UPPER BOUNDS FOR THE VOLUME AND DIAMETER OF m-DIMENSIONAL SECTIONS OF CONVEX BODIES

JESÚS BASTERO

Abstract. In this paper some upper bounds for the volume and diameter of central sections of symmetric convex bodies are obtained in terms of the isotropy constant of the polar body.

1. Introduction, Notation and Results

We consider $\mathbb{R}^n$ endowed with the canonical euclidian structure. K will be a convex body in $\mathbb{R}^n$. We shall say that K is in *isotropic position* if its centroid is the origin and there is a constant $\alpha > 0$ such that

$$\int_K \langle x, \theta \rangle^2 dx = \alpha$$

for all $\theta \in S^{n-1}$. Every convex body in $\mathbb{R}^n$ has a unique, up to orthogonal transformation, isotropic position (by a position of a convex body we mean here any affine transformation of the set, with determinant equal to 1). We recall the definition of the *isotropy constant* L_K of a convex body K

$$nL_K^2 |K|^{2/n} = \inf_{\substack{S \in SL(n) \\ t \in \mathbb{R}^n}} \frac{1}{|K|} \int_K |t + Sx|^2 \, dx, \tag{1.1}$$

so, if K is in in isotropic position and $|K| = 1$,

$$L_K^2 = \int_K \langle x, \theta \rangle^2 dx$$

for all $\theta \in S^{n-1}$.

In the sequel we will assume that $|K| = 1$. It is a major problem in asymptotic convex geometry to prove that the isotropy constant of any convex body K in $\mathbb{R}^n$, L_K, is bounded from above for an absolute constant (independent even on the dimension). It is well known that this fact is equivalent to the *slicing problem*: there exists an absolute constant $c > 0$ with the following property, if K is a convex body in $\mathbb{R}^n$ with there exists a hyperplane H such that $|K \cap H|_{n-1} > c$. The best known bound for the isotropy

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constant was given by Bourgain $L_K \leq Cn^{1/4}\log n$ (see [Bo1], [Bo2] and also [D] and [Pa] for the non symmetric case).

Let assume that the convex body K of volume one has its centroid in the origin. The *circumradius* $R(K)$ is the quantity

$$R(K) = \max\{|x|; x \in K\}.$$

A simple computation shows that $C\sqrt{n} \leq R(K)$. Moreover if K is isotropic $R(K) \leq (n+1)L_K$ (cf. [KLS]) and if instead of that, K° is in isotropic position then

$$R(K) \leq \frac{Cn}{L_{K^\circ}} \leq Cn$$

where C is an absolute constant (see Remark 3 bellow).

In this paper some upper bounds for the volume and circumradius of sections of symmetric convex bodies with volume one are obtained in terms of the isotropy constant of the polar body.

The proof we use in proving the part *ii)* of the Theorem 1.1 is based in the methods appearing in the paper [Bo2] in which Bourgain obtained some sharp estimates for isotropic ψ_2-convex bodies. Our result is equivalent to that appearing in [Bo2] (lemma in page 115), at least for high dimensional sections. However, the approach we use here presents some differences. We use the M-ellipsoid associated to every convex body and sharp estimates on covering numbers given by Pisier (see [Pi]) instead of using the ℓ-ellipsoid and Talagrand majoration theorem.

The main result we achieve is the following:

Theorem 1.1. *Let K be a centrally symmetric convex body in $\mathbb{R}^n$ with volume $|K| = 1$ such that K° is in isotropic position. Let E be any m-dimensional subspace, $1 \leq m \leq n$. Then*

i)

$$|K \cap E|_m^{1/m} \leq \frac{Cn\log(1+m)}{m^{3/4}L_{K^\circ}} \tag{1.2}$$

ii) *if $0 < \delta < 1/2$ there exists another subspace $F \subset E$, $(1-2\delta)m \leq dimF \leq (1-\delta)m$ such that*

$$R(K \cap F) \leq \frac{C}{\delta^3}\frac{\log(m+1)n^{5/4}}{m^{1/2}L_{K^\circ}}$$

for some absolute constant $C > 0$ (independent of K and of the dimensions m, n).

