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ELEMENTARY REVERSE HÖLDER TYPE INEQUALITIES WITH APPLICATION TO OPERATOR INTERPOLATION THEORY

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Abstract. We give a very elementary proof of the reverse Hölder type inequality for the classes of weights which characterize the boundedness on L^p of the Hardy operator for nonincreasing functions. The same technique is applied to Calderón operator involved in the theory of interpolation for general Lorentz spaces. This allows us to obtain further consequences for intermediate interpolation spaces.

0. Introduction

Ariño and Muckenhoupt characterized the class of weights, ω, such that the Hardy operator is bounded on $L^p(\omega)$ for nonnegative and nonincreasing functions (see [AM]). This class, say $(AM)_p$, is composed of those weights for which there is a constant $C > 0$ such that for every $t > 0$

$$\int_t^\infty \frac{\omega(x)}{x^p} dx \leq \frac{C}{t^p} \int_0^t \omega(x) dx$$

(see also [L] for an earlier version of this formula). A crucial step in their proof is the following reverse inequality: if $\omega \in (AM)_p$, then $\omega \in (AM)_{p-\epsilon}$ for some $\epsilon > 0$.

In this note we will see that this fact, the reverse type inequality, has a very elementary proof by means of some reiteration procedure. Furthermore, we will show how similar ideas can be applied in the context of operator interpolation theory for Lorentz spaces, in order to prove that weak type interpolation spaces are the same as restricted weak type interpolation ones. These concepts require some notations which will be introduced below but, briefly, in a very particular case, we say that a rearrangement invariant Banach space, X, is a *weak (p, q)-type interpolation space*, if every operator bounded from L^p into $L^{p,\infty}$ and from L^q into $L^{q,\infty}$ is also bounded on X. It is a *restricted weak (p, q)-type interpolation space*, if every operator bounded from $L^{p,1}$ into $L^{p,\infty}$ and from $L^{q,1}$ into $L^{q,\infty}$ is also bounded on X.

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As usual, a Banach space $(X, \|\cdot\|)$ of real-valued, locally integrable, Lebesgue measurable functions on I ($I = [0,1]$ or $[0,\infty)$) is said to be a r.i. space if it satisfies the following conditions:

i) If $g^* \leq f^*$ and $f \in X$, then $g \in X$ with $\|g\|_X \leq \|f\|_X$ (f^* denotes the nonincreasing rearrangement of the function f).
ii) If A is a Lebesgue measurable set of finite measure, then $\chi_A \in X$.
iii) $0 \leq f_n \uparrow$, $\sup_{n\in\mathbb{N}} \|f_n\|_X \leq M$, imply that $f = \sup f_n \in X$ and $\|f\|_X = \sup_{n\in\mathbb{N}} \|f_n\|_X$ (this property is called by some authors the *Fatou property*; see [LT]).

Let $\overline{A} = (A_0, A_1)$ and $\overline{B} = (B_0, B_1)$ be two compatible couples of Banach spaces (see [BL] and [BS]). Let A and B be intermediate spaces with respect to $\overline{A}$, respectively $\overline{B}$. We say that A and B are *relative interpolation spaces* with respect to $\overline{A}$ and $\overline{B}$ if every linear operator T satisfying $T : \overline{A} \to \overline{B}$ (i.e., $T : A_0 + A_1 \to B_0 + B_1$ and $\|T\|_{\overline{A}\to\overline{B}} = \max\{\|T\|_{A_0\to B_0}, \|T\|_{A_1\to B_1}\} < \infty$) maps boundedly A into B.

We are concerned in this paper with the case of couples $\overline{A}$ and $M(\overline{A})$, when $\overline{A}$ is a couple of compatible r.i. spaces and $M(\overline{A}) = (M(A_0), M(A_1))$ is the couple of the corresponding weak type spaces. For the definition of these spaces we begin by recalling that the fundamental function of a r.i. Banach space, X, is defined by $\Phi_X(t) = \|\chi_{[0,t]}\|_X$, $t \in I$. There is no loss of generality if we assume Φ_X to be positive, nondecreasing, absolutely continuous far from the origin, concave and to satisfy $\Phi_X(t)\Phi_{X'}(t) = t$ for all $t \in I$.

