THE VARIANCE CONJECTURE ON HYPERPLANE PROJECTIONS OF THE ℓ_p^n BALLS

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ABSTRACT. We show that for any $1 \leq p \leq \infty$, the family of random vectors uniformly distributed on hyperplane projections of the unit ball of ℓ_p^n verify the variance conjecture

$$\text{Var} |X|^2 \leq C \max_{\xi \in S^{n-1}} \mathbb{E}\langle X, \xi \rangle^2 \mathbb{E}|X|^2,$$

where C depends on p but not on the dimension n or the hyperplane. We will also show a general result relating the variance conjecture for a random vector uniformly distributed on an isotropic convex body and the variance conjecture for a random vector uniformly distributed on any Steiner symmetrization of it. As a consequence we will have that the class of random vectors uniformly distributed on any Steiner symmetrization of an ℓ_p^n-ball verify the variance conjecture.

1. INTRODUCTION AND NOTATIONS

A probability measure μ on $\mathbb{R}^n$ is said to be log-concave if it has a density with respect to the Lebesgue measure

$$d\mu(x) = e^{-V(x)}dx,$$

where $V : \mathbb{R}^n \to (-\infty, \infty]$ is a convex function. For instance, the uniform probability measure on a convex body and the Gaussian measure are examples of log-concave probabilities on $\mathbb{R}^n$. A log-concave random vector X is a random vector in $\mathbb{R}^n$ distributed according to a log-concave probability measure. A log-concave random vector X is called isotropic if the following two conditions hold:

- The barycenter is at the origin, i.e., $\mathbb{E}X = 0$,
- The covariance matrix is the identity I_n, i.e. $\mathbb{E}\langle X, e_i \rangle \langle X, e_j \rangle = \delta_{i,j}$,

where $\{e_i\}_{i=1}^n$ denotes the canonical basis in $\mathbb{R}^n$, $\delta_{i,j}$ is the Kronecker delta, $\langle \cdot, \cdot \rangle$ is the usual scalar product in $\mathbb{R}^n$, and $\mathbb{E}$ denotes the expectation. We will also denote by Var the variance. It is well known that for any log-concave random vector X there exists an affine map T, with non-zero determinant, such that TX is isotropic. If X is centered then T is non-degenerate linear map $T \in GL(n)$.

Given a centered log-concave random vector X, we will denote by λ_X^2 the largest eigenvalue of its covariance matrix M_X

$$\lambda_X^2 = \|M_X\|_{\ell_2^n \to \ell_2^n} = \max_{\xi \in S^{n-1}} \mathbb{E}\langle X, \xi \rangle^2,$$

where S^{n-1} denotes the Euclidean unit sphere in $\mathbb{R}^n$.

The variance conjecture was considered by Bobkov and Koldobsky in the context of the Central Limit Problem for isotropic convex bodies (see [BK]) and it states the following:

Conjecture 1.1. *There exists an absolute constant C such that for every isotropic log-concave random vector X*

$$Var |X|^2 \leq C\mathbb{E} |X|^2 = Cn.$$

It was conjectured before by Antilla, Ball, and Perissinaki (see [ABP]) that for an isotropic log-concave random vector X, $|X|$ is highly concentrated in a "thin shell" more than the trivial bound $Var |X| \leq \mathbb{E} |X|^2$ suggests.

The variance conjecture is a particular case of a stronger conjecture, due to Kannan, Lovász, and Simonovits (see [KLS]), concerning the spectral gap of log-concave probability measures. This conjecture can be stated in the following way due to the work of Cheeger, Maz'ya and Ledoux, among others:

Conjecture 1.2. *There exists an absolute constant C such that for any centered log-concave random vector X and for any locally Lipschitz function $g : \mathbb{R}^n \to \mathbb{R}$ such that the random variable $g(X)$ has finite variance*

$$Var g(X) \leq C\lambda_X^2 \mathbb{E} |\nabla g(X)|^2.$$

Notice that Conjecture 1.1 is the particular case of Conjecture 1.2 when $g(X) = |X|^2$ and X is isotropic. One can also consider the particular case in which $g(X) = |X|^2$ but X is not necessarily isotropic. This gives the following general variance conjecture

Conjecture 1.3. *There exists an absolute constant C such that for every centered log-concave random vector X*

$$Var |X|^2 \leq C\lambda_X^2 \mathbb{E} |X|^2.$$

This general variance conjecture was considered before in [AB1], where it was shown that uniform probability measures on hyperplane projections of B_1^n and B_∞^n (the unit balls of ℓ_1^n and ℓ_∞^n) verify it. In the particular case that we consider X isotropic this conjecture becomes Conjecture 1.1. However, it is not clear whether these conjectures are equivalent since the general case is not deduced from the isotropic case because we are considering only the function $g(X) = |X|^2$. Some estimates for the constant in Conjecture 1.3, when considering linear deformations of isotropic random vectors verifying Conjecture 1.1 were given in [AB1] and [AB2].

Not many examples are known to verify these conjectures. Conjecture 1.2 is known to be true for a Gaussian random vector and random vectors uniformly distributed on the ℓ_p^n-balls, some revolution bodies, the simplex, and, with an extra $\log n$ factor, on unconditional bodies and log-concave probabilities with many symmetries (see [BaC], [BaW], [B], [H], [K], [LW], [S]). The best general known result in Conjecture 1.2 adds a factor $n^{\frac{2}{3}} (\log n)^2$ and is due to Guédon-Milman, who proved the best known estimate in Conjecture 1.1 with an extra factor $n^{\frac{2}{3}}$ (see [GM]), and Eldan, who proved that the variance conjecture implies the Kannan-Lovász-Simonovits conjecture, up to a logarithmic factor (see [E]). Besides, Conjecture 1.3 (and thus, 1.1) is true for random vectors uniformly distributed on unconditional bodies [K] and, as mentioned before, hyperplane projections of B_1^n and B_∞^n (see

[AB1]), and increments of log-concave martingales (see [CG]). For more information on these conjectures and their relation with some other problems in asymptotic convex geometry we also refer the reader to the monographs [BGVV] and [AB2].

In this paper we approach the study of the general variance conjecture for the class of random vectors uniformly distributed on projections of B_p^n, the unit balls of ℓ_p^n $1 < p < \infty$, onto $(n-1)$-dimensional subspaces $H = \theta^\perp$, extending the results obtained for $p = 1, \infty$ in [AB1]. Namely, we will prove the following

Theorem 1.1. *There exists an absolute constant C such that for any hyperplane $H = \theta^\perp$, with $\theta \in S^{n-1}$, if X is a random vector uniformly distributed on $P_H(B_p^n)$ we have that if $p \leq n$*

$$Var\,|X|^2 \leq C \log(1+p) \lambda_X^2 \mathbb{E}\,|X|^2$$

and if $p > n$

$$Var\,|X|^2 \leq C \lambda_X^2 \mathbb{E}\,|X|^2.$$

Furthermore, if $1 \leq p \leq n$ the set of vectors $\theta \in S^{n-1}$ such that

$$Var\,|X|^2 \leq C \lambda_X^2 \mathbb{E}\,|X|^2$$

has Haar probability measure greater than $1 - \frac{1}{2^n}$.

Notice that the value of the constant in the theorem depends on p if $p \leq n$ and does not depend on p if $p > n$. The reason for this discontinuity in the value of the constant is just technical. Our proof gives a constant $C \log(1+p)$ for every value of $p \in [1, \infty]$ and, using a different method we were able to give a better estimate, independent of p, that holds for values of p greater than n.

We would like to remark that we are considering a random vectors uniformly distributed on projections of B_p^n and not the projections of random vectors uniformly distributed on B_p^n. When considering the projections of the random vectors the situation is much simpler. Even though it is probably straightforward for specialists, for the sake of completeness we will give in Section 5 a general result showing that an isotropic log-concave random vector verifies the variance conjecture if and only if any of its hyperplane projections does.

A convex body K is called isotropic if it has volume 1, $|K| = 1$, and for any vector $\theta \in S^{n-1}$ we have $\mathbb{E}\langle X, \theta \rangle = 0$ and $\mathbb{E}\langle X, \theta \rangle^2 = L_K^2$, where X is a random vector uniformly distributed on K and L_K does not depend on θ and is called the isotropic constant of K. Thus, K is isotropic if and only if the random vector uniformly distributed on $L_K^{-1} K$ is isotropic. Given a convex body K and a hyperplane $H = \theta^\perp$, with $\theta \in S^{n-1}$, the Steiner symmetrization of K with respect to H is the convex body defined as

$$S_\theta(K) = \left\{ x + t\theta \,:\, x \in P_{\theta^\perp} K, |t| \leq \frac{1}{2} |K \cap (x + \langle \theta \rangle)| \right\},$$

where $\langle \theta \rangle$ denotes the one-dimensional subspace spanned by θ. We will also study the relation between the variance conjecture for a random vector uniformly distributed on an isotropic convex body and a random vector uniformly distributed on the Steiner symmetrization of it with respect to any hyperplane. We will show the following general result, which shows that a random vector uniformly distributed on an isotropic body verifies the variance conjecture if and only if a random vector uniformly distributed on any of its Steiner symmetrizations does. As a consequence, if a random vector uniformly distributed on an isotropic convex body K verifies

the variance conjecture, then the class of random vectors uniformly distributed on any of its Steiner symmetrizations also verify the variance conjecture.

Theorem 1.2. *Let K be an isotropic convex body and $\theta \in S^{n-1}$. Let us denote by X a random vector uniformly distributed on K and by Y_θ a random vector uniformly distributed on $S_\theta(K)$, the Steiner symmetrization of K with respect to $H = \theta^\perp$. Then the following are equivalent*

- *There exists a constant C_1 such that*

$$Var\,|X|^2 \le C_1 \lambda_X^2 \mathbb{E}\,|X|^2.$$

- *There exists a constant C_2 such that*

$$Var\,|Y_\theta|^2 \le C_2 \lambda_{Y_\theta}^2 \mathbb{E}\,|Y_\theta|^2$$

for some $\theta \in S^{n-1}$.
- *There exists a constant C_3 such that*

$$Var\,|Y_\theta|^2 \le C_3 \lambda_{Y_\theta}^2 \mathbb{E}\,|Y_\theta|^2$$

for every $\theta \in S^{n-1}$,

where

$$C_2 \le C_3 \le 2(C_1 + C) \ and \ C_1 \le C_2 + C,$$

with C an absolute constant.

The paper is organized as follows: We will prove Theorem 1.1 in Section 4. In Section 2 we will present some known results that we will use and in Section 3 we will prove some technical lemmas we will need to prove Theorem 1.1. Finally, in Section 5 we will show the general results concerning the variance conjecture for projections of isotropic log-concave random vectors and for random vectors uniformly distributed on the Steiner symmetrizations of an isotropic convex body. We will always use the letters c, C, C' to denote absolute constants and will use $a \sim b$ to denote the existence of two positive absolute constants c, C such that $ca \le b \le Ca$.

