p-ADIC FAMILIES OF $\mathfrak{d}$-TH SHINTANI LIFTINGS

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Abstract. In this note we give a detailed construction of a Λ-adic $\mathfrak{d}$-th Shintani lifting. We derive a Λ-adic version of Kohnen's formula relating Fourier coefficients of half-integral weight modular forms and special values of twisted L-series. As a by-product we obtain a mild generalization of such classical formula.

Contents

1. Introduction

In his seminal paper [Shi73], Shimura unrevealed the first instance of what is nowadays called theta (or Howe) correspondence, which was systematically studied later by Waldspurger [Wal80, Wal91]. Shimura's work contained an in-depth study of half-integral weight modular forms, and provided Hecke-equivariant linear maps

$$\mathcal{S}_{k,N,\chi,\mathfrak{d}} : S^+_{k+1/2}(N,\chi) \longrightarrow S_{2k}(N,\chi^2)$$

from Kohnen's plus subspace of the space of half-integral weight modular forms to the space of classical modular forms of even weight. The construction depends on an auxiliary discriminant $\mathfrak{d}$ and different choices yield different maps.

By means of certain cycle integrals along geodesic paths on the complex upper half plane, one can define Hecke-equivariant linear maps

$$\theta_{k,N,\chi,\mathfrak{d}} : S_{2k}(N,\chi^2) \longrightarrow S^+_{k+2/1}(N,\chi)$$

which are adjoint to $\mathcal{S}_{k,N,\chi,\mathfrak{d}}$ with respect to the Petersson product, meaning that

$$\langle g, \theta_{k,N,\chi,\mathfrak{d}}(f)\rangle = \langle \mathcal{S}_{k,N,\chi,\mathfrak{d}}(g), f\rangle \quad \text{for all } f \in S_{2k}(N,\chi^2), g \in S^+_{k+1/2}(N,\chi).$$

This construction was first studied by Shintani [Shi75], and subsequently extended by Kohnen and Zagier [KZ81], Kohnen [Koh85], and Kojima–Tokuno [KT04], among others. The maps $\theta_{k,N,\chi,\mathfrak{d}}$ are referred to as $\mathfrak{d}$-*th Shintani liftings*. We will drop χ of the notation when it is the trivial character.

Under certain assumptions, for example when N is squarefree and χ is trivial, a theory of newforms of half-integral weight *à la* Atkin–Li–Miyake is available, and the $\mathfrak{d}$-th Shimura and $\mathfrak{d}$-th Shintani liftings establish a Hecke-equivariant isomorphism[1] between $S^{new}_{2k}(N,\chi^2)$ and $S^{+,new}_{k+1/2}(N,\chi)$: this is the so-called *Shimura–Shintani correspondence*. In this direction, one of the main motivations of the work of Kohnen and Zagier was to obtain an explicit Waldspurger-type formula relating Fourier coefficients of half-integral weight modular forms and twisted L-values of classical modular forms. For instance, suppose that N is odd and squarefree, and that $g \in S^{+,new}_{k+1/2}(N)$ and $f \in S^{new}_{2k}(N)$ are two non-zero new modular forms in Shimura–Shintani correspondence. Then,

Date: December 2019.

[1]We warn the reader that a single choice of $\mathfrak{d}$ does not provide the isomorphism on the full spaces. In general one has to consider a suitable combination of $\mathfrak{d}$-th liftings (cf. [Koh82, Theorem 2]).

Kohnen's formula [Koh85, Corollary 1] asserts that for any fundamental discriminant D with $(-1)^k D > 0$ and such that $(\frac{D}{\ell}) = w_\ell$ for all primes $\ell \mid D$, where w_ℓ is the eigenvalue of the Atkin–Lehner involution W_ℓ acting on f, one has

$$\frac{|a_{|D|}(g)|^2}{\langle g, g\rangle} = 2^{\nu(N)} \frac{(k-1)!}{\pi^k} |D|^{k-1/2} \frac{L(f, D, k)}{\langle f, f\rangle}. \tag{1}$$

Here, $a_{|D|}(g)$ denotes the $|D|$-th Fourier coefficient of g, and $\nu(N)$ is the number of prime divisors of N. We insist that fixed a newform $f \in S_{2k}^{new}(N)$, this formula is valid for *any* $g \in S_{k+1/2}^{+,new}(N)$ in Shimura–Shintani correspondence with f; any two such forms will be multiple one of each other, and the formula is clearly invariant under replacing g with a non-zero multiple of it.

One of the key ingredients in the proof of (1) is actually a formula that relates directly the $|\mathfrak{d}|$-th Fourier coefficient of the $\mathfrak{d}$-th Shintani lifting of f with the twisted special value $L(f, \mathfrak{d}, k)$. Indeed, assuming that $(-1)^k \mathfrak{d} > 0$ and that $(\frac{\mathfrak{d}}{\ell}) = w_\ell$ for all primes $\ell \mid N$, it is shown in [Koh85] that

$$a_{|\mathfrak{d}|}(\theta_{k,N,\mathfrak{d}}(f)) = (-1)^{[k/2]} 2^{\nu(N)+k} |\mathfrak{d}|^k (k-1)! \cdot \frac{L(f, \mathfrak{d}, k)}{(2\pi i)^k \mathfrak{g}(\chi_{\mathfrak{d}})}, \tag{2}$$

where $\mathfrak{g}(\chi_{\mathfrak{d}})$ is the Gauss sum attached to the quadratic character $\chi_{\mathfrak{d}}$. While Kohnen's formula in (1) depends notably on having a good theory of newforms as cited above (which in particular provides 'multiplicity one'), and therefore it does not extend easily when dropping the assumptions that N is squarefree and χ is trivial, the formula in (2) does generalize quite easily. We refer the reader to Kojima–Tokuno [KT04] for an extension of Kohnen's work and ideas, still under some mild assumptions on the pair (N, χ).

The pioneering work of Shimura and Waldspurger not only motivated the above mentioned works by Kohnen and Zagier, but it has also inspired many other investigations along several years. For instance, Gross–Kohnen–Zagier studied in [GKZ87] the relation between Fourier coefficients of half-integral weight modular forms (and actually, of Jacobi forms) and Heegner divisors. In turn, several variants of the Gross–Kohnen–Zagier formula have been proved so far. For example, Darmon–Tornaría [DT08] proved a Gross–Kohnen–Zagier type formula for Stark–Heegner points attached to real quadratic fields. This variant further allowed them to obtain a similar relationship as in Kohnen's formula for central critical *derivatives*, with the role of the Fourier coefficient $a_{|D|}(g)$ being played by the first derivative of the $|D|$-th Fourier coefficient of a p-adic family of half-integral forms. Also in this line, the p-adic variation of the Gross–Kohnen–Zagier theorem, including the existence of Λ-adic families of Jacobi forms, is studied in [LN19a, LN19b]. In a different direction and with a different flavour, there is also the work of Ono–Skinner [OS98], studying the divisibility of Fourier coefficients of half-integral weight modular forms by looking at the residual Galois representations of integral weight modular forms in correspondence with them. Their main result has interesting arithmetic consequences about special L-values of even integral weight eigenforms, twisted by quadratic characters, and about Tate–Shafarevich groups of elliptic curves.

In this paper we focus on the p-adic interpolation of the above liftings. Our main source of inspiration is the work [Ste94] of Stevens, who successfully described the p-adic interpolation of the first Shintani lifting. To achieve this, Stevens combined a cohomological interpretation of Shintani's cycle integrals with the theory of Λ-adic modular symbols developed in [GS93]. This was yet another arithmetic application of the widely successful theory initiated by Hida in [Hid86].

The first goal of this note is to describe a Λ-adic $\mathfrak{d}$-th Shintani lifting interpolating p-adically the classical $\mathfrak{d}$-th Shintani lifting. The main novelty of our approach is that we can derive a Λ-adic version of Kohnen's formula stated in (2). This formula, which will be further described below, relates the $|\mathfrak{d}|$-th Fourier coefficient of the Λ-adic $\mathfrak{d}$-th Shintani lifting to a suitable p-adic L-function interpolating twisted central L-values of p-ordinary eigenforms (built from the two-variable p-adic L-function of Greenberg–Stevens). The reader may intepret this formula as a cuspidal analogue of the well-known relation between the 0-th Fourier coefficient of a Λ-adic Eisenstein series and the relevant Kubota–Leopoldt p-adic L-function.

In more detail, given a Hida family $\mathbf{f}$ of ordinary p-stabilized newforms of tame level N and tame character χ^2, we construct a p-adic family of half-integral weight modular forms interpolating the $\mathfrak{d}$-th Shintani liftings of the classical specializations of $\mathbf{f}$. To describe our main results, suppose that $\mathbf{f}$ is given by a power series $\mathbf{f} \in \mathcal{R}[[q]]$, where $\mathcal{R}$ is a finite flat integral extension of the Iwasawa algebra $\Lambda = \mathbf{Z}_p[[1 + p\mathbf{Z}_p]] \simeq \mathbf{Z}_p[[T]]$. Let $\mathcal{U}^{\mathrm{cl}}$ denote the subset of classical points in the p-adic weight space $\mathcal{W} = \mathrm{Hom}(\mathcal{R}, \bar{\mathbf{Q}}_p)$, so that $\mathbf{f}(\kappa) \in S_{2k}(Np, \chi^2)$ is the q-expansion of an ordinary p-stabilized newform of level Np and character χ^2 for all $\kappa \in \mathcal{U}^{\mathrm{cl}}$. We show in Theorem 5.9 that the $\mathfrak{d}$-th Shintani liftings of the classical specializatons $\mathbf{f}(\kappa)$ are p-adically interpolated by a power series $\Theta_{\mathfrak{d}}(\mathbf{f}) \in \widetilde{\mathcal{R}}[[q]]$, where $\widetilde{\mathcal{R}}$ is the *metaplectic covering* of $\mathcal{R}$ (cf. Section 5). This induces a natural map $\pi : \widetilde{\mathcal{W}} \to \mathcal{W}$ on p-adic weight spaces that 'doubles' the signatures of the arithmetic points.

Theorem. *Let $\mathfrak{d}$ be a fundamental discriminant with $\mathfrak{d} \equiv 0 \pmod p$. There exists a power series*

$$\Theta_{\mathfrak{d}}(\mathbf{f}) = \sum_{m \geq 1} \mathbf{a}_m(\Theta_{\mathfrak{d}}(\mathbf{f}))q^m \in \widetilde{\mathcal{R}}[[q]]$$

and a subset of classical points $\widetilde{\mathcal{U}}^{\mathrm{cl}} \subset \widetilde{\mathcal{W}}$ above $\mathcal{U}^{\mathrm{cl}}$ such that

$$\Theta_{\mathfrak{d}}(\mathbf{f})(\tilde{\kappa}) = C(k, \chi, \mathfrak{d})^{-1} \frac{\Omega_\kappa}{\Omega^-_{\mathbf{f}(\kappa)}} \theta_{k,Np,\chi,\mathfrak{d}}(\mathbf{f}(\kappa))$$

for all $\tilde{\kappa} \in \widetilde{\mathcal{U}}^{\mathrm{cl}}$, where $\kappa = \pi(\tilde{\kappa}) \in \mathcal{U}^{\mathrm{cl}}$. Here, $C(k, \chi, \mathfrak{d})$ is a constant defined in (11)*, and $\Omega_{\mathbf{f}(\kappa)}$ and Ω_κ are complex and p-adic periods attached to $\mathbf{f}(\kappa)$ and κ, respectively (cf. Section 4.3).*

We refer the reader to Section 5 for the details of the construction and precise statements. We point out that one can construct a priori *two* different Λ-adic $\mathfrak{d}$-th Shintani liftings of a given Hida family $\mathbf{f}$, each of them satisfying an interpolation property on a different dense subset of classical points.