We denote by $M(X)$ the r.i. space of all measurable functions f for which there exists f^{**} and

$$\|f\|_{M(X)} = \sup_{t\in I} \Phi_X(t) f^{**}(t) < \infty$$

(recall that f^{**} is defined by $f^{**}(t) = \frac{1}{t}\int_0^t f^*$).

We will denote by $M^*(X)$ the space of all measurable functions for which

$$\|f\|_{M^*(X)} = \sup_{t\in I} \Phi_X(t) f^*(t) < \infty.$$

The function $\|\cdot\|_{M^*(X)}$ is a quasinorm on $M^*(X)$ and it is equivalent to $\|\cdot\|_{M(X)}$ if and only if $\frac{1}{\Phi_X} \in M(X)$. In this case $M(X) = M^*(X)$.

Since $\frac{1}{\Phi_X}$ is a decreasing function, it belongs to $M(X)$ if and only if it is in the A_1-class of Muckenhoupt (see [AM], [M]).

Let w be an a.e. positive locally integrable weight defined on $I = [0,\infty)$ and let $W(t)$ be defined by $W(t) = \int_0^t w < \infty, \forall t < \infty$. We assume that $\lim_{t\to\infty} W(t) = \infty$. We recall that the classical Lorentz space $\Lambda(W,p)$, $0 < p \leq \infty$, is the class of all real valued measurable functions on I such that

$$\|f\|_{\Lambda(W,p)} = \begin{cases} \left(\int_I f^*(t)^p dW(t)\right)^{1/p} < \infty & \text{if } 0 < p < \infty, \\ \sup_{t>0} f^*(t)W(t) < \infty & \text{if } p = \infty\ . \end{cases}$$

The space $\Lambda(W,p)$, $1 \leq p < \infty$, is normable if and only if $\omega \in (AM)_p$ (see [AM], and [R], [S] for other equivalent conditions).

The letter C denotes a numerical constant which may change in different occurrences.

1. Reverse type inequalities

We introduce the operators

$$Hf(t) = \frac{1}{t}\int_0^t \omega(x)dx, \qquad J_p f(t) = \int_t^\infty \omega(x)\left(\frac{t}{x}\right)^{p-1}\frac{dx}{x}.$$

In this framework, the condition $(AM)_p$ can be reformulated as: there exists a constant $C > 0$ so that $J_p\omega \le C\,H\omega$, a.e.

Theorem 1.1. *Let $1 \le p < \infty$. If $J_p\omega \le C\,H\omega$, a.e., then $J_{p-\epsilon}\omega \le C\,H\omega$, a.e., for some $\epsilon > 0$.*

Proof. A simple application of Fubini's theorem shows that

$$J_pH\omega = HJ_p\omega = p^{-1}(H\omega + J_p\omega)$$

and that the reiteration of J_p is

$$J_p^{(k)}\omega(t) = \frac{1}{(k-1)!}\int_t^\infty \omega(x)\left(\frac{t}{x}\right)^{p-1}\log^{k-1}\left(\frac{x}{t}\right)\frac{dx}{x}. \tag{1.1}$$

The hypothesis implies that

$$J_p^{(2)}\omega \le C\,J_pH\omega \le C\,(H\omega + J_p\omega) \le (1+C)^2\,H\omega.$$

Therefore, by induction for almost all points

$$J_p^{(k)}\omega \le (1+C)^k\,H\omega. \tag{1.2}$$

We take $\epsilon > 0$ such that $\epsilon(C+1) < 1$. We calculate the sum $\sum_1^\infty \epsilon^{k-1}J_p^{(k)}\omega$ by (1.1) and then we estimate it by using (1.2) to obtain

$$\int_t^\infty \omega(x)\left(\frac{t}{x}\right)^{p-1}\left(\frac{x}{t}\right)^{\epsilon}\frac{dx}{x} \le (C+1)\sum_0^\infty(\epsilon(C+1))^k H\omega \le CH\omega,$$

that is, condition $(AM)_{p-\epsilon}$.

