APPLICATIONS OF DEVIATION INEQUALITIES ON FINITE METRIC SETS

by

Jesús Bastero and Julio Bernués*

Departamento de Matemáticas
Facultad de Ciencias
50009 Zaragoza (Spain)

The problem of $(1+\epsilon)$-embedding the ℓ_∞^n-cube into finite dimensional normed spaces can be stated as follows: "Given X a normed space, $\dim X = n$ and $0 < \epsilon < 1$, estimate the highest cardinality $N(n,\epsilon) = N$ of the sets $T \subset X$ such that

$$1-\epsilon \le ||x-y|| \le 1+\epsilon \qquad\qquad \forall \;\; x \ne y \in T \quad " \tag{1}$$

The first step in this direction is a result by Johnson and Lindenstrauss (see Lemma 1 in [J-L]), which states that (1) holds for $X = \ell_2^n$, provided $n > C\epsilon^{-2}\log N$ (C numerical constant). Also, the same statement can be easily deduced from Theorem 1 in [J-S 1] for $X = \ell_p^n$, $1 \le p < 2$ and $n > C(\epsilon,p)\log N$.

In a recent paper by Bastero, Bernués and Kalton [B-B-K] the following result is proven: "There exists a numerical constant $C > 0$ such that (1) holds for every 1-subsymmetric n-dimensional normed space X, provided $n > C\epsilon^{-2}\ \log N$ ". Also, it is shown that estimates of the type $n \ge C(\epsilon)\log N$ are asymptotically in n and N the best possible.

It is worth noting that one actually achieves an embedding of the ℓ_∞^n -cube into the ℓ_p^N-cube, with $N \simeq C\epsilon^{-2}n$. This result is an improvement on the one given by [B-M-W] where an embedding of order $N \simeq C(\epsilon,p)n^3$ is obtained.

Unfortunately the method developed in that paper cannot be extended to a wider class of spaces (even to 2-symmetric spaces).

We will present here some extensions to the result in [B-B-K]. The key will be deviation inequalities by M.Talagrand, W.Johnson and Schechtman ([T], Theorem 3 and [J-S 2], Corollary 4) and by V.Milman and G.Schechtman ([M-S], Ch.7).

We will obtain good estimates for (1) for the 1-unconditional space $\ell_p^n(\ell_q^m)$, $1 \le p,q < \infty$ the study of which was suggested to us by N. Kalton as the first step beyond the 1-subsymmetric case, for cotype-2 spaces and for some K-symmetric spaces. In the cases considered the method gives the correct relation between n and N i.e. $n \ge C(\epsilon)\log N$, but in the ℓ_p^n case it produces a dependence on ϵ worse than $C\epsilon^{-2}$. This can be solved, in a different way than in [B-B-K], by using a sharp deviation inequality (6) for the convex function $||.||_p$.

Notation. For $i = 1,...,n$ let $(X_i, ||.||_i)$ be normed spaces, let Ω_i be a finite subset of X_i and let $I\!P_i$ be any probability measure on Ω_i. Define $(\Omega, \mathrm{d_p}, I\!P)$ as $\Omega = \prod_1^n \Omega_i$, d_p the distance induced on Ω by the norm in $X = \left(\sum_1^n \bigoplus X_i\right)_p$, $1 \le p \le \infty$ and $I\!P = \prod_1^n I\!P_i$ the product probability. x, x' will denote elements in X and η, η' elements in Ω. In case $(X_i, ||.||_i) = (I\!R, |.|)$, we will denote, as usual, $(X, \mathrm{d_p}) = (\ell_p^n, ||.||_p)$.

Given a function $f : \Omega \to I\!R$, we denote by M_f its median, by $\sigma_p(f) = \sigma_p$ its Lipschitz constant $\sigma_p = \sup\limits_{\eta\ne\eta'} \dfrac{|f(\eta)-f(\eta')|}{\mathrm{d}_p(\eta,\eta')}$ and by $\omega_f^p(\delta)$ its modulus of continuity $\omega_f^p(\delta) = \sup\limits_{\mathrm{d}_p(\eta,\eta')\le\delta} |f(\eta)-f(\eta')|$.