As a consequence of the existence of a Λ-adic $\mathfrak{d}$-th Shintani lifting, we prove in Theorem 5.13 a Λ-adic version of Kohnen's formula (2). In the simplified case in which χ is the trivial character Theorem 5.13 reduces to the following identity in $\widetilde{\mathcal{R}}$:

$$\mathbf{a}_{|\mathfrak{d}|}(\Theta_{\mathfrak{d}}(\mathbf{f})) = \mathrm{sgn}(\mathfrak{d}) \cdot 2^{\nu(N)} \cdot \tilde{a}_p(\mathbf{f}) \cdot \tilde{\mathcal{L}}_p^{\mathrm{GS}}(\mathbf{f}, \mathfrak{d}).$$

Here, the p-adic L-function $\tilde{\mathcal{L}}_p^{\mathrm{GS}}(\mathbf{f}, \mathfrak{d}) \in \widetilde{\mathcal{R}}$ interpolates the values $L(\mathbf{f}(\kappa), \mathfrak{d}, k)$ and it is built from a suitable restriction of the two-variable p-adic L-function studied by Greenberg–Stevens in [GS93], and $\tilde{a}_p(\mathbf{f})$ is the pull-back of $a_p(\mathbf{f})$ along the map $\pi : \widetilde{\mathcal{W}} \to \mathcal{W}$. As a by-product, we obtain a mild generalization of the classical formula in (2), and we also describe exceptional zero phenomena for the coefficients $\mathbf{a}_{|\mathfrak{d}|}(\Theta_{\mathfrak{d}}(\mathbf{f}))$.

We should mention a few works that are in the orbit of this article. First of all, we must say that Stevens' Λ-adic version of the first Shintani lifting was generalized by Park [Par10] to the non-ordinary finite slope case. Secondly, Park's approach has recently been extended to the case of the $\mathfrak{d}$-th Shintani lifting by Makiyama [Mak17], where $\mathfrak{d}$ is chosen such that $p \nmid \mathfrak{d}$. Our approach is then complementary to Makiyama's, and it is inspired by the preprint [Kaw] by Kawamura, who constructs certain p-adic families of Siegel cusp forms of arbitrary genus interpolating Duke–Imamoḡlu liftings of level $N = 1$. Our construction provides a much more general version of the Λ-adic $\mathfrak{d}$-th Shintani lifting than the one needed in Kawamura's discussion.

To close this introduction, let us briefly explain the organization of the article. In section 2 we set the basic definitions for integral binary quadratic forms, we describe Kojima–Tokuno's generalization of Kohnen's $\mathfrak{d}$-th Shintani lifting, and we also explain the exact relation with special values of L-functions. Section 3 is devoted to a classification result for integral binary quadratic forms from [GKZ87], which we use to derive the exact relations between the liftings in level N and Np, leading to a comparison of the Fourier coefficients of the lifting of a modular form with those of its p-stabilization. In chapter 4 we settle the language of Hida theory, and we describe classical modular symbols and their Λ-adic version *à la* Greenberg–Stevens. Section 5 contains the main results and applications. In particular, the definition of $\Theta_{\mathfrak{d}}$ is given in equation (30), the interpolation property is proved in Theorem 5.9, and the above mentioned Λ-adic Kohnen formula is proved in Theorem 5.13.

Acknowledgements. D.C. acknowledges financial support by Spanish Ministry of Economy and Competitiveness, through the Severo Ochoa programme for Centres of Excellence in R&D (SEV-2015-0554), and C.dVP. acknowledges financial support by a Junior Researcher Grant through the project AGAUR PDJ 2012. Both authors are also grateful to their host instutions for the warm hospitality during their respective visits, and to Professor L. Dieulefait for partial support through the project MTM2015-66716-P. This project has also received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No 682152).

1.1. **Notation.** We shall fix the following general notation throughout the entire paper. As usual $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$, and $\mathbf{C}$ will denote the ring of integers, the field of rational numbers, the field of real numbers, and the field of complex numbers, respectively. If $z \in \mathbf{C}^\times$ and $x \in \mathbf{C}$, we define $z^x = e^{x \log z}$, where $\log z = \log |z| + i \arg(z)$ with $-\pi < \arg(z) \leq \pi$. If ψ is a Dirichlet character of conductor c, we write

$$\mathfrak{g}(\psi) = \sum_{a \in (\mathbf{Z}/c\mathbf{Z})^\times} \psi(a) e^{2\pi i a/c}$$

for the Gauss sum of ψ. When ψ is primitive, one has $|\mathfrak{g}(\psi)|^2 = \psi(-1)\mathfrak{g}(\psi)\mathfrak{g}(\bar{\psi}) = c$. If ψ, ψ' are primitive Dirichlet characters of relatively prime conductors c, c', respectively, then $\mathfrak{g}(\psi\psi') = \psi(c')\psi'(c)\mathfrak{g}(\psi)\mathfrak{g}(\psi')$.

For any commutative ring R with unit, $\mathrm{SL}_2(R)$ will denote the special linear group over R. The group $\mathrm{SL}_2(\mathbf{R})$ acts as usual on the complex upper half plane $\mathfrak{H}$ via linear fractional transformations. If $r, M \geq 1$ are integers, and ψ is a Dirichlet character modulo M, we write $S_r(M, \psi)$ for the (complex) space of cusp forms of weight r, level M and character ψ. We define the Petersson product of two cusp forms $f, g \in S_r(M, \psi)$ by

$$\langle f, g \rangle = \frac{1}{i_M} \int_{\Gamma_0(M)\backslash\mathfrak{H}} f(z)\overline{g(z)} y^{r-2} dxdy,$$

where $z = x + iy$ and $i_M = [\mathrm{SL}_2(\mathbf{Z}) : \Gamma_0(M)]$. With this normalization, the Petersson product of f and g does not change if we replace M by a multiple of it and see f and g as forms of that level.

If $N \geq 1$ is an *odd* integer, $k \geq 0$ is an integer, and χ is a Dirichlet character modulo N, we write $\tilde{\chi}$ for the Dirichlet character modulo $4N$ given by $(\frac{4\epsilon}{\cdot})\chi$, where $\epsilon = \chi(-1)$, and write $S_{k+1/2}(4N, \tilde{\chi})$ for the (complex) space of cusp forms of half-integral weight $k + 1/2$, level $4N$ and character $\tilde{\chi}$, in the sense of Shimura [Shi73]. Observe that $\tilde{\chi}$ is an even character by construction. If $f, g \in S_{k+1/2}(4N, \tilde{\chi})$, their Petersson product is

$$\langle f, g \rangle = \frac{1}{i_{4N}} \int_{\Gamma_0(4N)\backslash\mathfrak{H}} f(z)\overline{g(z)} y^{k-3/2} dxdy.$$

We will denote by $S^+_{k+1/2}(N, \chi)$ the subspace of $S_{k+1/2}(4N, \tilde{\chi})$ consisting of those forms f whose q-expansion has the form

$$f(z) = \sum_{\substack{n \geq 1, \\ \epsilon(-1)^k n \equiv 0,1\,(4)}} a(n) q^n.$$

This is usually referred to as 'Kohnen's plus space'.

For every positive integer m, we denote by $\wp_{k,4N,m,\tilde{\chi}} \in S_{k+1/2}(4N, \tilde{\chi})$ the m-th Poincaré series characterized by the fact that

$$\langle g, \wp_{k,4N,m,\tilde{\chi}} \rangle = i_{4N}^{-1} \frac{\Gamma(k - 1/2)}{(4\pi m)^{k-1/2}} a_g(m) \quad \text{for all } g(z) = \sum_{m \geq 1} a_g(m) q^m \in S_{k+1/2}(4N, \tilde{\chi}),$$

and we denote by $P_{k,N,m,\chi}$ the projection of $\wp_{k,4N,m,\tilde{\chi}}$ in Kohnen's plus space $S^+_{k+1/2}(N, \chi)$.

We may recall the definition of Hecke operators acting on the space $S^+_{k+1/2}(N, \chi)$. For a prime $p \nmid N$, and $f = \sum a(n) q^n \in S^+_{k+1/2}(N, \chi)$, the action of the Hecke operator $T_{k+1/2,N,\chi}(p^2)$ is given by

$$T_{k+1/2,N,\chi}(p^2) f(z) = \sum_{\substack{n \geq 1, \\ \epsilon(-1)^k \equiv 0,1(4)}} \left(a(p^2 n) + \chi(p) \left(\frac{\epsilon(-1)^k n}{p} \right) p^{k-1} a(n) + \chi(p^2) p^{2k-1} a(n/p^2) \right) q^n,$$

where one reads $a(n/p^2) = 0$ if $p^2 \nmid n$. For primes $p \mid N$, one also defines operators $U(p^2)$ by setting

$$U(p^2) f(z) = \sum_{\substack{n \geq 1, \\ \epsilon(-1)^k \equiv 0,1(4)}} a(p^2 n) q^n.$$

2. $\mathfrak{d}$-th Shintani lifting

The aim of this section is to review the definition of the so-called $\mathfrak{d}$-th Shintani lifting, where $\mathfrak{d}$ is a fixed fundamental discriminant. This is a Hecke-equivariant linear map from integral weight modular forms to half-integral weight modular forms, studied in detail by Kohnen in [Koh85], building on the seminal work of Shintani [Shi75], and later generalized by others. We will briefly summarize the approach in Kojima–Tokuno [KT04], which essentially adapts Kohnen's work for arbitrary odd level and arbitrary nebentype character.

Before entering into the proper construction of the $\mathfrak{d}$-th Shintani lifting, we fix some notations concerning integral binary quadratic forms that will be of good use.

2.1. **Integral binary quadratic forms.** We write $\mathcal{Q}$ for the set of all integral binary quadratic forms

$$[a,b,c](X,Y) = aX^2 + bXY + cY^2, \quad a,b,c \in \mathbf{Z},$$

on which $\Gamma_0(1) = \mathrm{SL}_2(\mathbf{Z})$ acts by the rule

$$([a,b,c]\circ\gamma)(X,Y) = [a,b,c]((X,Y)^t\gamma), \quad \gamma \in \mathrm{SL}_2(\mathbf{Z}).$$

If one identifies the quadratic form $Q = [a,b,c]$ with the symmetric matrix

$$A_Q = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix},$$

then $[a,b,c]\circ\gamma$ corresponds to the matrix ${}^t\gamma A_Q \gamma$. Given $Q = [a,b,c] \in \mathcal{Q}$, its discriminant is by definition $b^2 - 4ac = \det(2A_Q)$. It is immediate that the discriminant is invariant under the above $\Gamma_0(1)$-action.