(See also [N] for a different approach to this result).

The same kind of ideas work when one is concerned with Calderón's operators involved in the theory of interpolation for r.i. spaces.

Let w be a weight as quoted before. We introduce the operators P, Q defined for measurable functions on $[0,\infty)$ by

$$Pf(t) = W^{-1/p}(t)\left(\int_0^t f^*(x)^p dW(x)\right)^{1/p}, \ \ t > 0,$$

and

$$Qf(t) = W^{-1/p}(t)\left(\int_t^\infty f^*(x)^p dW(x)\right)^{1/p}, \ \ t > 0$$

(these operators appear also in [Ma1]).

Theorem 1.2. *Let $1 \le p < \infty$ and let X be a r.i. space. If the operator P is defined and bounded in X, then there exists $\epsilon > 0$ such that the operator $\overline{P}$, defined by*

$$\overline{P}f(t) = \left(W(t)^{\epsilon-1}\int_0^t f^*(x)^p W(x)^{-\epsilon}dW(x)\right)^{1/p},$$

is well defined and bounded from X into X.

Proof. Suppose that $\|Pf\|_X \leq C\|f\|_X$ for $f \in X$. It is very easy to compute that $(Pf)^* = Pf$ and so, the reiteration of the operator P has the expression

$$P^{(n)}f(t)^p = W(t)^{-1}\int_0^t f^*(x)^p \frac{[\log(W(t)/W(x))]^{n-1}}{(n-1)!}dW(x).$$

We consider $\epsilon > 0$ such that $\epsilon C < 1$ and define

$$S_N(f)(t) = \left(\sum_{n=0}^{N}\left(\epsilon^n P^{(n+1)}(f(t))\right)^p\right)^{1/p}.$$

It is very easy to check that the sequence of functions $\{S_N(f)\}_{N=0}^{\infty}$ is nondecreasing and converges almost everywhere to the function $\overline{P}f$. Since

$$\|S_N(f)\|_A \leq \sum_{n=0}^{\infty}\epsilon^n C^{n+1} < \infty,$$

we conclude from the Fatou property that $\overline{P}f \in X$ and $\|\overline{P}f\|_X \leq C_1\|f\|_X$, and the theorem is proved.

The analogous result, with similar proof, for the operator Q is

Theorem 1.3. *If Q is defined and bounded in X, then there exists $\epsilon > 0$ such that the operator $\overline{Q}$, defined by*

$$\overline{Q}f(t) = \left(W(t)^{-\epsilon-1}\int_t^{\infty} f^*(x)^p W(x)^{\epsilon}dW(x)\right)^{1/p},$$

is well defined and bounded from X into X.

2. Theory of interpolation of operators

In order to see how Theorems 1.2 and 1.3 can be applied in the theory of interpolation of operators we should state some previous results.

Let A_0 and A_1 be two r.i. Banach spaces and let Φ_0, Φ_1 be their fundamental functions. We suppose that the following conditions are satisfied:

i) For $i = 0, 1$,

$$\frac{1}{\Phi_{A_i}} \in M(A_i). \tag{C.0}$$

ii) There exists a constant C such that, for all $t > 0$

$$\frac{\Phi_1(t)}{\Phi_0(t)}\left\|\frac{\chi_{[0,t]}}{\Phi_1}\right\|_{A_0} \leq C, \tag{C.1}$$

$$\frac{\Phi_0(t)}{\Phi_1(t)}\left\|\frac{\chi_{[t,\infty)}}{\Phi_0}\right\|_{A_1} \leq C. \tag{C.2}$$

