \frac{1}{I^{n-k}}\right)^{-1} A^{n-k} \omega_n M_{\mathbb{P}} \left(1 - \left(\frac{|s|}{IA}\right)^{n-k}\right)\end{aligned}$$

We choose  $I = (n/k)^{1/(n-k)}$  and we get

$$\int_{|s| \leq |x| \leq A} \frac{d\mathbb{P}(x)}{|x|^k} \leq \frac{k}{n-k} \left(\frac{n}{k}\right)^{n/(n-k)} A^{n-k} \omega_n M_{\mathbb{P}} \left(1 - \left(\frac{|s|}{A}\right)^{n-k} \frac{k}{n}\right)$$

We now optimize by taking  $A = (k/n)^{1/(n-k)} (\omega_n M_{\mathbb{P}})^{-1/n}$ , whenever  $|s| \leq (k/n)^{1/(n-k)} (\omega_n M_{\mathbb{P}})^{-1/n}$ , and we arrive at the result taking also into account that  $|S^{m-1}| = m \omega_m$ .

ii) Case  $k = n - 1$ .

$$\begin{aligned} \varphi_{\mathbb{P}}^{n-1}(s) &= \frac{2}{|S^{n-1}|} \int_{\{|s| \leq |x|\}} \frac{f(x) dx}{|x|^{n-2} \sqrt{|x|^2 - |s|^2}} \leq \\ &\leq \frac{2}{|S^{n-1}|} \left( \int_{|s| \leq |x| \leq |s|+A} + \int_{|s|+A \leq |x|} \right) \leq \\ &\leq \frac{2}{|S^{n-1}|} \left( |S^{n-1}| \|f\|_{\infty} \sqrt{(|s|+A)^2 - |s|^2} + \frac{1}{(|s|+A)^{n-2} \sqrt{2|s|A + A^2}} \right) \end{aligned}$$

Assume  $|s| \leq A$  then

$$\varphi_{\mathbb{P}}^{n-1}(s) \leq \frac{2}{|S^{n-1}|} \left( |S^{n-1}| \|f\|_{\infty} \sqrt{3}A + \frac{1}{A^{n-1}} \right)$$

We optimise by taking  $A = \left( \frac{n-1}{\sqrt{3}|S^{n-1}| \|f\|_{\infty}} \right)^{1/n}$  and then

$$\begin{aligned} \varphi_{\mathbb{P}}^{n-1}(s) &\leq \frac{2}{|S^{n-1}|} \left( |S^{n-1}| \|f\|_{\infty} \sqrt{3} \right)^{1-1/n} \left( (n-1)^{1/n} + \left( \frac{1}{n-1} \right)^{1-1/n} \right) \\ &\leq 2\sqrt{3} \|f\|_{\infty} \end{aligned}$$

whenever  $|s| \leq C\sqrt{n} \|f\|_{\infty}^{-1/n}$  for some absolute constant  $C > 0$ .  $\square$

*Remark 4.8.* Our result i) gives (assume  $M_{\mathbb{P}} = 1$  for simplicity) an upper bound in the range  $|s| \leq \left(\frac{k}{n}\right)^{1/(n-k)} \omega_n^{-1/n} (\leq c\sqrt{n})$ . By looking at the trivial estimate given by

$$\varphi_{\mathbb{P}}^k(s) \leq \left(1 - \frac{k}{n}\right) \omega_{n-k} \omega_n^{-1} \frac{1}{|s|^k}$$

we conclude that in the range  $|s| \geq C\omega_n^{-1/n} (\sim C\sqrt{n})$ ,

$$\varphi_{\mathbb{P}}^k(s) \leq \frac{\left(1 - \frac{k}{n}\right) \omega_{n-k}}{C^k \omega_n^{1-\frac{k}{n}}}$$

The computations in the beginning of this section show that for  $k = (1-\lambda)n$  or  $k = n - \ell$ ,  $2 \leq \ell$ , the function  $\varphi_{\mathbb{P}}^k(s)$  is bounded in the range  $|s| \geq C\sqrt{n}$  by  $c_1 e^{-cn}$ . Therefore, in both cases the distribution of  $\varphi_{\mathbb{P}}^k(s)$  is concentrated on  $|s| \leq c\sqrt{n}$  (constants depending only on  $\lambda$  or  $\ell$  respectively).

**Corollary 4.9.** *Under the hypothesis of Proposition 4.7 we have in particular,*

i) if  $k = (1-\lambda)n$ , then  $\varphi_{\mathbb{P}}^{(1-\lambda)n}(s) \lambda^{\lambda n/2} n^{(1-\lambda)/2} \leq C(\lambda) M_{\mathbb{P}}^{1-\lambda}$ , for all  $s \in \mathbb{R}^k$  and for some constant  $C(\lambda) > 0$  depending on  $\lambda$ ,

ii) if  $k = n - \ell$ ,  $2 \leq \ell$  fixed, then  $\varphi_{\mathbb{P}}^{n-\ell}(s) n^{-\ell/2} \leq C(\ell) M_{\mathbb{P}}^{1-\frac{\ell}{n}}$  for all  $s \in \mathbb{R}^k$  and for some constant  $C(\ell) > 0$  depending on  $\ell$ .

By comparing with the case of the Euclidean ball, we see that the bounds are sharp for all  $s$  in the case ii) and also in the range  $|s| \leq 1$  (say) for all values of  $k$ .

*Remark 4.10.* We can improve the numerical constants for central sections of star-shaped bodies: Let  $1 \leq k \leq n - 2$  and let  $K \subseteq \mathbb{R}^n$  be a star-shaped body of volume  $|K| = 1$ . Let  $rD_n$  be the Euclidean ball of volume 1 (of radius  $r = \omega_n^{-1/n}$ ). Then,

$$\varphi_K^k(s) \leq \varphi_K^k(0) \leq \varphi_{rD_n}^k(0) = \omega_{n-k} \omega_n^{\frac{k-n}{n}}, \quad \forall s \in \mathbb{R}^k$$

Indeed,

$$\begin{aligned} \varphi_K^k(s) &= \frac{|S^{n-k-1}|}{|S^{n-1}|} \int_{K \cap \{|x| \geq |s|\}} \left(1 - \frac{|s|^2}{|x|^2}\right)^{\frac{n-k-2}{2}} \frac{dx}{|x|^k} \\ &\leq \frac{|S^{n-k-1}|}{|S^{n-1}|} \int_K \frac{dx}{|x|^k} = \varphi_K^k(0) = \frac{\omega_{n-k}}{\omega_n} \tilde{W}_k(K) \end{aligned}$$

Where  $\tilde{W}_k(K)$  denotes the  $k$ -th dual mixed volume of  $K$  (see [BBR]). Now, by the dual Minkowski inequality  $\tilde{W}_k(K) \leq \omega_n^{\frac{k}{n}}$ , since  $|K| = 1$  and the result follows.

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