FACTORING SOBOLEV INEQUALITIES THROUGH CLASSES OF FUNCTIONS

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Abstract. We recall two approaches to recent improvements of the classical Sobolev inequality. The first one follows the point of view of Real Analysis, [21], [3], while the second one relies on tools from Convex Geometry, [32], [16]. In this paper we prove a (sharp) connection between them.

1. Introduction and notation

The classical Sobolev inequality states that for $1 \leq p < n$ and $\frac{1}{q} = \frac{1}{p} - \frac{1}{n}$, there exists a constant $C_{p,n} > 0$ such that for every $f\colon \mathbb{R}^n \to \mathbb{R}$ in the Sobolev space $W^{1,p}(\mathbb{R}^n)$,

$$\|\nabla f\|_p \geq C_{p,n} \|f\|_q \tag{1}$$

where $\|\cdot\|_q$ denotes the L_q-norm of the Euclidean norm of functions and ∇f is the gradient of f.

The best constant in the case $p = 1$ ($q = n/(n-1)$) was obtained by H. Federer, W. Fleming, [8] and independently by V. Maz'ja, [18] [19]. They proved $C_{1,n} = n\,\omega_n^{\frac{1}{n}}$, where ω_n denotes the Lebesgue measure of the Euclidean unit ball in $\mathbb{R}^n$, and showed that this fact is equivalent to the isoperimetric inequality (see for instance [5] for a survey). For the other values of $1 < p < n$, Aubin and Talenti got the best constants, [1], [28]. See also the recent approaches in [2] and [7].

We point out that one key step in classical proofs of (1) is the use of Polya-Szegö rearrangement inequality, see [26],

$$\|\nabla f\|_p \geq \|\nabla f^\circ\|_p, \qquad p \geq 1 \tag{2}$$

where $f^\circ(x) := f^*(\omega_n |x|^n)$, is a radial extension to $\mathbb{R}^n$ of the decreasing rearrangement of f, $f^*(t) = \inf\{\lambda > 0\ ;\ |\{|f| > \lambda\}|_n \leq t\}, t \geq 0$, $|\cdot|$ is the Euclidean distance in $\mathbb{R}^n$ and $|\cdot|_n$ is the Lebesgue measure on the (suitable) n-dimensional space. f° has the same distribution function as f and f^*. It is called the symmetric Schwarz rearrangement of f.

For $p = n$, the inequality with $q = \infty$ is not true. In the sixties Trudinger [31] and Moser [23] proposed an Orlicz space, $\mathcal{MT}(\Omega)$, of functions defined on open domains $\Omega \subset \mathbb{R}^n$ with $|\Omega|_n < \infty$ and showed the continuous inclusion $W_0^{1,n}(\Omega) \hookrightarrow \mathcal{MT}(\Omega)$, where $W_0^{1,n}(\Omega)$ is the closure of the space of $\mathcal{C}^1$ functions of compact support,

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$\mathcal{C}^1_{00}(\Omega)$, in the Sobolev space $W^{1,n}(\Omega)$. More precisely, they proved that there exists $C_n > 0$ such that for all $f \in W^{1,n}_0(\Omega)$

$$\|\nabla f\|_n \geq C_n \|f\|_{\mathcal{MT}}$$

and the constant (depending on $|\Omega|_n$) is sharp.

In the late seventies, Hansson [12] and Brezis-Wainger [4] improved the target space in the inclusion above. They introduced a rearrangement invariant function space, $H_n(\Omega)$, such that $W^{1,n}_0(\Omega) \hookrightarrow H_n(\Omega) \hookrightarrow \mathcal{MT}(\Omega)$. Moreover, $H_n(\Omega)$ was proved to be the optimal target space in the class of rearrangement invariant spaces. Equivalently, they obtained an inequality of the form $\|\nabla f\|_n \geq c_n \|f\|_{H_n} \geq c'_n \|f\|_{\mathcal{MT}}$ for some constants $c_n, c'_n > 0$ (depending on $|\Omega|_n$).