Condition i) is used in order to obtain the corresponding results for the space $M(A_i)$ instead of $M^*(A_i)$. Note that this condition, for $i=1$, follows from condition $(C.1)$. In fact (let $\Phi_0' = \Phi_{A_0'}$)

$$\begin{aligned}\frac{1}{t}\int_0^t \frac{ds}{\Phi_1(s)} &= \frac{1}{\Phi_0(t)\Phi_0'(t)}\int_0^\infty \frac{\chi_{[0,t]}(s)}{\Phi_1(s)}\chi_{[0,t]}(s)ds \\ &\leq \frac{1}{\Phi_0(t)}\left\|\frac{\chi_{[0,t]}}{\Phi_1}\right\|_{A_0}\frac{\|\chi_{[0,t]}\|_{A_0'}}{\Phi_0'(t)} \\ &= \frac{1}{\Phi_0(t)}\left\|\frac{\chi_{[0,t]}}{\Phi_1}\right\|_{A_0} \leq \frac{C}{\Phi_1(t)}.\end{aligned}$$

Conditions appearing in ii) actually mean equivalence conditions between the two members appearing in the corresponding inequalities. They are really significant and are satisfied for a lot of couples of classical function spaces (see [Ma2]).

The first consequence of $(C.1)$ and $(C.2)$ is that the peak (flat) part of functions are in A_0 (A_1) (it would also be a consequence of Theorem 1 in [Ma2]):

Lemma 2.1. *Let f be a function in $A_0 + A_1$. Then*

$$f^*\chi_{[0,t]} \in A_0, \qquad f^*\chi_{[t,\infty)} \in A_1.$$

Proof. In order to obtain the first part, by using standard arguments, we only have to suppose that $f \in A_1$ and show that $f^*\chi_{[0,t]} \in A_0$. Let h be an arbitrary decreasing function in the dual space A_0' with $\|h\|_{A_0'} = 1$. We have

$$\begin{aligned}\int_0^\infty f^*(s)\chi_{[0,t]}(s)h(s)ds &\leq \|f\|_{A_1}\int_0^t \frac{h(s)}{\Phi_1(s)}ds \leq \|f\|_{A_1}\left\|\frac{\chi_{[0,t]}}{\Phi_1}\right\|_{A_0} \\ &\leq C\|f\|_{A_1}\frac{\Phi_0(t)}{\Phi_1(t)},\end{aligned}$$

where we have used Holder's inequality and the fact that $f \in A_1 \Rightarrow f^*(s)\Phi_1(s) \leq \|f^*\|_{A_1}$. Now, taking the supremum in h's we have

$$\|f^*\chi_{[0,t]}\|_{A_0} \leq C\|f\|_{A_1}\frac{\Phi_0(t)}{\Phi_1(t)}. \tag{2.1}$$

For the second part, let f be in A_0 and show that $f^*\chi_{[t,\infty)} \in A_1$. Let $t > 0$ be a fixed real number. It is clear that $(f^*\chi_{[t,\infty)})^* \leq f^*(t)\chi_{[0,t]} + f^*\chi_{[t,\infty)}$. So, if h is an arbitrary decreasing function in the dual space A_1', with $\|h\|_{A_1'} = 1$, by using condition $(C.2)$ we have

$$\begin{aligned}\int_0^\infty (f^*\chi_{[t,\infty)})^*h &\leq \int_0^t f^*(t)h(s)ds + \int_t^\infty f^*(s)h(s)ds \\ &\leq f^*(t)\Phi_1(t) + \|f\|_{A_0}\int_t^\infty \frac{h(s)}{\Phi_0(s)}ds \\ &\leq \|f\|_{A_0}\left(\frac{\Phi_1(t)}{\Phi_0(t)} + \left\|\frac{\chi_{[t,\infty)}}{\Phi_0}\right\|_{A_1}\right) \leq C\frac{\Phi_1(t)}{\Phi_0(t)}\|f\|_{A_0},\end{aligned}$$

and taking the supremum on h's we get

$$\|f^*\chi_{[t,\infty)}\|_{A_1} \leq C\frac{\Phi_1(t)}{\Phi_0(t)}\|f\|_{A_0} \tag{2.2}$$

and the lemma is proved.