For an n-dimensional normed space E with K-symmetric basis $\{e_1,...,e_n\}$, i.e. $||\sum_1^n a_ie_i|| \le K\, ||\sum_1^n \epsilon_i a_i e_{\sigma(i)}||$ for every $\epsilon_i = \pm 1$, σ permutation of $\{1 \dots n\}$, denote $\lambda(n) = ||\sum_{i=1}^n e_i||$.

For E a normed space of dimension n and $1 \le p \le q \le \infty$, $\gamma_E(p,q) = \gamma_E = \inf\{||T||\,||S||$ such that $l_p^n \xrightarrow{T} X \xrightarrow{S} l_q^n$, $S \circ T = Id.\}$

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M will denote a convex Orlicz function satisfying Δ_2 condition at zero. The Simonenko coeficients for the function M, p_M, q_M are defined by $p_M = \inf_{t>0} \frac{tM'(t)}{M(t)}$, $q_M = \sup_{t>0} \frac{tM'(t)}{M(t)}$ where $M'(t)$ is the right hand side derivative of M (see [MA]).

The letter C will represent a numerical constant and different ocurrences of this letter may represent different values.

We are interested in bounding the deviation of f from its median, i.e. find upper bounds for $I\!\!P\{|f - M_f| > t\}$, $t > 0$.

The case $\Omega = \{0,1\}^n$ with probability $I\!\!P = \prod_1^n \left[\frac{\delta_0 + \delta_1}{2}\right]^n$ has been widely studied (δ_0 and δ_1 are Dirac measures). It is well known that for the Hamming distance ($p = 1$),

$$I\!\!P\{|f - M_f| > t\} \leq \exp -\frac{2t^2}{n\sigma_1^2} \tag{2}$$

(see [A-M], pg. 6).

The following two results are extensions to inequality (2) above.

Theorem 1 ([M-S]). *Consider* $(\Omega, \mathrm{d}_1, I\!\!P)$ *with* $I\!\!P_i$ *the counting probability on* Ω_i. *Let* $f : \Omega \to I\!\!R$ *be a function with median* M_f. *Then for every* $t > 0$,

$$I\!\!P\{|f - M_f| > t\} \leq 4\exp -\frac{t^2}{16\sigma_1^2\ell^2}$$

where ℓ^2 *is the length of* Ω *(see [M-S], Ch.7 for definition).*

Theorem 2 ([T] and [J-S 2]). *Let* $(\Omega, \mathrm{d}_p, I\!\!P)$ *with diameter* $\mathrm{diam}\ \Omega_i \leq 1, \quad i = 1, ..., n$. *Let* $f : \Omega \to I\!\!R$ *be a function with median* M_f *and define* $\overline{\sigma}_p$ *the infimun of the Lipschitz constants of all convex extension of f to* conv Ω . *Then for* $2 \leq p < \infty$

$$I\!\!P\{|f - M_f| > t\} \leq 4\ \exp -\frac{t^p}{4\overline{\sigma}_p^p}$$

For $1 \leq p < 2$ since $||.||_p \leq n^{1/p-1/2}||.||_2$ implies that $\overline{\sigma}_2 \leq \overline{\sigma}_p n^{1/p-1/2}$ we obtain

$$I\!\!P\{|f - M_f| > t\} \leq 4\ \exp -\frac{t^2}{4n^{2/p-1}\overline{\sigma}_p^2}$$

In the case $p = 1$ and $\Omega = \{0,1\}^n$, the estimates given by this last result and (2) are roughly the same. This may be also deduced from the fact that for every $A \subseteq \Omega$ and for every $\eta \in \Omega \setminus A$, $d(\eta, A) = d(\eta, \mathrm{conv}A)$.