Tartar [30], Maly-Pick [20] and Bastero-Milman-Ruiz [3], see also [13], refined those estimates using *classes of functions* as follows: For $1 \leq p < \infty$ denote

$$\mathcal{A}_{\infty,p}(\mathbb{R}^n) = \{f; \|f\|_{\infty,p} = \left(\int_0^\infty (f^{**}(t) - f^*(t))^p \frac{dt}{t^{p/n}}\right)^{1/p} < \infty\}$$

where f^{**} is the Hardy transform of f^* defined by $f^{**}(t) = \dfrac{1}{t}\displaystyle\int_0^t f^*(s)\, ds. W^{1,p}(\mathbb{R}^n)$

Then for all $f \in W^{1,n}_0(\Omega)$

$$\|\nabla f\|_n \geq (n-1)\, \omega_n^{\frac{1}{n}} \|f\|_{\infty,n} \geq c''_n \|f\|_{H_n} \tag{3}$$

for some $c''_n > 0$ (depending on $|\Omega|_n$). Observe that the constant in the first inequality depends neither on the measure of Ω nor on the support of f.

Once one considered classes of functions instead of vector spaces, Sobolev type inequalites could be extended further. At this point we recall the well known fact that $W^{1,p}_0(\mathbb{R}^n) = W^{1,p}(\mathbb{R}^n)$, [19]. In [21] the authors proved

$$\|\nabla f\|_p \geq c_{n,p} \|f\|_{\infty,p} \geq c'_{n,p} \|f\|_q, \qquad \forall\ f \in W^{1,p}(\mathbb{R}^n), \qquad 1 \leq p < n \tag{4}$$

and some constants $c_{n,p}, c'_{n,p} > 0$.

We now move on to a different philosophy. We start by recalling the so called Petty projection inequality, stated in [25] for convex bodies and extended by Zhang [32] to compact subsets $K \subset \mathbb{R}^n$

$$\frac{n\omega_n}{\omega_{n-1}} \left(\int_{S^{n-1}} \frac{du}{|P_{u^\perp}(K)|^n_{n-1}}\right)^{-\frac{1}{n}} \geq n\omega_n^{1/n} |K|^{\frac{n-1}{n}} \tag{5}$$

where $P_{u^\perp}$ is the the orthogonal projection onto the hyperplane $u^\perp$ and du is the normalized Haar probability on the unit sphere S^{n-1}. Petty projection inequality directly implies the isoperimetric inequality.

In 1999 Zhang [32] (see also [11] and the references therein) introduced a new class of functions

$$\mathcal{E}_p(\mathbb{R}^n) = \left\{ f \in W^{1,p}(\mathbb{R}^n); \mathcal{E}_p(f) := \frac{1}{I_p} \left(\int_{S^{n-1}} \|D_u f\|_p^{-n} du\right)^{-\frac{1}{n}} < \infty \right\}, \qquad p \geq 1$$

where $D_u f(x) := \langle \nabla f(x), u \rangle$ and $I_p^p := \displaystyle\int_{S^{n-1}} |u_1|^p du$ is a normalization constant so that $\mathcal{E}_p(f^\circ) = \|\nabla f^\circ\|_p$. The expression $\mathcal{E}_p(f)$ is an energy integral having applications in information theory. It is invariant under transformations of $\mathbb{R}^n$ of the

form $x \to x_0 + Ax, x_0 \in \mathbb{R}^n, A \in SL(n)$, [17]. Moreover, by Jensen's inequality and Fubini's theorem the following relation holds

$$\mathcal{E}_p(f) = \frac{1}{I_p}\left(\int_{S^{n-1}} \|D_u f\|_p^{-n} du\right)^{-1/n} \leq \frac{1}{I_p}\left(\int_{S^{n-1}} \|D_u f\|_p^p du\right)^{1/p} = \|\nabla f\|_p.$$

The following remarkable inequality

$$\mathcal{E}_p(f) \geq \mathcal{E}_p(f^\circ)\,, \qquad 1 \leq p < \infty \tag{6}$$

was proved in a series of papers: Zhang [32] initiated the approach by showing that his extension of the Petty projection inequality (5) is actually equivalent to (6) for $p = 1$. The general case was proved via the L_p-Brunn-Minkowski theory in [16], [17], [6]. The invariance of $\mathcal{E}_p(f)$ implies, by homogeneity, that (6) is affine-invariante i.e. invariant under transformations of $\mathbb{R}^n$ of the form $x \to x_0 + Ax, x_0 \in \mathbb{R}^n, A \in GL(n)$.

The inequality (6) is stronger than Polya-Szegö rearrangement inequality (2) and thus it yields to a new proof of Sobolev's inequality

$$\|\nabla f\|_p \geq \mathcal{E}_p(f) \geq \mathcal{E}_p(f^\circ) \geq C_{n,p}\|f\|_q, \qquad 1 \leq p < n. \tag{7}$$

See [32], [17] for such a proof of (1) with sharp constants. We remark the fact that $\mathcal{E}_p(f) \geq C_{n,p}\|f\|_q$ is affine-invariant while Sobolev's inequality $\|\nabla f\|_p \geq C_{n,p}\|f\|_q$ is invariant only under rigid motions.