We introduce the operator S, which could be named *Calderón operator* associated to our scheme, by

$$Sf(t) = \frac{1}{\Phi_0(t)}\|f^*\chi_{[0,t]}\|_{A_0} + \frac{1}{\Phi_1(t)}\|f^*\chi_{[t,\infty)}\|_{A_1}, \quad \forall t > 0.$$

This operator is well defined for functions in $A_0 + A_1$. The following result shows that the Calderón operator S is equivalent to the K-functional of interpolation relative to $\overline{A}$.

Lemma 2.2. *There exist constants $C_1, C_2 > 0$ such that if $f \in A_0 + A_1$, then*

$$C_1\Phi_0(t)S(f)(t) \le K\left(\frac{\Phi_0(t)}{\Phi_1(t)}, f; \overline{A}\right) \le C_2\Phi_0(t)S(f)(t), \quad \forall t > 0.$$

Proof. Let $t > 0$ be a fixed real number. If we take the decomposition

$$g = (f - f^*(t)\operatorname{sgn} f)\chi_E, \quad h = f - g,$$

where the set E satisfies $\{x; |f(x)| > f^*(t)\} \subseteq E \subseteq \{x : |f(x)| \ge f^*(t)\}$ and $m(E) = t$, we have that $K\left(\frac{\Phi_0(t)}{\Phi_1(t)}; f\right)$ is bounded by

$$\|f^*\chi_{[0,t]}\|_{A_0} + \frac{\Phi_0(t)}{\Phi_1(t)}\|f^*\chi_{[t,\infty)}\|_{A_1} \le \Phi_0(t)S(f)(t).$$

On the other hand, let $f \in A_0 + A_1$. For any decomposition $f = g + h$, $g \in A_0$, $h \in A_1$, since

$$(g+h)^*(s)\chi_{[0,t]}(s) \le (g^*(s/2) + h^*(s/2))\chi_{[0,t]}(s/2)$$

for all $s > 0$, by using (2.1) and (2.2) we have

$$\|(g+h)^*\chi_{[0,t]}\| \le C\left(\|g\|_{A_0} + \frac{\Phi_0(t)}{\Phi_1(t)}\|h\|_{A_1}\right)$$

(the constant C depends on the norm of the dilation operator D_2 on A_0 and A_1). In a similar way

$$(g+h)^*(s)\chi_{[t,\infty)}(s) \le (g^*(s/2) + h^*(s/2))\chi_{[t/2,\infty)}(s/2)$$

for all $s > 0$, and hence

$$\|(g+h)^*\chi_{[t,\infty)}\|_{A_1} \le C\left(\frac{\Phi_1(t)}{\Phi_0(t)}\|g\|_{A_0} + \|h\|_{A_1}\right)$$

(the constant C also depends on the norm of the dilation operator D_2 on A_0 and A_1). Then

$$S(f)(t)\Phi_0(t) \le C\left(\|g\|_{A_0} + \frac{\Phi_0(t)}{\Phi_1(t)}\|h\|_{A_1}\right)$$

and, taking the infimum over all possible decompositions, we get

$$S(f)(t)\Phi_0(t) \le CK\left(\frac{\Phi_0(t)}{\Phi_1(t)}; f\right).$$

The following result characterizes the pairs of spaces which are of interpolation with respect to the couples $\overline{A}$ and $M(\overline{A})$. The authors are indebted to Mieczyslaw Mastylo for showing them the reference [DK].

Theorem 2.3. *A pair A, B of intermediate spaces with respect to $\overline{A}$ and $M(\overline{A})$ is a pair of relative interpolation spaces with respect to $\overline{A}$ and $M(\overline{A})$ if and only if the Calderón operator S is bounded from A into B.*

Proof. The "if part" of the proof is a consequence of the preceding lemma and Lemma 1 of [DK]. For the other part we use the same lemma and Theorem 3 of [DK]. Note that in our case, since the condition $(C.0)$ is fulfilled, the weak interpolation (Φ_0, Φ_1)-property is exactly what we need.

