FROM JOHN TO GAUSS-JOHN POSITIONS VIA DUAL MIXED VOLUMES

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Abstract. We investigate the optimization of the dual mixed volumes $\{\tilde{W}_i(SK), 0 \in SK \subseteq D_n$ where $K \subseteq \mathbb{R}^n$ is a convex body, D_n the euclidian ball and SK runs over all positions of K. When S is linear we give necessary and sufficient conditions for K to be in extremal position in terms of a decomposition of the identity. We consider affine problems and we also present an approach involving parallel sections of K which can be understood as a dual fractional Kubota formula.

1. Introduction and Notation

A large number of relevant positions of convex bodies can be characterized as the solution of suitable defined optimization programs. These special positions have been used in different areas of mathematics and have been proven to be very useful tools for applications. Perhaps the first example is the classical F. John Theorem [J] which, for a convex body K in $\mathbb{R}^n$ and D_n the euclidian ball, characterizes the affine position SK that maximizes the function $\text{vol}(SK) = |SK|$ with the constraint $SK \subseteq D_n$.

In [GM], important positions of convex bodies such as, the ℓ-position, M-position, minimal surface area position and others, appear as solutions of optimization problems. For instance, problems involving mixed volumes such as $\min\{W_i(TK) \mid T \in GL(n)\}$ with constraint $\det(T) = 1$, were investigated. In this type of results, necessary and sufficient conditions are given in terms of either a decomposition of the identity or of isotropic properties of certain Borel measures.

More recently, in [LYZ] the extremal problem of minimizing the total L_p-curvature was considered and it was showed that some classic problems can be reformulated in this context. In [BR1] and [BR2], the authors started the study of optimization programs involving dual mixed volumes with constraint $\det(T) = 1$. They obtained a complete characterization of the MM^*-position and also of the solution of $\min\{\tilde{W}_i(TK) \mid T \in GL(n), \det T = 1\}$

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for $i \in (-\infty, 0) \cup (n, \infty)$, where $\tilde{W}_i(K)$ is the dual mixed volume defined as

$$\tilde{W}_i(K) = \frac{1}{n} \int_{S^{n-1}} \rho_K^{n-i}(u) d\sigma(u)$$

for any $i \in \mathbb{R}$, K containing the origin (see [L1], [BR2], [Ga]) and ρ_K is the *radial function* given by

$$\rho_K(x) = \max\{\lambda \geq 0 : \lambda x \in K\}$$

for $x \in \mathbb{R}^n \setminus \{0\}$. In this mixed dual volume framework, John's theorem can be understood as an optimization problem for $\tilde{W}_0(TK)$ with constraint $TK \subseteq D_n$.

In [GMR] the authors introduced the *Gauss-John position* which corresponds to the solution of the optimization problem $\tilde{W}_{n+1}(TK)$ with constraint $TK \subseteq D_n$. This is in fact the origin of this research since in that paper the authors give necessary conditions for a centrally symmetric convex body K to be in Gauss-John position.

In the second section of this paper we consider the optimization program

$$\big\{\tilde{W}_i(TK), 0 \in TK \subseteq D_n, T \in GL(n)\big\}, \qquad i \in \mathbb{R}$$

and we investigate necessary and sufficient conditions for I, the identity, to be the optimal solution.

By using a general optimization theorem by F. John [J], our first result (Theorem 2.1) gives, for all indices $i \in \mathbb{R}$, a necessary condition for K to be in extremal position. Next, for a range of indexes we obtain sufficient conditions and uniqueness up to orthogonal transformations (Theorem 2.4). Depending on the index we are considering, the methods we use vary from the simpler Lagrange multipliers technique to more delicate estimates involving the Laplace-Beltrami operator.

In particular, we deduce from our results that the necessary condition obtained in [GMR] for the Gauss-John position ($i = n+1$) for a symmetric convex body, is necessary *and* sufficient for general convex bodies containing the origin.

In the third section we study the role of translations and consider, for centrally symmetric convex bodies, the optimization of

$$\big\{\tilde{W}_i(a + TK),\ \ 0 \in a + TK \subseteq D_n,\ \ a \in \mathbb{R}^n,\ \ T \in GL(n)\big\}.$$

Note that since the dual mixed volumes are not affine invariants of K and there is no explicit formula that relates the radial functions of K and $a+K$, these affine optimization problems are different from the linear ones above. We show that affine optimization and linear optimization are equivalent for centrally symmetric convex bodies and for a certain range of indexes. Furthermore, we develop an alternative geometrical approach and we state a new formula for $\tilde{W}_i(L)$ in terms of its parallel sections (see Theorem 3.4 below) with the aid of some ideas by Koldobsky in [K]. The formula is also proved to be and extension of dual Kubota recursion formula and it is of independent interest.

In the last section we consider two questions that measure the distance from the extremal positions to the euclidian ball. For K centrally symmetric, we estimate their geometric distance and the parameter $\left(\frac{\tilde{W}_i(D_n)}{\tilde{W}_i(TK)}\right)^{\frac{1}{n-i}}$.

We denote by $|\cdot|_k$ the k-dimensional Lebesgue measure of a subset in $\mathbb{R}^n$ and for $k=n$ we put $|\cdot|_n = |\cdot|$. Notice that $|\cdot|$ may also represent the euclidian norm, but the context will avoid any confusion. The rest of the notation is standard and can be seen in [Ga], [L1] or [Sc].

2. Necessary and sufficient conditions

For any $i \in \mathbb{R}$ we consider the optimization of

$$\{\tilde{W}_i(TK), 0 \in TK \subseteq D_n, T \in GL(n)\}.$$

It is easy to see that a solution exists and that our problem is to maximize $W_i(TK)$ if $i < n$ and to minimize it for $i > n$. Note that for $i = n$, $\tilde{W}_i(TK) = |D_n|$ and there is nothing to prove.

Since $\tilde{W}_i(UK) = \tilde{W}_i(K)$ for any $U \in O(n)$ and any regular matrix T can be decomposed as $T = S_1U = VS_2$ with $U, V \in O(n)$ and S_1, S_2 symmetric and positive definite, we can suppose that the matrix T is symmetric and positive definite and identified with an element in $\mathbb{R}^{n(n+1)/2}$.

Observe that 0 must actually belong to the interior of K when $i > n$. Indeed, if $0 \in \partial K$ then $\rho_K(u) = 0$ at least for all u in a half sphere and so $\tilde{W}_i(K) = \infty$.