In [9] [10], [11], the authors investigated the space $\mathcal{E}_p^+(\mathbb{R}^n)$ defined analogously as before by

$$\mathcal{E}_p^+(f) := \frac{2^{1/p}}{I_p}\left(\int_{S^{n-1}} \|D_u^+ f\|_p^{-n} du\right)^{-\frac{1}{n}}$$

where $D_u^+ f(x) := \max\{\langle \nabla f(x), u\rangle, 0\}$ and proved

$$\mathcal{E}_p(f) \geq \mathcal{E}_p^+(f) \geq \mathcal{E}_p^+(f^\circ) = \mathcal{E}_p(f^\circ), \qquad p \geq 1 \tag{8}$$

which refined (7) for $1 \leq p < n$.

In the case $p \geq n$, affine-invariant inequalities of Sobolev type were studied in [6], [10] and [11] with the hypothesis of f having support of finite measure. In the limiting case $p = n$ they proved the sharp inequality

$$\|\nabla f\|_n \geq \mathcal{E}_n(f) \geq \mathcal{E}_n^+(f) \geq C_n\|f\|_{\mathcal{MT}} \tag{9}$$

while for $p > n$,

$$\mathcal{E}_p(f) \geq \mathcal{E}_p^+(f) \geq \left(\frac{p'}{|q|}\right)^{\frac{1}{p'}} n\omega_n^{1/n}\|f\|_\infty |\operatorname{supp} f|_n^{1/q} \qquad \text{where} \quad \frac{1}{p} + \frac{1}{p'} = 1 \tag{10}$$

where the constant *depending on the size of the support of* f is sharp (take $f^*(t) = (1 - t^{-p'/q})\chi_{[0,1]}(t)$).

In conclusion, the first approach so described looked for improvements of the right hand side of (1), while the second approach concerned the left hand side of (1). In this note we link these two approaches and show that for all $1 \leq p < \infty$ and $\frac{1}{q} := \frac{1}{p} - \frac{1}{n}$

$$\mathcal{E}_p^+(f) \geq \left(1 - \frac{1}{q}\right) n\ \omega_n^{1/n}\|f\|_{\infty,p} \qquad \forall f \in W^{1,p}(\mathbb{R}^n)$$

(see Theorem 2.1) where the constant is sharp. As a consequence, for $1 \leq p < n$ it gives the right constant in the first inequality in (4) and enables to connect (7),

(8) and (4). For $p = n$, Theorem 2.1 and its Corollary 2.1 connect (3) and (9), improving the first inequality in (3). In the case $p > n$ it links the first inequality in (4) and (7), (thanks to (8) and to the fact that they are also valid for $p > n$). Moreover, in Proposition 2.1 we see how it yields to lower estimates for $\mathcal{E}_p^+(f)$ better than (10).

In the third section we include a proof of the inequalities (6) and (8) which directly derives from Zhang's extension of Petty projection inequality, paying the penalty of an extra constant $\frac{I_p}{I_1}$ (which is independent of the dimension n). No use of the L_p-Brunn-Minkoswki theory and polytope approximation appearing in the papers [16], [17], [6] and [10] is made.

2. The results

The first result is the correct relation between $\mathcal{E}_p^+(f)$ and $\|f\|_{\infty,p}$.

Theorem 2.1. *Let $1 \le p < \infty$ and $\frac{1}{q} = \frac{1}{p} - \frac{1}{n}$, $q \in (-\infty, -n) \cup [\frac{n}{n-1}, \infty]$. Then*

$$\mathcal{E}_p^+(f) \ge \left(1 - \frac{1}{q}\right) n\, \omega_n^{1/n} \|f\|_{\infty,p} \qquad \forall f \in W^{1,p}(\mathbb{R}^n).$$

Moreover the constant is sharp.