Remarks. i) As a consequence of the same results of [DK] we obtain that the couples $\overline{A}$ and $M(\overline{A})$ are relative Calderón pairs (see [C]).

ii) In a very recent paper Kalton (see Theorem 5.3 of [K]) obtained the following result: Suppose A_0 and A_1 is a pair of r.i. spaces whose Boyd indices satisfy $p_1 > q_0$. Then $\overline{A}$ is a Calderón couple if and only if A_0 is stretchable and A_1 is compressible.

Let us assume now that $p_1 > q_0$, where

$$p_1 = p_{M(A_1)} = \lim_{s\to\infty} \frac{\log s}{\log \|D_s\|_{M(A_1)}}$$

and

$$q_0 = q_{M(A_0)} = \lim_{s\to 0} \frac{\log s}{\log \|D_s\|_{M(A_0)}}$$

are the corresponding Boyd indices, where D_s is the dilation operator defined by $D_s f(t) = f(t/s)$. Hence

$$\lim_{s\to\infty} \frac{\log\left(\|D_s\|_{M(A_1)}\|D_{1/s}\|_{M(A_0)}\right)}{\log s} < 0$$

and so, there exists a constant $C > 0$ such that for every $s \geq 1$

$$\|D_s\|_{M(A_1)}\|D_{1/s}\|_{M(A_0)} \leq C.$$

Therefore, if $s < t$,

$$\begin{aligned}
\frac{\Phi_0(s)}{\Phi_1(s)} &= \frac{\|D_{s/t}\chi_{[0,t)}\|_{M(A_0)}}{\Phi_1(s)} \\
&\leq \frac{\|D_{s/t}\|_{M(A_0)}\Phi_0(t)}{\|D_{t/s}\|^{-1}_{M(A_1)}\|D_{t/s}\chi_{[0,s)}\|_{M(A_1)}} \\
&\leq C\frac{\Phi_0(t)}{\Phi_1(t)}.
\end{aligned}$$

Now if $M(A_0)$ is convexifiable, then so is $M(A_1)$. Furthermore, the couple $M(\overline{A})$ is a Calderón couple, or in our notation the couples $M(\overline{A})$ and $M(\overline{A})$ are relative Calderón couples, and we do not need to assume extra conditions about Boyd indices. Indeed, it is clear that

$$\left\|\frac{\chi_{[0,t]}}{\Phi_1}\right\|_{M(A_0)} = \sup_{0<s\leq t} \frac{\Phi_0(s)}{\Phi_1(s)} \leq C\frac{\Phi_0(t)}{\Phi_1(t)},$$

which implies condition $(C.1)$, and in a similar way

$$\left\|\frac{\chi_{[t,\infty)}}{\Phi_0}\right\|_{M(A_1)} = \sup_{0<s<\infty} \frac{\Phi_1(s)}{\Phi_0(s+t)} \leq C\frac{\Phi_1(t)}{\Phi_0(t)}$$

implies condition $(C.2)$.

As a consequence by using the notations in [K] we obtain the following results:

Corollary 2.4. *Suppose that X is a r.i. Banach space.*

i) *If $M^*(X)$ is convexifiable and the Boyd index $q_{M(X)} < \infty$, then $M(X)$ is stretchable.*
ii) *If $p_{M(X)} > 1$, then $M^*(X)$ is convexifiable and $M(X)$ is compressible, in the sense of* [K].

Proof. We only have to consider $A_0 = M(X)$ and $A_1 = L^{q_{M(X)}+1,\infty}$ for i), and $A_0 = L^{p-\epsilon,\infty}$, $A_1 = M(X)$ for ii).

3. Application to Lorentz spaces

We apply the results of the preceding section to classical Lorentz spaces. Let w_i, $i = 0, 1$, be two weights and $1 \le p_i < \infty$ two real numbers. We will suppose that the weights satisfy the corresponding conditions (AM_{p_i}), $i = 0, 1$, quoted in the introduction.

