ON SHARP REITERATION THEOREMS AND WEIGHTED NORM INEQUALITIES

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Abstract. We prove sharp end forms of Holmstedt's reiteration theorem which are closely connected with a general form of Gehring's Lemma. Reverse type conditions for the Hardy-Littlewood-Polya order are considered and the maximal elements are shown to satisfy generalized Gehring conditions. The methods we use are elementary and based on variants of reverse Hardy inequalities for monotone functions.

1. Introduction

Given a fixed initial pair of compatible spaces, interpolation theory provides us with methods to construct scales of spaces with the interpolation property. The classical methods of interpolation all share the following reiteration principle: by iteration these constructions do not generate new spaces. Reiteration theorems thus play a central role in these theories. In particular reiteration simplifies the process of identification of interpolation spaces. Holmstedt's reiteration formula, for the real method of interpolation (cf. [Ho]), provides quantitative estimates and plays an important role in a manifold of applications to classical analysis and approximation theory.

Let $\bar{A}$ be a pair of compatible Banach spaces, $0 < \theta_0 < \theta_1 < 1, 0 < q_i \leq \infty, i = 0, 1, \eta = \theta_1 - \theta_0$, then Holmstedt's formula states that

$$K(t, f; \bar{A}_{\theta_0,q_0}, \bar{A}_{\theta_1,q_1}) \approx \left\{ \int_0^{t^{1/\eta}} (s^{-\theta_0} K(s, f; \bar{A}))^{q_0} \frac{ds}{s} \right\}^{1/q_0} + t \left\{ \int_{t^{1/\eta}}^{\infty} (s^{-\theta_1} K(s, f; \bar{A}))^{q_1} \frac{ds}{s} \right\}^{1/q_1}. \tag{1.1}$$

Holmstedt's formula is also valid if $\theta_0 = 0$ or $\theta_1 = 1$. For example, if $\theta_1 = 1$ we have

$$K(t, f; \bar{A}_{\theta_0,q_0}, A_1) \approx \left\{ \int_0^{t^{1/(1-\theta_0)}} (s^{-\theta_0} K(s, f; \bar{A}))^{q_0} \frac{ds}{s} \right\}^{1/q_0}. \tag{1.2}$$

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In [Mi], [BMR1], [MM] a connection between Holmstedt's reiteration formula and Gehring's Lemma (cf. [Ge] and [Iw] for a recent survey) was established and new methods to prove general forms of self improving inequalities, containing Gehring's Lemma as a particular case, were developed in the general context of real interpolation spaces. This general formulation is not only of theoretical interest but also gives new results in the classical setting. For example, it provides new methods to deal with the case of non-doubling measures through the use of suitable substitutes for the maximal operator of Hardy-Littlewood (for more on this we refer to [MM].)

In this note we reverse the flow and show how certain estimates for averages, that are naturally associated with reverse Hölder inequalities, can be used to give new sharp reiteration formulae of Holmstedt's type. We also apply reiteration formulae to obtain results related to the classical theory of weighted norm inequalities.

In [Ho] one can find estimates for the constants implicit in (1.1) and (1.2). This formulae often plays a crucial role in applications of interpolation theory to analysis, and has been extensively studied and extended in several directions by many authors. For detailed studies of reiteration theorems of Holmstedt type, as well as extensive lists of references, we refer the reader to [BK], [BL], [BS], [Ni] and [Ov].

If one becomes fuzzy about constants one can notice some slight defects of (1.1) or (1.2), which ironically are associated with some of the best features(!) of the formulae, namely its compactness and intuitive form. Indeed, Holmstedt's elegant formulation is achieved through collecting together terms with a consequent worsening of some constants of equivalence. In most applications this minor imperfection is of little consequence if any, but it does become a crucial issue for certain problems. For example, in the theory of extrapolation developed in [JM] careful control of the constants involved is necessary and the different terms implicit in (1.1) and (1.2) need to be kept under separate control. For example the following reiteration formula was obtained by Jawerth and Milman in [JM]

(1.3)

$$K(t,f;\bar{A}_{\theta_0,q},\bar{A}_{\theta_1,q}) \approx c_{\theta_0,q}\left\{\int_0^{t^{1/\eta}} (s^{-\theta_0}K(s,f;\bar{A}))^q \frac{ds}{s}\right\}^{1/q} + tc_{\theta_1,q}\left\{\int_{t^{1/\eta}}^{\infty} (s^{-\theta_1}K(s,f;\bar{A}))^q \frac{ds}{s}\right\}^{1/q} + t^{-\theta_0/\eta}K(t^{1/\eta},f;\bar{A}),$$

where $c_{\theta_i,q} = \Big((1-\theta_i)\theta_i q'\Big)^{1/q'}, i = 0,1.$

In this note we prove an end point version of (1.3) in the spirit of Holmstedt's original end point formula (1.2). Let us explain in more detail what we are going to do. Suppose that after having generated the space $\bar{A}_{\theta_0,p}$ we want to further interpolate between $(\bar{A}_{\theta_0,p}, A_1)_{\eta,q} = \bar{A}_{\theta_1,q}$, and A_1, with $\theta_1 = (1-\eta)\theta_0 + \eta$, i.e. $\eta = \frac{\theta_1-\theta_0}{1-\theta_0}$, then, according to (1.2) applied to the pair $(\bar{A}_{\theta_0,p}, A_1)$, we will have

$$K(t,f;\bar{A}_{\theta_1,q},A_1) \approx c_{\eta,p,q}\left\{\int_0^{t^{1/1-\eta}} \left[s^{-\eta}K(s,f;\bar{A}_{\theta_0,p},A_1)\right]^q \frac{ds}{s}\right\}^{1/q},$$

or alternatively we can apply (1.2) directly, i.e. use $\bar{A}$ as our base pair, in which case the constant of equivalence will depend on θ_1, q. In Theorem (2.11) below we shall prove that if $\theta_0 = 1/p'$, $\theta_1 = 1/q'$, then

$$\text{(HO)}\qquad K(t^{1-\theta_1}, f; \bar{A}_{\theta_1,q}, A_1) \leq c_{\eta,p,q,\theta_1} K(t^{1-\theta_1}, f; \bar{A}_{\theta_1,q}, A_1) + ct^{-\frac{\eta}{1-\eta}} K(t^{1/1-\eta}, f; \bar{A}_{\theta_0,p}, A_1),$$