Proof. Taking (8) into account it is enough to see that for any $f : \mathbb{R}^n \to \mathbb{R}$ compactly supported C^1 function

$$\mathcal{E}_p^+(f^\circ) = \frac{2^{1/p}}{I_p} \left(\int_{S^{n-1}} \|D_u^+ f^\circ\|_p^{-n} du \right)^{-1/n} \ge \left(1 - \frac{1}{q}\right) n\, \omega_n^{1/n} \|f\|_{\infty,p}.$$

Now, f° is Lipschitz and f^* locally Lipschitz and $f^{*\prime}$ integrable (see for instance [13] and [24]). Therefore since

$$f^{**}(t) - f^*(t) = \frac{1}{t} \int_0^t (f^*(s) - f^*(t)) ds = \frac{1}{t} \int_0^t \int_s^t -f^{*\prime}(u) du ds = \frac{1}{t} \int_0^t s |f^{*\prime}(s)| ds$$

we have

$$\|f^\circ\|_{\infty,p}^p = \|f\|_{\infty,p}^p = \int_0^\infty (f^{**}(t) - f^*(t))^p \frac{dt}{t^{p/n}} = \int_0^\infty \left(\frac{1}{t} \int_0^t s |f^{*\prime}(s)| ds \right)^p \frac{dt}{t^{p/n}}.$$

Apply Hardy's inequality $\int_0^\infty \left(\frac{1}{t} \int_0^t g(s) ds \right)^p \frac{dt}{t^{p/n}} \le \left(\frac{p}{p + \frac{p}{n} - 1} \right)^p \int_0^\infty g(s)^p \frac{ds}{s^{p/n}}$ to $g(s) = s|f^{*\prime}(s)|$ (see [19] for a reference on Hardy's inequalities with weights) to obtain

$$\|f^\circ\|_{\infty,p}^p \le \left(\frac{1}{1 - \frac{1}{q}} \right)^p \int_0^\infty s^{\frac{(n-1)p}{n}} |f^{*\prime}(s)|^p \, ds.$$

On the other hand, by definition of $f^\circ(x)$,

$$\langle \nabla f^\circ(x), u \rangle_+ = n\omega_n |x|^{n-1} |f^{*\prime}(\omega_n |x|^n)| \left\langle \frac{-x}{|x|}, u \right\rangle_+$$

and so by polar integration $x = r\theta$ and the change of variables $s = \omega_n r^n$

$$\|D_u^+ f^\circ\|_p^p = \int_{\mathbb{R}^n} \langle \nabla f^\circ(x), u \rangle_+^p \, dx = \frac{1}{2} I_p^p \left(n \omega_n^{1/n} \right)^p \int_0^\infty s^{\frac{(n-1)p}{n}} |f^{*\prime}(s)|^p ds \tag{11}$$

and the result follows.

In order to see that the constant is sharp consider truncations of the function $f^*(t) = t^{-1/q}$, whenever $p < n$, $f^*(t) = \log(1/t)$, for $p = n$ and $f^*(t) = (1 - t^{-1/q})\chi_{[0,1]}$, whenever $p > n$. □

Straightforward computations show that f^* and therefore $\|f\|_{\infty,p}$ are invariant under transformations of $\mathbb{R}^n$, $x \to x_0 + Ax, x_0 \in \mathbb{R}^n, A \in SL(n)$. Consequently, the inequality in Theorem 2.1 is affine-invariant.

Corollary 2.1. *Let $1 \le p < \infty$. For any $f \in W^{1,p}(\mathbb{R}^n)$*

$$\|\nabla f\|_p \ge n\ \omega_n^{1/n} \left(1 - \frac{1}{q}\right) \|f\|_{\infty,p}.$$

In particular, for $p = n$ we have $\|\nabla f\|_n \ge n\ \omega_n^{1/n} \|f\|_{\infty,n}$

Remark. The result refines the inequalities (4), (3) and (9).

Proof. It follows from the previous Theorem and the facts stated in the introduction

$$\left(1 - \frac{1}{q}\right) n\ \omega_n^{1/n} \|f\|_{\infty,p} \le \mathcal{E}_p^+(f^\circ) \le \mathcal{E}_p(f^\circ) \le \mathcal{E}_p(f) \le \|\nabla f\|_p$$

□

We pass to the case $p > n$. As we said in the introduction we shall see how Theorem 2.1 provides better estimates than inequality (10).

Proposition 2.1. *Let $p > n$, $\frac{1}{q} = \frac{1}{p} - \frac{1}{n}$ and f a compactly supported C^1 function. Then,*

$$\sup_{t>0} \{(\|f\|_\infty - f^*(t))\, t^{1/q}\} \le \alpha_{n,p} \|f\|_{\infty,p}$$

for some $\alpha_{n,p} > 0$ (independent of the support of f).

Remark. The proof gives $\alpha_{p,n} = \left(\left(p(1 - \frac{1}{q})\right)^{p'/p} + \frac{|q|}{p'}\right)^{\frac{1}{p'}}$.