In the sequel let $A_i = \Lambda(W_i, p_i)$, $i = 0, 1$, the corresponding Lorentz spaces. Since $\Phi_i(t) = W_i(t)^{1/p_i}$, condition $(C.0)$ is satisfied if and only if the nonincreasing weights W_i^{-1/p_i} satisfy

$$\frac{1}{t}\int_0^t \frac{dx}{W_i(x)^{1/p_i}} \le \frac{C}{W_i(t)^{1/p_i}}$$

for some constant $C > 0$ and for all $t > 0$.

In order to ensure that conditions $(C.1)$ and $(C.2)$ are fullfilled we suppose that there exists a real α, $0 < \alpha < 1$ and a $C > 0$ such that

$$(C_\alpha) \qquad C^{-1} W_0^{\alpha/p_0}(t) \le W_1^{1/p_1}(t) \le C W_0^{\alpha/p_0}(t).$$

Indeed

$$\frac{W_1^{1/p_1}(t)}{W_0^{1/p_0}(t)} \left\| \frac{\chi_{[0,t]}}{W_1^{1/p_1}} \right\|_{A_0} \le C W_0^{(\alpha-1)/p_1}(t) \left(\int_0^t \frac{dW_0(x)}{W_0^\alpha(x)} \right)^{1/p_0} \le C$$

and

$$\begin{aligned} \frac{W_0^{1/p_0}(t)}{W_1^{1/p_1}(t)} \left\| \frac{\chi_{[t,\infty)}}{W_0^{1/p_0}} \right\|_{A_1} &\le C W_1^{(1-\alpha)/p_1\alpha}(t) \left(\int_0^\infty \frac{dW_1(x)}{W_1^{1/\alpha}(x+t)} \right)^{1/p_1} \\ &\le W_1^{(1-\alpha)/p_1\alpha}(t) \left(\int_0^t \frac{dW_1(x)}{W_1^{1/\alpha}(t)} + \int_t^\infty \frac{dW_1(x)}{W_1^{1/\alpha}(x)} \right)^{1/p_1} \le C. \end{aligned}$$

It is worth noting that condition (C_α) and the fact that W_0^{-1/p_0} belongs to the A_1-class of Muckenhoupt imply that W_1^{-1/p_1} also belongs to the A_1-class.

We are going to prove that the Calderón operator associated to this scheme is equivalent to $P + Q$, where

$$Pf(t) = W_0^{-1/p_0}(t) \left(\int_0^t f^*(x)^{p_0} dW_0(x) \right)^{1/p_0}, \quad t > 0,$$

and

$$Qf(t) = W_1^{-1/p_1}(t)\left(\int_t^\infty f^*(x)^{p_1}\,dW_1(x)\right)^{1/q_1}, \ \ t>0.$$

Lemma 3.1. *There exists a constant $C>0$ such that*

$$C^{-1}Sf(t) \le Pf(t)+Qf(t) \le Sf(t).$$

Proof. It is clear that $Pf(t)+Qf(t) \le Sf(t)$, for all $t>0$. For the converse inequality

$$\begin{aligned} W_1^{-1/p_1}(t)\|f^*\chi_{[t,\infty)}\|_{A_1} &= W_1^{-1/p_1}(t)\left(\int_0^\infty f^*(x+t)^{p_1}\,dW_1(x)\right)^{1/p_1} \\ &\le C\,W_1^{-1/p_1}(t)\left[f^*(t)W_1^{1/p_1}(t)+\left(\int_t^\infty f^*(x)^{p_1}\,dW_1(x)\right)^{1/p_1}\right] \\ \le C\left[\frac{1}{W_0^{1/p_0}(t)}\left(\int_0^t f^*(x)^{p_0}\,dW_0\right)^{1/p_0}\right. &+ \left.\frac{1}{W_1^{1/p_1}(t)}\left(\int_t^\infty f^*(x)^{p_1}\,dW_1\right)^{1/p_1}\right]. \end{aligned}$$

Hence

$$Sf(t) \le C(Pf(t)+Qf(t))$$

and the lemma holds.