with $c_{\eta,p,q,\theta} \to 0$ when $\eta \to 0$, that is when $\theta_1 \to \theta_0$. In connection with the second term in (HO) note that

$$t^{-\frac{\eta}{1-\eta}} K(t^{1/1-\eta}, f; \bar{A}_{\theta_0,p}, A_1) = t\frac{K(t^{1/1-\eta}, f; \bar{A}_{\theta_0,p}, A_1)}{t^{1/1-\eta}},$$

and since $\frac{K(s,f;\bar{A}_{\theta_0,p},A_1)}{s}$ decreases,

$$t\frac{K(t^{1/1-\eta}, f; \bar{A}_{\theta_0,p}, A_1)}{t^{1/1-\eta}}$$

$$\leq (1-\eta)^{1/q} q^{1/q} \left\{ \int_0^{t^{1/1-\eta}} \left[\frac{K(s, f; \bar{A}_{\theta_0,p}, A_1)}{s} \right]^q s^{(1-\eta)q} \frac{ds}{s} \right\}^{1/q}$$

$$= (1-\eta)^{1/q} q1/q \left\{ \int_0^{t^{1/1-\eta}} \left[K(s, f; \bar{A}_{\theta_0,p}, A_1) s^{-\eta} \right]^q \frac{ds}{s} \right\}^{1/q},$$

which is (by Holmstedt's formula) again comparable with $K(t, f; \bar{A}_{\theta_1,q}, A_1)$. Therefore the second term of (HO) will vanish when $\theta_1 \to 1$, i.e. when $\eta \to 1$. In other words if constants were not an issue we could do just as well with Holmstedt's original formula. The decoupling that we have achieved in this fashion can be exploited to our advantage for certain crucial estimates for functions that satisfy reverse Hölder inequalities (cf. Example (2.7) below). The (minor) price we pay is that we need to assume that the right hand side is finite in order to be able to use the formula (cf. Example (2.7) for more on this.)

The proof of this result involves the use of some elementary variants of reverse Hardy inequalities valid only for monotone functions. The connection between reverse Hardy inequalities and Gehring's Lemma has been noted before. For example, in [Mi1] reverse Hardy inequalities inequalities given in [Be] and [Re] were used to prove a variant of Gehring's Lemma. Here we use closely related but different estimates (cf. (2.5) below). After completing the first draft of this paper we realized that some cases of the modified reverse Hardy inequalities we prove here were also contained in a paper by Franciosi-Moscariello [FM]. Moreover, these authors also use their estimates to give a proof of the classical version of Gehring's Lemma. We have thus chosen to discuss this application to Gehring's Lemma rather briefly in Example (2.7) (cf. also [FM]). The reiteration formulae we obtain here was conceived from a completely different point of view and seems to have wider applicability. Indeed we should note that the generalized setting afforded by interpolation theory has made it possible to deal with Gehring type Lemmas for non doubling measures (cf. [MM]) by means of replacing the maximal function of Hardy-Littlewood with maximal operators associated with packings ([AKMP], [MM]). The same remark applies for our results in this paper.

In retrospect, one could argue that the genesis of our argument is already present in the first proof of the usual Hardy inequality given by Hardy himself! In fact, if we follow mutatis mutandi the proof of Hardy's inequality in [HLP] page 242 adapted to a finite interval $(0,t)$, we find that the inequality contains an *extra* term which only dissapears when t tends to infinity, namely

(1.4)
$$\left(\int_0^t Pf(x)^p dx\right)^{1/p} + \frac{tPf(t)^p}{(p-1)\left(\int_0^t Pf(x)^p dx\right)^{1/p'}} \leq \frac{p}{p-1}\left(\int_0^t f^p(x)dx\right)^{1/p},$$

where $Pf(x) = \frac{1}{x}\int_0^x f(u)du$, and f is a measurable, bounded, positive function. The last formula should be compared with a special case of Proposition (2.1) below (cf. also [FM]) from which we deduce that the following estimate holds for decreasing f,

(1.5)
$$\left(\int_0^t f(x)^p dx\right)^{1/p} \leq \left(\frac{p-1}{p}\right)^{1/p}\left(\int_0^t Pf(x)^p dx\right)^{1/p} + \left(\frac{1}{p}\right)^{1/p} t^{\frac{1-p}{p}}\int_0^t f(x)dx.$$

Remarks. (i) For future use note that (1.4) obviously implies the Hardy inequality frequently applied in the literature, namely, for $t > 0$

$$\left(\int_0^t Pf(x)^p dx\right)^{1/p} \leq \frac{p}{p-1}\left(\int_0^t f^p(x)dx\right)^{1/p}. \tag{1.6}$$

(ii) One can further see the *reiterative* character of Hardy's original inequality by means of letting $P_p f = (Pf^p)^{1/p}$, in (1.4) to obtain

$$P_p Pf(t) + \frac{Pf(t)^p}{(p-1)\left(P_p(Pf)(t)\right)^{p/p'}} \leq \frac{p}{p-1} P_p f(t).$$

With this notation (1.5) can also be rewritten as

$$P_p f(t) \leq \left(\frac{p-1}{p}\right)^{1/p} P_p f(t) + \left(\frac{1}{p}\right)^{1/p} Pf(t).$$

These inequalities are associated with the pairs of spaces (L^1, L^∞) and (L^p, L^∞) and *propagate* naturally through the L^p scale as we have outlined above.

The plan of the paper is as follows. In section 2 we prove the reverse Hardy inequalities and reiteration formulae we have just outlined. In section 3 we show how the K functional can be used to bridge between certain classes of weights associated with Hardy type operators and Calderón's operator and reverse Hölder inequalities (cf. [AM], [Mu1], [BR], [BMR]..) which were briefly outlined in our previous work on this subject [BMR1]. In section 4 we show how the interpolation theoretic framework we have developed leads to an explicit connection between reverse Hölder inequalities and the Hardy-Littlewood-Polya (HLP) order. It is well known that the HLP order is preserved by convex functions and reversed by concave functions, we show that maximal elements that satisfy a reverse HLP condition for concave functions must satisfy a generalized reverse Hölder inequality. This fact is in turn connected to the classical theory of real interpolation of Calderón and the method of orbits (cf. [Ov] and [MO], and the references therein.)