(1) Following [M-S] and [T] it is not difficult to prove the following two statements:

(i) Let $(\Omega, \mathrm{d}_1, I\!\!P)$, with $I\!\!P_i$ *any* probability on Ω_i. Define

$$a_i = \sup_{\eta_i \in \Omega_i} \mathrm{Ave}_{\eta'_i \in \Omega_i} ||\eta_i - \eta'_i||_i$$

Write $A = (\sum a_i^2)^{1/2}$. Then for every function $f : \Omega \to I\!\!R$ and any $\delta > 0$

$$I\!\!P\{|f - M_f| > \omega_f^1(\delta)\} \leq 4\ \exp -\frac{\delta^2}{16A^2} \tag{3}$$

(ii) Let $(\Omega, \mathrm{d}_p, I\!\!P), 1 \leq p \leq \infty$. For every function $f : \Omega \to I\!\!R$ with median M_f, let $\overline{f}$ any convex extension of f to convΩ. Then for every $\delta > 0$,

$$I\!\!P\{|f - M_f| > \omega_{\overline{f}}^p(\delta)\} \leq 4\ \exp -\frac{\delta^p}{4(\max \mathrm{diam}\Omega_i)^p} \qquad 2 \leq p < \infty \tag{4}$$

and

$$\mathbb{P}\{|f-M_f|>\omega_f^p(\delta)\}\le 4\ \exp-\frac{\delta^2}{4n^{2/p-1}(\max\mathrm{diam}\Omega_i)^2}\qquad 1\le p\le 2 \tag{5}$$

(2) As said above, for the function $f=||.||_q$, $1\le q<\infty$, one can find somehow sharper estimates. More exactly,
Let $\Omega_i=\{0,1\}$ and $\mathbb{P}_i=\dfrac{\delta_0+\delta_1}{2}$. For every $t>0$ and for $f(\eta)=||\eta||_q$, $\eta\in\Omega$:

$$\mathbb{P}\{|f-M_f|>t\}\le \exp-\frac{t^2}{8n^{2/q-1}} \tag{6}$$

Indeed,
Note that for $f(\eta)=||\eta||_q$, $M_f=\left(\frac{n}{2}\right)^{1/q}$. Write $\eta=(\eta_i)_{i=1}^n\in\Omega$.

$$\mathbb{P}_n\{|f-M_f|>t\}=\mathbb{P}_n\left\{\left|\left(\frac{\sum_{k=1}^n\eta_i}{n}\right)^{1/q}-\left(\frac{1}{2}\right)^{1/q}\right|>\frac{t}{n^{1/q}}\right\}$$

This last expression suggests the use of the mean value theorem

$$\left|\left(\frac{\sum_{k=1}^n\eta_i}{n}\right)^{1/q}-\left(\frac{1}{2}\right)^{1/q}\right|=\left|\frac{\sum_{k=1}^n\eta_i}{n}-\frac{1}{2}\right|\frac{1}{q}\xi^{-1/q'}$$

$(\frac{1}{q}+\frac{1}{q'}=1)$ for some ξ in the open interval limited by $\frac{1}{2}$ and $\frac{\sum_{k=1}^n\eta_i}{n}$. If $t<\frac{1}{q}\left(\frac{n}{4}\right)^{1/q}$, let $s=\frac{1}{4}tq\left(\frac{4}{n}\right)^{1/q}<\frac{1}{4}$. Then $\left|\frac{\sum_{k=1}^n\eta_i}{n}-\frac{1}{2}\right|\le s$ implies that

$$\begin{aligned}\left|\left(\frac{\sum_{k=1}^n\eta_i}{n}\right)^{1/q}-\left(\frac{1}{2}\right)^{1/q}\right|&\le\frac{s}{q(\frac{1}{2}-s)^{1/q'}}\\&\le\frac{s4^{1/q'}}{q}=\frac{t}{n^{1/q}}\end{aligned}$$

and hence

$$\begin{aligned}\mathbb{P}_n\{|f-M_f|>t\}&\le\mathbb{P}_n\left\{\left|\frac{\sum_{k=1}^n\eta_i}{n}-\frac{1}{2}\right|>\frac{1}{4}tq\left(\frac{4}{n}\right)^{1/q}\right\}\\&\le\text{by (2)}\le\exp-\frac{2t^2q^2}{4^{2/q'}n^{2/q-1}}\\&\le\exp-\frac{t^2}{8n^{2/q-1}}\end{aligned}$$