Proof. For any $t > 0$ we have

$$\begin{aligned}
\|f\|_\infty - f^*(t) &= f^{**}(0) - f^{**}(t) + f^{**}(t) - f^*(t) = \int_0^t -f^{**\prime}(u)du + f^{**}(t) - f^*(t) \\
&= \int_0^t \frac{f^{**}(u) - f^*(u)}{u} du + f^{**}(t) - f^*(t) \\
&= \int_0^t \frac{f^{**}(u) - f^*(u)}{u} du \left(\frac{p'}{|q|}\right)^{\frac{1}{p'}} \left(\frac{|q|}{p'}\right)^{\frac{1}{p'}} + \frac{f^{**}(t) - f^*(t)}{\left(p\big(1 - \frac{1}{q}\big)\right)^{\frac{1}{p}}} \left(p\big(1 - \frac{1}{q}\big)\right)^{\frac{1}{p}}.
\end{aligned}$$

By Hölder's inequality, the latter expression is bounded from above by

$$\alpha_{p,n} \left(\left(\int_0^t \frac{f^{**}(u) - f^*(u)}{u} du\right)^p \left(\frac{p'}{|q|}\right)^{\frac{p}{p'}} + \frac{(f^{**}(t) - f^*(t))^p}{p\big(1 - \frac{1}{q}\big)}\right)^{\frac{1}{p}}.$$

Now, on one hand, for any $s > t > 0$ we have after integrating by parts

$$f^{**}(s) - f^*(s) = \frac{1}{s} \int_0^s u|f^{*\prime}(u)|du \ge \frac{1}{s} \int_0^t u|f^{*\prime}(u)|du = \frac{t}{s}(f^{**}(t) - f^*(t)).$$

Hence

$$\int_t^\infty (f^{**}(s)-f^*(s))^p s^{-p/n}ds \geq t^p (f^{**}(t)-f^*(t))^p \int_t^\infty \frac{ds}{s^{p+\frac{p}{n}}}$$

and consequently

$$\frac{(f^{**}(t)-f^*(t))^p}{p\big(1-\frac{1}{q}\big)} \leq t^{-\frac{p}{q}} \int_t^\infty (f^{**}(s)-f^*(s))^p s^{-p/n}ds.$$

On the other hand, by Hölder's inequality and since $p>n$,

$$\int_0^t \frac{f^{**}(s)-f^*(s)}{s} \leq \left(\int_0^t s^{(\frac{1}{n}-1)p'}ds\right)^{1/p'} \left(\int_0^t (f^{**}(s)-f^*(s))^p s^{-p/n}ds\right)^{1/p}$$

which implies

$$\left(\int_0^t \frac{f^{**}(s)-f^*(s)}{s}\right)^p \left(\frac{p'}{|q|}\right)^{\frac{p}{p'}} t^{\frac{p}{q}} \leq \int_0^t (f^{**}(s)-f^*(s))^p s^{-p/n}ds.$$

Thus, for any $t>0$ we have

$$t^{\frac{1}{q}}(\|f\|_\infty - f^*(t)) \leq \alpha_{p,n}\|f\|_{\infty,p}$$

which finishes the proof. □

Remark. $f^*(|\operatorname{supp} f|_n) = 0$ implies $\|f\|_\infty |\operatorname{supp} f|_n^{1/q} \leq \sup_{t>0}\{(\|f\|_\infty - f^*(t))\, t^{1/q}\}$ and so Proposition 2.1 shows that Theorem 2.1 is, up to constant, better than (10). The example $f^*(t) = (1-t^{-1/q})\chi_{[0,1]}(t)$ verifies $\sup_{t>0}\{(\|f\|_\infty - f^*(t))\, t^{1/q}\} = 1$ while $\|f\|_{\infty,p} = \infty$.

3. A simplified approach to affine Sobolev inequalities

In this part we will show a direct way to deduce the key inequalities (6) and (8) from Zhang's inequality, paying a penalty on the constant. We use similar ideas to those appearing in [24] which prove Polya-Szegö rearrangement inequality from the isoperimetric inequality.

Proposition 3.1. *Let $1 \leq p < \infty$. For all $f \in W^{1,p}(\mathbb{R}^n)$*

$$\mathcal{E}_p(f^\circ) \leq \frac{I_p}{I_1}\mathcal{E}_p(f) \qquad \text{and} \qquad \mathcal{E}_p^+(f^\circ) \leq \frac{I_p}{I_1}\mathcal{E}_p^+(f).$$

Remark. It is well known that $C_1\sqrt{p} \leq \frac{I_p}{I_1} \leq C_2\sqrt{p}$ with C_1, C_2 absolute constants.