2. Reverse Hardy inequalities and sharp reiteration formulae

The following result is a reverse Hardy inequality in the spirit of Hardy's original proof of his inequality (cf. also [FM] Lemma (3.4).)

Proposition 2.1. *Let f be a non negative measurable function in $(0,\infty)$.*
i) *If f is non increasing, $0<\alpha<\infty$ and $p>1$, then*

(2.2)
$$\left(\int_0^t x^{p(1-\alpha)} f(x)^p \frac{dx}{x}\right)^{1/p} \leq \left(\alpha \int_0^t x^{-\alpha p}\left(\int_0^x f(y)dy\right)^p \frac{dx}{x}\right)^{1/p} + \left(\frac{1}{p}\right)^{1/p} t^{-\alpha} \int_0^t f(x)dx.$$

ii) *If $tf(t)$ is non decreasing, $0<\beta<\infty$ and $p>1$, then*

(2.3)
$$\left(\int_t^\infty x^{p\beta} f(x)^p \frac{dx}{x}\right)^{1/p} \leq \left(\beta \int_t^\infty x^{\beta p}\left(\int_x^\infty f(y)\frac{dy}{y}\right)^p \frac{dx}{x}\right)^{1/p} + \left(\frac{1}{p}\right)^{1/p} t^{\beta} \int_t^\infty f(x)\frac{dx}{x}.$$

Proof. In order to prove (2.2) we assume that f is continuous (the general case follows by an easy approximation argument). Pick $\varepsilon>0$ small enough, then integrating by parts we find,

$$\int_\varepsilon^t x^{-\alpha p}\left(\int_\varepsilon^x f\right)^p \frac{dx}{x} = -\frac{t^{-\alpha p}}{\alpha p}\left(\int_\varepsilon^t f\right)^p + \frac{1}{\alpha}\int_\varepsilon^t x^{-\alpha p}\left(\int_\varepsilon^x f\right)^{p-1} f(x)dx.$$

Let $\varepsilon \to 0$, then by the monotone convergence theorem, and taking into account that since f is non increasing we have $\int_0^x f \geq xf(x)$, we get

$$\int_0^t x^{-\alpha p}\left(\int_0^x f\right)^p \frac{dx}{x} + \frac{t^{-\alpha p}}{\alpha p}\left(\int_0^t f\right)^p \geq \frac{1}{\alpha}\int_0^t x^{-\alpha p}\left(xf(x)\right)^{p-1} f(x)dx,$$

which readily implies (2.2).

To prove of (2.3) we can also assume that f is continuous. Let N be large, integrating by parts we find

$$\int_t^N x^{\beta p}\left(\int_x^N f(y)\frac{dy}{y}\right)^p \frac{dx}{x} = -\frac{t^{\beta p}}{\beta p}\left(\int_t^N f(y)\frac{dy}{y}\right)^p + \frac{1}{\beta}\int_t^N x^{\beta p}\left(\int_x^N f(y)\frac{dy}{y}\right)^{p-1} f(x)\frac{dx}{x}.$$

Let N tend to ∞, then taking into account that if $tf(t)$ is non decreasing we have

$$\int_x^\infty f(y)\frac{dy}{y} = \int_x^\infty f(y)y\frac{dy}{y^2} \geq f(x),$$

now the proof proceeeds in the same fashion as the proof of (2.2).

Note that (2.3) can also be obtained from (2.2) by means of a change of variables: take the non increasing function $f(1/t)/t$ in (2.2) and let $x = 1/y$.

Remarks. (i) If we let $\alpha = \frac{p-1}{p}$ in (2.2) then we get

$$\left(\int_0^t f(x)^p dx\right)^{1/p} \leq \left(\frac{p-1}{p}\right)^{1/p} \left(\int_0^t Pf(x)^p dx\right)^{1/p} + \left(\frac{1}{p}\right)^{1/p} t^{\frac{1-p}{p}} \int_0^t f(x)dx. \tag{2.4}$$

(ii) In order to deal with L^p norms on $(0, \infty)$ observe that if we let $t \to \infty$ in (2.4) we obtain the following inequality due to Bennett [Be] and Renaud [Re]

$$\|f\|_p \leq \left(\frac{p-1}{p}\right)^{1/p} \|Pf\|_p . \tag{2.5}$$

(iii) Applying part $i)$ of Proposition (2.1) to the non increasing function f^*, with exponent q/p (instead of p) and parameter $\alpha = 1 - (p/q)$ we obtain the following result (cf. [FM] Lemma (3.4))

$$\left(\int_0^t f^*(x)^q dx\right)^{p/q}$$

$$\leq \left(\frac{q-p}{q}\right)^{p/q} \left(\int_0^t \left(\frac{1}{x}\int_0^x (f^*)^p\right)^{q/p} dx\right)^{p/q} + \left(\frac{p}{q}\right)^{p/q} t^{\frac{p}{q}-1} \int_0^t f^*(x)^p dx. \tag{2.6}$$

Example (2.7) Using Herz's inequality (cf. [BS]) for the maximal operator of Hardy and Littlewood it is well known (cf. [Mi] and the references therein), and easy to see, that if a given positive function f satisfies the usual assumptions of Gehring's Lemma (cf. [Ge]), then its nonincreasing rearrangement f^* also satisfies a reverse Hölder inequality of the same order. In this context we can take as our starting point the existence of a constant M such that for some $1 < p < \infty$, $\forall t > 0$, it holds

$$\left(\frac{1}{t}\int_0^t f^*(x)^p dx\right)^{1/p} \leq \frac{M}{t}\int_0^t f^*(x)dx. \tag{2.8}$$

Let $q > p$ and assume that $f^* \in L^q$, we combine (2.7) and (2.8) and the usual Hardy inequality (1.6), to obtain

$$\left(\int_0^t f^*(x)^q dx\right)^{p/q}$$

$$\leq C\left(\frac{q-p}{q}\right)^{p/q} \left(\frac{1}{q-1}\right)^p \left(\int_0^t f^*(x)^q dx\right)^{p/q} + \left(\frac{p}{q}\right)^{p/q} t^{\frac{p}{q}-1} \int_0^t f^*(x)^p dx. \tag{2.9}$$

where C is a constant independent of p and q.