Next we are going to prove that relative interpolation spaces with respect to relative Calderón couples of Lorentz spaces coincide. This result extends previous results in [BR].

Proposition 3.2. *Let A be a r.i. space. The following are equivalent:*

3.2.i) *The pair A, A is of relative interpolation with respect to the couples $\overline{A}$, $M(\overline{A})$.*

3.2.ii) *There exists $\epsilon>0$, small enough, such that the pair A, A is of relative interpolation with respect to the relative Calderón couples $\overline{B}$, $M(\overline{B})$, where $B_0 = \Lambda(W_0^{1-\epsilon}, p_0)$ and $B_1 = \Lambda(W_1^{1+\epsilon}, p_1)$.*

3.2.iii) *The pair A, A is of relative interpolation with respect to the relative Calderón couples $\overline{C}$, $M(\overline{C})$, where $C_i = \Lambda(W_i^{1/p_i}, 1)$, $i=0,1$.*

Proof. We remark that $M(A_i) = M(C_i)$, $i=0,1$.

3.2.i) $\Rightarrow$ 3.2.ii). As a conclusion of Theorems 1.2 and 1.3, we can choose $\epsilon>0$, small enough, such that the operator $\overline{P}+\overline{Q}$ is bounded in A and besides that the couple $\overline{B}$ satisfies condition (C_β) for $\beta = \alpha(1+\epsilon)(1-\epsilon)^{-1} < 1$. We apply Theorem 2.3 and so we obtain 3.2.ii), since $\overline{P}+\overline{Q}$ is equivalent to the Calderón operator associated to the couples $\overline{B}$, $M(\overline{B})$.

3.2.ii) $\Rightarrow$ 3.2.iii). We only need to check that the Calderón operator associated to the couples $\overline{C}$ and $M(\overline{C})$ is bounded in A, when $p_0>1, p_1>1$. This fact is an inmediate consequence of Hölder's inequality

$$\begin{aligned} \int_0^t f^*\,dW_0^{1/p_0} &\le C\left(\int_0^t f^{*p_0}W_0^{-\epsilon}\,dW_0\right)^{1/p_0}\left(\int_0^t W_0^{-1+(\epsilon p_0')/p_0}\,dW_0\right)^{1/p_0'} \\ &\le C\,W_0(t)^{1/p_0}\overline{P}f(t) \end{aligned}$$

and

$$\int_t^\infty f^* dW_1^{1/p_1} \le C \left(\int_t^\infty f^{*p_1} W_0^\epsilon dW_1 \right)^{1/p_1} \left(\int_t^\infty W_1^{-1-(\epsilon p_1')/p_1} dW_0 \right)^{1/p_1'}$$
$$\le C\, W_1(t)^{1/p_1} \overline{Q} f(t).$$

3.2.iii) $\Rightarrow$ 3.2.i). This part is well known since each space C_i, $i = 0, 1$, is the corresponding Lorentz space $\Lambda(A_i)$ (see for instance [BS], Theorem 5.13).

As a corollary we get

Corollary 3.3. *Let A be a r.i. space and $1 < p_0 < p_1 < \infty$ and $1 \le q_0, q_1 < \infty$. The following assertions are equivalent:*

3.5.i) *The pair A, A is of relative interpolation with respect to the couples $(L^{p_0,1}, L^{p_1,1})$ and $(L^{p_0,\infty}, L^{p_1,\infty})$.*

3.5.ii) *The pair A, A is of relative interpolation with respect to the couples $(L^{p_0,q_0}, L^{p_1,q_1})$ and $(L^{p_0,\infty}, L^{p_1,\infty})$.*

3.5.iii) *The Boyd indices of A satisfy $p_A > p_0$ and $q_A < p_1$.*

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