Theorem 2.1 (Necessary condition)**.** *Let K be a convex body such that $0 \in K \subseteq D_n$ and let i be a real number. If K is in extremal position for the problem $\{\tilde{W}_i(TK), TK \subseteq D_n, T \in GL(n)\}$ then, there exist contact points $w_1, \ldots, w_N \in \partial K \cap S^{n-1}$ with $N \leq n(n+1)/2$ and $\lambda_1, \ldots, \lambda_N > 0$ with $\sum_{j=1}^N \lambda_i = 1$, such that*

$$I = i \int_{S^{n-1}} u \otimes u \, d\mu_i(u) + (n-i) \sum_{j=1}^N \lambda_j \, w_j \otimes w_j \tag{2.1}$$

where $d\mu_i(u)$ is the probability on S^{n-1} with normalized density

$$d\mu_i(u) = d\mu_{i,K}(u) = \rho_K^{n-i}(u) d\sigma(u) / \int_{S^{n-1}} \rho_K^{n-i}(u) d\sigma(u)$$

and $d\sigma(u)$ denotes the Lebesgue measure on S^{n-1}.

Proof. By using polar coordinates it is easy to check that

$$\tilde{W}_i(TK) = \frac{|n-i|}{n} \det T \int_{K^i} \frac{dx}{|Tx|^i},$$

where $K^i = \begin{cases} K, & \text{if } i < n, \\ \mathbb{R}^n \setminus K, & \text{otherwise} \end{cases}$.

Assume that K is in extremal position. It is clear that the compact $W = \partial K \cap S^{n-1}$ must be not empty. The result can be obtained as a direct consequence of a well known theorem of John (see [J]).

Theorem 2.2 (John). *Let $\Omega \subset \mathbb{R}^m, \Omega_1 \subset \mathbb{R}^l$ be (non empty) open sets and $S \subset \Omega_1$ compact. Let $F\colon \Omega \to \mathbb{R}$ and $G\colon \Omega \times \Omega_1 \to \mathbb{R}$ be $\mathcal{C}^{1)}$ functions. Let $A = \{x \in \Omega \mid G(x, y) \geq 0, \forall\, y \in S\}$. If F attains its minimum value at $x_0 \in A$, then there exist $y_1, \dots, y_s \in S$ and $\lambda_0, \lambda_1, \dots, \lambda_s \in \mathbb{R}$ such that*

- *$0 \leq s \leq m$ and $\lambda_0 \geq 0$, $\lambda_1, \dots, \lambda_s > 0$.*
- *$G(x_0, y_1) = \cdots = G(x_0, y_s) = 0$.*
- *The function $\Phi(x) = \lambda_0 F(x) - \sum_{j=1}^{s} \lambda_j G(x, y_j)$ verifies $\nabla\Phi(x_0) = 0$.*

Let $\Omega_1 = \mathbb{R}^n$ and $S = K$. Let $\Omega \subseteq \mathbb{R}^{n(n+1)/2}$ be defined by

$$\Omega = \{T \in \mathbb{R}^{n(n+1)/2} \mid \int_{K^i} \frac{dx}{|Tx|^i} < \infty\}$$

and $G\colon \Omega \times \Omega_1 \to \mathbb{R}$ be the function given by $G(T, x) = 1 - |T(x)|^2$. The set $A = \{T \in \Omega \mid G(T, x) \geq 0, \forall\, x \in K\}$ is just the set of elements $T \in \Omega$ such that $TK \subseteq D_n$. A is a compact convex set.

In the case $i > n$, we want to find necessary conditions for the identity I to be an minimum of $F(T) = \tilde{W}_i(TK)$ on A (the case $i < n$ is the same by just considering the function $-F$).

By direct computation, it is easy to show that

- $\dfrac{\partial G(T, x)}{\partial T}(I, x) = -2(x \otimes x)$.
- $\dfrac{\partial F(T)}{\partial T}(I) = \tilde{W}_i(K)I + \dfrac{i-n}{n} \displaystyle\int_{K^i} \frac{(-i)}{|x|^{i+2}}(x \otimes x)\, dx$.

So, by John's theorem there exist $y_1, \dots, y_s \in \partial K \cap S^{n-1}$ and $\lambda_0 \geq 0$, $\lambda_1, \dots, \lambda_s > 0$ such that

$$\lambda_0 \left(\tilde{W}_i(K)I + \frac{i-n}{n} \int_{K^i} \frac{(-i)}{|x|^{i+2}}(x \otimes x)\, dx \right) + \sum_{k=1}^{s} \lambda_k y_k \otimes y_k = 0.$$

By taking trace in the equation, $\lambda_0(n-i)\tilde{W}_i(K) + \displaystyle\sum_{k=1}^{s} \lambda_k = 0$ and so, if we write $t_k = \frac{\lambda_k}{\lambda_0(i-n)\tilde{W}_i(K)}$, we have that $\displaystyle\sum_{k=1}^{s} t_k = 1$ with $t_k > 0$ for all k and

$$\frac{I}{i-n} - \frac{i}{n} \int_{K^i} \frac{x \otimes x}{|x|^{i+2}} \frac{dx}{\tilde{W}_i(K)} + \sum_{k=1}^{s} t_k y_k \otimes y_k = 0.$$

If we finally take polar coordinates the result in the statement of the theorem holds. □

An alternative proof of Theorem 2.1 could have been made using separation theorems techniques in the spirit of [Ba].

Remark 2.3. Recall that a Borel measure μ on S^{n-1} is *isotropic* if there exists $c > 0$ such that its inertial matrix is multiple of the identity, that is

$$\int_{S^{n-1}} u \otimes u \, d\mu(u) = cI.$$

Now, condition (2.1) can be seen as the isotropy of a (real) measure whose absolutely continuous part (with respect to the Lebesgue measure on S^{n-1}) is $id\mu_i$ and its singular part is concentrated on contact points. For $i = 0$ (John's theorem) we only have singular part.

Theorem 2.4 (Sufficient condition)**.** *Assume that a convex body $0 \in K \subseteq D_n$ satisfies the condition (2.1). Then,*

1. *If $i \in [-2, 0] \cup [n+1, +\infty)$, K is in extremal position.*
2. *If $i \in (-\infty, -2) \cup (0, n)$ and the measure $d\mu_i$ is isotropic, K is in extremal position.*

Moreover, the position of K is unique up to orthogonal transformations.