If we choose q sufficiently close to p so that

$$C\left(\frac{q-p}{q}\right)^{p/q} \left(\frac{1}{q-1}\right)^p < 1,$$

we can move the first summand in (2.9) to the left hand side and we get

$$\left(\frac{1}{t}\int_0^t f^*(x)^q dx\right)^{1/q} \leq C_p\left(\frac{1}{t}\int_0^t f^*(x)^p dx\right)^{1/p}, \tag{2.10}$$

and we have obtained the self improving property of reverse Hölder inequalities for f^*.

There is a process of approximation that is needed to remove the extra assumption that $f^* \in L^q$. This is done as follows (cf. [Iw]). Let f be a positive, measurable, locally integrable function defined on a cube $Q_0 \subset \mathbb{R}^n$. Let f_s, $0 < s \leq 1$, be defined by

$$f_s(x) = \frac{1}{|Q_0|}\int_{Q_0} f((1-s)x + sy)dy,$$

for $x \in Q_0$.

It is clear that $f_s \in C(Q_0) \subset L^q(Q_0)$, for all $s \in (0,1)$, and for all $q \geq 1$ and it also follows readily that for $f \in L^p(Q_0)$,

$$\|f - f_s\|_{L^p(Q_0)} \to 0,$$

when $s \to 0$. It is also easy to see that if f satisfies a reverse Hölder inequality then so does f_s. We leave the details to the interested reader.

We now give a proof of (HO).

Theorem 2.11. *Let $\bar{A} = (A_0, A_1)$ be a pair of Banach spaces, and let a be an element in $A_0 + A_1$. Suppose that $1 \leq p < q < \infty$ and let $\theta_0 = 1/p'$, $\theta_1 = 1/q'$, where p' and q' are the corresponding conjugate exponents. Then if $a \in \bar{A}_{\theta_0,p} \cap \bar{A}_{\theta_1,q}$,*

$$\begin{aligned} K(t, a, \bar{A}_{\theta_1,q}, A_1) &\leq c_{\theta_1}\left(\frac{1}{t}\int_0^t \left(\frac{K(x,a;\overline{A})}{x}\right)^q dx\right)^{1/q} \\ &\leq c_{\theta_1} leq C\left(\frac{p}{q}\right)^{1/q} \frac{K(t^{1/p}, a; \bar{A}_{\theta_0,p}, A_1)}{t^{1/p}} \\ &+ C\left(\frac{q-p}{q}\right)^{1/q}\left(\frac{1}{t}\int_0^t \left(\frac{K(x^{1/p}, a; \bar{A}_{\theta_0,p}, A_1)}{x^{1/p}}\right)^q dx\right)^{1/q}. \end{aligned}$$

where c_{θ_1} is bounded away from 0 or 1, and C is an absolute constant independent of p and q.

Proof. The first inequality is simply Holmstedt's formula (1.2). Then we apply the proposition (2.1) to the function $(K(x,a;\overline{A})/x)^p$ with exponent q/p and parameter $\alpha = 1 - (p/q)$. We get

$$\begin{aligned} \left(\int_0^t \left(\frac{K(x,a;\overline{A})}{x}\right)^q dx\right)^{p/q} &\leq C\left(\frac{p}{q}\right)^{p/q} t^{-1+p/q}\int_0^t \left(\frac{K(x,a;\overline{A})}{x}\right)^p dx \\ &+ C\left(\frac{q-p}{q}\int_0^t \left(\int_0^x \left(\frac{K(y,a;\overline{A})}{y}\right)^p dy\right)^{q/p} \frac{dx}{x}\right)^{p/q}. \end{aligned}$$

Next, since

$$\left(\int_0^t \left(\frac{K(x,a;\overline{A})}{x}\right)^p dx\right)^{1/p} \leq CK(t^{1/p},a;\bar{A}_{\theta_0,p},A_1)$$

we obtain the result.

Remark. Recall that for the pair (L^p, L^∞), $1 \leq p < \infty$, we have (cf. [BL]),

$$K(t^{1/p},a,L^p,L^\infty) \approx \left\{\int_0^t a^*(s)^p ds\right\}^{1/p}.$$

Thus given a compatible pair of Banach spaces $\overline{A} = (A_0, A_1)$, for any $a \in A_0 + A_1$, we can consider $K(t^{1/p},a;\bar{A}_{\theta,p},A_1)/t^{1/p}$ as a generalized p average. In this context one can formulate reverse Hölder conditions in a very natural way and extend Example (2.7) in a substantial way. In particular one can formulate and prove a generalized version of Gehring's Lemma (cf. [Mi], [BMR1]). In the last quoted papers the self improving properties associated with reverse Hölder conditions are obtained using Holmstedt's classical formula (1.2) to arrive to elementary differential inequalities from which certain monotonicity properties follow. These monotonicity conditions can be expressed in several different ways. On the one hand, as we shall discuss in detail in the next section, they are related to properties of weights associated with Calderón operators (cf. [BR], [BMR]), and the literature quoted therein.) Monotonicity conditions can be also formulated using the theory of indices for submultiplicative functions as has been done in cf. [MM]. The new reiteration formula (HO) can also be used in this context as we have indicated in Example (2.7). Moreover, combining the results in this paper with those in [MM] we can prove Gehring type results for measures that do not satisfy doubling conditions. Interestingly, the key step in our approach for non doubling measures which is to consider maximal operators associated with packings (cf. [AKMP], [MM]) was suggested by the K functional method.

3. Reverse Hölder inequalities and M_p weights

In this section we explicitly show how the K functional provides also a way to bridge the theories for classes of weights associated with Hardy or Calderón operators and the A_p classes of Muckenhoupt.

We begin by recalling the definitions of certain classes of weights. Let $1 < p < \infty$, we say that a non negative locally integrable function on $(0,\infty)$ satisfies the condition M_p (resp. M^p) if there exists a constant C such that

$$\left(\int_t^\infty \frac{w(x)}{x^p}dx\right)^{1/p}\left(\int_0^t w(x)^{-p'/p}dx\right)^{1/p'} \leq C, \tag{M_p}$$

$$\left(\int_0^t w(x)dx\right)^{1/p}\left(\int_t^\infty \frac{w(x)^{-p'/p}}{x^{p'}}dx\right)^{1/p'} \leq C. \tag{M^p}$$

For $p = 1$ the classes M_1 or M^1 consist of those weights such that

$$Qw(t) \leq Cw(t) \quad \forall t > 0 \tag{M_1}$$

or

$$Pw(t) \leq Cw(t) \quad \forall t > 0. \tag{M^1}$$

We shall only need elementary forms of reverse Hölder inequalities which are associated to the classes M^1 and M_1 (cf. [BMR], [BMR1] and also [MM].)

$$w \in M^1 \implies \exists \varepsilon > 0 \ni t^{-\varepsilon} w(t) \in M^1, \tag{3.1}$$

$$w \in M_1 \implies \exists \varepsilon > 0 \ni t^{\varepsilon} w(t) \in M_1, \tag{3.2}$$

where ε depends on the weight w.