Remarks 1. The bound (1.2) is sharp at least for the high dimensions, since it gives the better known bound for the isotropy constant. In fact, it is clear for $m = n$ simply by taking $E = \mathbb{R}^n$. A similar situation happens for $m = n(1 - f(n))$ whenever $f(n) = O(1/\log n)$. Indeed, as a consequence of Hensley inequalities (cf. [H]) or Ball result (cf. [B2], lemma 6) we know that

$$L_K|E \cap K|_m^{1/(n-m)} \geq C$$

whenever that K is a symmetric convex body in $\mathbb{R}^n$ of $|K|_n = 1$ in isotropic position, E is any m-dimensional subspace of $\mathbb{R}`n$ and $C > 0$ is a numerical constant. Then if the convex body K is not assumed to be in isotropic position the inequality before is true at least for one E, m-dimensional subspace and so, taking into account (1.2) we have

$$\left(\frac{C_1}{L_K}\right)^{(n-m)/m} \leq \frac{C_2 n \log(1+m)}{m^{3/4} L_{K^\circ}},$$

and by using the rough estimates for the isotropy constant (for instance, $L_K \leq C\sqrt{n}$), we would have

$$\left(\frac{C_1}{\sqrt{n}}\right)^{(n-m)/m} \leq \frac{C_2 n \log(1+m)}{m^{3/4} L_{K^\circ}}.$$

When $m = n(1 - f(n))$ the inequality gives the better known estimate for the isotropy constant.

2. Apart from random estimates obtained recently by different authors, the most recent result in this line is that by Bourgain, Klartag and Milman (see [BKM]) who proved that if K_n is an isotropic convex body of volume one such that, L_{K_n} is the worst possible constant in dimension n, then for any subspace F of dimension $n-m$, $|K_n \cap F|_{n-m}^{1/m} \leq C$, for some absolute constant C. In our situation, if K is a symmetric convex body such that such that $L_{K^0} = L_{K_n}$ then by using that $L_m \leq C L_n$ we have that $|K \cap E|_m^{1/m} \leq C\frac{n}{m}$.

3. It is clear that no upper bounds for the circumradius of a general symmetric convex body can exist. However if K° is in isotropic position, then $R(K) \leq Cn$ and the expression (1.2) for $m = 1$ gives this estimate. We can also give a shoter proof based in the reverse Blaschke-Santaló's inequality (see [BM]). Indeed, by (2.3) we have

$$L_{K^\circ}^2 \leq C\frac{n^2}{|K^\circ|}\int_{K^\circ} \langle x, \theta \rangle^2 dx \leq Cn^2 \|\theta\|_K^2.$$

($\|\cdot\|_K$ represents the norm in $\mathbb{R}^n$ whose unit ball is K).

If K is a general convex body in $\mathbb{R}^n$ with its centroid in the origin, since $K \subset K - K$ and Rogers-Shephard's inequality we have that there exists a position of K, say $\tilde{K}$, for which

$$R(\tilde{K}) \leq Cn.$$

4. We can improve the estimate for the circumradius before only for high dimensional sections of convex bodies of volume one, whose polar is in isotropic position. For instance, by taking $m = n$ and $\delta = 1/4$ we obtain some $n/2$-dimensional section for a subspace F such that

$$R(K \cap F) \leq Cn^{3/4} \log n$$

for any symmetric convex body of volume one whose polar is in isotropic position.

Let us introduce eventually some notation. We represent ℓ_2^n the normed space $\mathbb{R}^n$ with the euclidean norm. Notice that $|\cdot|$ will represent also the euclidian norm as the volume $|K| = |K|_n$ (if we want to empathize the dimension). By C, c we will represent absolute constants which can vary from line to line. The expression $\sim$ denotes equivalence of two quantities up to an absolute factor.

2. Proof of the theorem

Proof. i) The case $m = n$ is just Bourgain's estimate for the isotropy constant. We only consider $1 \le m \le n-1$. By using Bourgain-Milman's reverse Blaschke-Santaló's inequality (see [BM]) we have that

$$\begin{aligned} |K \cap E|_m^{1/m} &\le \frac{C}{m|(K \cap E)^\circ|_m^{1/m}} \\ &= \frac{C}{m|P_E(K^\circ)|_m^{1/m}}. \end{aligned}$$

and also using Fubini's theorem and Brunn-Minkowski inequality

$$|K^\circ|_n^{1/n} \le |K^\circ \cap E^\perp|_{n-m}^{1/n} |P_E(K^\circ)|_m^{1/n}.$$