Proof. For simplicity, we express the necessary condition (2.1) as

$$I = i \int_{S^{n-1}} u \otimes u \, d\mu_i(u) + (n-i) \int_{S^{n-1}} w \otimes w \, d\nu(w)$$

where $d\nu$ is a probability measure concentrated on $W = \partial K \cap S^{n-1}$.

This implies that, for every diagonal matrix D,

$$\mathrm{tr}(D) = i \int_{S^{n-1}} \langle u, Du \rangle \; d\mu_i(u) + (n-i) \int_{S^{n-1}} \langle \omega, D\omega \rangle \; d\nu(\omega). \tag{2.2}$$

Let T be a symmetric positive definite matrix such that $TK \subseteq D_n$. Write $T = U^* D U$ where $U \in O(n)$ and D a diagonal matrix $D = (d_j)$ with $d_j > 0$. Denote $UK = K_1$ and observe that $DK_1 \subset D_n$ and, moreover, that it also verifies (2.1) and (2.2) with measures $d\tilde{\mu}_i = d\mu_{i,K_1}$ and $d\tilde{\nu}$ supported on $\partial K_1 \cap S^{n-1}$ (the image measure of $d\nu$ under U).

Case $i \geq n+1$.

We want to show that $\tilde{W}_i(DK_1) \geq \tilde{W}_i(K_1)$.

By using Laplace-Beltrami operator the following identity was stated in [BR2] (Proposition 2.2)

$$\begin{aligned} &(n-i) \int_{S^{n-1}} \langle \nabla h_{K_1^\circ}(u), D^{-1}u \rangle \rho_{K_1}^{n-i+1}(u) d\sigma(u) = \\ = \; &\mathrm{tr}(D^{-1}) \int_{S^{n-1}} \rho_{K_1}^{n-i}(u) d\sigma(u) - i \int_{S^{n-1}} \rho_{K_1}^{n-i}(u) \langle u, D^{-1}u \rangle d\sigma(u), \end{aligned}$$

where K_1° is the polar set of K_1 and $h_{K_1^\circ}(x)$ its support function.

Now, Holder's inequality for the conjugate indexes $p = i-n$ and $q = \frac{i-n}{i-n-1}$ yields (the case $i = n+1$ is a trivial equality),

$$\int_{S^{n-1}} h_{(DK_1)^\circ}(u)\rho_{K_1}^{n-i+1}(u)d\sigma(u) \leq \left(\int_{S^{n-1}} h_{(DK_1)^\circ}^{i-n}(u)d\sigma(u)\right)^{\frac{1}{i-n}} \left(\int_{S^{n-1}} \rho_{K_1}^{n-i}(u)d\sigma(u)\right)^{\frac{i-n-1}{i-n}}.$$

That is,

$$\tilde{W}_i(DK_1)^{\frac{1}{i-n}} \geq \tilde{W}_i(K_1)^{\frac{1}{i-n}-1}\frac{1}{n}\int_{S^{n-1}} h_{(DK_1)^\circ}(u).\rho_{K_1}^{n-i+1}(u)d\sigma(u).$$

By the inequality $h_{(DK_1)^\circ}(u) \geq \langle \bigtriangledown h_{K_1^\circ}(u), D^{-1}u\rangle$ (see [Sc], p. 40) and formula above we have

$$\tilde{W}_i(DK_1)^{\frac{1}{i-n}} \geq \tilde{W}_i(K_1)^{\frac{1}{i-n}-1}\frac{1}{n}\int_{S^{n-1}} \langle \bigtriangledown h_{K_1^\circ}(u), D^{-1}u\rangle \rho_{K_1}^{n-i+1}(u)d\sigma(u) = $$
$$= W_i(K_1)^{\frac{1}{i-n}}\left[\frac{1}{n-i}\mathrm{tr}(D^{-1}) - \frac{i}{n-i}\int_{S^{n-1}} \langle u, D^{-1}u\rangle d\tilde{\mu}_i(u)\right].$$

On the other hand, by the necessary condition (2.2),

$$\tilde{W}_i(DK_1)^{\frac{1}{i-n}} \geq \tilde{W}_i(K_1)^{\frac{1}{i-n}}\int_{S^{n-1}} \langle \omega, D^{-1}\omega\rangle \; d\tilde{\nu}(\omega).$$

Observe that for any $D \in GL(n)$ diagonal $\langle \omega, \omega\rangle^2 \leq \langle \omega, D\omega\rangle\langle \omega, D^{-1}\omega\rangle$. So,

$$1 = \int_{S^{n-1}} \langle \omega, \omega\rangle^2 \, d\tilde{\nu}(\omega) \leq \int_{S^{n-1}} \langle \omega, D\omega\rangle \; \langle \omega, D^{-1}\omega\rangle \, d\tilde{\nu}(\omega) \leq \int_{S^{n-1}} \langle \omega, D^{-1}\omega\rangle \, d\tilde{\nu}(\omega),$$

and since $DK_1 \subset D_n$, we have $\langle \omega, D\omega\rangle \leq 1$. Therefore, $\tilde{W}_i(DK_1) \geq \tilde{W}_i(K_1)$.

Case $i \in [-2, 0]$.

Clearly,

$$\frac{\tilde{W}_i(DK_1)}{\tilde{W}_i(K_1)} = \det(D)\int_{S^{n-1}} |Du|^{-i} \; d\tilde{\mu}_i(u) \leq \det(D)\left(\int_{S^{n-1}} |Du|^2 \; d\tilde{\mu}_i(u)\right)^{-\frac{i}{2}}$$
$$= \det(D)\left(\int_{S^{n-1}} \langle u, D^2(u)\rangle \; d\tilde{\mu}_i(u)\right)^{-\frac{i}{2}}.$$

On the other hand, by the necessary condition (2.2) and $DK_1 \subseteq D_n$,

$$\begin{aligned}\int_{S^{n-1}} \langle u, D^2(u)\rangle \, d\tilde{\mu}_i(u) &= \frac{\mathrm{tr}(D^2)}{i} - \frac{n-i}{i}\int_{S^{n-1}} \langle \omega, D^2\omega\rangle \, d\tilde{\nu}(\omega)\\ &= \frac{\mathrm{tr}(D^2)}{i} - \frac{n-i}{i}\int_{S^{n-1}} |D(\omega)|^2 \, d\tilde{\nu}(\omega)\\ &\le \frac{1}{i}(\mathrm{tr}(D^2) - (n-i)) \le \det(D)^{2/i}.\end{aligned}$$

The last inequality is a consequence of a linealization of Holder's inequality for negative exponents. Indeed, let $a_1, \dots, a_n > 0$ and $p > 0$. The two following facts are elementary

$$\Big(\frac{1}{n}\sum_{j=1}^n a_j^{-p}\Big)^{-1/p} \le \Big(\prod_{j=1}^n a_j\Big)^{1/n} \qquad \text{and} \qquad 1 - \frac{1}{p}(x-1) \le x^{-1/p}, \quad x > 0.$$

As a consequence we have that

$$1 - \frac{1}{p}\Big(\frac{1}{n}\sum_{j=1}^n a_j^{-p} - 1\Big) \le \Big(\prod_{j=1}^n a_j\Big)^{1/n}$$

and the result follows by taking $p = -i/n$, $a_j = d_j^{2n/i}$, $1 \le j \le n$.