For monotone weights, we have the following result

Lemma 3.3.

(1) *If w is a non increasing weight, then w satisfies M_p, for all $1 < p < \infty$.*
(2) *If $xw(x)$ is a non decreasing, then w^α satisfies M^p, for all $1 \leq p < \infty$ and for all $0 < \alpha < 1$*

Proof. (1) Let $t > 0$

$$\left(\int_t^\infty \frac{w(x)}{x^p} dx\right)^{1/p} \left(\int_0^t w(x)^{-p'/p} dx\right)^{1/p'}$$

$$\leq w(t)^{1/p} \left(\int_t^\infty \frac{dx}{x^p}\right)^{1/p} w(t)^{-1/p} \left(\int_0^t dx\right)^{1/p'} \leq C$$

(2) Let $t > 0$. Since $x < t$ implies that $x^\alpha w(x)^\alpha \leq t^\alpha w(t)^\alpha$ we have

$$\left(\int_0^t w(x)^\alpha dx\right)^{1/p} \left(\int_t^\infty \frac{w(x)^{-\alpha p'/p}}{x^{p'}} dx\right)^{1/p'}$$

$$\leq t^{\alpha/p} w(t)^{\alpha/p} \left(\int_0^t \frac{dx}{x^\alpha}\right)^{1/p} t^{-\alpha/p} w(t)^{-\alpha/p} \left(\int_t^\infty \frac{x^{\alpha p'/p}}{x^{p'}} dx\right)^{1/p'} \leq C.$$

The proof for $p = 1$ is even easier.

Given a compatible pair of Banach spaces $\overline{A} = (A_0, A_1)$, for any $a \in A_0 + A_1$, we can consider $K(t, a; \overline{A})/t$ as a weight. Note that since $K(t, a; \overline{A})$ increases and $K(t, a; \overline{A})/t$ is decreasing, Lemma (3.3) shows that $K(t, a; \overline{A})/t \in M_p$, $\forall p \in (1, \infty)$ and $(K(t, a; \overline{A})/t)^\alpha \in M^1$, $\forall \alpha \in (0, 1)$.

In (cf. [Mi], [BMR1]) the self improving properties of weights satisfying reverse Hölder conditions were obtained by means of showing monotonicity conditions of the type described by (3.1) and (3.2) for the corresponding K functionals, thus at this level reverse Hölder inequalities imply M^1. We shall now ellaborate more explicitly the converse: M^1 conditions imply the usual reverse Hölder inequalities.

It will be convenient to organize things around averages and tie the values of the θ and q parameters although this is not necessary.

Definition 3.4. *Let $\overline{A} = (A_0, A_1)$ be a compatible pair of Banach spaces. An element $a \in A_0 + A_1$ will be called* **left-Gehring** *(respectively* **right-Ghering***) if the non increasing function $K(t, a; \overline{A})/t$ verifies the condition M^1 (respectively M_1). We shall say that a is* **Gehring** *if it is simultaneously left and right-Gehring.*

Remark. An element a is Ghering if and only if the corresponding weight function $w(t) = K(t, a; \overline{A})/t$ is a quasipower. In fact, we have the following inequalities

$$Sw = Pw + Qw \leq Cw \leq CPw \leq CSw,$$

where the second inequality holds because w is non increasing and the constant C is the sum of the constants appearing in the definitions of M_1 and M^1.

For the converse note that if w is a quasipower, then $w \in L^1 + L^\infty$ and w is a Gehring element for the pair (L^1, L^∞).

The relationship of all this with reverse Hölder inequalities is now given by

Proposition 3.5. *Let a be an element in $A_0 + A_1$. Then,*

(1) *a is* **left-Gehring** *if and only if there exists $p_0 > 1$ such that $a \in \bar{A}_{\theta_0,p_0} + A_1$, where $1 = (1/p_0) + \theta_0$, and*

$$\frac{K(t^{1/p_0}, a; \bar{A}_{\theta_0,p_0}, A_1)}{t^{1/p_0}} \sim \frac{K(t, a; \overline{A})}{t}.$$

(2) *a is* **right-Gehring** *if and only if there exists $p_1 > 1$ such that $a \in A_0 + \bar{A}_{\theta_1,p_1}$, where $\theta_1 = 1/p_1$, and*

$$K(t^{1/p_1}, a; A_0, \bar{A}_{\theta_1,p_1}) \sim K(t, a; \overline{A}).$$

(3) *a is* **Gehring** *if and only if there exist $p_0, p_1 > 1$, with $1 < (1/p_0) + (1/p_1)$ such that $a \in \bar{A}_{\theta_0,p_0} + \bar{A}_{\theta_1,p_1}$, where $1 = (1/p_0) + \theta_0$, $\theta_1 = 1/p_1$, and*

$$K(t^{\theta_1 - \theta_0}, a; \bar{A}_{\theta_0,p_0}, \bar{A}_{\theta_1,p_1}) \sim t^{-\theta_0} K(t, a; \overline{A}).$$

Under these conditions we shall say that p_0 (resp. p_1) is a left-exponent (resp.right-exponent) of the element a.

Proof.
(1) Suppose that a is left-Gehring.The nonincreasing function $K(t, a; \overline{A})/t$ satisfies the condition (M^1) and therefore satisfies (3.1) for some ε. Consequently, by the embedding properties of the $L^{p,q}$ spaces, with $p_0 = 1/1 - \varepsilon$, we have,

$$\left(\frac{1}{t}\int_0^t \left(\frac{K(x, a, \overline{A})}{x}\right)^p dx\right)^{1/p} \leq C\frac{K(t, a, \overline{A})}{t}, \tag{3.6}$$

for all $t > 0$. Let θ_0 be such that $1 = (1/p_0) + \theta_0$, then Holmstedt's formula (1.2) implies that

$$K(t^{1/p_0}, a; \bar{A}_{\theta_0,p_0}, A_1) \leq C_{p_0}\left(\int_0^t \left(\frac{K(x, a; \overline{A})}{x^\theta}\right)^{p_0} \frac{dx}{x}\right)^{1/p}$$

$$= C_{p_0}\left(\int_0^t \left(\frac{K(x,a;\overline{A})}{x}\right)^{p_0} dx\right)^{1/p_0} \le C_{p_0} t^{-1/p_0}\frac{K(t,a,\overline{A})}{t}.$$

The reverse inequality follows readily from Holmstedt's formula and the fact that $K(t,a;\overline{A})/t$ is non increasing.