If $\frac{1}{q}\left(\frac{n}{4}\right)^{1/q}\le t\le\left(\frac{n}{2}\right)^{1/q}$, as before $\left|\frac{\sum_{k=1}^n\eta_i}{n}-\frac{1}{2}\right|\le\frac{1}{4}$ implies that

$$\left|\left(\frac{\sum_{k=1}^n\eta_i}{n}\right)^{1/q}-\left(\frac{1}{2}\right)^{1/q}\right|\le\frac{1}{q4^{1/q}}$$

Thus

$$\begin{aligned}\mathbb{P}_n\{|f-M_f|>t\}&\le\mathbb{P}_n\left\{|f-M_f|>\frac{1}{q}\left(\frac{n}{4}\right)^{1/q}\right\}\le\mathbb{P}_n\left\{\left|\frac{\sum_{k=1}^n\eta_i}{n}-\frac{1}{2}\right|>\frac{1}{4}\right\}\\&\le\exp-\frac{n}{8}\le\exp-\frac{t^2}{2\ 4^{1/q'}n^{2/q-1}}\le\exp-\frac{t^2}{8n^{2/q-1}}\end{aligned}$$

Eventually we realize that for $t > \left(\frac{n}{2}\right)^{1/q}$, $\{|f - M_f| > t\} = \emptyset$.

Comparing these estimates with the ones given by (4) and (5) one gets that both bounds are roughly the same for $1 \leq q \leq 2$, and for the natural convex extension of f to conv Ω i.e $\overline{f}(x) = ||x||_q$. But the same convex extension $||x||_q$ for $q > 2$ plus formula (5) leads us to a worse estimate than the one given by (6). We do not know whether there exist a convex extension of f which, by applying Theorem 2, yields to our estimate.

In the sequel applications we will use the most convenient metric in Ω so that we obtain better estimates for the event $\{|f - M_f| > \omega_f^p(\delta)\}$.

We remark that, in general, inequality (3) would be sufficient in order to achieve the right estimates for n in the problem we are considering, but it would lead to worse ones for the dependence on ϵ.

Theorem 3. *There exists a numerical constant $C > 0$, such that for every $0 < \epsilon < 1$, property (1) holds for:*

(i) Any n-dimensional K-symmetric space E such that $\psi(\delta) = \sup_{n \in I\!N} \frac{\lambda([\delta n])}{\lambda(n)} \to 0$ as $\delta \to 0$, provided

$$n > C(\epsilon) \log N$$

(ii) Any n-dimensional normed space E K-isomorphic to ℓ_M^n provided

$$n > C \left(\frac{CK^2}{\epsilon}\right)^{q_M \max\{1, \frac{2}{p_M}\}} \log N$$

(iii) Any normed space K-isomorphic to $\ell_p^n(\ell_q^m)$, $1 \leq p, q < \infty$ $n, m \in I\!N$, provided

$$nm > C \left(\frac{2K}{\epsilon}\right)^{\max\{2,p,q\}} \log N$$

(iv) Any n-dimensional normed space E, provided

$$n^{1+\max\{2,p\}(1/q-1/p)} > C \left(\frac{2\gamma_E}{\epsilon}\right)^{\max\{2,p\}} \log N$$

Proof:

(i) Let $X(\omega) = \sum_{i=1}^n \epsilon_i(\omega) x_i$ an E-valued random variable where $I\!P\{\epsilon_i = +1\} = I\!P\{\epsilon_i = -1\} = \frac{1}{2}$, $i = 1, ..., n$. Let $Y(\omega) = \sum_{i=1}^n \epsilon_i'(\omega) x_i$ an independent copy of X. We are interested in the distribution of $||X - Y||$.

$$\frac{||X - Y||}{2} = \frac{||\sum_{i=1}^n (\epsilon_i(\omega) - \epsilon_i'(\omega)) x_i||}{2}$$
$$\stackrel{\mathrm{d}}{=} f(\eta)$$

where $\eta \in \Omega = \{0, 1, -1\}^n \subset I\!R^n$, $I\!P_i\{\eta_i = 0\} = \frac{1}{2}$ and $I\!P_i\{\eta_i = 1\} = I\!P_i\{\eta_i = -1\} = \frac{1}{4}$ and $f : \Omega \to I\!R$, defined by $f(\eta) = ||\sum_{i=1}^n \eta_i x_i||$.