Case $i \in (\infty, -2) \cup (0, n)$.

We use once again the fact that if a Borel measure $d\mu$ on S^{n-1} is isotropic, then $U(d\mu)$, its image measure under $U \in O(n)$ is also isotropic. This implies that, as in the previous cases, we can restrict ourselves to D diagonal matrices such that $DK \subseteq D_n$.

If $DK \subseteq D_n$ then $\int_{S^{n-1}} |D\omega|^2 \, d\nu(\omega) \le 1$ and so, by the necessary condition (2.2) and since $d\mu_i$ is isotropic, it is enough to show

$$\frac{\tilde{W}_i(DK)}{\tilde{W}_i(K)} = \Big(\prod_{j=1}^n d_j\Big) \int_{S^{n-1}} |Du|^{-i} d\mu_i(u) \le 1$$

under the (weaker) constraint $\sum_{j=1}^n d_j^2 \le n$. This function has a maximum value but, by differentiating, it cannot be attained at interior point.

It remains to study $\max\{\frac{\tilde{W}_i(DK)}{\tilde{W}_i(K)}\}$ under the constraint $\sum_{j=1}^n d_j^2 = n$. By the AM-GM inequality, $\Big(\prod_{j=1}^n d_j^2\Big)^{1/n} \le \frac{1}{n}\Big(\sum_{j=1}^n d_j^2\Big) \le 1$ so, $\prod_{j=1}^n d_j \le 1$. Hence, it suffices to show $g(D) = \int_{S^{n-1}} |Du|^{-i} d\mu_i(u) \le 1$ under the constraint

$\sum_{j=1}^{n} d_j^2 = n$. By using Lagrange multipliers we get that every extreme point satisfies

$$-i \int_{S^{n-1}} |Du|^{-i-2} u_j^2 d\mu_i(u) = 2\lambda$$

for all $j = 1, \dots, n$ and some $\lambda \in \mathbb{R}$. Therefore,

$$g(D) = \int_{S^{n-1}} |Du|^{-i} d\mu_i(u) = \sum_{j=1}^{n} d_j^2 \int_{S^{n-1}} |Du|^{-i-2} u_j^2 d\mu_i(u) = \frac{2\lambda n}{-i}.$$

Hence,

$$\begin{aligned}
\int_{S^{n-1}} |Du|^{-i} d\mu_i(u) &= n \int_{S^{n-1}} |Du|^{-i-2} u_j^2 d\mu_i(u) \quad \text{(for each } j\text{)} \\
&= \int_{S^{n-1}} |Du|^{-i-2} \sum_{j=1}^{n} u_j^2 d\mu_i(u) \\
&= \int_{S^{n-1}} |Du|^{-i-2} d\mu_i(u).
\end{aligned}$$

If $i < -2$ by Holder's inequality ($p = -i/(-i-2) > 1$) we get that

$$\begin{aligned}
\int_{S^{n-1}} |Du|^{-i-2} d\mu_i(u) &\leq \left(\int_{S^{n-1}} |Du|^{-i} d\mu_i(u) \right)^{1+2/i} \\
&= \left(\int_{S^{n-1}} |Du|^{-i-2} d\mu_i(u) \right)^{1+2/i},
\end{aligned}$$

which implies $\left(\int_{S^{n-1}} |Du|^{-i-2} d\mu_i(u) \right)^{-i/2} \leq 1$ and therefore $g(D) \leq 1$.

If $i \in (0, n)$ we take $p = (-i-2)/(-i) > 1$ and we use again Hölder inequality to obtain that

$$\begin{aligned}
\int_{S^{n-1}} |Du|^{-i} \, d\mu_i(u) &\leq \left(\int_{S^{n-1}} |Du|^{-i-2} d\mu_i(u) \right)^{-i/(-i-2)} \\
&= \left(\int_{S^{n-1}} |Du|^{-i} d\mu_i(u) \right)^{-i/(-i-2)},
\end{aligned}$$

which implies $g(D) = \int_{S^{n-1}} |Du|^{-i} d\mu_i(u) \leq 1$.

The uniqueness (up to orthogonal transformations) is obvious in all cases due to the consequences of having an equality in the corresponding inequalities. □

Remark 2.5. In the particular case when K is the unit ball of a 1-symmetric norm, both the absolutely continuous and the singular parts are isotropic, it verifies the necessary condition (2.1) for all i and so K is in extremal position for *all* indexes in theorem 2.4.

3. From the linear problem to the affine problem for centrally symmetric convex bodies

The results in the previous section characterize, for a range of indexes, the position of a convex body that is the solution of certain linear problem. Since the dual mixed volumes are not affine invariant one can wonder if we can also characterize the solution of a affine extremal problem of the type

$$\max\{\tilde{W}_i(a+TK);\ \ 0\in a+TK\subseteq D_n, a\in\mathbb{R}^n, T\in GL(n)\}\qquad (i<n),$$
$$\min\{\tilde{W}_i(a+TK);\ \ 0\in a+TK\subseteq D_n, a\in\mathbb{R}^n, T\in GL(n)\}\qquad (i>n).$$

By making use of John's theorem it is possible to state some necessary conditions of the extremal positions but, in general, we can't prove that they are sufficient. Despite this inconvenience, if K is centrally symmetric we go further and prove that, for positive indexes, the linear extremal problem and the affine one have the same solution.

We start with a simple observation that shows that the origin plays an special role for symmetric convex bodies.

Lemma 3.1. *Let $L\subseteq D_n$, symmetric with respect to a point $a\in\mathbb{R}^n$. Then,*

(i) $L-a\subseteq D_n$

(ii) $\partial(L-a)\cap S^{n-1}\neq\emptyset$ *implies that* $a=0$.