2. Preliminaries

In this Section we present the tools we use to prove the aforementioned results. We will use the techniques developed in [BaN]. We will denote by σ_p^n the uniform area measure (Hausdorff measure) on ∂B_p^n, the boundary of B_p^n, and by μ_p^n the cone probability measure on ∂B_p^n, defined by

$$\mu_p^n(A) = \frac{|\{ta \in \mathbb{R}^n; a \in A, 0 \le t \le 1\}|}{|B_p^n|} \qquad A \subseteq \partial B_p^n.$$

A relation between these two measures was proved in [NR]. For the sake of completeness we include a short proof of it in the following lemma:

Lemma 2.1. *Let σ_p^n and μ_p^n be the uniform area measure and the cone probability measure on ∂B_p^n. Then*

$$\frac{d\sigma_p^n(x)}{d\mu_p^n(x)} = n|B_p^n|\,|\nabla(\|\cdot\|_p)(x)|$$

for almost every point $x \in \partial B_p^n$.

Proof. Let $g : \partial B_p^n \to \mathbb{R}$ be an integrable function with respect to μ_p^n. Denoting by $\sigma_{t\partial B_p^n}$ the uniform area measure on $t\partial B_p^n$ and using the co-area formula, we have that

$$
\begin{aligned}
\int_{\partial B_p^n} g(y) d\mu_p^n(y) &= \frac{1}{|B_p^n|} \int_{B_p^n} g\left(\frac{x}{\|x\|_p}\right) dx \\
&= \frac{1}{|B_p^n|} \int_0^1 \int_{t\partial B_p^n} \frac{g\left(\frac{x}{\|x\|_p}\right)}{|\nabla(\|\cdot\|_p)(x)|} d\sigma_{t\partial B_p^n}(x) dt \\
&= \frac{1}{|B_p^n|} \int_0^1 t^{n-1} \int_{\partial B_p^n} \frac{g(y)}{|\nabla(\|\cdot\|_p)(y)|} d\sigma_p^n(y) dt \\
&= \int_{\partial B_p^n} \frac{1}{n|B_p^n|} \frac{g(y)}{|\nabla(\|\cdot\|_p)(y)|} d\sigma_p^n(y).
\end{aligned}
$$

$\square$

Consequently, by using Cauchy's formula, if $H = \theta^\perp$, X is a random vector uniformly distributed on $K = P_H B_p^n$ and $f : K \to \mathbb{R}$ is a Borel integrable function

$$
\begin{aligned}
\mathbb{E} f(X) &= \frac{1}{|K|} \int_K f(x) dx \\
&= \frac{1}{2|K|} \int_{\partial B_p^n} f(P_H(y)) \frac{|\langle \nabla \|\cdot\|_p(y), \theta\rangle|}{|\nabla\|\cdot\|_p(y)|} d\sigma_p^n(y) \\
&= \frac{\int_{\partial B_p^n} f(P_H(y)) |\langle \nabla(\|\cdot\|_p)(y), \theta\rangle| d\mu_p^n}{\int_{\partial B_p^n} |\langle \nabla(\|\cdot\|_p)(y), \theta\rangle| d\mu_p^n} \\
&= \frac{\int_{\partial B_p^n} f(P_H(y)) \left|\sum_{i=1}^n |y_i|^{p-1} \operatorname{sgn}(y_i)\theta_i\right| d\mu_p^n(y)}{\int_{\partial B_p^n} \left|\sum_{i=1}^n |y_i|^{p-1} \operatorname{sgn}(y_i)\theta_i\right| d\mu_p^n(y)}.
\end{aligned}
$$

We will use the following probabilistic description of the measure μ_p^n (see, for instance, [SZ1], [BaN], [NR]): Let $g_1, \ldots, g_n$ be independent copies of a random variable g with density with respect to the Lebesgue measure

$$
\frac{e^{-|t|^p}}{2\Gamma(1 + 1/p)}
$$

for every $t \in \mathbb{R}$ and denote by

$$
S = \left(\sum_{i=1}^n |g_i|^p\right)^{\frac{1}{p}}.
$$

Then

- The random vector $\dfrac{G}{S} := \left(\dfrac{g_1}{S}, \ldots, \dfrac{g_n}{S}\right)$ and the random variable S are independent.
- $\dfrac{G}{S}$ is distributed on ∂B_p^n according to the cone measure μ_p^n.

Hence

$$\mathbb{E}\,f(X) = \frac{\mathbb{E}f\left(P_H\left(\frac{g_1}{S},\ldots,\frac{g_n}{S}\right)\right)\left|\sum_{i=1}^{n}\frac{|g_i|^{p-1}}{S^{p-1}}\,\mathrm{sgn}(g_i)\theta_i\right|}{\mathbb{E}\left|\sum_{i=1}^{n}\frac{|g_i|^{p-1}}{S^{p-1}}\,\mathrm{sgn}(g_i)\theta_i\right|}.$$

By the independence of $\frac{G}{S}$ and S, we have

$$\begin{aligned}\mathbb{E}\,f(X) &= \frac{\mathbb{E}f\left(P_H\left(\frac{g_1}{S},\ldots,\frac{g_n}{S}\right)\right)\left|\sum_{i=1}^{n}|g_i|^{p-1}\,\mathrm{sgn}(g_i)\theta_i\right|}{\mathbb{E}\left|\sum_{i=1}^{n}|g_i|^{p-1}\,\mathrm{sgn}(g_i)\theta_i\right|}\\[2mm] &= \frac{\mathbb{E}f\left(P_H\left(\frac{G}{S}\right)\right)\psi_\theta}{\mathbb{E}\psi_\theta},\end{aligned}$$

where ψ_θ is defined as

$$(1)\qquad\qquad \psi_\theta = \left|\sum_{i=1}^{n}|g_i|^{p-1}\,\mathrm{sgn}(g_i)\theta_i\right|.$$

We will sometimes use the notation ψ instead of ψ_θ when there is no possibility of confusion.

The following theorem, which will be used to obtain some estimates for the expected value of ψ, was proved in [ACCP]:

Theorem 2.1. *Let $1 < q < \infty$, $X_1,\ldots,X_n$ be independent identically distributed integrable random variables. For every $s \geq 0$ define*

$$M(s) = \frac{q}{q-1}\int_0^s\left(\int_{|X_1|\leq\frac{1}{t}}t^{q-1}\,|X_1|^q\,d\mathbb{P} + \int_{|X_1|>1/t}|X_1|d\mathbb{P}\right)dt.$$

Then, for every $x \in \mathbb{R}^n$,

$$c_1(q-1)^{1/q}\|x\|_M \leq \mathbb{E}\left(\sum_{i=1}^{n}|x_iX_i|^q\right)^{\frac{1}{q}} \leq c_2\|x\|_M,$$

where c_1, c_2, are positive absolute constants and $\|x\|_M$ denotes the Luxemburg norm given by the Orlicz function M, which is defined by

$$\|x\|_M = \inf\left\{\rho > 0 : \sum_{i=1}^{n}M\left(\frac{|x_i|}{\rho}\right)\leq 1\right\}.$$

We will also make use of the following theorem, which was proved in [KS]:

Theorem 2.2. *Let $1 \leq q \leq \infty$ and $a \in \mathbb{R}^{n\times n}$. Then*

$$Ave_\pi\left(\sum_{i=1}^{n}|a_{i,\pi(i)}|^q\right)^{\frac{1}{q}} \sim \frac{1}{n}\sum_{k=1}^{n}(a_{i,j}^*)_k + \left(\frac{1}{n}\sum_{k=n+1}^{n^2}(a_{i,j}^*)_k^q\right)^{\frac{1}{q}},$$

where $a_{i,j}^* \in \mathbb{R}^{n^2}$ is the decreasing rearrangement of a and π runs over all the permutations of $\{1,\ldots,n\}$.

In the same paper the authors showed that when $q = 2$ this estimate can be estimated by using an Orlicz function.

3. Some probabilistic estimates

In this section we will prove several technical lemmas we will need in order to prove Theorem 1.1. The following lemma is well known:

Lemma 3.1. *Let $\alpha \geq 0$ and let $g_1, \ldots, g_n$ be independent copies of a random variable g, with density with respect to the Lebesgue measure $\frac{e^{-|t|^p}}{2\Gamma(1+1/p)}$, and $S = \left(\sum_{i=1}^n |g_i|^p\right)^{\frac{1}{p}}$. Then*

$$\mathbb{E}|g|^\alpha = \frac{\Gamma\left(\frac{\alpha+1}{p}\right)}{\Gamma\left(\frac{1}{p}\right)}$$

and

$$\mathbb{E}S^\alpha = \frac{\Gamma\left(\frac{n+\alpha}{p}\right)}{\Gamma\left(\frac{n}{p}\right)}$$

Proof. The value of $\mathbb{E}|g|^\alpha$ can be computed directly. Let us compute $\mathbb{E}S^\alpha$.

$$\mathbb{E}S^\alpha = \mathbb{E}\left(\sum_{i=1}^n |g_i|^p\right)^{\alpha/p} = \int_{\mathbb{R}^n} \|x\|_p^\alpha \frac{e^{-\|x\|_p^p}}{(2\Gamma(1+1/p))^n} dx.$$

Changing to polar coordinates

$$\mathbb{E}S^\alpha = \frac{n|B_p^n|}{(2\Gamma(1+1/p))^n} \int_0^\infty r^{n+\alpha-1} e^{-r^p} dr$$

and this expression implies the result. $\qquad\square$

This lemma implies the following:

Lemma 3.2. *Let $X_1, \ldots, X_n$ be independent copies of $X = g^2 - \bar{g}^2$, where $\bar{g}$ is an independent copy of g, defined as before. Then, for any $2 \leq \alpha \leq e^p$ we have*

$$\left(\mathbb{E}\left|\sum_{i=1}^n X_i\right|^\alpha\right)^{\frac{1}{\alpha}} \leq C\sqrt{\alpha n}.$$

Proof. By the triangle inequality

$$(\mathbb{E}|X|^\alpha)^{\frac{1}{\alpha}} \leq 2\left(\mathbb{E}|g|^{2\alpha}\right)^{\frac{1}{\alpha}} = 2\left(\frac{\Gamma\left(\frac{1+2\alpha}{p}\right)}{\Gamma\left(\frac{1}{p}\right)}\right)^{\frac{1}{\alpha}}.$$

Using Stirling's formula

$$(\mathbb{E}|X|^\alpha)^{\frac{1}{\alpha}} \leq C\alpha^{\frac{2}{p}} \leq C_1,$$

since $\alpha \leq e^p$. Now, since the random variables X_i are symmetric, taking $\varepsilon_1, \ldots, \varepsilon_n$ independent Bernoulli random variables, which are also independent of the random variables X_i, we have

$$\mathbb{E}\left|\sum_{i=1}^n X_i\right|^\alpha = \mathbb{E}\mathbb{E}_\varepsilon \left|\sum_{i=1}^n \varepsilon_i X_i\right|^\alpha$$

and, by Khintchine's inequality (see [HA] for the best value of the constant in Khintchine's inequality)

$$\mathbb{E}_\varepsilon \left| \sum_{i=1}^n \varepsilon_i X_i \right|^\alpha \leq (C_2 \sqrt{\alpha})^\alpha \left(\sum_{i=1}^n |X_i|^2 \right)^{\frac{\alpha}{2}} \leq (C_2 \sqrt{\alpha})^\alpha n^{\frac{\alpha}{2}-1} \sum_{i=1}^n |X_i|^\alpha.$$