In this case, consider $\{0, 1, -1\}^n \subset l_1^n$. It is straightforward to show that $A \leq n^{1/2}$. Now using the fact that there is an 1-symmetric norm $||.||_0$ in E such that $||x|| \leq ||x||_0 \leq K||x||$ $\forall x \in X$ it is easy to see: a) $M_f \geq CK^{-1}\lambda(n)$, and b) $\omega_f^1(\delta) \leq CK\lambda([\delta])$, $\delta > 0$.

Indeed, write $\lambda_0(n) = ||\sum_{i=1}^n x_i||_0$. a) Since $f(\eta) \geq K^{-1}||\sum_{i=1}^n |\eta_i| x_i||_0$, then $M_f \geq K^{-1}\lambda_0([n/2]) \geq CK^{-1}\lambda(n)$.

b) Let $\xi_i = \eta_i - \eta_i'$ ($\xi_i = 0, \pm 1, \pm 2$) $A = \{i : |\xi_i| = 2\}$ and $B = \{i : |\xi_i| = 1\}$. Note that $||\eta - \eta'||_1 = 2|A| + |B|$ ($|.|$ denotes the cardinality of a subset). Now

$$|f(\eta) - f(\eta')| \leq ||\sum_{i=1}^n |\eta_i - \eta_i'| x_i||_0 \leq 2\lambda_0(|A| + |B|)$$
$$\leq CK\lambda(||\eta - \eta'||_1)$$

and so, $\omega_f^1(\delta) \le CK\lambda([\delta])$.

Given $\epsilon > 0$, let $0 < \delta < 1$ s.t. $\psi(\delta) \le \dfrac{\epsilon}{CK^2}$. Then

$$\frac{\omega_f^1(\delta n)}{M_f} \le CK^2 \frac{\lambda([\delta n])}{\lambda(n)} \le CK^2 \psi(\delta) < \epsilon$$

So $\epsilon M_f \ge \omega_f^1(\delta n)$ and

$$\begin{aligned} I\!\!P_n\{|f - M_f| > \epsilon M_f\} &\le I\!\!P_n\{|f - M_f| > \omega_f^1(\delta n)\} \\ &\le (\text{by } (3)) \le 4 \ \exp -Cn\delta^2. \end{aligned}$$

Now take N independent copies (N to be determined) of X, $X_1, \ldots, X_N$. It follows from the computations above:

$$\begin{aligned} I\!\!P_n\{| \ ||X_i - X_j|| - 2M_f| > \epsilon 2M_f, \forall\ 1 \le i \ne j \le N\} \\ \le \binom{N}{2} 4 \ \exp -Cn\delta^2 \end{aligned}$$

So if the last number is strictly smaller than 1 we can assure the existence of some ω in the probability space which verifies

$$2M_f(1-\epsilon) \le ||X_i(\omega) - X_j(\omega)|| \le 2M_f(1+\epsilon)$$

whenever $1 \le i \ne j \le N$. Thus the result (i) follows inmediately.

(ii) There is no loss of generality if we suppose $K = 1$. The only modifications which we would need are similar to the ones which appear in the previous case and so, they only affect the numerical constant.

We need only consider the set $\Omega = \{0,1\}^n$ with $I\!\!P = \prod_1^n \left[\dfrac{\delta_0 + \delta_1}{2}\right]^n$. Metricize the set $\{0,1\}^n$ with the $\ell_{p_M}^n$-norm and let $f(\eta) = ||\eta||_{\ell_M^n}$ for $\eta \in \Omega$. Denote $\overline{f}$ the natural extension of f to conv Ω, $\overline{f}(x) = ||x||_{\ell_M^n}$. Note that $\lambda(n) = \frac{1}{M^{-1}\left(\frac{1}{n}\right)}$. Then we can write, as before, $M_f \ge \frac{C}{M^{-1}\left(\frac{1}{n}\right)}$. Also $|f(x) - f(x')| \le ||x - x'||_{\ell_M^n}$ and use the well known fact that $\forall\ \ 0 \le \alpha \le 1$

$$\alpha^{q_M} M(t) \le M(\alpha t) \le \alpha^{p_M} M(t)$$

to obtain $\epsilon M_f \ge \omega_{\overline{f}}^{p_M}(\delta)$. Indeed, since $x, x' \in$ conv Ω, we have that

$$1 = \sum_{i=1}^n M\left(\frac{|x_i - x_i'|}{||x - x'||_{\ell_M^n}}\right) \le \sum_{i=1}^n |x_i - x_i'|^{p_M} M\left(\frac{1}{||x - x'||_{\ell_M^n}}\right).$$