For the "only if part" observe that by Holmsted's formula, (3.6) is equivalent with the hypothesis. Therefore, since 1 averages are dominated by p averages it follows that $K(t,a;\overline{A})/t$ satisfies the condition (M^1).

(2) The proof is very similar to the preceeding one, using (3.2) and Holmstedt's formula.

(3) It is an easy consequence of (1), (2) and the fact that Holmstedt's formula (1.1) can be rewritten as

$$K(t^{\theta_1-\theta_0}, a; \bar{A}_{\theta_0,p_0}, \bar{A}_{\theta_1,p_1}) \sim K(t^{1-\theta_0}, a; A_{\theta_0,p_0}, A_1) + t^{-\theta_0}K(t^{\theta_1}, a; A_0, A_{\theta_1,p_1}).$$

Remark. Holmstedt's formula has the following classical application: Let $\overline{A} = (A_0, A_1)$ and denote by $\overline{X}$ an intermediate pair, say $\overline{X} = (\bar{A}_{\theta_0,p_0}, \bar{A}_{\theta_1,p_1})$. Taking $a, b \in A_0 + A_1$, then

$$K(t,b,\overline{A}) \le CK(t,a,\overline{A}),\ \forall t>0\ \Rightarrow K(t,b,\overline{X}) \le CK(t,a,\overline{X}),\ \forall t>0 \qquad (3.7)$$

For Gehring elements we can reverse assertion (3.7).

Corollary 3.8. *Let a be a left-Gehring element with left-exponent p_0 (respectively right-Ghering with exponent p_1 or Gehring with exponents (p_0,p_1)). Denote by $\overline{X}$ the pair $(\bar{A}_{\theta_0,p_0}, A_1)$, (respectively $(A_0, \bar{A}_{\theta_1,p_1})$ or $(\bar{A}_{\theta_0,p_0}, \bar{A}_{\theta_1,p_1})$). Suppose that b is any other element in $A_0 + A_1$ such that*

$$K(t,b;\bar{X}) \le C_1 K(t,a;\bar{X}) \quad \forall t>0.$$

Then,

$$K(t,b;\overline{A}) \le C_2 K(t,a;\overline{A}) \quad \forall t>0.$$

Proof. We shall only deal with the case $\overline{X} = (\bar{A}_{\theta_0,p_0}, A_1)$, the remaining cases can be proved in a similar way.

We use once again Holmstedt's formula. Indeed, since the function $K(t,b;\overline{A})/t$ is non increasing, we have

$$\begin{aligned} K(t^{1/p}, b; \overline{A}) &\le t^{(1/p)-1}\left(\int_0^{t^{1/p}} \left(\frac{K(x,b;\overline{A})}{x^\theta}\right)^p \frac{dx}{x}\right)^{1/p} \\ &\le C_p t^{(1/p)-1} K(t,b;\bar{A}_{\theta_0,p_0}, A_1) \\ &\le C_p C_1 t^{(1/p)-1} K(t,a;\bar{A}_{\theta_0,p_0}, A_1) \\ &\le CK(t^{1/p}, a; \overline{A}). \end{aligned}$$

as we wished to show.

4. Hardy-Littlewood-Polya Order

Recall that given f and g positive, measurable functions on $\mathbb{R}^n$ we say that $f \prec g$ if

$$\int_0^t f^* \leq \int_0^t g^*,$$

for all $t > 0$, where f^* denotes the non increasing rearrangement of f. The order relation of Hardy-Littlewood-Polya [HLP] appears naturally in different contexts. In interpolation theory it was used by Calderón and Mityagin (cf. [Ca], [BS]) to characterize all the interpolation spaces between L^1 and L^∞. Indeed if $\overline{A} = (L^1, L^\infty)$, then relation $f \prec g$ is equivalent to $K(t, f; L^1, L^\infty) \leq K(t, g; L^1, L^\infty)$, for all $t > 0$.

The HLP order is preserved by convex functions (cf. [HLP]):

$$f \prec g \Rightarrow \varphi(f) \prec \varphi(g)$$

for all convex function φ.

We consider the following question: What happens when the HLP order is reversed?

The last remark shows that $f \prec g$ implies $f^p \prec g^p$, for all $p > 1$. Corollary (3.8) will allow us to reverse this relation for left-Gehring elements.

Proposition 4.1. *Suppose that $g \in L^1 + L^\infty$ is a left-Gehring element with left-exponent p. Let f be another function in $L^1 + L^\infty$ such that $f^p \prec g^p$, then $f \prec g$.*

Proof. Note only that $(\int_0^t f^p)^{1/p} \sim K(t^{1/p}, f; L^p, L^\infty)$ and the proof follows readily.

Next we shall prove that this property is essentially only satisfied by left-Gehring elements!

We shall say that a compatible pair of Banach spaces satisfies the hypothesis (H) if

$$\forall t_0 > 0,\ \exists b_0 \in A_0 + A_1 \ \ni K(t, b_0; \overline{A}) = \min\{t, t_0\}\ \forall t > 0. \tag{4.2}$$

Proposition 4.3. *Let $\overline{A}$ be a compatible pair of Banach spaces satisfying the condition (H). Denote by $\overline{X}$ the pair $(\bar{A}_{\theta_0,p_0}, A_1)$, (resp., $(A_0, \bar{A}_{\theta_1,p_1})$ or $(\bar{A}_{\theta_0,p_0}, \bar{A}_{\theta_1,p_1})$). Let a be an element in $A_0 + A_1$ such that, for all $b \in A_0 + A_1$ satisfying $K(t, b; \overline{X}) \leq C_1 K(t, a; \overline{X}) < \infty$ for all $t > 0$, it follows that $K(t, b; \overline{A}) \leq C_2 K(t, a; \overline{A})$, for all $t > 0$, i.e. is maximal. Then a is a left-Gehring element (resp. right-Gehring or Gehring).*