Then

$$|K \cap E|_m^{1/m} \le \frac{C}{m|K^\circ|^{1/m}} \, |K^\circ \cap E^\perp|_{n-m}^{1/m}$$

Now we use a fact which can be deduced from a well known result by Milman-Pajor (cf. [MP] Proposition 3.11) (this result can be found implicit in K. Ball ([B1])): if T in $\mathbb{R}^n$ is an isotropic symmetric convex body then

$$|T \cap E^\perp|_{n-m}^{1/m} \le C \frac{L_m}{L_T} \, |T|_n^{(n-m)/nm}$$

where C is an absolute constant and L_m is the supremun of the isotropy constants of all m-dimensional convex bodies. Hence, we apply this result to $T = K^\circ$, taking into account that $|K^\circ|_n^{1/n} \sim C/n$ by reverse Blaschke-Santaló's inequality. Eventually Bourgain's estimate for the isotropy constant gives the result.

ii) Since the convex body K° is in isotropic position and its volume is $|K^\circ|_n^{1/n} \sim 1/n$, the isotropic constant $L = L_{K^\circ}$ satisfies

$$L^2 \sim \frac{n}{|K^\circ|_n} \int_{K^\circ} |x|^2 dx, \tag{2.3}$$

according to (1.1).

Let E be any m-dimensional subspace of $\mathbb{R}^n$. We denote by X the m-dimensional normed space $X = (E, \|\cdot\|_{K\cap E})$, where $\|\cdot\|_{K\cap E}$ is the norm on E whose unit ball is $K \cap E$. We denote by B the polar set of $K \cap E$ in E so the dual space is $X^* = (E, \|\cdot\|_B)$ and it is well known that $B = P_E(K^\circ)$ where P_E denotes the orthogonal projection on E.

We consider the M-ellipsoid associated to the symmetric convex body (in E), $K\cap E$, (see [Pi]), i. e. given any $0<p<2$ there exists an isomorphism $u:\ell_2^m\to X$ such that

(2.4)
$$\max\{d_k(u),c_k(u^{-1}),e_k(u),e_k(u^*),e_k(u^{-1}),e_k(u^{-1*})\}\leq\frac{C}{\sqrt{2-p}}\left(\frac{m}{k}\right)^{1/p}$$

for some absolute constant $C>0$ and for all $k=1,\dots,m$, where the d_k,c_k,e_k's are the corresponding Kolmogorov, Gelfand and entropy numbers. Furthermore

$$\max\{\log N(u(D_m),tK\cap E),\log N(u^*(B),tD_m), \tag{2.5}$$
$$\log N(K\cap E,tu(D_m)),\log N(D_m,tu^*(B))\} \tag{2.6}$$
$$\leq\left(\frac{C}{\sqrt{2-p}}\right)^p\frac{m}{t^p} \tag{2.7}$$

for all $t\geq C/\sqrt{2-p}$, where $C>0$ is an absolute constant (see, for instance [Pi], theorem 7.13 corollaries 7.15. and 7.16).

We can assume that the isomorphism u has a diagonal expression if we take the canonical basis $\{\varepsilon_i\}_{i=1}^m$ in ℓ_2^m and some orthogonal basis $\{e_i\}_{i=1}^m$ (with respect to the euclidian structure of $\mathbb{R}^n$) in E such that $u(\varepsilon_i)=\lambda_ie_i$, for some $\lambda_i>0$, $1\leq i\leq m$ in such a way that $\lambda_1\leq\lambda_2\leq\cdots\leq\lambda_m$. Let $1\leq k<m$. Since

$$c_k(u^{-1})\leq\frac{C}{\sqrt{2-p}}\left(\frac{m}{k}\right)^{1/p}$$

(see, (2.4)), by definition of the Gelfand numbers, there exists a subspace $F_1\subseteq E$ with $\dim F_1>m-k$, such that

$$|u^{-1}(x)|\leq\frac{C}{\sqrt{2-p}}\left(\frac{m}{k}\right)^{1/p}\|x\|_K$$

for all $x\in F_1$. Let $F_2=\operatorname{span}\{e_1,\dots,e_{m-k}\}$. It is clear that $F=F_1\cap F_2$ verifies that $m-2k<\dim F\leq m-k$ and