If Δ is a discriminant, we write $\mathcal{Q}(\Delta)$ for the subset of quadratic forms in $\mathcal{Q}$ having discriminant Δ. There is an induced $\Gamma_0(1)$-action on $\mathcal{Q}(\Delta)$. If $\mathfrak{d}$ is a fundamental discriminant dividing Δ and $Q = [a,b,c] \in \mathcal{Q}(\Delta)$, then we set

$$\omega_{\mathfrak{d}}(Q) = \begin{cases} 0 & \text{if } \gcd(a,b,c,\mathfrak{d}) > 1, \\ \left(\frac{\mathfrak{d}}{r}\right) & \text{if } \gcd(a,b,c,\mathfrak{d}) = 1 \text{ and } Q \text{ represents } r, \gcd(r,\mathfrak{d}) = 1. \end{cases}$$

One can easily check that this definition does not depend on the choice of the integer r, when $\gcd(a,b,c,\mathfrak{d}) = 1$. In addition, the value of $\omega_{\mathfrak{d}}(Q)$ depends only on the $\Gamma_0(1)$-equivalence class of Q. Besides the definition, when $\Delta > 0$ one can compute $\omega_{\mathfrak{d}}(Q)$ by using the following explicit formula (cf. [Koh85, p.263, Proposition 6])

$$\omega_{\mathfrak{d}}([a,b,c]) = \prod_{q^\nu || a} \left(\frac{\mathfrak{d}/q^*}{q^\nu}\right)\left(\frac{q^*}{ac/q^\nu}\right). \tag{3}$$

Here, q runs over the prime factors of a, $q^\nu || a$ means that $q^\nu \mid a$ and $\gcd(q, a/q^\nu) = 1$, and $q^* := \left(\frac{-1}{q}\right) q$.

We denote by $\mathcal{Q}^0(\Delta)$ the subset of *primitive* forms in $\mathcal{Q}(\Delta)$, namely those forms for which $\gcd(a,b,c) = 1$. The induced function $\omega_{\mathfrak{d}} : \mathcal{Q}^0(\Delta)/\Gamma_0(1) \to \{\pm 1\}$ is usually referred to as a *genus character*. When endowing $\mathcal{Q}^0(\Delta)/\Gamma_0(1)$ with its natural group structure, $\omega_{\mathfrak{d}}$ becomes a group homomorphism; and conversely, every group homomorphism $\mathcal{Q}^0(\Delta)/\Gamma_0(1) \to \{\pm 1\}$ is of the form $\omega_{\mathfrak{d}'}$ for some fundamental discriminant $\mathfrak{d}'$ dividing Δ, the only relations being that $\omega_{\mathfrak{d}} = \omega_{\mathfrak{d}'}$ if $\Delta = \mathfrak{d}\mathfrak{d}' m^2$ for some natural number m.

If $M \geq 1$ is an integer, we also denote by $\mathcal{Q}_M(\Delta)$ the subset of forms $Q = [a,b,c] \in \mathcal{Q}(\Delta)$ such that $a \equiv 0 \pmod{M}$. One can easily check that the congruence subgroup $\Gamma_0(M)$ acts on $\mathcal{Q}_M(\Delta)$. If $t > 0$ is a divisor of M, then the map

$$Q = [a,b,c] \mapsto Q_t := Q \circ \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} = [at^2, bt, c] \tag{4}$$

yields a bijection $\mathcal{Q}_{M/t}(\Delta) \to \mathcal{Q}_{Mt}(\Delta t^2)$. If $\mathfrak{d}$ is a fundamental discriminant dividing Δ as above, one has

$$\omega_{\mathfrak{d}}(Q_t) = \chi_{\mathfrak{d}}(t^2)\omega_{\mathfrak{d}}(Q), \tag{5}$$

which equals $\omega_{\mathfrak{d}}(Q)$ if $\gcd(t,\mathfrak{d}) = 1$. When we do not want to specify the discriminant, $\mathcal{Q}_M$ will denote the union of all the sets $\mathcal{Q}_M(\Delta)$.

2.2. **The $\mathfrak{d}$-th Shimura and Shintani liftings.** Fix through all this paragraph an odd integer $N \geq 1$ and an integer $k \geq 1$, and fix also a Dirichlet character χ modulo N. Let $N_0 \geq 1$ be the conductor of χ, χ_0 be the primitive character modulo N_0 associated with χ, and put $N_1 = N/N_0$ and $\epsilon = \chi(-1)$. Although it is not always needed, for simplicity we assume through all our discussion that $\gcd(N_0, N_1) = 1$. Also, for technical reasons that will be apparent below, we make the following hypothesis.

$(\star)$ if $k = 1$, either N is squarefree or χ is trivial and N is cubefree.

Fix also a fundamental discriminant $\mathfrak{d}$ satisfying $\gcd(N_0, \mathfrak{d}) = 1$.

If $Q = [a, b, c] \in \mathcal{Q}$ is an integral binary quadratic form, we set $\chi_0(Q) := \chi_0(c)$. For an integer $u \geq 1$ such that $\gcd(N_0, u) = 1$, and $\mathfrak{d}'$ a discriminant with $\mathfrak{d}\mathfrak{d}' > 0$, we consider the set of integral binary quadratic forms

$$\mathcal{Q}_{N_0 u}(N_0^2\mathfrak{d}\mathfrak{d}') = \{Q = [a, b, c] \in \mathcal{Q} : b^2 - 4ac = N_0^2\mathfrak{d}\mathfrak{d}',\ N_0 u \mid a\},$$

as defined in the previous paragraph. Recall that there is a natural action of $\Gamma_0(N_0u)$ on this set. We consider now the subset

$$\mathcal{L}_{N_0 u}(N_0^2\mathfrak{d}\mathfrak{d}') := \{Q = [a, b, c] \in \mathcal{Q}_{N_0u}(N_0^2\mathfrak{d}\mathfrak{d}') : \gcd(N_0, c) = 1\},$$

which is also invariant under $\Gamma_0(N_0u)$. For $k \geq 2$, define a function $f^{k,\chi_0}_{N_0,u}(z; \mathfrak{d}, \mathfrak{d}')$ of $z \in \mathfrak{H}$ by

$$f^{k,\chi_0}_{N_0,u}(z; \mathfrak{d}, \mathfrak{d}') := \sum_{Q \in \mathcal{L}_{N_0u}(N_0^2\mathfrak{d}\mathfrak{d}')} \chi_0(Q)\omega_{\mathfrak{d}}(Q)Q(z, 1)^{-k}. \tag{6}$$

These functions converge absolutely uniformly on compact sets, and further enjoy the following properties:

i) $f^{k,\chi_0}_{N_0,u}(g \cdot z; \mathfrak{d}, \mathfrak{d}') = \bar{\chi}_0^2(\delta)(\gamma z + \delta)^{2k} f^{k,\chi_0}_{N_0,u}(z; \mathfrak{d}, \mathfrak{d}')$ for all
$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_0(N_0u);$$
ii) $f^{k,\chi_0}_{N_0,u}(-z; \mathfrak{d}, \mathfrak{d}') = f^{k,\chi_0}_{N_0,u}(z; \mathfrak{d}, \mathfrak{d}')$ (the map $[a, b, c] \mapsto [a, -b, c]$ gives a bijection of $\mathcal{L}_{N_0u}(N_0^2\mathfrak{d}\mathfrak{d}')$ onto itself);
iii) $f^{k,\chi_0}_{N_0,u}(\bar{z}; \mathfrak{d}, \mathfrak{d}') = \overline{f^{k,\bar{\chi}_0}_{N_0,u}(z; \mathfrak{d}, \mathfrak{d}')}$.

The functions $f^{k,\chi_0}_{N_0,u}(z; \mathfrak{d}, \mathfrak{d}')$ yield cusp forms of weight $2k$, level N_0u, and character $\bar{\chi}_0^2$. For $k = 1$, the series in (6) is not absolutely convergent. However, one can apply "Hecke's convergence trick" to define $f^{k,\chi_0}_{N_0,u}(z; \mathfrak{d}, \mathfrak{d}')$ in a similar manner (cf. [Koh85, p. 239]). In this case, hypothesis $(\star)$ ensures that these functions are cusp forms as well. An explicit description of their Fourier coefficients (for $k \geq 2$) can be found in [KT04, Proposition 1.2].

Remark 2.1. *The functions $f^{k,\chi_0}_{N_0,u}(z; \mathfrak{d}, \mathfrak{d}')$ as above coincide with those denoted by $f_{k,N_0^2,u}(z; \mathfrak{d}, \mathfrak{d}', \chi_0)$ in* [KT04]. *Indeed, the sum in equation* (6) *could be taken over the sets $\mathcal{Q}_{N_0u}(N_0^2\mathfrak{d}\mathfrak{d}')$ and even $\mathcal{Q}_{N_0^2u}(N_0^2\mathfrak{d}\mathfrak{d}')$ remaining unchanged, due to the presence of the term $\chi_0(Q)$.*

Next consider the 'kernel function' $\Omega_{k,N,\chi}(z, \tau; \mathfrak{d})$ of $(z, \tau) \in \mathfrak{H} \times \mathfrak{H}$ defined by

$$\Omega_{k,N,\chi}(z, \tau; \mathfrak{d}) = i_N c^{-1}_{k,\mathfrak{d},\chi} \sum_{\substack{m \geq 1, \\ \epsilon(-1)^k m \equiv 0,1 (4)}} m^{k-1/2} \left(\sum_{t \mid N_1} \mu(t)\chi_{\mathfrak{d}}\bar{\chi}_0(t)t^{k-1} f^{k,\chi_0}_{N_0,N_1/t}(tz; \mathfrak{d}, \epsilon(-1)^k m) \right) e^{2\pi i m\tau},$$

where

$$i_N = [\Gamma_0(1) : \Gamma_0(N)], \quad c_{k,\mathfrak{d},\chi} = (-1)^{[k/2]} |\mathfrak{d}|^{-k+1/2} \pi \binom{2k-2}{k-1} 2^{-3k+2} \epsilon^{k-1/2} N_0^{1-k} \frac{\mathfrak{g}(\chi_{\mathfrak{d}})}{\mathfrak{g}(\chi_{\mathfrak{d}}\bar{\chi}_0)}.$$

For a fixed $\tau \in \mathfrak{H}$, the function $\Omega_{k,N,\chi}(\cdot, \tau; \mathfrak{d})$ on $\mathfrak{H}$ is a cusp form of weight $2k$, level N and character $\bar{\chi}_0^2$ (for $k = 1$, one needs again hypothesis $(\star)$). One has the basic identity (cf. [KT04, Theorem 2.2])

$$\Omega_{k,N,\chi}(z, \tau; \mathfrak{d}) = C \sum_{n \geq 1} n^{k-1} \left(\sum_{d \mid n} \chi_{\mathfrak{d}}\bar{\chi}(d)(n/d)^k P_{k,N,n^2|\mathfrak{d}|/d^2,\chi}(\tau) \right) e^{2\pi i n z},$$

where the $P_{k,N,n^2|\mathfrak{d}|/d^2,\chi}$ are the Poincaré series as defined above, and

$$C = i_N c_{k,\mathfrak{d},\chi}^{-1} \frac{(-1)^{[k/2]} 3(2\pi)^k}{(k-1)!} \epsilon^{k-1/2} N_0^{1-k} \frac{\mathfrak{g}(\chi_{\mathfrak{d}})}{\mathfrak{g}(\chi_{\mathfrak{d}}\bar{\chi}_0)}.$$

With our running assumptions on N, k, χ and $\mathfrak{d}$, for each cusp form

$$g(\tau) = \sum_{\substack{n\geq 1,\\ \epsilon(-1)^k n \equiv 0,1(4)}} c(n) e^{2\pi i n\tau} \in S^+_{k+1/2}(N,\chi)$$

in Kohnen's plus space, one can define a function

$$\mathcal{S}_{k,N,\chi,\mathfrak{d}}(g)(z) = \sum_{n\geq 1} \left(\sum_{d|n} \chi_{\mathfrak{d}}\chi(d) d^{k-1} c(n^2|\mathfrak{d}|/d^2) \right) e^{2\pi i n z}.$$

which satisfies the following property:

$$\mathcal{S}_{k,N,\chi,\mathfrak{d}}(g)(z) = \langle g, \Omega_{k,N,\chi}(-\bar{z},\cdot;\mathfrak{d})\rangle.$$