Proof. We consider first the case $\overline{X} = (\bar{A}_{\theta_0,p_0}, A_1)$. Let $t_0 > 0$ fixed. Let $b_0 \in A_0 \cap A_1$ such that (4.2) holds. Let $\lambda > 0$. Then, by Holmstedt's formula (1.2) we have

$$\begin{aligned} K(t^{1/p_0}, \lambda b_0; A_{\theta_0,p_0}, A_1) &\leq C \left(\int_0^t \left(\frac{K(x, \lambda b_0; \overline{A})}{x_0^\theta} \right)^{p_0} \frac{dx}{x} \right)^{1/p_0} \\ &= C\lambda \left(\int_0^t \left(\frac{\min\{x, t_0\}}{x} \right)^{p_0} dx \right)^{1/p_0} \\ &= C\lambda \left(\int_0^t \left(P(\chi_{[0,t_0]})(x) \right)^{p_0} dx \right)^{1/p_0} \\ &\text{(by Hardy's inequality)} \\ &\leq C\lambda \left(\int_0^t \left(\chi_{[0,t_0]}(x) \right)^{p_0} dx \right)^{1/p_0} \\ &= C\lambda \min\{t, t_0\}^{1/p_0}. \end{aligned}$$

If we now take

$$\lambda_0 = \left(\frac{1}{t_0} \int_0^{t_0} \left(\frac{K(x, a; \overline{A})}{x} \right)^{p_0} dx \right)^{1/p_0},$$

then

$$K(t^{1/p_0}, \lambda_0 b_0; \bar{A}_{\theta_0,p_0}, A_1) \leq C \min\{t, t_0\}^{1/p_0} \left(\frac{1}{t_0} \int_0^{t_0} \left(\frac{K(x, a; \overline{A})}{x} \right)^{p_0} dx \right)^{1/p_0}.$$

Consider two cases: if $t \leq t_0$, then since the function $(K(x, a; \overline{A})/x)^{p_0}$ is non increasing $P((K(x, a; \overline{A})/x)^{p_0})$ is also non increasing and therefore

$$\left(\frac{1}{t_0} \int_0^{t_0} \left(\frac{K(x, a; \overline{A})}{x} \right)^{p_0} dx \right)^{1/p_0} \leq \left(\frac{1}{t} \int_0^{t} \left(\frac{K(x, a; \overline{A})}{x} \right)^{p_0} dx \right)^{1/p_0}.$$

If on the other hand $t_0 < t$ then,

$$\left(\int_0^{t_0} \left(\frac{K(x, a; \overline{A})}{x} \right)^{p_0} dx \right)^{1/p_0} \leq \left(\int_0^{t} \left(\frac{K(x, a; \overline{A})}{x} \right)^{p_0} dx \right)^{1/p_0}.$$

Therefore for all $t > 0$ we have

$$\begin{aligned} K(t^{1/p_0}, \lambda_0 b_0; \bar{A}_{\theta_0,p_0}, A_1) &\leq C \left(\int_0^t \left(\frac{K(x,\ a; \overline{A})}{x} \right)^{p_0} dx \right)^{1/p_0} \\ &\leq CK(t^{1/p_0}, a; \bar{A}_{\theta_0,p_0}, A_1). \end{aligned}$$

Therefore by hypothesis we have

$$K(t, \lambda_0 b_0; \overline{A}) \leq C' K(t, a; \overline{A}),$$

for all $t>0$. In particular if we take $t=t_0$ we get

$$\left(\frac{1}{t_0}\int_0^{t_0}\left(\frac{K(x,a;\overline{A})}{x}\right)^{p_0}dx\right)^{1/p_0}t_0\le C'K(t_0,a;\overline{A}).$$

and this is true for all t_0. Therefore by Proposition (3.5) a is left-Gehring with left-exponent p_0.

The case $\overline{X}=(A_0,\bar{A}_{\theta_1,p_1})$ is dual and the arguments are similar.

We finally consider the case $\overline{X}=(\bar{A}_{\theta_0,p_0},\bar{A}_{\theta_1,p_1})$. Let b such that

$$K(t,b;\bar{A}_{\theta_0,p_0},A_1)\le C_1K(t,a;\bar{A}_{\theta_0,p_0},A_1)<\infty$$

for all $t>0$. Since $\theta_0<\theta_1$, by the reiteration theorem we have

$$\bar{A}_{\theta_1,p_1}=(\bar{A}_{\theta_0,p_0},A_1)_{\eta,p_1},$$

with $(1-\eta)\theta_0+\eta=\theta_1$. Holmstedt's formula implies that

$$K(t,b;\bar{A}_{\theta_0,p_0},\bar{A}_{\theta_1,p_1})\sim t\left(\int_{1/\eta}^{\infty}\left(\frac{K(x,b;\bar{A}_{\theta_0,p_0},A_1)}{x^{\eta}}\right)^{p_1}\frac{dx}{x}\right)^{1/p_1}$$

which leads to

$$K(t,b;\bar{A}_{\theta_0,p_0},\bar{A}_{\theta_1,p_1})\le C_1K(t,a;\bar{A}_{\theta_0,p_0},\bar{A}_{\theta_1,p_1}),$$

for all $t>0$. By hypothesis we must have $K(t,b;\overline{A})\le C_2K(t,a;\overline{A})$, for all $t>0$. We now can apply the first part of this proposition to deduce that a is left-Gehring. In a similar way we conclude that a is right-Gehring as well, and the result follows.

Remarks. (i) We now show that pairs of L^p spaces or more generally pairs of rearrangement invariant spaces verify condition (H) (cf. [BR]). Indeed, suppose that A_0 and A_1 are rearrangement invariant spaces (r.i. spaces) such that the function

$$\phi_0/\phi_1:(0,\infty)\longrightarrow(0,\infty)$$

is onto, where $\phi_i(t)=\|\chi_{[0,t]}\|_{A_i}$ $(i=0,1)$, is the fundamental function of A_i.