Hence

$$\left(\mathbb{E} \left| \sum_{i=1}^n X_i \right|^\alpha \right)^{\frac{1}{\alpha}} \leq C_1 C_2 \sqrt{\alpha n}.$$

$\square$

Let us recall that for every $\theta \in S^{n-1}$, ψ_θ was defined like

$$\psi_\theta = \left| \sum_{i=1}^n |g_i|^{p-1} \operatorname{sgn}(g_i) \theta_i \right|.$$

We will also call

$$\phi_\theta = \left(\sum_{i=1}^n |g_i|^{2p-2} \theta_i^2 \right)^{\frac{1}{2}}.$$

Notice that since the random variables g_i are symmetric with respect to the origin, for any choice of signs $\varepsilon_i = \pm 1$ we have

$$\mathbb{E}\psi_\theta = \mathbb{E} \left| \sum_{i=1}^n |g_i|^{p-1} \operatorname{sgn}(g_i) \theta_i \right| = \mathbb{E} \left| \sum_{i=1}^n |\varepsilon_i g_i|^{p-1} \operatorname{sgn}(\varepsilon_i g_i) \theta_i \right|.$$

Thus, taking $\varepsilon_1, \ldots, \varepsilon_n$ independent Bernoulli random variables, by Khintchine's inequality we have

$$\mathbb{E}\psi_\theta = \mathbb{E}_\varepsilon \mathbb{E}_g \left| \sum_{i=1}^n |\varepsilon_i g_i|^{p-1} \operatorname{sgn}(\varepsilon_i g_i) \theta_i \right| = \mathbb{E}_g \mathbb{E}_\varepsilon \left| \sum_{i=1}^n \varepsilon_i |g_i|^{p-1} \operatorname{sgn}(g_i) \theta_i \right|$$

$$\sim \mathbb{E} \left(\sum_{i=1}^n |g_i|^{2p-2} \theta_i^2 \right)^{\frac{1}{2}} = \mathbb{E}\phi_\theta.$$

The following lemma gives estimates for the value of $\mathbb{E}\psi_\theta$, independent of the direction θ, in terms of the $\|\theta\|_1$, or in terms of the value of $\mathbb{E}\psi_{\theta_0}$, where θ_0 is the diagonal direction.

Lemma 3.3. *Let* $\theta_0 = \left(\frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}} \right)$. *Then*

a) *There exist absolute constants* C_1, C_2 *such that for any* $1 \leq p < \infty$ *and* $\theta \in S^{n-1}$

$$\frac{C_1}{p} \leq \mathbb{E}\,\psi_\theta \leq \frac{C_2}{\sqrt{p}}.$$

Furthermore, for any $1 \leq p < \infty$ *and* $\theta \in S^{n-1}$

$$\frac{C_1}{p} \leq \mathbb{E}\,\psi_\theta^2 \leq \frac{C_2}{p}.$$

b) *There exist two absolute constants* C_1, C_2 *such that for any* $1 \leq p < \infty$

$$\frac{C_1}{\sqrt{n}} \mathbb{E}\psi_{\theta_0} \|\theta\|_1 \leq \mathbb{E}\psi_\theta \leq \frac{C_2}{p} \|\theta\|_1.$$

c) *There exists an absolute constant C such that*

$$\mathbb{E}\psi_\theta \leq C\mathbb{E}\psi_{\theta_0}.$$

Furthermore, there exists an absolute constant c such that

$$\sigma\left\{\theta \in S^{n-1} : \mathbb{E}\psi_\theta \geq c\mathbb{E}\psi_{\theta_0}\right\} \geq 1 - \frac{1}{2^n}.$$

Proof. Let us first prove a):

By Jensen's inequality we have

$$\mathbb{E}\psi_\theta \quad \sim \quad \mathbb{E}\phi_\theta \geq \mathbb{E}\sum_{i=1}^n |g_i|^{p-1}\theta_i^2 = \mathbb{E}|g|^{p-1} = \frac{1}{\Gamma\left(\frac{1}{p}\right)} = \frac{1}{p}\frac{1}{\Gamma\left(1+\frac{1}{p}\right)} \sim \frac{1}{p}.$$

On the other hand, by Hölder's inequality

$$\mathbb{E}\psi_\theta \quad \sim \quad \mathbb{E}\phi_\theta \leq \left(\mathbb{E}\sum_{i=1}^n |g_i|^{2p-2}\theta_i^2\right)^{\frac{1}{2}} = \left(\mathbb{E}|g|^{2p-2}\right)^{\frac{1}{2}} = \left(\frac{\Gamma\left(\frac{2p-1}{p}\right)}{\Gamma\left(\frac{1}{p}\right)}\right)^{\frac{1}{2}}$$

$$= \quad \left(\frac{1}{p}\frac{\Gamma\left(\frac{2p-1}{p}\right)}{\Gamma\left(1+\frac{1}{p}\right)}\right)^{\frac{1}{2}} \sim \frac{1}{\sqrt{p}}.$$

In the same way, taking independent Bernoulli random variables and using Khintchine's inequality

$$\mathbb{E}\psi_\theta^2 \quad \sim \quad \mathbb{E}\phi_\theta^2 = \mathbb{E}|g|^{2p-2} = \frac{\Gamma\left(\frac{2p-1}{p}\right)}{\Gamma\left(\frac{1}{p}\right)} = \frac{1}{p}\frac{\Gamma\left(\frac{2p-1}{p}\right)}{\Gamma\left(1+\frac{1}{p}\right)} \sim \frac{1}{p}.$$

Let us now prove b):

Notice that if $p = 1$, by Khintchine's inequality $\mathbb{E}\psi_\theta \sim 1$ for every $\theta \in S^{n-1}$ and then the result follows. Assume that $p > 1$. On the one hand, by Lemma 3.1

$$\mathbb{E}\psi_\theta \leq \mathbb{E}\left(\sum_{i=1}^n |g_i|^{p-1}|\theta_i|\right) \leq \frac{c_2}{p}\|\theta\|_1.$$

On the other hand,

$$\mathbb{E}\psi_\theta \sim \mathbb{E}\phi_\theta = \mathbb{E}\|(|g_i|^{p-1}\theta_i)_{i=1}^n\|_2.$$

Thus, applying Theorem 2.1 with $X_i = |g_i|^{p-1}$ and $q = 2$, we have that

$$\mathbb{E}\psi_\theta \sim \|\theta\|_M,$$

with

$$M(s) = 2\int_0^s \left(\int_0^{t^{-\frac{1}{p-1}}} tx^{2p-2}\frac{e^{-x^p}}{\Gamma\left(1+\frac{1}{p}\right)}dx + \int_{t^{-\frac{1}{p-1}}}^\infty x^{p-1}\frac{e^{-x^p}}{\Gamma\left(1+\frac{1}{p}\right)}dx\right)dt$$

$$= \frac{2}{p\Gamma\left(1+\frac{1}{p}\right)}\int_0^s \left(\int_0^{t^{-p^*}} tr^{\frac{p-1}{p}}e^{-r}dr + \int_{t^{-p^*}}^\infty e^{-r}dr\right)dt,$$

where $p^* = \frac{p}{p-1}$ is the dual exponent of p. Let B_M be the unit ball of $\|\cdot\|_M$. Taking into account that the norm $\|\cdot\|_M$ is 1-symmetric we have that

$$B_M \subseteq \frac{n}{\|(1,\ldots,1)\|_M} B_1^n.$$

Thus, for any $\theta \in S^{n-1}$

$$\|\theta\|_M \geq \frac{\|\theta_0\|_M}{\sqrt{n}}\|\theta\|_1$$

and so

$$\mathbb{E}\psi_\theta \geq \frac{c_1}{\sqrt{n}}\mathbb{E}\psi_{\theta_0}\|\theta\|_1.$$

Finally, we prove c):

Since for any permutation π of $\{1,\ldots,n\}$

$$\mathbb{E}\psi_\theta \sim \mathbb{E}\phi_\theta = \mathbb{E}\left(\sum_{k=1}^{n}|g_k|^{2p-2}\theta_k^2\right)^{\frac{1}{2}} = \mathbb{E}\left(\sum_{k=1}^{n}|g_k|^{2p-2}\theta_{\pi(k)}^2\right)^{\frac{1}{2}}$$

we have that this expectation equals

$$\mathbb{E}\mathrm{Ave}_\pi\left(\sum_{k=1}^{n}|g_k|^{2p-2}\theta_{\pi(k)}^2\right)^{\frac{1}{2}}$$

which, by Theorem 2.2 applied to $a_{i,j} = |g_i|^{p-1}\theta_j$, is equivalent to

$$\mathbb{E}\left(\frac{1}{n}\sum_{k=1}^{n}(|g_i|^{p-1}\theta_j)_k^* + \left(\frac{1}{n}\sum_{k=n+1}^{n^2}(|g_i|^{2p-2}\theta_j^2)_k^*\right)^{\frac{1}{2}}\right).$$

Now, since by Hölder's inequality

$$\frac{1}{n}\sum_{k=1}^{n}(|g_i|^{p-1}\theta_j)_k^* \leq \left(\frac{1}{n}\sum_{k=1}^{n}(|g_i|^{2p-2}\theta_j^2)_k^*\right)^{\frac{1}{2}}$$

we have that

$$\left(\frac{1}{n}\sum_{k=1}^{n}(|g_i|^{p-1}\theta_j)_k^* + \left(\frac{1}{n}\sum_{k=n+1}^{n^2}(|g_i|^{2p-2}\theta_j^2)_k^*\right)^{\frac{1}{2}}\right)$$

$$\leq \left(\left(\frac{1}{n}\sum_{k=1}^{n}(|g_i|^{2p-2}\theta_j^2)_k^*\right)^{\frac{1}{2}} + \left(\frac{1}{n}\sum_{k=n+1}^{n^2}(|g_i|^{2p-2}\theta_j^2)_k^*\right)^{\frac{1}{2}}\right)$$

$$\leq \sqrt{2}\left(\frac{1}{n}\sum_{i,j=1}^{n}|g_i|^{2p-2}\theta_j^2\right)^{\frac{1}{2}} = \sqrt{2}\phi_{\theta_0}$$

and taking expectation and using Khintchine's inequality again we obtain

$$\mathbb{E}\psi_\theta \leq c_2\mathbb{E}\psi_{\theta_0}.$$

Besides, by Markov's inequality for any $A \geq 0$

$$\frac{|B_1^n|}{|B_2^n|} = \int_{S^{n-1}}\frac{1}{\|\theta\|_1^n}d\sigma(\theta) \geq \frac{1}{A^n}\sigma\{\theta \in S^{n-1} : \|\theta\|_1 \leq A\}.$$

Thus, since $\left(\frac{|B_1^n|}{|B_2^n|}\right)^{\frac{1}{n}} \leq \frac{C}{\sqrt{n}}$, taking $A = \frac{1}{2C}\sqrt{n}$, we obtain that

$$\sigma\left\{\theta \in S^{n-1} : \|\theta\|_1 \leq \frac{1}{2C}\sqrt{n}\right\} \leq \frac{1}{2^n}$$

and, by part b) in this lemma, there exists an absolute constant c such that

$$\sigma\{\theta \in S^{n-1} : \mathbb{E}\psi_\theta \geq c\mathbb{E}\psi_{\theta_0}\} \geq 1 - \frac{1}{2^n},$$

which finishes the proof. $\qquad\square$

In both parts b) and c) in Lemma 3.3 we have related $\mathbb{E}\psi_\theta$ with $\mathbb{E}\psi_{\theta_0}$. In the following lemma we are going to estimate the value of $\mathbb{E}\psi_{\theta_0}$.