Proof. Suppose f is a $\mathcal{C}^1$ function of compact support. Let $\Phi(t)$ represent either $|t|$ or $\max\{t,0\}$. By Sard's theorem, for almost all $t>0$ the level set $\{|f| \geq t\}$ is compact with $\mathcal{C}^1$ boundary $\{|f|=t\}$ and $\nabla f(x) \neq 0, \forall x \in \{|f|=t\}$. By Federer's co-area formula

$$\int_{\mathbb{R}^n} \Phi(\langle \nabla f(x), u\rangle)^p dx = \int_0^\infty \left(\int_{\{|f|=t\}} \Phi(\langle \nabla f(x), u\rangle)^p d\mu(x)\right) dt$$

where, for almost every $t>0$, $d\mu(x)=\dfrac{dH_{n-1}(x)}{|\nabla f(x)|}$ being $dH_{n-1}(x)$ the corresponding Hausdorff measure on $\{|f|=t\}$. Next we use Jensen inequality

$$\int_{\mathbb{R}^n}\Phi(\langle\nabla f(x),u\rangle)^p dx \geq \int_0^\infty\left(\int_{\{|f|=t\}}\Phi(\langle\nabla f(x),u\rangle)\frac{d\mu(x)}{\int_{\{|f|=t\}}d\mu(x)}\right)^p\left(\int_{\{|f|=t\}}d\mu(x)\right)dt$$
$$=\int_0^\infty\left(\int_{\{|f|=t\}}\Phi(\langle\nabla f(x),u\rangle)d\mu(x)\right)^p\left(\int_{\{|f|=t\}}d\mu(x)\right)^{1-p}dt.$$

Denote $M(t)=\displaystyle\int_{\{|f|\geq t\}}dx$. For almost every $t>0$, another use of the co-area formula yields to

$$\int_{\{|f|=t\}}d\mu(x)=\left(-\int_t^\infty\left(\int_{\{|f|=s\}}\frac{dH_{n-1}(x)}{|\nabla f(x)|}\right)ds\right)'=-M'(t)=|M'(t)|$$

and so

$$\left(\int_{S^{n-1}}\left(\int_{\mathbb{R}^n}\Phi(\langle\nabla f(x),u\rangle)^p dx\right)^{-n/p}du\right)^{-p/n}$$
$$\geq\left(\int_{S^{n-1}}\left(\int_0^\infty\left(\int_{\{|f|=t\}}\Phi(\langle\nabla f(x),u\rangle)d\mu(x)\right)^p|M'(t)|^{1-p}dt\right)^{-n/p}du\right)^{-p/n}.$$

Use Minkowski's integral inequality to bound the previous formula from below

$$\geq\int_0^\infty\left(\int_{S^{n-1}}\left(\int_{\{|f|=t\}}\Phi(\langle\nabla f(x),u\rangle)d\mu(x)\right)^{-n}du\right)^{-p/n}|M'(t)|^{1-p}dt$$
$$=\int_0^\infty\left(\int_{S^{n-1}}\left(\int_{\{|f|=t\}}\Phi(\langle\nu(x),u\rangle)dH_{n-1}(x)\right)^{-n}du\right)^{-p/n}|M'(t)|^{1-p}dt$$

where for a.e. $t>0$, $\nu(x)$ is the outer normal vector to $\{|f|=t\}$ (w.r.t. $\{|f|\geq t\}$) at the point x.

For every "good" $t>0$ from Sard's theorem, one can easily see, [32], that the linear functional $g\in\mathcal{C}(S^{n-1})\to\int_{\{|f|=t\}}g(\nu(x))dH_{n-1}(x)$ can be represented by a finite measure $d\mu_t$ on S^{n-1}. That is

$$\int_{S^{n-1}}g(v)d\mu_t(v)=\int_{\{|f|=t\}}g(\nu(x))dH_{n-1}(x)\qquad\forall\, g\in\mathcal{C}(S^{n-1}).$$

Recall (see for instance [27]) that every convex body $K\subset\mathbb{R}^n$ determines a *surface area measure* on S^{n-1} denoted by S_K.

It is also proved in [32] that, by Minkowski existence theorem (see [27]), there exists a unique up to translations convex body K_t in $\mathbb{R}^n$ whose surface area measure S_{K_t} is μ_t. For this reason μ_t is also called the surface area measure of (the compact set) $\{|f|\geq t\}$.