It follows that, for a fixed τ, $z \mapsto \overline{\Omega_{k,N,\chi}(-\bar{z},\tau;\mathfrak{d})}$ defines a cusp form of weight $2k$, level N, and character χ^2. As a consequence, $g \mapsto \mathcal{S}_{k,N,\chi,\mathfrak{d}}(g)$ yields a linear map

$$\mathcal{S}_{k,N,\chi,\mathfrak{d}} : S^+_{k+1/2}(N,\chi) \longrightarrow S_{2k}(N,\chi^2)$$

with kernel function $\Omega_{k,N,\chi}(-\bar{z},\cdot;\mathfrak{d})$. In addition, this map commutes with Hecke operators (meaning that $T_{k+1/2,N,\chi}(p^2)$ corresponds to $T_{2k,N,\chi^2}(p)$ for $p \nmid N$ and $U(q^2)$ corresponds to $T_{2k,N,\chi^2}(q)$ for $q \mid N$). The linear map $\mathcal{S}_{k,N,\chi,\mathfrak{d}}$ is the so-called *$\mathfrak{d}$-th Shimura lifting.*

We denote by $\theta_{k,N,\chi,\mathfrak{d}} : S_{2k}(N,\chi^2) \to S^+_{k+1/2}(N,\chi)$ the adjoint map with respect to the Petersson product, meaning that for all $g \in S^+_{k+1/2}(N,\chi)$ and $f \in S_{2k}(N,\chi^2)$

$$\langle g, \theta_{k,N,\chi,\mathfrak{d}}(f)\rangle = \langle \mathcal{S}_{k,N,\chi,\mathfrak{d}}(g), f\rangle.$$

Thus for any $f \in S_{2k}(N,\chi^2)$ we have that:

$$\theta_{k,N,\chi,\mathfrak{d}}(f) = \langle f(z), \overline{\Omega_{k,N,\chi}(-\bar{z},\tau;\mathfrak{d})}\rangle = i_N^{-1} \int_{\Gamma_0(N)\backslash\mathfrak{H}} f(z)\Omega_{k,N,\chi}(-\bar{z},\tau;\mathfrak{d}) y^{2k-2} dxdy =$$

$$= i_N c_{k,\mathfrak{d},\chi}^{-1} \sum_{\substack{m\geq 1,\\ \epsilon(-1)^k m \equiv 0,1(4)}} m^{k-1/2} \left(\sum_{t|N_1} \mu(t)\chi_{\mathfrak{d}}\bar{\chi}_0(t) t^{k-1} \langle f, f^{k,\bar{\chi}_0}_{N_0,N_1/t}(-tz;\mathfrak{d},\epsilon(-1)^k m)\rangle \right) q^m.$$

In particular, an explicit expression for $\theta_{k,N,\chi,\mathfrak{d}}(f)$ can be determined by computing the Petersson products

$$\langle f, f^{k,\bar{\chi}_0}_{N_0,N_1/t}(-tz;\mathfrak{d},\epsilon(-1)^k m)\rangle,$$

for $m \geq 1$ with $\epsilon(-1)^k m \equiv 0,1 \pmod 4$. Using property ii) listed above for the functions $f_{k,N_0,u}$, we see that

$$f^{k,\bar{\chi}_0}_{N_0,N_1/t}(-tz;\mathfrak{d},\epsilon(-1)^k m) = f^{k,\bar{\chi}_0}_{N_0,N_1/t}(tz;\mathfrak{d},\epsilon(-1)^k m).$$

Secondly, using the bijection in (4) and that $\omega_{\mathfrak{d}}(Q_t) = \chi_{\mathfrak{d}}(t^2)\omega_{\mathfrak{d}}(Q)$ by (5), we deduce that

$$f^{k,\bar{\chi}_0}_{N_0,N_1 t}(z;\mathfrak{d},\epsilon(-1)^k m t^2) = \chi_{\mathfrak{d}}(t^2) f^{k,\bar{\chi}_0}_{N_0,N_1/t}(-tz;\mathfrak{d},\epsilon(-1)^k m) \tag{7}$$

for all divisors t of N_1. Therefore, we may rewrite the above expression for $\theta_{k,N,\chi,\mathfrak{d}}(f)$ as

$$\theta_{k,N,\chi,\mathfrak{d}}(f) = i_N c_{k,\mathfrak{d},\chi}^{-1} \sum_{\substack{m\geq 1,\\ \epsilon(-1)^k m \equiv 0,1(4)}} m^{k-1/2} \left(\sum_{t|N_1} \mu(t)\chi_{\mathfrak{d}}\bar{\chi}_0(t) t^{k-1} \langle f, f^{k,\bar{\chi}_0}_{N_0,N_1 t}(z;\mathfrak{d},\epsilon(-1)^k m t^2)\rangle \right) q^m.$$

Finally, proceeding similarly as in [Koh85, p. 265-266], one can check that for $t \mid N_1$

$$\langle f, f^{k,\bar{\chi}_0}_{N_0,N_1 t}(z;\mathfrak{d},\epsilon(-1)^k m t^2)\rangle = i_{Nt}^{-1}\pi \binom{2k-2}{k-1} 2^{-2k+2} (|\mathfrak{d}| m t^2)^{-k+1/2} r_{k,Nt,\chi}(f;\mathfrak{d},\epsilon(-1)^k m t^2), \tag{8}$$

where for any discriminant $\mathfrak{d}'$ with $\mathfrak{d}\mathfrak{d}' > 0$ we set

$$r_{k,Nt,\chi}(f;\mathfrak{d},\mathfrak{d}') := \sum_{Q\in\mathcal{L}_{Nt}(N_0^2\mathfrak{d}\mathfrak{d}')/\Gamma_0(Nt)} \omega_{\mathfrak{d}}(Q)\chi_0(Q)\int_{C_Q} f(z)Q(z,1)^{k-1}dz.$$

Here, for each $Q=[a,b,c]$, C_Q denotes the image in $\Gamma_0(N)\backslash\mathfrak{H}$ of a geodesic in the upper half plane associated with Q. Namely, consider the semicircle S_Q in the complex upper half plane defined by the equation $a|z|^2 + b\mathrm{Re}(z) + c = 0$, and denote by ω_Q, $\omega_Q' \in \mathbf{P}^1(\mathbf{R})$ the pair of points

$$(\omega_Q,\omega_Q') := \begin{cases} \left(\frac{b-\sqrt{\mathrm{disc}(Q)}}{2a}, \frac{b+\sqrt{\mathrm{disc}(Q)}}{2a}\right) & \text{if } a\neq 0,\\ (-c/b, i\infty) & \text{if } a=0, b>0,\\ (i\infty, -c/b) & \text{if } a=0, b<0.\end{cases}$$

Notice that ω_Q and ω_Q' are the endpoints of the semicircle. When $\mathrm{disc}(Q)$ is a perfect square, we let C_Q be the image of S_Q (oriented from ω_Q to ω_Q') in $\Gamma_0(N)\backslash\mathfrak{H}$. If $\mathrm{disc}(Q)$ is not a perfect square, then let $\Gamma_0(N)_Q$ be the stabilizer of Q in $\Gamma_0(N)$, $\Gamma_0(N)_Q^+$ be its index two subgroup of positive trace elements, and

$$\gamma_Q = \begin{pmatrix} r & s\\ t & u\end{pmatrix} \in \Gamma_0(N)_Q^+$$

be the unique generator such that $r - t\omega_Q > 1$. Then, we let C_Q be the image in $\Gamma_0(N)\backslash\mathfrak{H}$ of the oriented geodesic path from $\gamma_Q(i\infty)$ to $i\infty$. Note that with this construction, the endpoints of C_Q are always rational cusps. We will write

$$I_{k,\chi}(f,Q) := \chi_0(Q)\int_{C_Q} f(z)Q(z,1)^{k-1}dz,$$

so that

$$r_{k,Nt,\chi}(f;\mathfrak{d},\mathfrak{d}') = \sum_{Q\in\mathcal{L}_{Nt}(N_0^2\mathfrak{d}\mathfrak{d}')/\Gamma_0(Nt)} \omega_{\mathfrak{d}}(Q) I_{k,\chi}(f,Q). \tag{9}$$

With this, one eventually concludes that

$$\theta_{k,N,\chi,\mathfrak{d}}(f) = C(k,\chi,\mathfrak{d}) \sum_{\substack{m\geq 1,\\ \epsilon(-1)^k m\equiv 0,1(4)}} \left(\sum_{t\mid N_1} \mu(t)\chi_{\mathfrak{d}}\bar{\chi}_0(t)t^{-k-1} r_{k,Nt,\chi}(f;\mathfrak{d},\epsilon(-1)^k mt^2)\right) q^m, \tag{10}$$

where

$$C(k,\chi,\mathfrak{d}) := c_{k,\mathfrak{d},\chi}^{-1}\pi\binom{2k-2}{k-1}2^{-2k+2}|\mathfrak{d}|^{1/2-k} = (-1)^{[k/2]}\epsilon^{k+1/2}2^k N_0^{k-1}\frac{\mathfrak{g}(\chi_{\mathfrak{d}}\bar{\chi}_0)}{\mathfrak{g}(\chi_{\mathfrak{d}})}. \tag{11}$$

When the character χ is trivial, we will denote this constant by $C(k,\mathfrak{d}) = (-1)^{[k/2]}2^k$.

Remark 2.2. *One can easily check that the quantities $r_{k,Nt,\chi}(f;\mathfrak{d},\mathfrak{d}')$ in* (9) *are all zero when $\epsilon(-1)^k\mathfrak{d} < 0$, so that $\theta_{k,N,\chi,\mathfrak{d}}(f)$ vanishes identically. Therefore we may assume that $\mathfrak{d}$ is chosen such that $\epsilon(-1)^k\mathfrak{d} > 0$.*

Remark 2.3. *The explicit expression for $\theta_{k,N,\chi,\mathfrak{d}}$ in* [KT04, Theorem 3.2] *reads slightly different, with t^{-k} instead of t^{-k-1}, since they use a slight variation of the sum of cycle integrals $r_{k,Nt,\chi}$, considering equivalence by $\Gamma_0(N)$ instead of $\Gamma_0(Nt)$. The two sums yield the same result, and the reason for the extra factor t^{-1} showing up in our expression is due to the fact that $i_{Nt} = t\cdot i_N$. For trivial character,* (10) *recovers the expression in* [Koh85, Eq. (8)]*, where the constant $C(k,\mathbf{1},\mathfrak{d}) = (-1)^{[k/2]}2^k$ seems to be missing.*

When $f\in S_{2k}(N,\chi^2)$ is *new*, the expression in (10) gets simplified. Indeed, the identity in (7) shows that the forms $f_{N_0,N_1t}^{k,\bar{\chi}_0}(z;\mathfrak{d},\epsilon(-1)^k mt^2)$ are *old* when $t>1$, and hence the left hand side of (8) vanishes, so that $r_{k,Nt,\chi}(f;\mathfrak{d},\epsilon(-1)^k mt^2) = 0$ for $t>1$. Therefore, when $f\in S_{2k}(N,\chi^2)$ is *new* one finds

$$\theta_{k,N,\chi,\mathfrak{d}}(f) = C(k,\chi,\mathfrak{d}) \sum_{\substack{m\geq 1,\\ \epsilon(-1)^k m\equiv 0,1(4)}} r_{k,N,\chi}(f;\mathfrak{d},\epsilon(-1)^k m)q^m. \tag{12}$$

In particular, if $m \geq 1$ is such that $\epsilon(-1)^k m \equiv 0, 1 \pmod 4$, then the m-th coefficient of $\theta_{k,N,\chi,\mathfrak{d}}(f)$ is just

$$a_m(\theta_{k,N,\chi,\mathfrak{d}}(f)) = C(k,\chi,\mathfrak{d}) r_{k,N,\chi}(f;\mathfrak{d},\epsilon(-1)^k m).$$

2.3. **Fourier coefficients and L-values.** It is well-known that Fourier coefficients of half-integral weight modular forms encode special values of (twisted) L-series of integral weight modular forms. We will review this phenomenon in this paragraph, assuming for simplicity of exposition that N and $\mathfrak{d}$ are relatively prime.