Lemma 3.4. *Let* $\theta_0 = \left(\frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}}\right)$. *Then, if* $1 \leq p \leq n$

$$\mathbb{E}\psi_{\theta_0} \sim \frac{1}{\sqrt{p}}$$

and, if $p = n^\gamma$ *with* $\gamma > 1$,

$$\mathbb{E}\psi_{\theta_0} \sim \frac{\sqrt{n}}{p} = \frac{1}{p^{1-\frac{1}{2\gamma}}}.$$

Proof. By Lemma 3.3 $\mathbb{E}\psi_{\theta_0} \leq \frac{C}{\sqrt{p}}$. Let us prove $\mathbb{E}\psi_{\theta_0} \geq \frac{c}{\sqrt{p}}$. We have seen that, by Khintchine's inequality,

$$\mathbb{E}\psi_{\theta_0} \sim \frac{1}{\sqrt{n}}\|(|g_i|^{p-1})_{i=1}^n\|_2.$$

Thus, applying Theorem 2.1 with $X_i = |g_i|^{p-1}$ and $q = 2$, we have that

$$\mathbb{E}\psi_{\theta_0} \sim \frac{1}{\sqrt{n}}\|(1, \ldots, 1)\|_M,$$

with

$$
\begin{aligned}
M(s) &= 2\int_0^s \left(\int_0^{t^{-\frac{1}{p-1}}} tx^{2p-2}\frac{e^{-x^p}}{\Gamma\left(1+\frac{1}{p}\right)}dx + \int_{t^{-\frac{1}{p-1}}}^\infty x^{p-1}\frac{e^{-x^p}}{\Gamma\left(1+\frac{1}{p}\right)}dx\right)dt \\
&= \frac{2}{p\Gamma\left(1+\frac{1}{p}\right)}\int_0^s \left(\int_0^{t^{-p^*}} tr^{\frac{1}{p^*}}e^{-r}dr + \int_{t^{-p^*}}^\infty e^{-r}dr\right)dt \\
&= \frac{2\left(1-\frac{1}{p}\right)}{p\Gamma\left(1+\frac{1}{p}\right)}\int_0^s t\int_0^{t^{-p^*}} r^{-\frac{1}{p}}e^{-r}drdt,
\end{aligned}
$$

where the last identity follows from integration by parts and $p^* = \frac{p}{p-1}$ is the dual exponent of p.

On the one hand, since

$$
\begin{aligned}
M(s) &\geq \frac{2}{p\Gamma\left(1+\frac{1}{p}\right)}\int_{\frac{s}{2}}^s \int_0^{t^{-p^*}} tr^{\frac{1}{p^*}}e^{-r}drdt \\
&\geq \frac{s}{p\Gamma\left(1+\frac{1}{p}\right)}\int_{\frac{s}{2}}^s \int_0^{s^{-p^*}} r^{\frac{1}{p^*}}e^{-r}drdt
\end{aligned}
$$

$$= \frac{s^2}{2p\Gamma\left(1+\frac{1}{p}\right)} \left(\Gamma\left(2-\frac{1}{p}\right) - \int_{s^{-p^*}}^{\infty} r^{\frac{1}{p^*}} e^{-r} dr \right),$$

we have that if $\rho = c\sqrt{\frac{n}{p}}$ and $p \le \frac{c^2}{\alpha^{\frac{2}{p^*}}} n$ with $\alpha \ge 1$

$$
\begin{aligned}
M\left(\frac{1}{\rho}\right) & \ge \frac{1}{c^2 n \Gamma\left(1+\frac{1}{p}\right)} \left(\Gamma\left(2-\frac{1}{p}\right) - \int_{\rho^{p^*}}^{\infty} r^{\frac{1}{p^*}} e^{-r} dr \right) \\
& \ge \frac{1}{c^2 n \Gamma\left(1+\frac{1}{p}\right)} \left(\Gamma\left(2-\frac{1}{p}\right) - \int_{\alpha}^{\infty} r^{\frac{1}{p^*}} e^{-r} dr \right) \\
& \ge \frac{1}{c^2 n \Gamma\left(1+\frac{1}{p}\right)} \left(\Gamma\left(2-\frac{1}{p}\right) - \int_{\alpha}^{\infty} r e^{-r} dr \right) \\
& = \frac{1}{c^2 n \Gamma\left(1+\frac{1}{p}\right)} \left(\Gamma\left(2-\frac{1}{p}\right) - (\alpha+1)e^{-\alpha} \right)
\end{aligned}
$$

Taking α a constant big enough and then c a constant small enough we have that if $p \le Cn$ for some absolute constant $C < 1$,

$$M\left(\frac{1}{c\sqrt{\frac{n}{p}}}\right) \ge \frac{1}{n}$$

and so

$$\|(1,\ldots,1)\|_M \ge c\sqrt{\frac{n}{p}}.$$

Consequently,

$$\mathbb{E}\psi_{\theta_0} \ge \frac{c}{\sqrt{p}}.$$

On the other hand, since

$$
\begin{aligned}
M(s) & = \frac{2\left(1-\frac{1}{p}\right)}{p\Gamma\left(1+\frac{1}{p}\right)} \int_0^s t \int_0^{t^{-p^*}} r^{-\frac{1}{p}} e^{-r} dr dt \\
& \ge \frac{2\left(1-\frac{1}{p}\right)}{p\Gamma\left(1+\frac{1}{p}\right)} \int_0^s t \int_0^{t^{-p^*}} t^{\frac{1}{p-1}} e^{-r} dr dt \\
& = \frac{2\left(1-\frac{1}{p}\right)}{p\Gamma\left(1+\frac{1}{p}\right)} \int_0^s t^{1+\frac{1}{p-1}} \left(1 - e^{-t^{-p^*}}\right) dt \\
& \ge \frac{2\left(1-\frac{1}{p}\right)}{p\Gamma\left(1+\frac{1}{p}\right)} \int_0^s t^{1+\frac{1}{p-1}} \left(1 - e^{-s^{-p^*}}\right) dt \\
& = \frac{2\left(1-\frac{1}{p}\right) s^{2+\frac{1}{p-1}}}{p\left(2+\frac{1}{p-1}\right)\Gamma\left(1+\frac{1}{p}\right)} \left(1 - e^{-s^{-p^*}}\right).
\end{aligned}
$$

we have that if $Cn \leq p \leq n$, $\rho = \alpha\sqrt{\frac{n}{p}}$ with $\alpha \leq 1$, there is an absolute constant c such that

$$
\begin{aligned}
M\left(\frac{1}{\rho}\right) &\geq \frac{cp^{\frac{1}{2p-2}}}{\alpha^{2+\frac{1}{p-1}}n^{1+\frac{1}{2p-2}}}\left(1 - e^{-\alpha^{p^*}}\right) \\
&\geq \frac{cp^{\frac{1}{2p-2}}}{\alpha^{2+\frac{1}{p-1}}n^{1+\frac{1}{2p-2}}}\left(1 - e^{-\alpha^{p^*}}\right) \\
&\geq \frac{c}{\alpha n},
\end{aligned}
$$

since $p \sim n$. If we take α a constant small enough,

$$
M\left(\frac{1}{\alpha\sqrt{\frac{n}{p}}}\right) \geq \frac{1}{n}
$$

and so

$$
\|(1,\ldots,1)\|_M \geq \alpha\sqrt{\frac{n}{p}}.
$$

Consequently,

$$
\mathbb{E}\psi_{\theta_0} \geq \frac{c}{\sqrt{p}}
$$

also if $Cn \leq p \leq n$.

By Lemma 3.3,

$$
\mathbb{E}\psi_{\theta_0} \leq \frac{c_2}{p}\|\theta_0\|_1 = \frac{c_2\sqrt{n}}{p}
$$

Consequently, if $p = n^\gamma$ with $\gamma > 1$,

$$
\mathbb{E}\psi_{\theta_0} \leq \frac{c_2\sqrt{n}}{p} = \frac{c_2}{p^{1-\frac{1}{2\gamma}}}.
$$

On the other hand, since $p \geq n$, if $n \geq 2$

$$
\begin{aligned}
M(s) &= \frac{2\left(1-\frac{1}{p}\right)}{p\Gamma\left(1+\frac{1}{p}\right)}\int_0^s t\int_0^{t^{-p^*}} r^{-\frac{1}{p}}e^{-r}\,dr\,dt \\
&\geq \frac{2\left(1-\frac{1}{p}\right)}{p\Gamma\left(1+\frac{1}{p}\right)}\int_0^s t\int_0^{t^{-p^*}} r^{-\frac{1}{p}}e^{-t^{-p^*}}\,dr\,dt \\
&= \frac{2}{p\Gamma\left(1+\frac{1}{p}\right)}\int_0^s e^{-t^{-p^*}}\,dt \\
&\geq \frac{2}{p\Gamma\left(1+\frac{1}{p}\right)}\int_{s2^{-\frac{1}{p^*}}}^s e^{-t^{-p^*}}\,dt \\
&\geq \frac{2s}{p\Gamma\left(1+\frac{1}{p}\right)}\left(1 - 2^{-\frac{1}{p^*}}\right)e^{-2s^{-p^*}} \\
&\geq \frac{2s}{p\Gamma\left(1+\frac{1}{p}\right)}\left(1 - 2^{-\frac{n-1}{n}}\right)e^{-2s^{-p^*}} \\
&\geq \frac{\sqrt{2}(\sqrt{2}-1)s}{p\Gamma\left(1+\frac{1}{p}\right)}e^{-2s^{-p^*}}.
\end{aligned}
$$

and then

$$M\left(\frac{1}{s}\right) \geq \frac{\sqrt{2}(\sqrt{2}-1)}{ps}e^{-2s^{p^*}}.$$

Thus, if $p = n^\gamma$ and we take $s = \alpha n^{1-\gamma}$, with $\alpha \leq 1$

$$\begin{aligned}
M\left(\frac{1}{\alpha n^{1-\gamma}}\right) &\geq \frac{\sqrt{2}(\sqrt{2}-1)}{n}e^{-2\alpha\frac{n^\gamma}{n^\gamma-1}}n^{\frac{(1-\gamma)n^\gamma}{n^\gamma-1}} \\
&\geq \frac{\sqrt{2}(\sqrt{2}-1)}{n}e^{-2\alpha\frac{n^\gamma}{n^\gamma-1}} \\
&\geq \frac{\sqrt{2}(\sqrt{2}-1)}{n}e^{-2\alpha} \\
&\geq \frac{1}{n}
\end{aligned}$$

if we take $\alpha \leq \frac{1}{2}\log\left(\sqrt{2}(\sqrt{2}-1)\right)$. Consequently,

$$\mathbb{E}\psi_{\theta_0} \geq cn^{\frac{1}{2}-\gamma} = \frac{c\sqrt{n}}{p} = \frac{c}{p^{1-\frac{1}{2\gamma}}}.$$