Let ΠK_t be the projection body associated to K_t, i.e. the convex body defined by its support function as

$$h(\Pi K_t, u) = \frac{1}{2}\int_{S^{n-1}} |\langle v, u\rangle| dS_{K_t}(v) = |P_{u^\perp}(K_t)|_{n-1}, \qquad u \in S^{n-1}.$$

Let $\Pi^* K_t$ be the polar projection body of K_t. Its volume is

$$\begin{aligned} 2^{-n}|\Pi^* K_t|_n &= \int_{S^{n-1}} h(\Pi K_t, u)^{-n} du = \int_{S^{n-1}} \left(\int_{S^{n-1}} |\langle u, v\rangle| dS_{K_t}(v)\right)^{-n} du \\ &= \int_{S^{n-1}} \left(\int_{\{|f|=t\}} |\langle \nu(x), u\rangle| dH_{n-1}(x)\right)^{-n} du. \end{aligned}$$

Finally, Petty's projection inequality (5) and the fact proved in [32], $M(t) = |\{|f| \geq t\}|_n \leq |K_t|_n$, show that for almost all $t > 0$

$$\left(\int_{S^{n-1}} \left(\int_{\{|f|=t\}} |\langle \nu(x), u\rangle| dH_{n-1}(x)\right)^{-n} du\right)^{-1/n} \geq 2\frac{\omega_{n-1}}{\omega_n^{1-1/n}} M(t)^{\frac{(n-1)}{n}}.$$

Consider the case $\Phi(t) = |t|$. M and f^* are differentiable (except, possibly, on some set N of zero measure) and $M = (f^*)^{-1}$ on the set $(f^*)^{-1}\big((0,\infty) \setminus N\big)$, therefore

$$\begin{aligned} \left(\int_{S^{n-1}} \|D_u f\|_p^{-n} du\right)^{-p/n} &\geq 2^p \left(\frac{\omega_{n-1}}{\omega_n^{1-1/n}}\right)^p \int_0^\infty M(t)^{\frac{(n-1)p}{n}} |M'(t)|^{1-p} dt \\ &= \left(\frac{2\omega_{n-1}}{\omega_n^{1-1/n}}\right)^p \int_0^\infty s^{\frac{p(n-1)}{n}} |f^{*\prime}(s)|^p ds. \end{aligned}$$

In the other case, $\Phi(t) = \max\{t, 0\}$, since $\int_{S^{n-1}} \langle u, v\rangle d\mu_t(v) = 0$ we have

$$h(\Pi K_t, u) = \frac{1}{2}\int_{S^{n-1}} |\langle v, u\rangle| d\mu_t(v) = \int_{S^{n-1}} \langle v, u\rangle_+ d\mu_t(v).$$

Hence

$$\left(\int_{S^{n-1}} \|D_u^+ f\|_p^{-n} du\right)^{-p/n} \geq \left(\frac{\omega_{n-1}}{\omega_n^{1-1/n}}\right)^p \int_0^\infty s^{\frac{p(n-1)}{n}} |f^{*\prime}(s)|^p ds.$$

Finally we recall (11) and analogously

$$\left(\int_{S^{n-1}} \|D_u f^\circ\|_p^{-n} du\right)^{-p/n} = \left(n\omega_n^{1/n}\right)^p \left(\int_0^\infty s^{\frac{(n-1)p}{n}} |f^{*\prime}(s)|^p ds\right) I_p^p.$$

Therefore

$$\mathcal{E}_p(f^\circ) \leq \frac{n\omega_n}{2\omega_{n-1}} I_p \mathcal{E}_p(f) = \frac{I_p}{I_1}\mathcal{E}_p(f) \quad \text{and} \quad \mathcal{E}_p^+(f^\circ) \leq \frac{n\omega_n}{2\omega_{n-1}} I_p \mathcal{E}_p^+(f) = \frac{I_p}{I_1}\mathcal{E}_p^+(f).$$

□

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References

[1] T.Aubin, *Problemes isopérimétriques et espaces de Sobolev*, J. Differential. Geom. **11** (1976), pp 573–598.

[2] S. Bobkov, M. Ledoux, *From Brunn-Minkowski to sharp Sobolev inequalities*, Ann. Mat. Pura Appl.(4) **187** (2008), pp 369–384.

[3] J. Bastero, M. Milman, F.J. Ruiz, *A note on $L(\infty, q)$ spaces and Sobolev embeddings*, Indiana U. Math. J. **52** (2003), pp 1215–1229.

[4] H. Brézis & S. Wainger, *A note on limiting cases of Sobolev embeddings and convolution inequalities*, Comm. Partial Diff. Eq. **5** (1980), pp 773–789.