Indeed, suppose first that N is odd and squarefree, and let $f \in \sum a_n(f)q^n \in S_{2k}(N)$ be a normalized Hecke eigenform of weight $2k$, level $\Gamma_0(N)$, and trivial nebentype character. For each prime divisor ℓ of N, let W_ℓ denote the ℓ-th Atkin–Lehner involution, and $w_\ell \in \{\pm 1\}$ be the Atkin–Lehner eigenvalue of f at ℓ, so that $f|W_\ell = w_\ell f$. Let $\mathfrak{d}$ be a fundamental discriminant with $(-1)^k\mathfrak{d} > 0$ and such that $\gcd(N,\mathfrak{d}) = 1$. Let $L(f,\chi_{\mathfrak{d}},s)$ be the complex L-series of f twisted by the quadratic character $\chi_{\mathfrak{d}}$. This L-series has holomorphic continuation to the whole complex plane, yielding a completed L-series $\Lambda(f,\chi_{\mathfrak{d}},s)$ that satisfies the functional equation

$$\Lambda(f,\chi_{\mathfrak{d}},s) = (-1)^k \chi_{\mathfrak{d}}(-N) w_N \Lambda(f,\chi_{\mathfrak{d}},2k-s),$$

where $w_N \in \{\pm 1\}$ is the product of all the w_ℓ for $\ell \mid N$ prime. In this setting, Kohnen's formula asserts that if $g = \sum a_n(g)q^n \in S^+_{k+1/2}(4N)[f]$ is any non-zero half-integral weight modular form of weight $k+1/2$ and level $4N$ in Shimura–Shintani correspondence with f, and $\chi_{\mathfrak{d}}(\ell) = w_\ell$ for all prime divisors ℓ of N, then (see [Koh85, Corollary 1])

$$\frac{|a_{|\mathfrak{d}|}(g)|^2}{\langle g,g\rangle} = 2^{\nu(N)} \frac{(k-1)!}{\pi^k} |\mathfrak{d}|^{k-1/2} \frac{L(f,\chi_{\mathfrak{d}},k)}{\langle f,f\rangle}, \tag{13}$$

where $\nu(N)$ is the number of prime divisors of N. A key point in the proof of this formula is the identity

$$r_{k,N}(f;\mathfrak{d},\mathfrak{d}) = 2^{\nu(N)} |\mathfrak{d}|^k (k-1)! \cdot \frac{L(f,\chi_{\mathfrak{d}},k)}{(2\pi i)^k \mathfrak{g}(\chi_{\mathfrak{d}})}, \tag{14}$$

which the reader can check in p. 243 of op. cit. (note that $\mathfrak{g}(\chi_{\mathfrak{d}}) = \chi_{\mathfrak{d}}(-1)|\mathfrak{d}|/\mathfrak{g}(\chi_{\mathfrak{d}}) = (-1)^k|\mathfrak{d}|/\mathfrak{g}(\chi_{\mathfrak{d}})$). Assuming that f is new, we have $a_{|\mathfrak{d}|}(\theta_{k,N,\mathfrak{d}}(f)) = C(k,\mathfrak{d}) r_{k,N}(f;\mathfrak{d},\mathfrak{d})$, and hence we deduce that

$$a_{|\mathfrak{d}|}(\theta_{k,N,\mathfrak{d}}(f))) = C(k,\mathfrak{d}) 2^{\nu(N)} |\mathfrak{d}|^k (k-1)! \cdot \frac{L(f,\chi_{\mathfrak{d}},k)}{(2\pi i)^k \mathfrak{g}(\chi_{\mathfrak{d}})}. \tag{15}$$

Kohnen's formula in (13) has been generalized by Kojima–Tokuno to the case where f has non-squarefree level and non-trivial nebentype character, under a suitable multiplicity one assumption. We refer the reader to [KT04, Theorems 4.1, 4.2] for the details. We will rather focus on the identity analogous to (14), which is also a key step in the proof of the generalization of (13) but it holds unconditionally. In order to describe such identity, we need to introduce some notation.

Let $f \in S^{\mathrm{new}}_{2k}(N,\chi^2)$ be a normalized newform of weight $2k$, odd level $N \geq 1$, and nebentype character χ^2. As before, N_0 will denote the conductor of χ, χ_0 the primitive character associated with χ, and $\epsilon = \chi(-1)$. We continue to assume that $\gcd(N_0,N_1) = 1$, where $N_1 = N/N_0$, and hypothesis $(\star)$. With this, let $\mathfrak{d}$ be a fundamental discriminant such that $\epsilon(-1)^k\mathfrak{d} > 0$, and assume further that[2] $\gcd(N,\mathfrak{d}) = 1$. We will briefly explain how does one relate $r_{k,N,\chi}(f;\mathfrak{d},\mathfrak{d})$, as defined in (9), to the special value $L(f,\chi_{\mathfrak{d}}\bar{\chi}_0,k)$ of the L-series of f twisted by $\chi_{\mathfrak{d}}\bar{\chi}_0$, yielding the analogous identity to (14) above.

For each positive divisor d of N_1, with $\gcd(d,N_1/d) = 1$ (we write $d||N_1$), let W_d be the d-th Atkin–Lehner element in $\mathrm{GL}_2^+(\mathbf{R})$ defined by any matrix

$$W_d = \frac{1}{\sqrt{d}} \begin{pmatrix} d & \alpha_d \\ N & \beta_d d \end{pmatrix} \quad \text{where } \alpha_d, \beta_d \in \mathbf{Z} \text{ are such that } \beta_d d^2 - \alpha_d N = d.$$

Since d divides exactly N_1, observe from the definition that $\beta_d \equiv d^{-1}$ modulo N/d.

Since we are considering divisors of N_1, the above elements W_d act as automorphisms of $S_{2k}(N,\chi^2)$ via the weight $2k$ slash operator. Furthermore, since f is a normalized newform,

[2] Notice that this is stronger than our previous assumption $\gcd(N_0,\mathfrak{d}) = 1$.

for each d as above there exists a normalized newform $f_d \in S_{2k}(N,\chi^2)$ and a non-zero constant $w_d(f)$ such that

$$f|_{2k}W_d = w_d(f)f_d.$$

These constants are multiplicative, meaning that if $dd'||N_1$ with $\gcd(d,d')=1$, then $w_{dd'}(f) = w_d(f)w_{d'}(f)$.

The elements W_d also act on quadratic forms, by the rule

$$Q \circ W_d := {}^tW_dQW_d, \quad Q \in \mathcal{L}_N(N_0^2\mathfrak{d}^2).$$

It is straightforward to check that $Q\circ W_d$ belongs again to $\mathcal{L}_N(N_0^2\mathfrak{d}^2)$. Since W_d normalizes $\Gamma_0(N)$, the map $Q\mapsto Q\circ W_d$ establishes a bijection from $\mathcal{L}_N(N_0^2\mathfrak{d}^2)/\Gamma_0(N)$ to itself. In addition, one can take the set

$$\left\{Q_\mu \circ W_d : \mu \in \mathbf{Z}/\mathfrak{d}\mathbf{Z}\times(\mathbf{Z}/N_0\mathbf{Z})^\times,\, d||N_1\right\}, \quad Q_\mu = \begin{pmatrix} 0 & \mathfrak{d}N_0/2 \\ \mathfrak{d}N_0/2 & \mu \end{pmatrix},$$

as a complete set of representatives for $\mathcal{L}_N(N_0^2\mathfrak{d}^2)/\Gamma_0(N)$. One can therefore rewrite the sum defining $r_{k,N,\chi}(f;\mathfrak{d},\mathfrak{d})$ as a sum over the above set, and use the explicit representatives to compute the involved cycle integrals and eventually deduce the following formula (cf. [KT04, (4-21), (4-22)]). We sketch a proof for completeness.

Proposition 2.4 (Kojima–Tokuno)**.** *Let $k\geq 1$ be an integer and $N\geq 1$ be an odd integer. Let χ be a Dirichlet character modulo N, with conductor N_0 and associated primitive character χ_0, and let $\epsilon=\chi(-1)$. Assume $\gcd(N_0,N_1)=1$, where $N_1=N/N_0$, and let $\mathfrak{d}$ be a fundamental discriminant such that $\gcd(N,\mathfrak{d})=1$ and $\epsilon(-1)^k\mathfrak{d}>0$. If $f\in S_{2k}(N,\chi^2)$ is a normalized Hecke eigenform, then*

$$r_{k,N,\chi}(f;\mathfrak{d},\mathfrak{d}) = \chi_{\mathfrak{d}}(-1)R_{\mathfrak{d}}(f)(-1)^k|\mathfrak{d}|^kN_0^k(k-1)!\cdot\frac{L(f,\chi_{\mathfrak{d}}\bar\chi_0,k)}{(2\pi i)^k\mathfrak{g}(\chi_{\mathfrak{d}}\bar\chi_0)}, \tag{16}$$

where

$$R_{\mathfrak{d}}(f) = \prod_{\ell^e||N_1}\left(1+\chi_{\mathfrak{d}}\bar\chi_0(\ell^e)w_{\ell^e}(f)\frac{1-\chi_{\mathfrak{d}}\bar\chi_0(\ell)a_\ell(f)\ell^{-k}}{1-\chi_{\mathfrak{d}}\chi_0(\ell)\overline{a_\ell(f)}\ell^{-k}}\right).$$

Here, the product is over the prime divisors ℓ of N_1, and for each such prime, with $\ell^e||N_1$, $w_{\ell^e}(f)$ denotes the constant associated with W_{ℓ^e} as above.