Proof. Let $x\in L$. Since $2a-x\in L$, by the convexity and symmetry of D_n we have $x-a=x/2+(x-2a)/2\in D_n$.

Let $x\in L$ such that $x-a\in\partial(L-a)\cap S^{n-1}$. Then,

$$1=|x-a|\leq\left|\frac{x}{2}\right|+\left|\frac{x-2a}{2}\right|\leq 1$$

and so this forces $x-2a=\lambda x$, for some $\lambda\geq 0$ and $|x|=|x-2a|=1$ which means $\lambda=1$ and $a=0$. □

In the next proposition the first assertion in the previous lemma will allow us to reduce the affine problem to the comparison of $\tilde{W}_i(TK)$ and $\tilde{W}_i(a+TK)$. The second one will imply that the extreme values are attained only when $a=0$.

Proposition 3.2. *Let $K\subseteq\mathbb{R}^n$ be a centrally symmetric convex body. Then,*

(i) If $i>n$,

$$\min\{\tilde{W}_i(TK);\ TK\subseteq D_n\}$$
$$=\min\{\tilde{W}_i(a+TK);\ \ 0\in a+TK\subseteq D_n\}.$$

(ii) If $0\leq i<n$,

$$\max\{\tilde{W}_i(TK);\ TK\subseteq D_n\}$$
$$=\max\{\tilde{W}_i(a+TK);\ \ 0\in a+TK\subseteq D_n\}.$$

Proof. *(i)* If we let $0 \in a + TK \subseteq D_n$, in order to prove that

$$\tilde{W}_i(a + TK) \geq \min\{\tilde{W}_i(TK);\ \ TK \subseteq D_n,\ \ T \in GL(n)\}$$

it is enough to show $\tilde{W}_i(a+TK) \geq \tilde{W}_i(TK)$, since, by Lemma 3.1., $a+TK \subseteq D_n$ implies $TK \subseteq D_n$.

If 0 is not an interior point of $a + TK$, $\tilde{W}_i(a + TK) = +\infty \geq \tilde{W}_i(TK)$, while if 0 is an interior point, since $f(x) = x^{n-i}$ is a convex function on $(0, +\infty)$,

$$\begin{aligned}\tilde{W}_i(a + TK) &= \frac{1}{n}\int_{S^{n-1}} \frac{1}{2}\rho_{a+TK}^{n-i}(u) + \frac{1}{2}\rho_{a+TK}^{n-i}(-u)\, d\sigma(u) \\ &\geq \frac{1}{n}\int_{S^{n-1}} \left(\frac{1}{2}\rho_{a+TK}(u) + \frac{1}{2}\rho_{a+TK}(-u)\right)^{n-i} d\sigma(u),\end{aligned}$$

but if we denote by E_u the 1-dimensional subspace given by $u \in S^{n-1}$, then $\rho_{a+TK}(u) + \rho_{a+TK}(-u) = (a + TK) \cap E_u$. Therefore, since TK is centrally symmetric and by the 1-dimensional Brunn-Minkowski inequality, we get that

$$\begin{aligned}\tilde{W}_i(a + TK) &\geq \frac{1}{n}\int_{S^{n-1}} \left(\frac{|(a + TK) \cap E_u|}{2}\right)^{n-i} d\sigma(u) \\ &\geq \frac{1}{n}\int_{S^{n-1}} \left(\frac{|TK \cap E_u|}{2}\right)^{n-i} d\sigma(u) \\ &= \frac{1}{n}\int_{S^{n-1}} \rho_{TK}^{n-i}(u)\, d\sigma(u) = \tilde{W}_i(TK).\end{aligned}$$

(ii) As before, it is enough to prove that if $0 \in a + TK \subseteq D_n$, then

$$W_i(a + TK) \leq W_i(TK). \tag{3.3}$$

Recall that

$$W_i(a + TK) = \frac{n-i}{n}\int_{TK} \frac{dx}{|x - a|^i}$$

so the inequality (3.3) is a direct consequence of the following lemma, since $d\mu(x) = \chi_{TK}(x)dx$ is symmetric and log-concave and the function $|\cdot|^{-i}$ is quasiconcave (and so unimodal, see [Bh]).

Lemma 3.3. *Let μ be a symmetric log-concave measure on $\mathbb{R}^n$ and let f be a unimodal function on $\mathbb{R}^n$ (i.e. a function which is an increasing pointwise limit of positive linear combinations of indicator functions on centrally symmetric convex in $\mathbb{R}^n$, see* [Bh]*), then*

$$\int_{\mathbb{R}^n} f(x - a)d\mu(x) \leq \int_{\mathbb{R}^n} f(x)d\mu(x)$$

for all $a \in \mathbb{R}^n$.

Proof of the Lemma. It is well known that $\mu(a+D) \leq \mu(D)$ for any $a \in \mathbb{R}^n$ and D a centrally symmetric bounded convex set in $\mathbb{R}^n$. Since f is unimodal we have

$$f_m(x) = \sum_{j=1}^{N_m} a_{m,j} \chi_{D_{m,j}}(x) \uparrow f(x)$$

for all $x \in \mathbb{R}^n$, where $a_{m,j} > 0$, $D_{m,j}$ are symmetric bounded convex sets for $m = 1, 2, \dots$ and $j = 1, 2, \dots, N_m$. It is now clear that

$$\int_{\mathbb{R}^n} f_m(x-a) d\mu(x) \leq \int_{\mathbb{R}^n} f_m(x) d\mu(x)$$

and therefore the monotone convergence theorem gives the result.

□

3.1. **A geometric tomography approach.** Taking into account the proof of the case $i > n$ in proposition 3.2, we wonder if an alternative geometrical argument involving sections (in the spirit of the geometric tomography) could be used for proving (3.3) in the range $0 < i < n$.

The answer is affirmative when $i \in (n-1, n)$ and $i = 1, \dots, n-1$ simply by computing central sections while for the other values of $i > 0$ we need to deal with averages of parallel sections. Let us describe this approach in this subsection.

Since the function $f(x) = x^{n-i}$ is concave we could proceed for $i \in [n-1, n)$ as in the case *(i)* in proposition 3.2.