$$|u^{-1}(x)|=\left|\sum_{i=1}^{m-k}\lambda_i^{-1}x_i\varepsilon_i\right|\geq\frac{1}{\lambda_{m-k}}\left(\sum_{i=1}^{m-k}x_i^2\right)^{1/2}=\frac{|x|}{\lambda_{m-k}}$$

for all $x=\sum_{i=1}^{m-k}x_ie_i\in F$. So,

$$\begin{aligned}|x|&\leq\lambda_{m-k}\frac{C}{\sqrt{2-p}}\left(\frac{m}{k}\right)^{1/p}\|x\|_K\\&\leq\frac{1}{k}\left(\sum_{i=m-k+1}^{m}\lambda_i\right)\frac{C}{\sqrt{2-p}}\left(\frac{m}{k}\right)^{1/p}\|x\|_K\\&\leq\frac{1}{k}\left(\sum_{i=1}^{m}\lambda_i\right)\frac{C}{\sqrt{2-p}}\left(\frac{m}{k}\right)^{1/p}\|x\|_K\end{aligned}$$

for all $x \in F$. We will assume the following

Claim.

$$\sum_{i=1}^{m} \lambda_i \leq \frac{C n^{(3/2-1/2p)} m^{1/p}}{L\sqrt{2-p}}.$$

We follow the proof. If we assume that $k = \delta m$, for some $0 < \delta < 1/2$ then

$$|x| \leq \frac{C}{(2-p)\delta^{1+1/p}} \frac{n^{3/2-1/2p}}{L\, m^{1-1/p}}$$

for all $x \in F \cap K$ and we achieve the result, by taking $p = 2 - 1/\log m$. $\square$

Proof of the Claim.

We consider ℓ_2^m canonically embedded in ℓ_2^n and we denote by $\{\varepsilon_i\}_{i=1}^n$ its canonical basis; in the same way, we extend the orthogonal basis $\{e_i\}_{i=1}^m$ from E to all $\mathbb{R}^n$. Let $P_E : \mathbb{R}^n \to E$ the orthogonal projection on E.

Let $K_1 \subseteq \ell_2^n$ the convex set defined by

$$K_1 = \{x = \sum_{i=1}^{n} x_i \varepsilon_i \in \ell_2^n;\ \sum_{i=1}^{n} x_i e_i \in K^\circ\}.$$

K_1 is an orthogonal copy of K°, therefore they have the same volume and isotropy constant, and hence, by (2.3),

$$\frac{C}{n^2} L^2 \leq \frac{1}{|K_1|_n} \int_{K_1} x_i^2 dx$$

for an absolute constant and for all $1 \leq i \leq n$. Hence

$$\frac{C}{n^2} L^2 \sum_{i=1}^{m} \lambda_i \leq \frac{1}{|K_1|_n} \int_{K_1} \sum_{i=1}^{m} \lambda_i x_i^2\, dx \tag{2.8}$$

$$= \frac{1}{|K_1|_n} \int_{K_1} \left\langle \sum_{i=1}^{n} x_i \varepsilon_i, u^* \left(\sum_{i=1}^{m} x_i e_i \right) \right\rangle dx \tag{2.9}$$

$$= \frac{1}{|K_1|_n} \int_{K_1} \left\langle x, u^* \circ P_E \left(\sum_{i=1}^{n} x_i e_i \right) \right\rangle dx \tag{2.10}$$

$$\leq \frac{1}{|K_1|_n} \int_{K_1} \max_{z \in u^*(B)} \langle x, z \rangle\, dx \tag{2.11}$$

since B, which is the polar of $K \cap E$ in E, can be expressed as $B = P_E(K^\circ)$ and $u^* : (E, \|\cdot\|_B) \to \ell_2^m$.

Now we follow the method appearing in [D], [Gi], [Pa] in order to estimate from above the last integral. If we assume that $u^*(B) \subset R D_m$, for some $R > 0$ large enough, we use the Dudley-Fernique's technique. Let $N \in \mathbb{N}$