Proof. By definition of $r_{k,N,\chi}(f;\mathfrak{d},\mathfrak{d})$, and choosing the above set of representatives, we must compute

$$r_{k,N,\chi}(f;\mathfrak{d},\mathfrak{d}) = \sum_{\mu,d}\omega_{\mathfrak{d}}(Q_\mu\circ W_d)\chi_0(Q_\mu\circ W_d)\int_{C_{Q_\mu\circ W_d}} f(z)(Q_\mu\circ W_d)(z,1)^{k-1}dz.$$

To do so, first observe that for an arbitrary Q

$$\omega_{\mathfrak{d}}(Q\circ W_d) = \chi_{\mathfrak{d}}(d)\omega_{\mathfrak{d}}(Q) \quad \text{and} \quad \chi_0(Q\circ W_d) = \chi_0(\beta_d^2 d)\chi_0(Q) = \bar\chi_0(d)\chi_0(Q),$$

where in the last equality we use that $\beta_d\equiv d^{-1}$ modulo N_0. Also, notice that $\omega_{\mathfrak{d}}(Q_\mu)=\chi_{\mathfrak{d}}(\mu)$ because μ is represented by Q_μ, and $\chi_0(Q_\mu)=\chi_0(\mu)$. Finally, for a quadratic form Q and $\gamma\in\mathrm{GL}_2^+(\mathbf{R})$, one has

$$\int_{C_Q} f(z)Q(z,1)^{k-1}dz = \int_{C_{Q\circ\gamma}}(f|_{2k}\gamma)(z)(Q\circ\gamma)(z,1)^{k-1}dz.$$

With this, and recalling that $f|_{2k}W_d = w_d(f)f_d$, we see that

$$\begin{aligned} r_{k,N,\chi}(f;\mathfrak{d},\mathfrak{d}) &= \sum_d \chi_{\mathfrak{d}}(d)\chi_0(\beta_d^2 d)w_d(f)\sum_\mu \chi_{\mathfrak{d}}(\mu)\chi_0(\mu)\int_{C_{Q_\mu}} f_d(z)Q_\mu(z,1)^{k-1}dz = \\ &= \sum_d \chi_{\mathfrak{d}}\bar{\chi}_0(d)w_d(f)\sum_\mu \chi_{\mathfrak{d}}\chi_0(\mu)\int_{-\mu/\mathfrak{d}N_0}^{i\infty} f_d(z)(\mathfrak{d}N_0 z+\mu)^{k-1}dz = \\ &= \sum_d \chi_{\mathfrak{d}}\bar{\chi}_0(d)w_d(f)\sum_\mu \chi_{\mathfrak{d}}\chi_0(\mu)(i\mathfrak{d}N_0)^{k-1}i\int_0^\infty f_d\left(it-\frac{\mu}{\mathfrak{d}N_0}\right)t^{k-1}dt = \\ &= (i\mathfrak{d}N_0)^{k-1}i\sum_d \chi_{\mathfrak{d}}\bar{\chi}_0(d)w_d(f)\sum_\mu \chi_{\mathfrak{d}}\chi_0(-\mathrm{sgn}(\mathfrak{d})\mu)\int_0^\infty f_d\left(it+\frac{\mu}{|\mathfrak{d}|N_0}\right)t^{k-1}dt. \end{aligned}$$

Now, notice that $\chi_{\mathfrak{d}}(\mathrm{sgn}(\mathfrak{d})) = \chi_{\mathfrak{d}}(-1)$, hence $\chi_{\mathfrak{d}}(-\mathrm{sgn}(\mathfrak{d})\mu) = \chi_{\mathfrak{d}}(\mu)$. Besides, by our choice of $\mathfrak{d}$ we have $\mathrm{sgn}(\mathfrak{d}) = \epsilon(-1)^k$. Therefore, $\chi_0(-\mathrm{sgn}(\mathfrak{d})\mu) = \epsilon^k\chi_0(\mu)$. Thus,

$$\begin{aligned} r_{k,N,\chi}(f;\mathfrak{d},\mathfrak{d}) &= i^k\epsilon^k(\mathfrak{d}N_0)^{k-1}\sum_d \chi_{\mathfrak{d}}\bar{\chi}_0(d)w_d(f)\int_0^\infty \sum_\mu \chi_{\mathfrak{d}}\chi_0(\mu)f_d\left(it+\frac{\mu}{|\mathfrak{d}|N_0}\right)t^{k-1}dt = \\ &= i^k\epsilon^k(\mathfrak{d}N_0)^{k-1}\sum_d \chi_{\mathfrak{d}}\bar{\chi}_0(d)w_d(f)\mathfrak{g}(\chi_{\mathfrak{d}}\chi_0)\int_0^\infty f_{d,\chi_{\mathfrak{d}}\bar{\chi}_0}(it)t^{k-1}dt = \\ &= i^k\epsilon^k\chi_{\mathfrak{d}}(-1)^{k-1}(|\mathfrak{d}|N_0)^{k-1}\mathfrak{g}(\chi_{\mathfrak{d}}\chi_0)\frac{(k-1)!}{(2\pi)^k}\sum_d \chi_{\mathfrak{d}}\bar{\chi}_0(d)w_d(f)L(f_d,\chi_{\mathfrak{d}}\bar{\chi}_0,k), \end{aligned}$$

where we have used Birch's Lemma and the integral representation of $L(f_d,\chi_{\mathfrak{d}}\bar{\chi}_0,s)$. Now, using [Miy06, Theorem 4.6.16], one checks that the sum over d equals

$$R_{\mathfrak{d}}(f)\cdot L(f,\chi_{\mathfrak{d}}\bar{\chi}_0,k).$$

In addition, using that $i^k = (-1)^k i^{-k}$, $\epsilon^k = \chi_{\mathfrak{d}}(-1)^k(-1)^k$, and that

$$\mathfrak{g}(\chi_{\mathfrak{d}}\chi_0) = \chi_{\mathfrak{d}}(-1)\epsilon\frac{|\mathfrak{d}|N_0}{\mathfrak{g}(\chi_{\mathfrak{d}}\bar{\chi}_0)} = (-1)^k\frac{|\mathfrak{d}|N_0}{\mathfrak{g}(\chi_{\mathfrak{d}}\bar{\chi}_0)},$$

we can rewrite the above expression as

$$r_{k,N,\chi}(f;\mathfrak{d},\mathfrak{d}) = \chi_{\mathfrak{d}}(-1)R_{\mathfrak{d}}(f)(-1)^k|\mathfrak{d}|^k N_0^k(k-1)!\cdot\frac{L(f,\chi_{\mathfrak{d}}\bar{\chi}_0,k)}{(2\pi i)^k\mathfrak{g}(\chi_{\mathfrak{d}}\bar{\chi}_0)}.$$

□

Note that the identity in (16) reduces to (14) when one assumes that χ is trivial, N is squarefree and $\chi_{\mathfrak{d}}(\ell) = w_\ell(f)$ for all primes $\ell \mid N$. Indeed, when χ is trivial we have $N_0 = 1$ and $\chi_{\mathfrak{d}}(-1)(-1)^k = 1$, and the Fourier coefficients of f are real, thus the assumption that N is squarefree yields $R_{\mathfrak{d}}(f) = 2^{\nu(N)}$. Finally, recalling that when f is new we have $a_{|\mathfrak{d}|}(\theta_{k,N,\chi,\mathfrak{d}}(f)) = C(k,\chi,\mathfrak{d})r_{k,N,\chi}(f;\mathfrak{d},\mathfrak{d})$, with $C(k,\chi,\mathfrak{d})$ as in (11), we immediately deduce the following consequence.

Corollary 2.5. *With the same assumptions as in the theorem, if f is new then*

$$a_{|\mathfrak{d}|}(\theta_{k,N,\chi,\mathfrak{d}}(f)) = C(k,\chi,\mathfrak{d})\chi_{\mathfrak{d}}(-1)R_{\mathfrak{d}}(f)(-1)^k|\mathfrak{d}|^k N_0^k(k-1)!\cdot\frac{L(f,\chi_{\mathfrak{d}}\bar{\chi}_0,k)}{(2\pi i)^k\mathfrak{g}(\chi_{\mathfrak{d}}\bar{\chi}_0)}. \tag{17}$$

Again, note that (17) reduces to (15) when χ is trivial, N is squarefree, and the same assumption on Atkin–Lehner eigenvalues as above.

3. $\mathfrak{d}$-th Shintani liftings of p-stabilized newforms

We now investigate the relation between the Fourier coefficients of the $\mathfrak{d}$-th Shintani lifting of a newform $f \in S_{2k}(N,\chi^2)$ and those of its ordinary p-stabilization in $S_{2k}(Np,\chi^2)$. To do so, we need first to discuss a detailed study of the classification of integral binary quadratic forms up to equivalence by congruence subgroups.

3.1. **Discussion on integral binary quadratic forms.** Let $N \geq 1$ be an odd integer, and let Δ be a discriminant. Recall that we have introduced the set

$$\mathcal{Q}_N(\Delta) = \{[a,b,c] \in \mathcal{Q}(\Delta) : \, a \equiv 0 \pmod N\},$$

on which $\Gamma_0(N)$ acts. Our aim is to classify the $\Gamma_0(N)$-orbits in this set, following the discussion in [GKZ87] and emphasizing some aspects that will be of special interest for us. As a matter of notation, we will write forms in $\mathcal{Q}_N(\Delta)$ as $[aN, b, c]$, where a, b, c are integers. Yet another invariant for the action of $\Gamma_0(N)$ on $\mathcal{Q}_N(\Delta)$ is the residue class of b modulo $2N$. Notice that not every residue class is allowed: one must have $b^2 \equiv \Delta \pmod{4N}$. Setting

$$R_N(\Delta) := \{\varrho \in \mathbf{Z}/2N\mathbf{Z} : \, \varrho^2 \equiv \Delta \pmod{4N}\},$$

and defining

$$\mathcal{Q}_{N,\varrho}(\Delta) = \{[a,b,c] \in \mathcal{Q}(\Delta) : \, a \equiv 0 \pmod N), \, b \equiv \varrho \pmod{2N}\}$$

for each $\varrho \in R_N(\Delta)$, observe that one has a $\Gamma_0(N)$-invariant decomposition

$$\mathcal{Q}_N(\Delta) = \bigsqcup_{\varrho \in R_N(\Delta)} \mathcal{Q}_{N,\varrho}(\Delta).$$

The sets $\mathcal{Q}_{N,\varrho}(\Delta)$ being $\Gamma_0(N)$-invariant, we are reduced to study their $\Gamma_0(N)$-orbits. We further define the subset of $\Gamma_0(N)$-*primitive* forms in $\mathcal{Q}_{N,\varrho}(\Delta)$ by

$$\mathcal{Q}^0_{N,\varrho}(\Delta) = \{[aN,b,c] \in \mathcal{Q}_{N,\varrho}(\Delta) : \, \gcd(a,b,c) = 1\}.$$

Remark 3.1. *Observe that we consider the greatest common divisor of a, b, and c, not of aN, b, and c.*

One has a $\Gamma_0(N)$-invariant bijection of sets

$$\mathcal{Q}_{N,\varrho}(\Delta) = \bigsqcup_{d^2 \mid \Delta} \bigsqcup_{\substack{\ell \in R_N(\Delta/d^2), \\ d\ell \equiv \varrho \, (2N)}} d \cdot \mathcal{Q}^0_{N,\ell}(\Delta/d^2),$$

where d varies over the positive integers such that d^2 divides Δ, and ℓ varies over the elements in $R_N(\Delta/d^2)$ such that $d\ell \equiv \varrho$ modulo $2N$. Via this last decomposition, it is enough to study the $\Gamma_0(N)$-orbits in the sets of the form $\mathcal{Q}^0_{N,\varrho}(\Delta)$, where $N \geq 1$ is an integer, Δ is a discriminant, and $\varrho \in R_N(\Delta)$.