Thus,

$$\omega_{\overline{f}}^{p_M}(\delta) \le \frac{1}{M^{-1}\left(\frac{1}{\delta^{p_M}}\right)}.$$

Now for every $0 < \epsilon < 1$, let $\delta > 0$ such that $\delta^{p_M} = (C\epsilon)^{q_M} n$.

$$M\left[C\epsilon M^{-1}\left(\frac{1}{(C\epsilon)^{q_M} n}\right)\right] \ge \frac{1}{n}$$

If $p_M \ge 2$

$$\begin{aligned} I\!\!P_n\{|f - M_f| > \epsilon M_f\} &\le I\!\!P_n\{|f - M_f| > \omega_{\overline{f}}^{p_M}(\delta)\} \\ &\le (\text{by } (4)) \le 4 \ \exp -Cn(C\epsilon)^{q_M}. \end{aligned}$$

Eventually we conclude as in (i).

In the case $p_M \leq 2$ we use the corresponding deviation inequality (5). (We will obtain a better dependence on ϵ when $q_M \leq 2$, by using that ℓ_M^n is cotype 2 (see Corollary 4 below)).

(iii) Suppose that $K = 1$. As before, since our space is 1-unconditional, we need only study the set $\Omega = \{0,1\}^{nm}$ with $I\!\!P = \prod_1^n \left[\frac{\delta_0 + \delta_1}{2}\right]^{nm}$. Metricize the set $\{0,1\}^{nm}$ with the $\ell^{nm}_{\max\{p,q\}}$-norm and let $f(\eta) = ||\eta||_{\ell_p^n(\ell_q^m)}$ for $\eta \in \Omega$. Denote $\overline{f}$ the natural extension of f to convΩ, $\overline{f}(x) = ||x||_{\ell_p^n(\ell_q^m)}$. Since

$$\frac{||\eta||_{\ell_p^n(\ell_q^m)}}{n^{1/p}m^{1/q}} \geq \frac{||\eta||_1}{nm}$$

then

$$I\!\!P\left\{\frac{||\eta||_{\ell_p^n(\ell_q^m)}}{n^{1/p}m^{1/q}} \leq \frac{1}{3}\right\} < \frac{1}{2}$$

and so $M_f \geq \dfrac{n^{1/p}m^{1/q}}{3}$.

Using the inequality $||x||_r \leq n^{1/r-1/s}||x||_s$ for every $1 \leq r \leq s \leq \infty$, we get $\omega_{\overline{f}}^{\max\{p,q\}}(\delta) \leq m^{1/q-1/p}\delta$, if $q \leq p$ and $\omega_{\overline{f}}^{\max\{p,q\}}(\delta) \leq n^{1/p-1/q}\delta$, otherwise. Indeed, call $I_i = (|x_{i1} - x'_{i1}|, ..., |x_{im} - x'_{im}|)$, $1 \leq i \leq n$.

If $q \leq p$,

$$|\overline{f}(x) - \overline{f}(x')| = |\ ||x|| - ||x'||\ | \leq ||x - x'|| = \left(\sum_{i=1}^n ||I_i||_q^p\right)^{1/p}$$

$$\leq \left(\sum_{i=1}^n \left(||I_i||_p m^{1/q-1/p}\right)^p\right)^{1/p} = m^{1/q-1/p}||x - x'||_p$$

The case $p \leq q$ is analagous.

Now apply (4) for $\max\{p,q\} \geq 2$ and (5) for $\max\{p,q\} \leq 2$ and conclude as in (i).