(to be chosen later). By using (2.7), it is clear that for every $j = 1, \dots, N$ there exist points $y_1^{(j)}, \dots, y_{N_j}^{(j)}$ such that

$$u^*(B) \subset \bigcup_{i=1}^{N_j} \left(y_i^{(j)} + \frac{R}{2^j} D_m \right),$$

where

$$\log N_j \le \left(\frac{C}{\sqrt{2-p}} \right)^p \frac{m 2^{jp}}{R^p}.$$

Therefore, for every $z \in u^*(B)$ and $1 \le j \le N$ we choose the points $y^{(j)}$ such that $|z - y^{(j)}| \le R/2^j$. Hence

$$\begin{aligned} z &= 0 + (y^{(1)} - 0) + (y^{(2)} - y^{(1)}) + \cdots + (y^{(N)} - y^{(N-1)}) + (z - y^{(N)}) \\ &= w_1 + \cdots + w_N + w, \end{aligned}$$

where

$$\begin{aligned} |w_j| &\le |y^{(j)} - z| + |z - y^{(j-1)}| \le \frac{3R}{2^j} \\ |w| &= |z - y^{(N)})| \le \frac{R}{2^N}. \end{aligned}$$

Each vector w_j belongs to a finite set $\mathcal{F}_j$ of cardinality $|\mathcal{F}_j| \le N_j N_{j-1}$, so

$$\log |\mathcal{F}_j| \le 2 \left(\frac{C}{\sqrt{2-p}} \right)^p \frac{m 2^{jp}}{R^p}.$$

Therefore

$$\int_{K_1} \max_{z \in u^*(B)} \langle x, z \rangle \, dx \le \quad \sum_{j=1}^N \int_{K_1} \max_{w_j \in \mathcal{F}_j} \langle x, w_j \rangle dx + \int_{K_1} \max_{w \in \frac{R}{2^N} D_m} \langle x, w \rangle dx$$

Since K_1 is in isotropic position, the last summand verifies

$$\begin{aligned} \frac{1}{|K_1|_n} \int_{K_1} \max_{w \in \frac{R}{2^N} D_m} \langle x, w \rangle dx &\le \frac{R}{2^N} \frac{1}{|K_1|_n} \int_{K_1} |x| dx \\ &\le \frac{R}{2^N} \left(\frac{1}{|K_1|_n} \int_{K_1} |x|^2 dx \right)^{1/2} \\ &\le \text{ by (2.3) } \le \frac{CR}{2^N} \frac{L}{\sqrt{n}}. \end{aligned}$$

We use the following well known fact (which is a consequence of Borel's lemma, see [MS]): if $\tilde{K}$ is an isotropic convex body with volume $|\tilde{K}|_n = 1$ then

$$\int_{\tilde{K}} \max_{i=1}^N |\langle x, y_i \rangle| dx \le C \, L_{\tilde{K}} \log N \max_{i=1}^N |y_i| \tag{2.12}$$

for any finite family of unit vectors $\{y_1,\dots,y_N\}$, where $C>0$ is an absolute constant. Then we get that

$$\begin{aligned}\frac{n}{|K_1|_n}\int_{K_1}\max_{w_j\in\mathcal{F}_j}|\langle x,w_j\rangle|dx &\le C\frac{|K_1|_n^{-1/n}}{|K_1|_n}\int_{K_1}\max_{w_j\in\mathcal{F}_j}|\langle x,w_j\rangle|dx\\ &\le C_1L\log|\mathcal{F}_j|\max_{w_j\in\mathcal{F}_j}|w_j|\\ &\le C_1L\left(2\left(\frac{C}{\sqrt{2-p}}\right)^p\frac{m2^{jp}}{R^p}\right)\frac{3R}{2^j}\\ &\le \frac{CC_1Lm}{(2-p)^{p/2}}\,\frac{2^{j(p-1)}}{R^{p-1}}.\end{aligned}$$

Taking into account (2.11) and (2.12) we have

$$\begin{aligned}L\sum_{i=1}^m\lambda_i &\le Cn\left(\frac{R}{2^N}\sqrt{n}+\frac{CC_1m}{(2-p)^{p/2}}\sum_{j=1}^N\frac{2^{j(p-1)}}{R^{p-1}}\right)\\ &\le Cn\left(\frac{R}{2^N}\sqrt{n}+\frac{CC_1m}{(2-p)^{p/2}(p-1)}\left(\frac{2^N}{R}\right)^{p-1}\right).\end{aligned}$$

We optimize by taking

$$\frac{R}{2^N}=\left(\frac{CC_1m}{\sqrt{n}(2-p)^{p/2}}\right)^{1/p}$$

and hence the claim holds.

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Departamento de Matemáticas, Facultad de Ciencias, Universidad de Zaragoza, 50009 Zaragoza, Spain

E-mail address, (Jesús Bastero): bastero@unizar.es