$\square$

Hence, we obtain the following

Corollary 3.1. *If* $1 \leq p \leq n$*. Then,*

$$\sigma\left\{\theta \in S^{n-1} : \mathbb{E}\psi_\theta \sim \frac{1}{\sqrt{p}}\right\} \geq 1 - \frac{1}{2^n}.$$

If $p > n$*, then for every* $\theta \in S^{n-1}$

$$\mathbb{E}\psi_\theta \sim \frac{1}{p}\|\theta\|_1.$$

Proof. The first estimate is a consequence of part c) in Lemma 3.3 and Lemma 3.4. The second estimate is a consequence of part b) in Lemma 3.3 and Lemma 3.4. $\square$

Remark. Actually, it can be proved that for any $n \in \mathbb{N}$ and any fixed $\theta \in S^{n-1}$, $\lim_{p\to\infty} p\mathbb{E}\psi_\theta = \|\theta\|_1$.

Lemma 3.5. *Let* $I \subseteq \{1,\ldots,n\}$ *be any set of indices and* $\theta \in S^{n-1}$*. Then,*

$$\frac{\mathbb{E}\left|\sum_{i\in I}|g_i|^{p-1}\operatorname{sgn}(g_i)\theta_i\right|}{\mathbb{E}\left|\sum_{i=1}^n |g_i|^{p-1}\operatorname{sgn}(g_i)\theta_i\right|} \leq 1$$

and

$$\frac{\mathbb{E}\left(\sum_{i\in I}|g_i|^{2p-2}\theta_i^2\right)^{\frac{1}{2}}}{\mathbb{E}\left(\sum_{i=1}^n |g_i|^{2p-2}\theta_i^2\right)^{\frac{1}{2}}} \leq C,$$

where C *is an absolute constant.*

Proof. By the triangle inequality, we have that

$$2\left|\sum_{i\in I}|g_i|^{p-1}\operatorname{sgn}(g_i)\theta_i\right| \leq \left|\sum_{i\in I}|g_i|^{p-1}\operatorname{sgn}(g_i)\theta_i + \sum_{i\in I^c}|g_i|^{p-1}\operatorname{sgn}(g_i)\theta_i\right|$$
$$+ \left|\sum_{i\in I}|g_i|^{p-1}\operatorname{sgn}(g_i)\theta_i - \sum_{i\in I^c}|g_i|^{p-1}\operatorname{sgn}(g_i)\theta_i\right|.$$

Since the random variables g_i are symmetric, the expected value of the second term equals the expected value of the first term and then

$$2\mathbb{E}\left|\sum_{i\in I}|g_i|^{p-1}\operatorname{sgn}(g_i)\theta_i\right| \leq 2\mathbb{E}\left|\sum_{i=1}^n|g_i|^{p-1}\operatorname{sgn}(g_i)\theta_i\right| = 2\mathbb{E}\psi,$$

which proves the first inequality. The second inequality is a consequence of the first one and Khintchine's inequality. $\qquad\square$

4. The variance conjecture on hyperplane projections of B_p^n

In this section we prove Theorem 1.1.

Proof. First of all, notice that, by Proposition 4 in [AB1], for any $\xi \in S^{n-1} \cap H$ we have that if X is a random vector uniformly distributed on $P_H(B_p^n)$

$$\mathbb{E}\langle|B_p^n|^{-\frac{1}{n}}X,\xi\rangle^2 \sim L_{B_p^n}^2 \sim 1.$$

Thus,

$$\mathbb{E}\langle X,\xi\rangle^2 \sim |B_p^n|^{\frac{2}{n}} \sim \frac{1}{n^{\frac{2}{p}}}$$

and so

$$\lambda_X^2 \mathbb{E}|X|^2 \sim n^{1-\frac{4}{p}}.$$

Now, using the probabilistic representation of X mentioned in Section 2, we have that

$$\begin{aligned}
\operatorname{Var}|X|^2 &= \mathbb{E}|X|^4 - (\mathbb{E}|X|^2)^2 \\
&= \frac{1}{\mathbb{E}\psi}\mathbb{E}\left|P_H\left(\frac{G}{S}\right)\right|^4\psi - \frac{1}{(\mathbb{E}\psi)^2}\left(\mathbb{E}\left|P_H\left(\frac{G}{S}\right)\right|^2\psi\right)^2 \\
&= \frac{1}{\mathbb{E}\psi}\mathbb{E}\left(\left|\frac{G}{S}\right|^2 - \left\langle\frac{G}{S},\theta\right\rangle^2\right)^2\psi \\
&\quad - \frac{1}{(\mathbb{E}\psi)^2}\left(\mathbb{E}\left|\frac{G}{S}\right|^2\psi - \mathbb{E}\left\langle\frac{G}{S},\theta\right\rangle^2\psi\right)^2 \\
&\leq \frac{1}{\mathbb{E}\psi}\mathbb{E}\left|\frac{G}{S}\right|^4\psi - \left(\frac{1}{\mathbb{E}\psi}\mathbb{E}\left|\frac{G}{S}\right|^2\psi\right)^2 \\
&\quad + \frac{1}{\mathbb{E}\psi}\mathbb{E}\left\langle\frac{G}{S},\theta\right\rangle^4\psi + 2\frac{1}{(\mathbb{E}\psi)^2}\mathbb{E}\left|\frac{G}{S}\right|^2\psi\,\mathbb{E}\left\langle\frac{G}{S},\theta\right\rangle^2\psi \\
&= \sum_{i=1}^n\left(\frac{1}{\mathbb{E}\psi}\mathbb{E}\frac{g_i^4}{S^4}\psi - \left(\frac{1}{\mathbb{E}\psi}\mathbb{E}\frac{g_i^2}{S^2}\psi\right)^2\right)
\end{aligned}$$

$$+ \quad \sum_{i\neq j} \left(\frac{1}{\mathbb{E}\psi}\mathbb{E}\frac{g_i^2 g_j^2}{S^4}\psi - \frac{1}{(\mathbb{E}\psi)^2}\mathbb{E}\frac{g_i^2}{S^2}\psi\mathbb{E}\frac{g_j^2}{S^2}\psi \right)$$

$$+ \quad \frac{1}{\mathbb{E}\psi}\mathbb{E}\left\langle \frac{G}{S},\theta \right\rangle^4 \psi + \frac{2}{(\mathbb{E}\psi)^2}\mathbb{E}\left| \frac{G}{S} \right|^2 \psi\,\mathbb{E}\left\langle \frac{G}{S},\theta \right\rangle^2 \psi.$$

We are going to bound from above each one of the four summands in the last expression. The upper bound of the first, third, and fourth term will be of the order that would give an absolute constant in the variance conjecture. The estimate we obtain for the second term will be the one that will cause the constant to depend on p if $p \leq n$.

4.1. **Upper bound for the last term.**
By the independence of $\frac{G}{S}$ and S we have that for any $\theta \in S^{n-1}$

$$\frac{1}{\mathbb{E}\psi}\mathbb{E}\left\langle \frac{G}{S},\theta \right\rangle^2 \psi \quad = \quad \frac{\mathbb{E}\left\langle \frac{G}{S},\theta \right\rangle^2 \left| \sum_{i=1}^n \frac{|g_i|^{p-1}}{S^{p-1}}\operatorname{sgn}(g_i)\theta_i \right|}{\mathbb{E}\left| \sum_{i=1}^n \frac{|g_i|^{p-1}}{S^{p-1}}\operatorname{sgn}(g_i)\theta_i \right|}$$

$$= \quad \frac{\mathbb{E}S^{p-1}}{\mathbb{E}S^{p+1}}\frac{\mathbb{E}(\sum_{i=1}^n g_i\theta_i)^2 \left| \sum_{i=1}^n |g_i|^{p-1}\operatorname{sgn}(g_i)\theta_i \right|}{\mathbb{E}\left| \sum_{i=1}^n |g_i|^{p-1}\operatorname{sgn}(g_i)\theta_i \right|}.$$

Taking $\varepsilon_1,\ldots,\varepsilon_n$ independent Bernoulli random variables also independent with respect to the g_i's we have that

$$\frac{\mathbb{E}S^{p-1}}{\mathbb{E}S^{p+1}}\frac{\mathbb{E}(\sum_{i=1}^n g_i\theta_i)^2 \left| \sum_{i=1}^n |g_i|^{p-1}\operatorname{sgn}(g_i)\theta_i \right|}{\mathbb{E}\left| \sum_{i=1}^n |g_i|^{p-1}\operatorname{sgn}(g_i)\theta_i \right|}$$

$$= \quad \frac{\mathbb{E}S^{p-1}}{\mathbb{E}S^{p+1}}\frac{\mathbb{E}_\varepsilon \mathbb{E}_g(\sum_{i=1}^n \varepsilon_i g_i\theta_i)^2 \left| \sum_{i=1}^n |g_i|^{p-1}\operatorname{sgn}(g_i)\varepsilon_i\theta_i \right|}{\mathbb{E}_g\left| \sum_{i=1}^n |g_i|^{p-1}\operatorname{sgn}(g_i)\theta_i \right|}$$

$$\leq \quad \frac{\mathbb{E}S^{p-1}}{\mathbb{E}S^{p+1}}\frac{\mathbb{E}_g\left(\mathbb{E}_\varepsilon(\sum_{i=1}^n \varepsilon_i g_i\theta_i)^4\right)^{\frac{1}{2}}\left(\mathbb{E}_\varepsilon\left| \sum_{i=1}^n |g_i|^{p-1}\operatorname{sgn}(g_i)\varepsilon_i\theta_i \right|^2\right)^{\frac{1}{2}}}{\mathbb{E}_g\left| \sum_{i=1}^n |g_i|^{p-1}\operatorname{sgn}(g_i)\theta_i \right|}.$$