Let $i = 1, \dots, n-1$. If $d\mu$ denotes the Haar measure on the Grassman manifold $\mathcal{G}(n, n-i)$ of all $(n-i)$-dimensional subspaces of $\mathbb{R}^n$ and $d\mu_E$ the Haar measure on the sphere in the subspace $E \subseteq \mathbb{R}^n$, it is easy to check (see, for example [Ga], [L2], [GV]) that

$$\begin{aligned} \tilde{W}_i(a+TK) &= C_n \int_{\mathcal{G}(n,n-i)} \int_{S^{n-1}\cap E} \rho_{a+TK}^{n-i}(u)\, d\mu_E(u)\, d\mu(E) \\ &= C_n \int_{\mathcal{G}(n,n-i)} |(a+TK)\cap E|_{n-i}\, d\mu(E), \end{aligned}$$

where C_n is a renorming constant. This formula is known as the dual Kubota formula. By Brunn-Minkowski inequality and since K is centrally symmetric, $|(a+TK)\cap E|_{n-i} \leq |TK \cap E|_{n-i}$ and so this fact would directly imply that

$$\tilde{W}_i(a+TK) \leq C_n \int_{\mathcal{G}(n,n-i)} |TK \cap E|_{n-i}\, d\mu(E) = \tilde{W}_i(TK).$$

In order to go further with other indexes we first prove the formula (3.4) that relates the dual mixed volumes $\tilde{W}_i(L)$ and the parallel sections of L. This is shown to be an extension of dual Kubota formula for the range $i < n$ and it has an independent interest.

We need to introduce some notation. Consider the orthogonal group $O(n)$ equipped with its normalized Haar measure $d\nu$. We identify each

$U \in O(n)$ with the n-tuple $(\xi_1, \dots, \xi_n)$ of orthonormal vectors in $\mathbb{R}^n$ such that $Ue_j = \xi_j, j = 1 \dots n$, where (e_j) is the canonical basis in $\mathbb{R}^n$. We denote by $U_k = (\xi_1, \dots, \xi_k)$ the $n \times k$ matrix composed by the k first columns of U.

For $1 \le k \le n$, $L \subset \mathbb{R}^n$ a star body, $U \in O(n)$ and $s = (s_1, \dots, s_k)$, we denote,

$$A_{U_k}^L(s) = \Big| L \cap (s_1\xi_1 + \dots s_k\xi_k + \{\xi_1, \dots, \xi_k\}^\perp) \Big|_{n-k}.$$

For $k = n$ this definition should be understood as a characteristic function, that is, $A_{U_n}^L(s) = 1$ if $\sum s_j\xi_j \in L$ and $A_{U_n}^L(s) = 0$ otherwise.

Theorem 3.4. *Let $L \subset \mathbb{R}^n$ be a star body and let $i \in \mathbb{R}$ and $k \in \mathbb{N}$ with $i < k \le n, k \in \mathbb{N}$. Then*

$$\tilde{W}_i(L) = c(n,k,i) \int_{\mathbb{R}^k} \left(\int_{O(n)} A_{U_k}^L(s)\, d\nu(U) \right) \frac{ds}{|s|^i} \tag{3.4}$$

where $c(n,k,i) = \dfrac{(n-i)}{n} \dfrac{\Gamma(\frac{k}{2})\Gamma(\frac{n-i}{2})}{\Gamma(\frac{n}{2})\Gamma(\frac{k-i}{2})}$.

Proof. Observe that, since the function $A_{U_k}^L$ is of bounded support and $i < k$, the right hand side integral is finite.

Assume that $U \in O(n)$. By Fubini's theorem is clear that

$$\int_{\mathbb{R}^k} A_{U_k}^L(s) \frac{ds}{|s|^i} = \int_{\mathbb{R}^n} |(\langle x, \xi_1 \rangle, \dots, \langle x, \xi_k \rangle)|^{-i} \chi_L(x) dx.$$

Now, again by Fubini's theorem,

$$\int_{O(n)} \int_{\mathbb{R}^k} A_{U_k}^L(s) \frac{ds}{|s|^i}\, d\nu(U) = \int_{O(n)} \int_L |(\langle x, \xi_1 \rangle, \dots, \langle x, \xi_k \rangle)|^{-i}\, dx\, d\nu(U)$$
$$= \int_L \frac{dx}{|x|^i} \int_{O(n)} \left| \left(\langle \frac{x}{|x|}, \xi_1 \rangle, \dots, \langle \frac{x}{|x|}, \xi_k \rangle \right) \right|^{-i} d\nu(U).$$

By the orthogonal invariance of $d\nu$ and $\langle \cdot, \cdot \rangle$, the inner integral is independent of x, that is, $\displaystyle\int_{O(n)} |(\langle e, \xi_1 \rangle, \dots, \langle e, \xi_k \rangle)|^{-i}\, d\nu(U)$ is constant on $e \in S^{n-1}$. If we denote this integral as $I(k,n)$ we have thus proved so far that

$$I(k,n) \frac{n}{n-i} \tilde{W}_i(L) = \int_{\mathbb{R}^k} \left(\int_{O(n)} A_{U_k}^L(s)\, d\nu(U) \right) \frac{ds}{|s|^i}.$$

Since the formula above holds for any L we consider $L = D_n$. Clearly, $\tilde{W}_i(D_n) = |D_n|$. On the other hand, $A_{U_k}^{D_n}(s)$ is independent of $\xi_1, \dots, \xi_k$ and $A_{U_k}^{D_n}(s) = \chi_{D_k}(s)(1 - |s|^2)^{\frac{n-k}{2}} |D_{n-k}|$.

Therefore,

$$\int_{\mathbb{R}^k}\left(\int_{O(n)} A^L_{U_k}(s)\,d\nu(U)\right)\frac{ds}{|s|^i}=|D_{n-k}|\int_{D_k}(1-|s|^2)^{\frac{n-k}{2}}\frac{ds}{|s|^i}.$$

And, by integrating in polar coordinates and φ being normalized, this is equal to

$$k|D_k|\,|D_{n-k}|\left(\int_0^1 r^{k-1-i}(1-r^2)^{\frac{n-k}{2}}\,dr\right).$$

By direct computation, $\int_0^1 r^{k-1-i}(1-r^2)^{\frac{n-k}{2}}\,dr=\frac{1}{2}\beta(\frac{n-k}{2}+1,\frac{k-i}{2})$ and so, $c^{-1}(n,k,i)=\frac{k|D_k|\,|D_{n-k}|}{2|D_n|}\beta(\frac{n-k}{2}+1,\frac{k-i}{2})$.

Finally, recall that $|D_m|=\frac{\pi^{m/2}}{\Gamma(1+(m/2))}$ and $\beta(a,b)=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}, a,b>0$ to simplify $c(n,k,i)$. □

The following corollary shows that, as $i\to k^-$, we recover the formula for $\tilde{W}_k(L)$ in terms of the central sections of L, that is, the dual Kubota formula (see [L2], [GV]).