Continue to fix parameters N and Δ as before. For each $\varrho \in R_N(\Delta)$, associated with the set $\mathcal{Q}^0_{N,\varrho}(\Delta)$ there is the integer

$$m_\varrho := \gcd\left(N, \varrho, \frac{\varrho^2 - \Delta}{4N}\right),$$

which is well-defined even if ϱ is only defined modulo $2N$. Indeed, replacing ϱ by $\varrho + 2N$ replaces $\frac{\varrho^2-\Delta}{4N}$ by $\frac{\varrho^2-\Delta}{4N} + \varrho + N$. Thus m_ϱ is an invariant of the subset $\mathcal{Q}^0_{N,\varrho}(\Delta)$. One can check that

$$m_\varrho = \gcd(N, b, ac) \qquad \text{for any } Q = [aN, b, c] \in \mathcal{Q}^0_{N,\varrho}(\Delta).$$

In particular, notice that $m_\varrho \mid \gcd(N, \Delta)$. Using this, one can further decompose m_ϱ as

$$m_\varrho = m_1 m_2, \qquad \text{where} \qquad m_1 = \gcd(N, b, a), \quad m_2 = \gcd(N, b, c).$$

Notice that $\gcd(m_1, m_2) = 1$ because $\gcd(a, b, c) = 1$. In view of this, we have a $\Gamma_0(N)$-invariant decomposition

$$\mathcal{Q}^0_{N,\varrho}(\Delta) = \bigsqcup_{m_1, m_2} \mathcal{Q}^0_{N,\varrho,m_1,m_2}(\Delta),$$

where m_1, m_2 run over the pairs of positive divisors of m_ϱ which satisfy $m_1 m_2 = m_\varrho$ and $\gcd(m_1, m_2) = 1$. There are 2^ν such pairs, where ν denotes the number of prime factors of m_ϱ.

The following statement is the Proposition in p. 505 of [GKZ87]:

Proposition 3.2. *Let m_ϱ be defined as above and fix any decomposition $m_\varrho = m_1m_2$ with $m_1, m_2 > 0$ integers such that $\gcd(m_1, m_2) = 1$. Fix also any decomposition $N = N_1N_2$ of N into coprime factors satisfying $\gcd(m_1, N_2) = \gcd(m_2, N_1) = 1$. Then the map $Q = [aN, b, c] \mapsto \tilde{Q} = [aN_1, b, cN_2]$ yields a one-to-one correspondence*

$$\mathcal{Q}^0_{N,\varrho,m_1,m_2}(\Delta)/\Gamma_0(N) \xrightarrow{1:1} \mathcal{Q}^0(\Delta)/\Gamma_0(1).$$

In particular, $|\mathcal{Q}^0_{N,\varrho}(\Delta)/\Gamma_0(N)| = 2^\nu|\mathcal{Q}^0(\Delta)/\Gamma_0(1)|$ where ν is the number of prime factors of m_ϱ.

Notice that this proposition finishes the precise description of the set of integral binary quadratic forms $\mathcal{Q}_N(\Delta)$, up to $\Gamma_0(N)$-action. Indeed, as a summary of the above discussion we have a disjoint union of $\Gamma_0(N)$-invariant sets

$$\mathcal{Q}_N(\Delta) = \bigsqcup_{d^2|\Delta} \bigsqcup_{\varrho\in R_N(\Delta/d^2)} \bigsqcup_{(m_1,m_2)} d\cdot \mathcal{Q}^0_{N,\varrho,m_1,m_2}(\Delta/d^2), \tag{18}$$

where d varies over the positive integers such that $d^2 \mid \Delta$, and for each $\varrho \in R_N(\Delta/d^2)$ the pair (m_1, m_2) ranges over the pairs of coprime positive integers with $m_1m_2 = m_\varrho$. For each tuple (d, ϱ, m_1, m_2), the set of classes $\mathcal{Q}^0_{N,\varrho,m_1,m_2}(\Delta/d^2)/\Gamma_0(N)$ is in bijection with $\mathcal{Q}^0(\Delta/d^2)/\Gamma_0(1)$, which is a class group *à la Gauss*.

Let us close this paragraph by pointing out some particular instances of the above proposition that will be useful for our later discussion. We will look at sets of the form

$$\mathcal{L}_N(N_0^2\Delta)/\Gamma_0(N) \quad\text{and}\quad \mathcal{L}_{Np}(N_0^2\Delta)/\Gamma_0(Np),$$

where N_0 divides exactly N (meaning that $\gcd(N_0, N/N_0) = 1$), p is an odd prime not dividing N, and Δ is a discriminant. Recall also that the sets '$\mathcal{L}$' consist of forms in the sets '$\mathcal{Q}$' with $\gcd(N_0, c) = 1$. We will naturally write $\mathcal{L}^0_{N,\varrho}(\cdot)$, $\mathcal{L}^0_{N,\varrho,m_1,m_2}(\cdot)$, etc., for the intersection of $\mathcal{Q}^0_{N,\varrho}(\cdot)$, $\mathcal{Q}^0_{N,\varrho,m_1,m_2}(\cdot)$, etc. with $\mathcal{L}_N(\cdot)$. An important observation for the following statements is that, via the decomposition as in (18), the subset $\mathcal{L}_N(N_0^2\Delta)$ (resp. $\mathcal{L}_{Np}(N_0^2\Delta)$) of $\mathcal{Q}_N(N_0^2\Delta)$ (resp. $\mathcal{Q}_{Np}(N_0^2\Delta)$), corresponds to the union of those subsets $d\cdot\mathcal{Q}^0_{N,\varrho,m_1,m_2}(N_0^2\Delta/d^2)$ (resp. $d\cdot\mathcal{Q}^0_{Np,\varrho,m_1,m_2}(N_0^2\Delta/d^2)$) with $\gcd(N_0, m_2) = 1$. We assume for simplicity that

$$\gcd(N, N_0^2\Delta) = N_0 \qquad \text{(equivalently, } \gcd(N_1,\Delta) = 1). \tag{†}$$

Corollary 3.3. *Under the above assumptions,*

$$\mathcal{L}_N(N_0^2\Delta) = \bigsqcup_{d^2|N_0^2\Delta} \bigsqcup_{\varrho\in R_N(N_0^2\Delta/d^2)} d\cdot\mathcal{L}^0_{N,\varrho}(N_0^2\Delta/d^2), \tag{19}$$

and moreover the identity map $[aN, b, c] \mapsto [aN, b, c]$ yields a bijection between each of the sets $\mathcal{L}^0_{N,\varrho}(N_0^2\Delta/d^2)/\Gamma_0(N)$ and $\mathcal{L}^0(N_0^2\Delta/d^2)/\Gamma_0(1)$.

Proof. For every $Q = [aN, b, c] \in \mathcal{L}_N(N_0^2\Delta)$ one has $\gcd(N_0, c) = 1$. Since $m_\varrho \mid \gcd(N, N_0^2\Delta)$ and m_2 divides both c and m_ϱ, condition (†) implies $m_2 = 1$. In particular, we must have $m_1 = m_\varrho$, and $\mathcal{L}^0_{N,\varrho}(N_0^2\Delta/d^2) = \mathcal{L}^0_{N,\varrho,m_\varrho,1}(N_0^2\Delta/d^2)$ for all $\varrho \in R_N(N_0^2\Delta/d^2)$, hence we can take $N_1 = N$ and $N_2 = 1$ in Proposition 3.2. □

Corollary 3.4. *Let N, N_0 and Δ be as in the above corollary. If $p > 2$ is a prime such that $p \,||\, \Delta$, then*

$$\mathcal{L}_{Np}(N_0^2\Delta) = \bigsqcup_{d^2|N_0^2\Delta} \bigsqcup_{\varrho\in R_{Np}(N_0^2\Delta/d^2)} d\cdot\mathcal{L}^0_{Np,\varrho}(N_0^2\Delta/d^2), \tag{20}$$

and both maps $[aNp, b, c] \mapsto [aNp, b, c]$ and $[aNp, b, c] \mapsto [aN, b, cp]$ yield bijections between each of the sets $\mathcal{L}^0_{Np,\varrho}(N_0^2\Delta/d^2)$ and $\mathcal{L}^0(N_0^2\Delta/d^2)/\Gamma_0(1)$.

Proof. This follows similarly as the previous corollary, noticing that the assumption $p \,||\, \Delta$ implies that one still has $m_2 = 1$ and $m_1 = m_\varrho$. Therefore, we can use both maps in the statement to establish the claimed bijections, by virtue of Proposition 3.2. □

In the above corollaries, the fact that the invariant m_2 is always 1 when restricted to the decompositions of the sets $\mathcal{L}_N(N_0^2\Delta)$ and $\mathcal{L}_{Np}(N_0^2\Delta)$ simplifies significantly the discussion. The situation becomes a bit more involved if $p^2 \mid \Delta$. Indeed, let N and Δ be as above, and fix a prime $p > 2$ with $p^2 \mid \Delta$. Then we can write a $\Gamma_0(Np)$-invariant decomposition

$$\mathcal{L}_{Np}(N_0^2\Delta) = \mathcal{L}_{Np}^{p}(N_0^2\Delta) \sqcup \mathcal{L}_{Np}^{(p)}(N_0^2\Delta),$$

where

$$\mathcal{L}_{Np}^{p}(N_0^2\Delta) = \bigsqcup_{\substack{d^2 \mid N_0^2\Delta, \\ p \mid d}} \bigsqcup_{\varrho \in R_{Np}(\frac{N_0^2\Delta}{d^2})} d \cdot \mathcal{L}_{Np,\varrho}^{0}(N_0^2\Delta/d^2),$$

$$\mathcal{L}_{Np}^{(p)}(N_0^2\Delta) = \bigsqcup_{\substack{d^2 \mid N_0^2\Delta, \\ p \nmid d}} \bigsqcup_{\varrho \in R_{Np}(\frac{N_0^2\Delta}{d^2})} d \cdot \mathcal{L}_{Np,\varrho}^{0}(N_0^2\Delta/d^2).$$

Notice that $\mathcal{L}_{Np}^{p}(N_0^2\Delta) = p \cdot \mathcal{L}_{Np}(N_0^2\Delta/p^2)$, and hence one could apply the first of the above two corollaries to describe the decomposition of this set into primitive subsets. Thus we may focus our attention on the set $\mathcal{L}_{Np}^{(p)}(N_0^2\Delta)$. To do so, first notice that if $[aNp, b, c] \in \mathcal{L}_{Np,\varrho}^{0}(N_0^2\Delta/d^2)$ for some d and ϱ arising in the union defining $\mathcal{L}_{Np}^{(p)}(N_0^2\Delta)$, then either

- i) $p \mid a$ and $p \nmid c$, or
- ii) $p \nmid a$ and $p \mid c$.

Indeed, this follows easily from the fact that p^2 divides the discriminant of $[aNp, b, c]$ and $p \nmid \gcd(a, b, c)$. We call $\mathcal{L}_{Np,\varrho}^{0,a}(N_0^2\Delta/d^2)$, resp. $\mathcal{L}_{Np,\varrho}^{0,c}(N_0^2\Delta/d^2)$, the subset of forms in $\mathcal{L}_{Np,\varrho}^{0}(N_0^2\Delta/d^2)$ falling in case i), resp. ii). In a natural way, we also define $\mathcal{L}_{Np}^{(p),a}(N_0^2\Delta)$ and $\mathcal{L}_{Np}^{(p),c}(N_0^2\Delta)$, so that

$$\mathcal{L}_{Np}^{(p)}(N_0^2\Delta) = \mathcal{L}_{Np}^{(p),a}(N_0^2\Delta) \sqcup \mathcal{L}_{Np}^{(p),c}(N_0^2\Delta).$$

Corollary 3.5. *Let N, Δ, and p be as above. For $\star \in \{a, c\}$, there is a $\Gamma_0(Np)$-invariant decomposition*

$$\mathcal{L}_{Np}^{(p),\star}(N_0^2\Delta) = \bigsqcup_{\substack{d^2 \mid N_0^2\Delta, \\ p \nmid d}} \bigsqcup_{\varrho \in R_{Np}(N_0^2\Delta/d^2)} d \cdot \mathcal{L}_{Np,\varrho}^{0,\star}(N_0^2\Delta/d^2).$$

A bijection between each of the sets $\mathcal{L}_{Np,\varrho}^{0,\star}(N_0^2\Delta/d^2)/\Gamma_0(Np)$ and $\mathcal{L}^{0}(N_0^2\Delta/d^2)/\Gamma_0(1)$ is induced by the identity map $[aNp, b, c] \mapsto [aNp, b, c]$ if $\star = a$, and by the map $\tau : [aNp, b, c] \mapsto [aN, b, pc]$ if $\star = c$.