By Khintchine's inequality, Lemma 3.1, Lemma 3.3 and Lemma 3.5

$$\frac{1}{\mathbb{E}\psi}\mathbb{E}\left\langle \frac{G}{S},\theta \right\rangle^2 \psi \quad \leq \quad C\frac{\mathbb{E}S^{p-1}}{\mathbb{E}S^{p+1}}\frac{\mathbb{E}_g\left(\sum_{i=1}^n g_i^2\theta_i^2\right)\left(\sum_{j=1}^n |g_j|^{2p-2}\theta_j^2\right)^{\frac{1}{2}}}{\mathbb{E}_g\left| \sum_{j=1}^n |g_j|^{p-1}\operatorname{sgn}(g_j)\theta_j \right|}$$

$$= \quad \frac{\mathbb{E}S^{p-1}}{\mathbb{E}S^{p+1}}\frac{\sum_{i=1}^n \theta_i^2\mathbb{E}_g g_i^2\left(\sum_{j=1}^n |g_j|^{2p-2}\theta_j^2\right)^{\frac{1}{2}}}{\mathbb{E}_g\left| \sum_{j=1}^n |g_j|^{p-1}\operatorname{sgn}(g_j)\theta_j \right|}$$

$$\leq \quad \frac{\mathbb{E}S^{p-1}}{\mathbb{E}S^{p+1}}\frac{\sum_{i=1}^n \theta_i^2\mathbb{E}_g g_i^2\left(|g_i|^{p-1}|\theta_i| + \left(\sum_{j\neq i} |g_j|^{2p-2}\theta_j^2\right)^{\frac{1}{2}}\right)}{\mathbb{E}_g\left| \sum_{j=1}^n |g_j|^{p-1}\operatorname{sgn}(g_j)\theta_j \right|}$$

$$= \quad \frac{\mathbb{E}S^{p-1}}{\mathbb{E}S^{p+1}}\frac{\sum_{i=1}^n \theta_i^2\left(\mathbb{E}_g|g_i|^{p+1}|\theta_i| + \mathbb{E}_g g_i^2\mathbb{E}_g\left(\sum_{j\neq i} |g_j|^{2p-2}\theta_j^2\right)^{\frac{1}{2}}\right)}{\mathbb{E}_g\left| \sum_{j=1}^n |g_j|^{p-1}\operatorname{sgn}(g_j)\theta_j \right|}$$

$$\leq \quad \frac{\mathbb{E}S^{p-1}}{\mathbb{E}S^{p+1}}\sum_{i=1}^n \theta_i^2(C_1|\theta_i| + C_2) \leq C\frac{\mathbb{E}S^{p-1}}{\mathbb{E}S^{p+1}}.$$

By Lemma 3.1 we have $\frac{\mathbb{E}S^{p-1}}{\mathbb{E}S^{p+1}} \sim \frac{1}{n^{\frac{2}{p}}}$. Thus

$$\frac{1}{\mathbb{E}\psi}\mathbb{E}\left\langle \frac{G}{S}, \theta \right\rangle^2 \psi \quad \leq \quad \frac{C}{n^{\frac{2}{p}}}.$$

Also, as before,

$$\begin{aligned}
\frac{1}{\mathbb{E}\psi}\mathbb{E}\left|\frac{G}{S}\right|^2 \psi &= \frac{\mathbb{E}S^{p-1}}{\mathbb{E}S^{p+1}}\frac{1}{\mathbb{E}\psi}\sum_{i=1}^{n}\mathbb{E}g_i^2\psi \\
&\leq \frac{C}{n^{\frac{2}{p}}\mathbb{E}\psi}\sum_{i=1}^{n}\left(\mathbb{E}|g_i|^{p+1}|\theta_i| + \mathbb{E}_g g_i^2 \mathbb{E}_g\left(\sum_{j\neq i}|g_j|^{2p-2}\theta_j^2\right)^{\frac{1}{2}}\right) \\
&\leq \frac{C}{n^{\frac{2}{p}}}\left(\|\theta\|_1 + n\right) \\
&\leq Cn^{1-\frac{2}{p}}
\end{aligned}$$

and so

$$\frac{1}{(\mathbb{E}\psi)^2}\mathbb{E}\left|\frac{G}{S}\right|^2 \psi \mathbb{E}\left\langle \frac{G}{S}, \theta \right\rangle^2 \psi \leq Cn^{1-\frac{4}{p}}.$$

4.2. Upper bound for the first and third term.

Similarly, by the independence of $\frac{G}{S}$ and S, Hölder's inequality, Khintchine's inequality, Lemma 3.1 and Lemma 3.5 we have

$$\frac{1}{\mathbb{E}\psi}\mathbb{E}\left\langle \frac{G}{S}, \theta \right\rangle^4 \psi = \frac{\mathbb{E}S^{p-1}}{\mathbb{E}S^{p+3}\mathbb{E}\psi}\mathbb{E}_g\mathbb{E}_\varepsilon\left(\sum_{i=1}^{n}g_i\,\mathrm{sgn}(g_i)\varepsilon_i\theta_i\right)^4\left|\sum_{j=1}^{n}|g_j|^{p-1}\mathrm{sgn}(g_j)\varepsilon_j\theta_j\right|$$

$$\leq \frac{\mathbb{E}S^{p-1}}{\mathbb{E}S^{p+3}\mathbb{E}\psi}\mathbb{E}_g\left(\mathbb{E}_\varepsilon\left(\sum_{i=1}^{n}g_i\,\mathrm{sgn}(g_i)\varepsilon_i\theta_i\right)^8\right)^{\frac{1}{2}}\left(\mathbb{E}_\varepsilon\left|\sum_{j=1}^{n}|g_j|^{p-1}\mathrm{sgn}(g_j)\varepsilon_j\theta_j\right|^2\right)^{\frac{1}{2}}$$

$$\sim \frac{\mathbb{E}S^{p-1}}{\mathbb{E}S^{p+3}\mathbb{E}\psi}\mathbb{E}\left(\sum_{i=1}^{n}g_i^2\theta_i^2\right)^2\left|\sum_{j=1}^{n}|g_j|^{2p-2}\theta_j^2\right|^{\frac{1}{2}}$$

$$\leq \frac{\mathbb{E}S^{p-1}}{\mathbb{E}S^{p+3}\mathbb{E}\psi}\mathbb{E}\sum_{i=1}^{n}g_i^4\theta_i^2\left|\sum_{j=1}^{n}|g_j|^{2p-2}\theta_j^2\right|^{\frac{1}{2}}$$

$$= \frac{\mathbb{E}S^{p-1}}{\mathbb{E}S^{p+3}\mathbb{E}\psi}\sum_{i=1}^{n}\theta_i^2\mathbb{E}g_i^4\left|\sum_{j=1}^{n}|g_j|^{2p-2}\theta_j^2\right|^{\frac{1}{2}}$$

$$\leq \frac{\mathbb{E}S^{p-1}}{\mathbb{E}S^{p+3}\mathbb{E}\psi}\sum_{i=1}^{n}\theta_i^2\mathbb{E}g_i^4\left(|g_i|^{p-1}|\theta_i| + \left(\sum_{j\neq i}|g_j|^{2p-2}\theta_j^2\right)^{\frac{1}{2}}\right)$$

$$= \frac{\mathbb{E}S^{p-1}}{\mathbb{E}S^{p+3}\mathbb{E}\psi}\sum_{i=1}^{n}\theta_i^2\left(\mathbb{E}|g_i|^{p+3}|\theta_i| + \mathbb{E}g_i^4\mathbb{E}\left(\sum_{j\neq i}|g_j|^{2p-2}\theta_j^2\right)^{\frac{1}{2}}\right)$$

$$\leq \frac{\mathbb{E}S^{p-1}}{\mathbb{E}S^{p+3}}C\sum_{i=1}^{n}\theta_i^2$$

$$\leq \quad Cn^{-\frac{4}{p}}$$

since, by Lemma 3.1, $\frac{\mathbb{E}S^{p-1}}{\mathbb{E}S^{p+3}} \sim n^{-\frac{4}{p}}$. This bounds the third term. Besides, this estimate implies the following bound on the first term:

$$\sum_{i=1}^{n}\left(\frac{1}{\mathbb{E}\psi}\mathbb{E}\frac{g_i^4}{S^4}\psi - \left(\frac{1}{\mathbb{E}\psi}\mathbb{E}\frac{g_i^2}{S^2}\psi\right)^2\right) \leq \sum_{i=1}^{n}\frac{1}{\mathbb{E}\psi}\mathbb{E}\frac{g_i^4}{S^4}\psi \leq Cn^{1-\frac{4}{p}}.$$

4.3. Upper bound for the second term.

It remains to bound the second term

$$\sum_{i\neq j}\left(\frac{1}{\mathbb{E}\psi}\mathbb{E}\frac{g_i^2 g_j^2}{S^4}\psi - \frac{1}{(\mathbb{E}\psi)^2}\mathbb{E}\frac{g_i^2}{S^2}\psi\mathbb{E}\frac{g_j^2}{S^2}\psi\right).$$

For any $i \neq j$ we have

$$\frac{1}{\mathbb{E}\psi}\mathbb{E}\frac{g_i^2 g_j^2}{S^4}\psi - \frac{1}{(\mathbb{E}\psi)^2}\mathbb{E}\frac{g_i^2}{S^2}\psi\mathbb{E}\frac{g_j^2}{S^2}\psi = \frac{\mathbb{E}S^{p-1}}{\mathbb{E}S^{p+3}}\frac{\mathbb{E}g_i^2 g_j^2\psi}{\mathbb{E}\psi} - \left(\frac{\mathbb{E}S^{p-1}}{\mathbb{E}S^{p+1}}\right)^2\frac{\mathbb{E}g_i^2\psi\mathbb{E}g_j^2\psi}{(\mathbb{E}\psi)^2}$$

$$= \frac{(\mathbb{E}S^{p-1})^2}{(\mathbb{E}S^{p+1})^2(\mathbb{E}\psi)^2}\left(\frac{(\mathbb{E}S^{p+1})^2}{\mathbb{E}S^{p-1}\mathbb{E}S^{p+3}}\mathbb{E}g_i^2 g_j^2\psi\mathbb{E}\psi - \mathbb{E}g_i^2\psi\mathbb{E}g_j^2\psi\right).$$

By Hölder's inequality, $(\mathbb{E}S^{p+1})^2 \leq \mathbb{E}S^{p-1}\mathbb{E}S^{p+3}$. Then, we have

$$\frac{1}{\mathbb{E}\psi}\mathbb{E}\frac{g_i^2 g_j^2}{S^4}\psi - \frac{1}{(\mathbb{E}\psi)^2}\mathbb{E}\frac{g_i^2}{S^2}\psi\mathbb{E}\frac{g_j^2}{S^2}\psi \leq \frac{(\mathbb{E}S^{p-1})^2}{(\mathbb{E}S^{p+1})^2(\mathbb{E}\psi)^2}\left(\mathbb{E}g_i^2 g_j^2\psi\mathbb{E}\psi - \mathbb{E}g_i^2\psi\mathbb{E}g_j^2\psi\right).$$