Indeed, by taking conditional expectations, it follows readily that

$$\begin{aligned}K(t,\chi_{[0,t_0]};\overline{A})&=\inf_{0<a<1}a\phi_{A_0}(t_0)+t(1-a)\phi_{A_1}(t_0)\\&=\min\{\phi_{A_0}(t_0),t\phi_{A_1}(t_0)\}\\&=\phi_{A_1}(t_0)\min\{\frac{\phi_{A_0}(t_0)}{\phi_{A_1}(t_0)},t\},\end{aligned}$$

and we only need to find one point t_1 such that $\phi_0(t_1)/\phi_1(t_1)=t_0$.

(ii) We can understand these results in the framework of Calderón pairs. We define the orbit spaces $\mathrm{Orb}_{\overline{A}}(a)=\{Ta\}$, where $a\in A_0+A_1$ and T runs over all bounded operators from $A_0\to A_0$ and $A_1\to A_1$ (cf. [Ov]).

Let $\overline{X}$ be the pair $(\bar{A}_{\theta_0,p_0}, A_1)$, (resp., $(A_0, \bar{A}_{\theta_1,p_1})$ or $(\bar{A}_{\theta_0,p_0}, \bar{A}_{\theta_1,p_1})$). It is clear that if $a \in \overline{X}$ then

$$\mathrm{Orb}_{\overline{X}}(a) \supseteq \mathrm{Orb}_{\overline{A}}(a).$$

On the other hand, supposing that $\overline{A}$ is Calderón pair, Corollary 3.8. says that if a is a left-Gehring (resp., right-Gehring or Gehring) element with left-exponent p_0 then

$$\mathrm{Orb}_{\overline{X}}(a) = \mathrm{Orb}_{\overline{A}}(a).$$

For pairs $\overline{A}$ satisfying condition (H) and supposing that $\overline{X} = (\bar{A}_{\theta_0,p_0}, A_1)$ is a Calderón pair we can prove the reverse statement, i.e., $\mathrm{Orb}_{\overline{X}}(a) = \mathrm{Orb}_{\overline{A}}(a)$ implies that a is a left-Gehring element.

Indeed, let b an element such that $K(t, b; \overline{X}) \leq C_1 K(t, a\overline{X})$, for all $t > 0$. Then there exists an operator T such that $Ta = b$ and $T : \overline{X} \to \overline{X}$, so $b \in \mathrm{Orb}_{\overline{X}}(a)$ and therefore we can find another operator S bounded from $A_0 \to A_0$ and $A_1 \to A_1$ satisfying $Sa = b$. We achieve $K(t, b; \overline{A}) \leq CK(t, b; \overline{A})$, for all $t > 0$ and hence, by Proposition 4.3., a is a left-Gehring element.

References

[AKMP] I. Asekritova, N. Ya. Kruglajk, L. Maligranda and L. E. Persson, *Distribution and rearrangement estimates of the maximal function and interpolation*, Studia Math. **124** (1997), 107-132.

[AM] M. A. Ariño and B. Muckenhoupt, *Maximal Functions on Classical Lorentz Spaces and Hardy's Inequality with Weights for Nonincreasing Functions*, Transactions A.M.S. **320** **(2)** (1990), 727-735.

[BMR] J. Bastero, M. Milman and F. Ruiz, *On the connection between weighted norm inequalities, commutators and real interpolation (preprint).*

[BMR1] J. Bastero, M. Milman and F. Ruiz, *Reverse Hölder Inequalites and Interpolation,*, Proc. Haifa Conference on Interpolation Theory and Related Topic **to appear**.

[BR] J. Bastero and F. Ruiz, *Elementary reverse Hölder type inequalities with application to operator interpolation theory*, Proceedings A.M.S. **124** (1996), 3183-3192.

[Be] G. Bennett, *Lower bounds for matrices*, Linear Algebra and Appl **82** (1986), 81-98.

[BS] C. Bennett and R. Sharpley, *Interpolation of operators*, Academic Press, 1988.

[BL] J. Bergh and J. Löfström, *Interpolation Spaces. An Introduction*, Springer-Verlag, New York, 1976.

[BK] Y. Brundyi and N. Krugljak, *Interpolation functors and interpolation spaces.*, North-Holland., 1991.

[Ca] A. P. Calderón, *Spaces between L^1 and L^∞ and the theorem of Marcinkiewicz*, Studia Math **26** (1966), 273-299.

[Cw] M. Cwikel, *Monotonicity properties of interpolation spaces II*, Arkiv för Matematik **19** **(1)** (1981), 123-136.

[FM] M. Franciosi and G. Moscariello, *Higher integrability results*, Manuscripta Math **52** (1985), 151-170.

[Ge] F. W. Gehring, *The L^p integrability of partial derivatives of a quasiconformal mapping*, Acta Math. **130** (1973), 123-136.

[HLP] Hardy-Littlewood-Polya, *Inequalities*, Cambridge Univ. Press, 1951.

[Ho] T. Holmstedt, *Interpolation of quasi-normed spaces*, Math. Scand. **26** (1970), 177-199.

[Iw] T. Iwaniec, *Quasiconformal mappings and analysis, Ann Arbor, MI*, Springer, New York, 1998, pp. 181–204.

[JM] B. Jawerth and M. Milman, *Extrapolation Theory with Applications*, Memoirs of the Amer. Math. Soc. **440** (1991).

[MM] M. Mastylo and M. Milman, *A new approach to Gehring's Lemma*, (preprint).

[MO] M. Mastylo and V. Ovchinnikov, *On the relation between complex and real methods of interpolation*, Studia Math **125** (1997), 201–218.

[Mi] M. Milman, *A note on Gehring's Lemma*, Ann. Acad. Sci. Fenn. Ser. Math. **21** (1996), 389-398.

[Mi1] M. Milman, *A Note on Reversed Hardy Inequalities and Gehring's Lemma*, Comm. on Pure and Appl. Math **50** (1997), 311-315.

[Mu1] B. Muckenhoupt, *Weighted norm inequalities for the Hardy maximal function*, Trans. Amer. Math. Soc. **165** (1972), 207-226.

[Mu2] B. Muckenhoupt, *Hardy's inequalities with weights*, Studia Math. **44** (1972), 31-38.

[Ni] P. Nilsson, *Reiteration theorems for real interpolation and approximation spaces*, Ann. Mat. Pura Appl. **132** (1982), 291-330.

[Ov] V. I. Ovchinnikov, *The method of orbits in interpolation theory*, Math. Reports **1** (1984), 349-515.

[Re] P. F. Renaud, *A reversed Hardy inequality*, Bull. Austral. Math. So **34** (1986), 225-232.

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