(iv) By hypothesis there are vectors $x_1, ..., x_n \in E$ such that

$$\left(\sum_{i=1}^n |a_i|^q\right)^{1/q} \leq ||\sum_{i=1}^n a_i x_i|| \leq \gamma_E(p,q)\left(\sum_{i=1}^n |a_i|^p\right)^{1/p} \tag{7}$$

Let $\Omega = \{0,1,-1\}^n$, diam $\Omega = 2$, $I\!\!P\{\eta_i = 0\} = \frac{1}{2}$, $I\!\!P\{\eta_i = 1\} = I\!\!P\{\eta_i = -1\} = \frac{1}{4}$, $f : \Omega \to I\!\!R$ defined by $f(\eta) = ||\sum_{i=1}^n \eta_i x_i||$ and $\overline{f}$ its natural convex extension to conv Ω. Write $r = \max\{2,p\}$ and consider $\Omega \subset \ell_r^n$. Using (7) it is easy to see that $M_f \geq \left(\frac{n}{2}\right)^{1/q}$ and $\omega_{\overline{f}}^r(\delta) \leq \gamma_E n^{1/p-1/r}\delta$.

$$\begin{aligned} I\!\!P\{|\ ||X - Y|| - 2M_f| > \epsilon 2M_f\} &= I\!\!P\{|f(\eta) - M_f| > \epsilon M_f\} \\ &\leq I\!\!P\{|f(\eta) - M_f| > \epsilon\left(\frac{n}{2}\right)^{1/q}\} \\ &\leq \text{by (4) and (5)} \leq 4\exp -\left[C\left(\frac{\epsilon}{2\gamma_E}\right)^r n^{1+r(1/q-1/p)}\right] \end{aligned}$$

and conclude as in (i).

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- Spaces verifying the condition in (i) are for instance the Lorentz sequence spaces $E = \ell^n_{d(\omega,p)}$, when $1 \leq p < \infty$ and $\omega = (n^{-r})_n$, $0 < r < 1$, etc..

- Result (ii) can easily be extended to the space $\ell_{p_1}^{n_1}(\ell_{p_2}^{n_2}(\ldots(\ell_{p_k}^{n_k}))\ldots)$, $1 \leq p_1, \ldots, p_k < \infty$.

- Result (iii) provides an estimate of $N = N(n,\epsilon)$ for general normed spaces E which can be improved (as shown below) by adding extra conditions to the space, namely cotype conditions etc....

Corollary 4. *There exists a numerical constant $C > 0$, such that for every $0 < \epsilon < 1$, property (1) holds for:*

(1) Any n-dimensional cotype q normed space E with cotype q constant C_q provided

$$n^{2/q} > \frac{C\ C_q^2}{\epsilon^2} \log N$$

(2) Any n-dimensional weak cotype 2 normed space with weak cotype 2 constant wC_2 provided

$$n > \frac{C\ wC_2^2(1 + \log CwC_2)^2}{\epsilon^2} \log N$$

Proof: (2) Apply the result by [F-L-M] to find a subspace $Y \subseteq X$ of dimension $k = \left[\frac{cn^{2/q}}{C_2^q}\right]$ such that $Y \overset{2}{\cong} \ell_2^k$ and use (i) with $p = q$.

(3) Apply the results in [M-P] and conclude as before.

- The case $X = \ell_p,\ 1 \leq p < \infty$, can be deduced either from Theorem 3 (ii) or Corollary 4(1). The former gives us a better dependence on ϵ; more precisely, $C(\epsilon) = C\epsilon^{-\max\{2,p\}}$. For $2 \leq p < \infty$ a sharper estimate, namely $C\epsilon^{-2}$ can be obtained by using inequality (6).

- Corollary 4.(2) is actually a consecuence of the theorems in [F-L-M] and [J-L] quoted above, because $\ell_2^{c(\epsilon)n^{2/q}} \overset{1+\epsilon}{\hookrightarrow} X$. But a direct application of these two results leads us to an inferior estimate for the dependence on ϵ (namely $C(\epsilon) = C\epsilon^{-4}$).

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Jesús Bastero and Julio Bernués
Departamento of Matematicas
Facultad de Ciencias
Universidad de Zaragoza
50009 Zaragoza (Spain)