Corollary 3.5. *Let $L\subset\mathbb{R}^n$ be a star body, $i<k\le n-1, k,n\in\mathbb{N}$. Then,*

$$\lim_{i\to k^-} c(n,k,i)\int_{\mathbb{R}^k}\left(\int_{O(n)} A^L_{U_k}(s)\,d\nu(U)\right)\frac{ds}{|s|^i}=c_{n,k}\int_{G(n,n-k)}|E\cap L|_{n-k}\,d\mu(E)$$

Proof. Let ϕ be $\mathcal{C}^{1)}$-function of compact support on $\mathbb{R}^k$. Then, by integrating in polar coordinates,

$$\frac{1}{\Gamma\left(\frac{k-i}{2}\right)}\int_{\mathbb{R}^k}\frac{\phi(s)}{|s|^i}ds=\frac{1}{\Gamma\left(\frac{k-i}{2}\right)}\int_0^\infty r^{k-1-i}\int_{S^{k-1}}\phi(r\theta)\,d\sigma(\theta)\,dr.$$

Denote $\Phi(r)=\int_{S^{k-1}}\phi(r\theta)\,d\sigma(\theta)$ which is also $\mathcal{C}^{1)}$-function of compact support on $\mathbb{R}$. By integration by parts we have

$$\frac{1}{\Gamma\left(\frac{k-i}{2}\right)}\int_0^\infty r^{k-1-i}\Phi(r)\,dr=\frac{1}{(k-i)\Gamma\left(\frac{k-i}{2}\right)}\int_0^\infty r^{k-i}\Phi'(r)\,dr.$$

Now, since $\lim_{i\to k^-}(k-i)\Gamma(\frac{k-i}{2})=2$,

$$\lim_{i\to k^-}\frac{1}{\Gamma\left(\frac{k-i}{2}\right)}\int_{\mathbb{R}^k}\frac{\phi(s)}{|s|^i}ds=\frac{1}{2}\sigma(S^{k-1})\phi(0).$$

And so, by an approximation argument we have that

$$\lim_{i\to k^-}\int_{\mathbb{R}^k}\left(\int_{O(n)} A^L_{U_k}(s)\,d\nu(U)\right)\frac{ds}{|s|^i}=c_{n,k}\int_{O(n)} A^L_{U_k}(0)\,d\nu(U).$$

Now, by the integration formula in [K] Lemma 1 which is consequence of the conditional expectation theorem,

$$\int_{O(n)} A^L_{U_k}(0)d\nu(U) = \int_{G(n,k)} d\mu(E) \int_{\xi_1,\dots,\xi_k\in E} A^L_{U_k}(0)d\nu(U_k)$$

where $d\nu(U_k)$ is the Haar measure in $O(k)$. Finally, since $A^L_{U_k}(0) = |L \cap \{\xi_1,\dots,\xi_k\}^\perp|_{n-k}$ and the uniqueness properties of $d\mu$

$$\int_{O(n)} A^L_{U_k}(0)d\nu = \int_{G(n,k)} |L\cap E^\perp|_{n-k}d\mu(E) = \int_{G(n,n-k)} |L\cap H|_{n-k}d\mu(H).$$

□

Remark 3.6. Let f be a (say) continuous non-negative function on S^{n-1}. Let $i \in \mathbb{R}$, $k \in \mathbb{N}$ as in 3.4. Consider the set $L = \{x \in \mathbb{R}^n; \tilde f(x) \geq 1\}$, where $\tilde f$ denotes the $(i-n)$-homogeneous extension of f on $\mathbb{R}^n\setminus\{0\}$. L is clearly a star shaped body and its corresponding radial function is $\rho_L(u) = f(u)^{1/(n-i)}$, for all $u \in S^{n-1}$. Then, by the same arguments as in the proof of theorem 3.4,

$$\int_{S^{n-1}} f(u)d\mu(u) = c(n,k,i)\int_{\mathbb{R}^k}\left(\int_{O(n)} A^L_{U_k}(s)d\nu(U)\right)\frac{ds}{|s|^i}.$$

Moreover, by taking limits as in 3.5 and integrating in polar coordinates we recover the formula

$$\begin{aligned}\int_{S^{n-1}} f(x)d\mu(x) &= c_{n,k}\int_{G(n,n-k)} |L\cap H|_{n-k}d\mu(H)\\ &= \int_{G(n,n-k)} d\mu(H)\int_{S^{n-1}\cap H} f(x)\,d\mu_H(x),\end{aligned}$$

which in particular also implies the recursion dual Kubota formula ([Ga], [L2], [GV]).

We return to the inequality (3.3). Let $0 < i < k \leq n-1$. Now it is clear that $\tilde W_i(a+TK) \leq \tilde W_i(TK)$ since

$$\begin{aligned}\tilde W_i(a+TK) &= c_{n,k,i}\int_{O(n)}\left(\int_{\mathbb{R}^k} A^{a+TK}_{U_k}(s)\frac{ds}{|s|^i}\right)d\nu(U)\\ &= c_{n,k,i}\int_{O(n)}\left(\int_{\mathbb{R}^k} A^{TK}_{U_k}(s)\frac{ds}{|s-b_U|^i}\right)d\nu(U)\end{aligned}$$

where $b_U = (\langle a,\xi_1\rangle,\dots,\langle a,\xi_k\rangle)$ for any $U(\xi_1,\dots,\xi_n) \in O(n)$. Since the function $A^{TK}_{U_k}(s)$ is even and log-concave by Brunn-Minkowski inequality, we easily deduce (3.3).

4. Further remarks.

The results in section 2 allow us estimate two parameters that measure the distance between the extremal positions and the euclidian ball.