Note that if $\{\bar{g}_i\}_{i=1}^{n}$ are independent copies of g, independent of $\{g_i\}_{i=1}^{n}$, and $\bar{\psi} = \left|\sum_{i=1}^{n}|\bar{g}_i|^{p-1}\operatorname{sgn}(\bar{g}_i)\theta_i\right|$, we have that

$$\begin{aligned}
\mathbb{E}g_i^2 g_j^2\psi\mathbb{E}\psi - \mathbb{E}g_i^2\psi\mathbb{E}g_j^2\psi &= \mathbb{E}_{g\otimes\bar{g}}g_i^2(g_j^2 - \bar{g}_j^2)\psi\bar{\psi} \\
&= \mathbb{E}_{g\otimes\bar{g}}\bar{g}_i^2(\bar{g}_j^2 - g_j^2)\psi\bar{\psi}.
\end{aligned}$$

Thus,

$$\mathbb{E}g_i^2 g_j^2\psi\mathbb{E}\psi - \mathbb{E}g_i^2\psi\mathbb{E}g_j^2\psi = \frac{1}{2}\mathbb{E}_{g\otimes\bar{g}}(g_i^2 - \bar{g}_i^2)(g_j^2 - \bar{g}_j^2)\psi\bar{\psi}$$

and so,

$$\sum_{i\neq j}\frac{1}{\mathbb{E}\psi}\mathbb{E}\frac{g_i^2 g_j^2}{S^4}\psi - \frac{1}{(\mathbb{E}\psi)^2}\mathbb{E}\frac{g_i^2}{S^2}\psi\mathbb{E}\frac{g_j^2}{S^2}\psi \leq \frac{(\mathbb{E}S^{p-1})^2}{2(\mathbb{E}S^{p+1})^2(\mathbb{E}\psi)^2}\mathbb{E}\psi\bar{\psi}\sum_{i\neq j}(g_i^2 - \bar{g}_i^2)(g_j^2 - \bar{g}_j^2)$$

$$\leq \frac{(\mathbb{E}S^{p-1})^2}{2(\mathbb{E}S^{p+1})^2(\mathbb{E}\psi)^2}\mathbb{E}\psi\bar{\psi}\left(\sum_{i=1}^{n}(g_i^2 - \bar{g}_i^2)\right)^2 \sim \frac{n^{-\frac{4}{p}}}{(\mathbb{E}\psi)^2}\mathbb{E}\psi\bar{\psi}\left(\sum_{i=1}^{n}(g_i^2 - \bar{g}_i^2)\right)^2.$$

Now, for any $\alpha \geq 1$, this is bounded by

$$\leq n^{-\frac{4}{p}}\left(\frac{\mathbb{E}\psi\bar{\psi}\left(\sum_{i=1}^{n}(g_i^2 - \bar{g}_i^2)\right)^{2\alpha}}{(\mathbb{E}\psi)^2}\right)^{\frac{1}{\alpha}} \leq n^{-\frac{4}{p}}\frac{(\mathbb{E}\psi^2)^{\frac{1}{\alpha}}}{(\mathbb{E}\psi)^{\frac{2}{\alpha}}}\left(\mathbb{E}\left(\sum_{i=1}^{n}(g_i^2 - \bar{g}_i^2)\right)^{4\alpha}\right)^{\frac{1}{2\alpha}}.$$

By Lemma 3.3, $\frac{\left(\mathbb{E}\psi^2\right)^{\frac{1}{\alpha}}}{(\mathbb{E}\psi)^{\frac{2}{\alpha}}} \leq Cp^{\frac{1}{\alpha}}$ and, taking $\alpha \sim \log p$ we have by Lemma 3.2 that

$$\left(\mathbb{E}\left(\sum_{i=1}^{n}(g_i^2 - \bar{g}_i^2)\right)^{4\alpha}\right)^{\frac{1}{2\alpha}} \sim n\log p$$

and so

$$\sum_{i \neq j}\left(\frac{1}{\mathbb{E}\psi}\mathbb{E}\frac{g_i^2 g_j^2}{S^4}\psi - \frac{1}{(\mathbb{E}\psi)^2}\mathbb{E}\frac{g_i^2}{S^2}\psi\mathbb{E}\frac{g_j^2}{S^2}\psi\right) \leq Cn^{1-\frac{4}{p}}\log(1+p).$$

Besides, by Corollary 3.1, if $1 \leq p \leq n$, $\frac{\left(\mathbb{E}\psi^2\right)^{\frac{1}{\alpha}}}{(\mathbb{E}\psi)^{\frac{2}{\alpha}}} \leq C$ for a set of directions of measure greater than $1 - \frac{1}{2^n}$. Taking $\alpha = 2$ when θ belongs to this set we obtain

$$\sum_{i \neq j}\left(\frac{1}{\mathbb{E}\psi}\mathbb{E}\frac{g_i^2 g_j^2}{S^4}\psi - \frac{1}{(\mathbb{E}\psi)^2}\mathbb{E}\frac{g_i^2}{S^2}\psi\mathbb{E}\frac{g_j^2}{S^2}\psi\right) \leq Cn^{1-\frac{4}{p}}.$$

This finishes the proof in the case $p \leq n$.

In the case that $p > n$ we take into account that, since the random variables g_i are independent and identically distributed

$$
\begin{aligned}
\mathbb{E}\psi\bar{\psi}\left(\sum_{i=1}^{n}(g_i^2 - \bar{g}_i^2)\right)^2 &= \mathbb{E}\psi\bar{\psi}\left(\sum_{i=1}^{n}(g_i^2 - \mathbb{E}g_i^2) + \sum_{i=1}^{n}(\mathbb{E}\bar{g}_i^2 - \bar{g}_i^2)\right)^2 \\
&\leq \sqrt{2}\mathbb{E}\psi\bar{\psi}\left[\left(\sum_{i=1}^{n}(g_i^2 - \mathbb{E}g_i^2)\right)^2 + \left(\sum_{i=1}^{n}(\bar{g}_i^2 - \mathbb{E}\bar{g}_i^2)\right)^2\right] \\
&= 2\sqrt{2}\mathbb{E}\psi\mathbb{E}\psi\left(\sum_{i=1}^{n}(g_i^2 - \mathbb{E}g_i^2)\right)^2 \\
&\leq 2\sqrt{2}\mathbb{E}\psi\mathbb{E}\sum_{j=1}^{n}|g_j|^{p-1}|\theta_j|\left(\sum_{i=1}^{n}(g_i^2 - \mathbb{E}g_i^2)\right)^2 \\
&= 2\sqrt{2}\|\theta\|_1\mathbb{E}\psi\mathbb{E}|g_1|^{p-1}\left(\sum_{i=1}^{n}(g_i^2 - \mathbb{E}g_i^2)\right)^2 \\
&= 2\sqrt{2}\|\theta\|_1\mathbb{E}\psi\mathbb{E}|g_1|^{p-1}\sum_{i=1}^{n}(g_i^2 - \mathbb{E}g_i^2)^2 \\
&\leq \frac{Cn\|\theta\|_1\mathbb{E}\psi}{p}.
\end{aligned}
$$

Since by part b) in Lemma 3.3

$$\mathbb{E}\psi_\theta \geq \frac{c_1}{\sqrt{n}}\mathbb{E}\psi_{\theta_0}\|\theta\|_1,$$

we have that

$$\frac{n^{-\frac{4}{p}}}{(\mathbb{E}\psi)^2}\mathbb{E}\psi\bar{\psi}\left(\sum_{i=1}^{n}(g_i^2 - \bar{g}_i^2)\right)^2 \leq C\frac{\sqrt{n}}{p\mathbb{E}\psi_{\theta_0}}n^{1-\frac{4}{p}}$$

and, since $p \geq n$, by Lemma 3.4 $\mathbb{E}\psi_{\theta_0} \sim \frac{\sqrt{n}}{p}$ and we obtain the result. $\qquad\square$

5. Hyperplane projections of isotropic random vectors and Steiner
symmetrization

In this section we will show how the variance conjecture for an isotropic log-concave random vector relates to the variance conjecture for its hyperplane projections or for its Steiner symmetrizations (when the vector is uniformly distributed on an isotropic body).

Proposition 5.1. *Let μ be a log-concave probability on $\mathbb{R}^n$ and X a random vector distributed according to μ. Then for any linear subspace E*

$$\left|\sqrt{Var|X|^2} - \sqrt{Var|P_E X|^2}\right| \leq \sqrt{Var|P_{E^\perp} X|^2}.$$

Proof. For any linear subspace E,

$$|X|^2 = |P_E(X)|^2 + |P_{E^\perp}(X)|^2.$$

Thus,

$$
\begin{aligned}
\mathrm{Var}|X|^2 \ &= \ \mathrm{Var}|P_E X|^2 + \mathrm{Var}|P_{E^\perp} X|^2 \\
&+ \ 2\left(\mathbb{E}|P_E X|^2 |P_{E^\perp} X|^2 - \mathbb{E}|P_E X|^2 \mathbb{E}|P_{E^\perp} X|^2\right) \\
&= \ \mathrm{Var}|P_E X|^2 + \mathrm{Var}|P_{E^\perp} X|^2 \\
&+ \ 2\left(\mathbb{E}|P_E X|^2(|X|^2 - |P_E X|^2) - \mathbb{E}|P_E X|^2 \mathbb{E}(|X|^2 - |P_E X|^2)\right) \\
&= \ \mathrm{Var}|P_E X|^2 + \mathrm{Var}|P_{E^\perp} X|^2 \\
&+ \ 2\mathbb{E}|P_E X|^2(|X|^2 - \mathbb{E}|X|^2) - 2\mathrm{Var}|P_E X|^2 \\
&= \ \mathrm{Var}|P_{E^\perp} X|^2 - \mathrm{Var}|P_E X|^2 \\
&+ \ 2\mathbb{E}(|P_E X|^2 - \mathbb{E}|P_E X|^2)(|X|^2 - \mathbb{E}|X|^2) \\
&\leq \ \mathrm{Var}|P_{E^\perp} X|^2 - \mathrm{Var}|P_E X|^2 \\
&+ \ 2\sqrt{\mathrm{Var}|P_E X|^2}\sqrt{\mathrm{Var}|X|^2}.
\end{aligned}
$$

Consequently,

$$\mathrm{Var}|X|^2 - 2\sqrt{\mathrm{Var}|P_E X|^2}\sqrt{\mathrm{Var}|X|^2} - \mathrm{Var}|P_{E^\perp} X|^2 + \mathrm{Var}|P_E X|^2 \leq 0.$$

Since the roots of the polyomial

$$p(x) = x^2 - 2\sqrt{\mathrm{Var}|P_E X|^2}\,x - \mathrm{Var}|P_{E^\perp} X|^2 + \mathrm{Var}|P_E X|^2$$

are

$$\sqrt{\mathrm{Var}|P_E X|^2} \pm \sqrt{\mathrm{Var}|P_{E^\perp} X|^2},$$

we obtain the result.   □

As a consequence, we have the following

Theorem 5.1. *Let X be an isotropic log-concave random vector. Then the following are equivalent*

- *There exists a constant C_1 such that*

$$Var|X|^2 \leq C_1 n.$$

- *There exists a constant C_2 such that*

$$Var|P_E X|^2 \leq C_2(n-1)$$

for some hyperplane E.