[5] I. Chavel, *Isoperimetric inequalities. Differential Geometric and Analytic Perspectives in Mathematical Physics*, Cambridge Tracts in Mathematics **145** Cambridge Univ. Press, 2001.

[6] A. Cianchi, E. Lutwak, D. Yang & G. Zhang, *Affine Moser-Trudinger and Morrey-Sobolev inequalities*, Calc. Var. Partial Differ. Equ. **36** (2009), pp 419–436.

[7] D. Cordero-Erausquin, B. Nazaret, C. Villani, *A mass-transportation approach to sharp Sobolev and Gagliardo-Nirenberg inequalities*, Adv. Math. **182** (2004), pp 307–332.

[8] H. Federer & W. H. Fleming, *Normal integral currents.* Ann. Math.**72** (1960), pp 458–520.

[9] C. Haberl & F.E. Schuster, *Asymmetric affine L_p Sobolev inequalities*, J. Funct. Anal **257** (2009), pp 641–658.

[10] C. Haberl & F.E. Schuster, *General L_p affine isoperimetric inequalties*, J. Differential Geom. **83** (2009), pp 1–26.

[11] C. Haberl, F.E. Schuster & J. Xiao, *An asymmetric affine Pólya-Szegö principle*, preprint.

[12] K. Hansson, *Imbedding theorems of Sobolev type in potential theory*, Math. Scand.**45** (1979), pp 77–102.

[13] S. Kesavan, *Symmetrization and Applications. Series in Analysis vol. 3.* World Sc. Publ. (2006).

[14] V. Kolyada, *Rearrangements of functions and embedding theorems*, Russ. Math. Surveys Eq. **44** (1989), pp 73–117.

[15] E. Lutwak, *The Brunn-Minkowski-Firey Theory I: Mixed volumes and the Minkowski problem.* J. Differential Geom. **38** (1993), pp 131–150.

[16] E. Lutwak, D. Yang & G. Zhang, *On Lp affine isoperimetric inequalities*, J. Differential Geom. **56** (2000), pp 111–132.

[17] E. Lutwak, D. Yang & G. Zhang, *Sharp affine L_p Sobolev inequalities*, J. Differential Geom. **62** (2002), pp 17–38.

[18] W. G. Maz'ya, *Classes of domains and embedding theorems for functional spaces*, Soviet Math. Dokl. **1** (1961), pp 882–885.

[19] W. G. Maz'ya, *Sobolev spaces*, Springer Series in Soviet Mathematics **145** Springer-Verlag, Berlin, 1985.

[20] J. Malý, L. Pick, *An elementary proof of sharp Sobolev embeddings*, Proc. Amer. Math. Soc., **13** (2002), pp 555–563.

[21] J. Martin, M. Milman, *Pointwise symmetrization inequalities for Sobolev functions and applications* , Adv. Math., **225** (2010), pp 121–199.

[22] J. Martin, M. Milman, E. Pustylnik, *Sobolev inequalities symmetrization and self improvement via truncation*, J. Funct. Anal. **252** (2), (2007), pp 677–695.

[23] J. Moser, *A sharp form of an inequality by Trudinger*, Indiana Univ. Math. J. **20** (1971), pp 1077- 1092.

[24] J. Mossino, *Inégalités Isopérimétriques et Applications en Physique*, Travaux en Cours, Hermann, Paris, 1984.

[25] C. M. Petty, *Centroid surfaces*, Pacific J. math. **11** (1961), pp 1535–1547.

[26] G. Pólya & G. Szegö , *Isoperimetric inequalities in Mathematical Physics*, Ann. Math. Stu. **27** Princeton University Press, Princeton 1951.

[27] R. Schneider, *Convex Bodies: The Brunn-Minkowski Theory*, Cambridge University Press, Cambridge, 1993.

[28] G. Talenti, *Best constants in Sobolev inequality*, Ann. Math. Pura Appl. **110** (1976), pp 353–372.

[29] G. Talenti, *Inequalities in rearrangement-invariant function spaces*, Non Linear Analysis, Function spaces ans Applications, Prometheus, Prague **5**, (1995), pp 177–230.

[30] L. Tartar, *Imbedding theorems of Sobolev spaces into Lorentz spaces*, Boll. Unione Mat. Ital. Sez B Artic. Ric. Mat. **8**, 1 (1998), 479–500.
[31] N.S. Trudinger, *On imbeddings into Orlicz spaces and some applications*, J. Math. Mech. **17** (1967), 473-483.
[32] G. Zhang, *The affine Sobolev inequality*, J. Differential Geom. **53** (1999), pp 183–202.

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