For $K, L \subset \mathbb{R}^n$, their *geometric distance* is defined as

$$d_G(K,L) = \inf\{ab \mid \frac{1}{a}K \subseteq L \subset bK, a, b > 0\}.$$

Theorem 4.1. *Let $K \subseteq D_n$ be a centrally symmetric convex body and $i \neq n$ and assume that K is in extremal position for the functional $\tilde{W}_i(TK)$. Then*

$$a_{n,i} D_n \subseteq K \subseteq D_n,$$

where

$$a_{n,i} = \begin{cases} \frac{1}{\sqrt{n-i}} & \text{for } i < 0, \\ \sqrt{\frac{1-i}{n-i}} & \text{for } 0 \leq i < 1, \\ \frac{1}{\sqrt{n}} \left(\frac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{i-n+1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{i}{2}\right)} \right)^{1/(i-n)} & \text{for } i \in (n, \infty). \end{cases} \tag{4.5}$$

Proof. By Theorem 2.1, K verifies

$$I_n = i \int_{S^{n-1}} u \otimes u \, d\mu_i(u) + (n-i) \sum_{j=1}^{N} \lambda_j \, w_j \otimes w_j.$$

Let $y \in K^\circ$. Then $|\langle y, w_j \rangle| \leq 1$, for all $1 \leq j \leq N$ and so

$$\frac{1}{n-i} \int_{S^{n-1}} \left(|y|^2 - i|\langle u, y\rangle|^2\right) d\mu_i(u) \leq \sum_{j=1}^{N} \lambda_j = 1.$$

If $0 \leq i < 1$, we have $\frac{1-i}{n-i} \int_{S^{n-1}} |y|^2 d\mu_i(u) \leq \sum_{j=1}^{N} \lambda_j = 1$, which implies $|y| \leq \sqrt{\frac{n-i}{1-i}}$ and so

$$\sqrt{\frac{1-i}{n-i}}\, D_n \subseteq K \subseteq D_n.$$

If $i < 0$, we have in this case $1 \geq \frac{1}{n-i} \int_{S^{n-1}} |y|^2 d\mu_i(u) = \frac{|y|^2}{n-i}$ and

$$\frac{1}{\sqrt{n-i}}\, D_n \subseteq K \subseteq D_n.$$

If $i > n$, use the inequality $|\langle y, w_j\rangle| \leq |y|$, for all $1 \leq j \leq N$ and so

$$\frac{1}{n-i} \int_{S^{n-1}} \left(|y|^2 - i|\langle u, y\rangle|^2\right) d\mu_i(u) \leq |y|^2 \sum_{j=1}^{N} \lambda_j = |y|^2$$

or equivalently, $(i-n+1)|y|^2 \geq i \int_{S^{n-1}} |\langle u, y\rangle|^2 d\mu_i(u)$.

Since $\rho_K(u)^{-1} = \|u\|_K \geq |\langle u, y\rangle|$ we have

$$\int_{S^{n-1}} |\langle u, y\rangle|^2 \rho_K(u)^{n-i} d\sigma(u) \geq \int_{S^{n-1}} |\langle u, y\rangle|^{i-n+2} d\sigma(u)$$

$$= \frac{2\pi^{\frac{n-1}{2}}\,\Gamma\left(\frac{i-n+3}{2}\right)}{\Gamma\left(\frac{i+2}{2}\right)} |y|^{i-n+2}.$$

Assume that K_1 is John position for K. Then $\frac{1}{\sqrt{n}} D_n \subseteq K_1 \subseteq D_n$ and $1/\sqrt{n} \leq \rho_{K_1}(u) \leq 1$, for all $u \in S^{n-1}$. Thus

$$\tilde{W}_i(K) \leq \tilde{W}_i(K_1) \leq \left(\frac{1}{\sqrt{n}}\right)^{n-i} |D_n|$$

and therefore

$$\int_{S^{n-1}} |\langle u, y\rangle|^2 d\mu_i(u) \geq \frac{(\sqrt{n})^{n-i}}{n|D_n|} \frac{2\pi^{\frac{n-1}{2}}\,\Gamma\left(\frac{i-n+3}{2}\right)}{\Gamma\left(\frac{i+2}{2}\right)} |y|^{i-n+2}$$

$$\geq \frac{n^{(n-i)/2}(i-n+1)\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{i-n+1}{2}\right)}{\pi^{\frac{1}{2}}\, i\,\Gamma\left(\frac{i}{2}\right)} |y|^{i-n+2}.$$

Hence, $|y| \leq \sqrt{n}\left(\frac{\sqrt{\pi}\,\Gamma\left(\frac{i}{2}\right)}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{i-n+1}{2}\right)}\right)^{1/(i-n)}$ and the result follows.

□

Finally, the second parameter is $\left(\frac{\tilde{W}_i(D_n)}{\tilde{W}_i(TK)}\right)^{\frac{1}{n-i}}$, a natural extension of the volume ratio ($i = 0$), for the extremal position of K. It turns out that the estimate using John's position provides sharp upper bound, being the reason that for unit balls of 1-symmetric norms all positions coincide.

Proposition 4.2. *Let $K \subseteq \mathbb{R}^n$ a centrally symmetric convex body and $i \neq n$. There exists position $TK \subseteq D_n$ such that*

$$1 \leq \left(\frac{\tilde{W}_i(D_n)}{\tilde{W}_i(TK)}\right)^{1/(n-i)} \leq \sqrt{n}.$$

Proof. We consider $TK \subseteq D_n$ the maximal volume ($i = 0$) position of K contained in D_n. It is well known that $\frac{1}{\sqrt{n}} D_n \subseteq TK \subseteq D_n$, which is equivalent to $\frac{1}{\sqrt{n}}\rho_{D_n}(x) \leq \rho_{TK}(x) \leq \rho_{D_n}(x)\, \forall x \in \mathbb{R}^n \setminus \{0\}$ and so,

$$1 \leq \left(\frac{\tilde{W}_i(D_n)}{\tilde{W}_i(TK)}\right)^{1/(n-i)} \leq \sqrt{n}.$$

□

Remark 4.3. For a range of indexes we can prove that these bounds are sharp for $K = B_1^n$ the ℓ_1^n-ball. Indeed, it is clear that $B_1^n \subseteq D_n$ satisfies

(2.1) and a standard computation gives

$$\tilde{W}_i(K) = |D_n| \frac{2^{n/2}\Gamma\left(\frac{n}{2}+1\right)}{n2^{\frac{i-2}{2}}\Gamma\left(\frac{i}{2}\right)} \mathbb{E} \left\| \sum_{j=1}^{n} g_j e_j \right\|_1^{i-n}$$

where the g_j's are i.i.d. normalized Gaussian variables. Fix $p = i - n \geq 1$. By Hölder's inequality,

$$\mathbb{E} \left\| \sum_{j=1}^{n} g_j e_j \right\|_1^{p} \geq n^p \left(\mathbb{E}|g_1|\right)^p .$$

Now, by using Stirling's formula it is easy to see that, asymptotically in n,

$$\tilde{W}_i(K) \geq C^p n^{p/2} |D_n|,$$

which this gives the result.

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