- *There exists a constant C_3 such that*

$$Var|P_E X|^2 \leq C_3(n-1)$$

   for every hyperplane E,

where

$$C_2 \leq C_3 \leq 4\left(C_1 + \frac{C}{n}\right) \ \text{ and } \ C_1 \leq 2\left(C_2 + \frac{C}{n}\right),$$

with C an absolute constant.

Proof. Let $E = \theta^\perp$ be a hyperplane and X an isotropic log-concave random vector. Since X is isotropic, also $P_E X$ is isotropic. Thus, if X verifies the variance conjecture with constant C_1 then, for every hyperplane $E = \theta^\perp$

$$
\begin{aligned}
\sqrt{Var|P_E X|^2} \ &\leq \ \sqrt{Var|X|^2} + \sqrt{Var\langle X,\theta\rangle^2} \\
&\leq \ \sqrt{C_1 n} + \sqrt{\mathbb{E}\langle X,\theta\rangle^4} \\
&\leq \ \sqrt{2}\sqrt{C_1 n + \mathbb{E}\langle X,\theta\rangle^4}.
\end{aligned}
$$

By Borell's inequality

$$\sqrt{Var|P_E X|^2} \leq \sqrt{2}\sqrt{C_1 n + C'} = \sqrt{2}\sqrt{C_1 + \frac{C}{n}}\sqrt{n}.$$

Thus, there exists an absolute constant C such that

$$Var|P_E X|^2 \leq 2\left(C_1 + \frac{C}{n}\right)n \leq 4\left(C_1 + \frac{C}{n}\right)(n-1).$$

In the same way, if there exists a hyperplane $E = \theta^\perp$ such that $Var|P_E X|^2 \leq C_2(n-1)$, then

$$
\begin{aligned}
\sqrt{Var|X|^2} \ &\leq \ \sqrt{Var|P_E X|^2} + \sqrt{Var\langle X,\theta\rangle^2} \\
&\leq \ \sqrt{C_2(n-1)} + \sqrt{\mathbb{E}\langle X,\theta\rangle^4} \\
&\leq \ \sqrt{2}\sqrt{C_2(n-1) + \mathbb{E}\langle X,\theta\rangle^4}
\end{aligned}
$$

and, by Borell's inequality,

$$\sqrt{Var|X|^2} \leq \sqrt{2}\sqrt{C_2 n + C} = \sqrt{2}\sqrt{C_2 + \frac{C}{n}}\sqrt{n}.$$

Thus, there exists an absolute constant C such that

$$Var|X|^2 \leq 2\left(C_2 + \frac{C}{n}\right)n.$$

$\square$

Now we will prove Theorem 1.2. It will be a consequence of the following

Proposition 5.2. *Let K be an isotropic convex body, $\theta \in S^{n-1}$ and $S_\theta(K)$ its Steiner symmetrization with respect to the hyperplane $H = \theta^\perp$. Let Y be a random vector uniformly distributed on $S_\theta(K)$ and X a random vector uniformly distributed on K. Then there exists an absolute constant C such that*

$$\left| Var|Y|^2 - Var|X|^2 \right| \leq CnL_K^4.$$

Proof. Without loss of generality we can assume that $\theta = e_n$. We have that

$$
\begin{aligned}
\mathrm{Var}|Y|^2 &= \sum_{i=1}^{n} \left(\mathbb{E}\langle Y, e_i \rangle^4 - (\mathbb{E}\langle Y, e_i \rangle^2)^2 \right) \\
&+ \sum_{i \neq j} \left(\mathbb{E}\langle Y, e_i \rangle^2 \langle Y, e_j \rangle^2 - \mathbb{E}\langle Y, e_i \rangle^2 \mathbb{E}\langle Y, e_j \rangle^2 \right).
\end{aligned}
$$

Notice that if $i \neq n$

$$
\begin{aligned}
\mathbb{E}\langle Y, e_i \rangle^4 &= \int_{P_H(K)} \langle y, e_i \rangle^4 |S_\theta(K) \cap (y + \langle e_n \rangle)| dy \\
&= \int_{P_H(K)} \langle y, e_i \rangle^4 |K \cap (y + \langle e_n \rangle)| dy = \mathbb{E}\langle X, e_i \rangle^4.
\end{aligned}
$$

If $i = n$ and for every $y \in P_H(K)$ we have that $K \cap (y + \langle e_n \rangle)$ is the segment $[a(y), b(y)]e_n$, which has length $2l(y)$

$$
\begin{aligned}
\mathbb{E}\langle Y, e_n \rangle^4 &= \int_{P_H(K)} \int_{-l(y)}^{l(y)} t^4 dt dy \leq \int_{P_H(K)} \int_{a(y)}^{b(y)} t^4 dt dy \\
&= \mathbb{E}\langle X, e_n \rangle^4.
\end{aligned}
$$

In the same way, if $i \neq n$

$$
\mathbb{E}\langle Y, e_i \rangle^2 = \mathbb{E}\langle X, e_i \rangle^2
$$

and if $i = n$

$$
\mathbb{E}\langle Y, e_n \rangle^2 \leq \mathbb{E}\langle X, e_n \rangle^2.
$$

Besides, if $i, j \neq n$

$$
\mathbb{E}\langle Y, e_i \rangle^2 \langle Y, e_j \rangle^2 = \mathbb{E}\langle X, e_i \rangle^2 \langle X, e_j \rangle^2
$$

and if $i \neq n$

$$
\begin{aligned}
\mathbb{E}\langle Y, e_i \rangle^2 \langle Y, e_n \rangle^2 &= \int_{P_H(K)} \langle y, e_i \rangle^2 \int_{-l(y)}^{l(y)} t^2 dt dy \leq \int_{P_H(K)} \langle y, e_i \rangle^2 \int_{a(y)}^{b(y)} t^2 dt dy \\
&= \mathbb{E}\langle X, e_i \rangle^2 \langle X, e_n \rangle^2.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\mathrm{Var}|Y|^2 = \mathrm{Var}|X|^2 \;&+\; \mathbb{E}\langle Y, e_n \rangle^4 - \mathbb{E}\langle X, e_n \rangle^4 \\
&+\; (\mathbb{E}\langle X, e_n \rangle^2)^2 - (\mathbb{E}\langle Y, e_i \rangle^2)^2 \\
&+\; 2\sum_{i=1}^{n-1} \mathbb{E}\langle Y, e_i \rangle^2 \langle Y, e_n \rangle^2 - \mathbb{E}\langle X, e_i \rangle^2 \langle X, e_n \rangle^2 \\
&+\; 2\sum_{i=1}^{n-1} \mathbb{E}\langle X, e_i \rangle^2 (\mathbb{E}\langle X, e_n \rangle^2 - \mathbb{E}\langle Y, e_n \rangle^2).
\end{aligned}
$$

Consequently

$$
\begin{aligned}
\mathrm{Var}|Y|^2 \;\leq\;\; &\mathrm{Var}|X|^2 \\
&+\; (\mathbb{E}\langle X, e_n \rangle^2)^2 - (\mathbb{E}\langle Y, e_i \rangle^2)^2 \\
&+\; 2\sum_{i=1}^{n-1} \mathbb{E}\langle X, e_i \rangle^2 (\mathbb{E}\langle X, e_n \rangle^2 - \mathbb{E}\langle Y, e_n \rangle^2).
\end{aligned}
$$

Now, if K is isotropic

$$
\mathrm{Var}|Y|^2 \;\leq\;\; \mathrm{Var}|X|^2
$$

$$
\begin{aligned}
+ \quad & L_K^4 - (\mathbb{E}\langle Y, e_i\rangle^2)^2 \\
+ \quad & 2(n-1)L_K^2(L_K^2 - \mathbb{E}\langle Y, e_n\rangle^2) \\
\leq \quad & \mathrm{Var}|X|^2 + (2n-1)L_K^4.
\end{aligned}
$$

On the other hand, by Hölder's inequality and Borell's lemma

$$
\begin{aligned}
\mathrm{Var}|Y|^2 \;\geq\;& \mathrm{Var}|X|^2 - \mathbb{E}\langle X, e_n\rangle^4 \\
& - 2\sum_{i=1}^{n-1} \mathbb{E}\langle X, e_i\rangle^2 \langle X, e_n\rangle^2 \\
\geq\;& \mathrm{Var}|X|^2 - \mathbb{E}\langle X, e_n\rangle^4 \\
& - 2\sum_{i=1}^{n-1} \mathbb{E}(\langle X, e_i\rangle^4)^{\frac{1}{2}} \mathbb{E}(\langle X, e_n\rangle^4)^{\frac{1}{2}} \\
\geq\;& \mathrm{Var}|X|^2 - C(\mathbb{E}\langle X, e_n\rangle^2)^2 \\
& - C\sum_{i=1}^{n-1} \mathbb{E}\langle X, e_i\rangle^2 \mathbb{E}\langle X, e_n\rangle^2
\end{aligned}
$$

Thus, if K is isotropic

$$
\mathrm{Var}|Y|^2 \geq \mathrm{Var}|X|^2 - CnL_K^4.
$$

$\square$

As a consequence, we have Theorem 1.2:

Proof of Theorem 1.2. Let K be an isotropic convex body and let Y_θ be a random vector on $S_\theta(K)$. Then

$$
\lambda_{Y_\theta}^2 = L_K^2
$$

and

$$
\mathbb{E}|Y_\theta|^2 = (n-1)L_K^2 + \mathbb{E}\langle Y_\theta, \theta\rangle^2.
$$

Thus $(n-1)L_K^2 \leq \mathbb{E}|Y_\theta|^2 \leq nL_K^2$ and so, by the previous proposition, if X verifies the variance conjecture with constant C_1 then for any $\theta \in S^{n-1}$

$$
\mathrm{Var}\,|Y_\theta|^2 \leq \mathrm{Var}\,|X|^2 + CnL_K^4 \leq (C_1 + C)nL_K^4 \leq 2(C_1 + C)\lambda_{Y_\theta}^2 \mathbb{E}|Y|^2
$$

and if for some $\theta \in S^{n-1}$ Y_θ verifies the variance conjecture with constant C_2 then

$$
\mathrm{Var}\,|X|^2 \leq \mathrm{Var}\,|Y_\theta|^2 + CnL_K^4 \leq (C_2 + C)nL_K^4 = (C_2 + C)\lambda_X^2 \mathbb{E}|X|^2.
$$

$\square$

6. Acknowledgements

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