Proof. In this case, the assumption (†) together with the definition of m_ϱ imply that $p \mid\mid m_\varrho$ for all $\varrho \in R_{Np}(N_0^2\Delta/d^2)$, with $d^2 \mid N_0^2\Delta$, $p \nmid d$. In addition, m_2 can only be either 1 or p because $\gcd(N_0, c) = 1$. Thus the only possibilities for the pair (m_1, m_2) are $(m_\varrho, 1)$ and $(m_\varrho/p, p)$. By construction, one easily checks that the set $\mathcal{L}_{Np,\varrho}^{0,a}(N_0^2\Delta/d^2)$ is exactly $\mathcal{L}_{Np,\varrho,m_\varrho,1}^{0}(N_0^2\Delta/d^2)$, while the set $\mathcal{L}_{Np,\varrho}^{0,c}(N_0^2\Delta/d^2)$ coincides with $\mathcal{L}_{Np,\varrho,m_\varrho/p,p}^{0}(N_0^2\Delta/d^2)$, thus the result follows by applying Proposition 3.2. □

3.2. **Relating level N and level Np.** Continue to assume that $\gcd(N_1, \Delta) = 1$ as before (see (†)), and let p be an odd prime with $p \nmid N$. After our careful study of the previous paragraph, we now want to compare the sets

$$\mathcal{L}_{Np}(N_0^2\Delta)/\Gamma_0(Np) \quad \text{and} \quad \mathcal{L}_{N}(N_0^2\Delta)/\Gamma_0(N).$$

To begin with, we need to understand how the sets $R_N(N_0^2\Delta)$ and $R_{Np}(N_0^2\Delta)$ are related. Observe that $R_N(N_0^2\Delta)$ is the subset of $\mathbf{Z}/2N\mathbf{Z}$ consisting of the residue classes $\varrho \in \mathbf{Z}/2N\mathbf{Z}$ such that $\varrho^2 \equiv N_0^2\Delta \pmod{4N}$ (in particular, notice that ϱ must be congruent to 0 modulo N_0). And similarly for $R_{Np}(N_0^2\Delta)$. Of course, these sets could be empty.

The natural projection morphism

$$\mathbf{Z}/2Np\mathbf{Z} = \mathbf{Z}/2N\mathbf{Z} \times \mathbf{Z}/p\mathbf{Z} \twoheadrightarrow \mathbf{Z}/2N\mathbf{Z}$$

induces a surjective map $R_{Np}(N_0^2\Delta) \twoheadrightarrow R_N(N_0^2\Delta)$, and we have

$$\sharp R_{Np}(N_0^2\Delta) = \left(1 + \left(\frac{N_0^2\Delta}{p}\right)\right) \sharp R_N(N_0^2\Delta) = \left(1 + \left(\frac{\Delta}{p}\right)\right) \sharp R_N(N_0^2\Delta).$$

Suppose that both $R_N(N_0^2\Delta)$ and $R_{Np}(N_0^2\Delta)$ are non-empty. Then the above map is a bijection if $(\frac{\Delta}{p}) = 0$, and it is a 2-to-1 map if $(\frac{\Delta}{p}) = 1$. For a given $\varrho \in R_N(N_0^2\Delta)$, observe that $\mathcal{L}^0_{Np,\varrho'}(N_0^2\Delta) \subseteq \mathcal{L}_{N,\varrho}(N_0^2\Delta)$ for all $\varrho' \in R_{Np}(N_0^2\Delta)$ mapping to ϱ. However, notice that if $\varrho' \equiv 0 \pmod p$ (equivalently, if p divides Δ, so that there is a unique ϱ' mapping to ϱ), it could happen that a $\Gamma_0(Np)$-primitive form $Q = [aNp, b, c]$ is not $\Gamma_0(N)$-primitive[3] as a form in $\mathcal{L}_{N,\varrho}(N_0^2\Delta)$. If p does not divide Δ and $(\frac{\Delta}{p}) = 1$, we will certainly have

$$\bigsqcup_{\substack{\varrho' \in R_{Np}(N_0^2\Delta),\\ \varrho' \mapsto \varrho}} \mathcal{L}^0_{Np,\varrho'}(N_0^2\Delta) \subseteq \mathcal{L}^0_{N,\varrho}(N_0^2\Delta),$$

where on the left hand side there are exactly two ϱ' mapping to ϱ.

When $p^2 \nmid \Delta$, we have the following:

Proposition 3.6. *Suppose that $p \nmid N$, $p^2 \nmid \Delta$, and assume (†). If $(\frac{\Delta}{p}) = -1$, then the set $\mathcal{L}_{Np}(N_0^2\Delta)/\Gamma_0(Np)$ is empty. Otherwise, both maps $[aNp, b, c] \mapsto [(ap)N, b, c]$ and $\tau : [aNp, b, c] \mapsto [aN, b, pc]$ from $\mathcal{L}_{Np}(N_0^2\Delta)$ to $\mathcal{L}_N(N_0^2\Delta)$ yield surjective maps*

$$\mathcal{L}_{Np}(N_0^2\Delta)/\Gamma_0(Np) \longrightarrow \mathcal{L}_N(N_0^2\Delta)/\Gamma_0(N).$$

If $(\frac{\Delta}{p}) = 0$, these maps are bijections, whereas if $(\frac{\Delta}{p}) = 1$, these maps are 2-to-1.

Proof. The assertion in the case $(\frac{\Delta}{p}) = -1$ is clear from our above discussion, as all of the sets $R_{Np}(N_0^2\Delta/d^2)$ will be empty. Otherwise, recall that we have natural surjective maps

$$R_{Np}(N_0^2\Delta/d^2) \longrightarrow R_N(N_0^2\Delta/d^2),\ \varrho' \longmapsto \varrho,$$

for each positive d with $d^2 \mid N_0^2\Delta$. By Corollaries 3.3, 3.4 applied with levels N and Np, for every ϱ' above ϱ, we have bijections

$$\mathcal{L}^0_{Np,\varrho'}(N_0^2\Delta/d^2)/\Gamma_0(Np) \xrightarrow{\beta} \mathcal{L}^0(N_0^2\Delta/d^2)/\Gamma_0(1) \xleftarrow{\text{id}} \mathcal{L}^0_{N,\varrho}(N_0^2\Delta/d^2)/\Gamma_0(N),$$

where β can be induced by either the identity map or by τ. Choosing β to be induced by the identity map (resp. by τ) for all choices of d and ϱ', yields the desired result, noticing that the map $R_{Np}(N_0^2\Delta/d^2) \to R_N(N_0^2\Delta/d^2)$ is a bijection when $(\frac{\Delta}{p}) = 0$ and 2-to-1 when $(\frac{\Delta}{p}) = 1$. □

When $p^2 \mid \Delta$, we find the following:

Proposition 3.7. *Suppose that $p \nmid N$, $p^2 \mid \Delta$, and assume (†). Then, for each $\star \in \{a, c\}$, we have a bijection*

$$\mathcal{L}^{(p),\star}_{Np}(N_0^2\Delta)/\Gamma_0(Np) \longrightarrow \mathcal{L}^{(p)}_N(N_0^2\Delta)/\Gamma_0(N).$$

Such a bijection is induced by the identity map if $\star = a$, and by the map τ if $\star = c$.

Proof. Using Corollaries 3.3 and 3.5 one can deduce that for each positive d such that $d^2 \mid \Delta$ and $p \nmid d$, and for each pair $\varrho' \mapsto \varrho$, there are bijections

$$\mathcal{L}^{0,\star}_{Np,\varrho'}(N_0^2\Delta/d^2)/\Gamma_0(Np) \xrightarrow{\beta} \mathcal{L}^0(N_0^2\Delta/d^2)/\Gamma_0(1) \xleftarrow{\text{id}} \mathcal{L}^0_{N,\varrho}(N_0^2\Delta/d^2)/\Gamma_0(N),$$

where β can be induced by the identity map when $\star = a$ and by the map τ when $\star = c$. Since $p \mid \Delta/d^2$ we also have bijections $R_{Np}(N_0^2\Delta/d^2) \to R_N(N_0^2\Delta/d^2)$, $\varrho' \mapsto \varrho$, and hence we deduce the desired result. □

3.3. **$\mathfrak{d}$-th Shintani lifting and p-stabilization.** Fix an odd integer $N \geq 1$, an integer $k \geq 1$, and a Dirichlet character χ modulo N. Let $f \in S^{new}_{2k}(N, \chi^2)$ be a newform, and p be an odd prime not dividing N for which f is p-ordinary. Let $\alpha = \alpha_p(f)$ and $\beta = \beta_p(f)$ be the roots of the p-th Hecke polynomial of f, labelled so that α is a p-adic unit. The ordinary p-stabilization $f_\alpha \in S_{2k}(Np)$ of f is defined by

$$f_\alpha = f - \beta V_p f = (1 - \beta V_p)f,$$

where the operator $V_p : S_{2k}(N, \chi^2) \to S_{2k}(Np, \chi^2)$ is given by $V_p f(z) = f(pz)$. Notice that $f_\alpha \in S_{2k}(Np, \chi^2)$ is an old form by construction (however, it is only old at p, and new at all primes dividing N).

[3]The form $[aNp, b, c]$ is seen as $[(ap)N, b, c]$. Thus, even if $\gcd(a, b, c) = 1$ it could be that $\gcd(ap, b, c) = p$. This will happen if both b and c are multiple of p and a is not, although this implies that Δ is divisible by p.

Continue to use N_0, N_1 χ_0, ϵ with the same meaning and assumptions as in Section 2, and fix a fundamental discriminant $\mathfrak{d}$ satisfying the following conditions:

(i) $\gcd(N, \mathfrak{d}) = 1$, $\epsilon(-1)^k\mathfrak{d} > 0$;
(ii) $\mathfrak{d} \equiv 0 \pmod p$;
(iii) $\theta_{k,N,\chi,\mathfrak{d}}(f) \neq 0$.

Notice that we need to assume $\gcd(N_0, \mathfrak{d}) = 1$ for the construction of the $\mathfrak{d}$-th Shintani lifting. In order to apply the result from the previous paragraph, we will also need to assume $\gcd(N_1, \mathfrak{d}) = 1$ to fulfill hypothesis (†) when considering discriminants divisible by $\mathfrak{d}$.

Remark 3.8. *With our working assumptions, the existence of such a discriminant $\mathfrak{d}$ is guaranteed by non-vanishing results for special values of twisted L-series (combine, for example,* [BFH90] *and* [Wal91, Théorème 4]*) together with Corollary 2.5.*

Having made the above choice for $\mathfrak{d}$, we want to relate the $\mathfrak{d}$-th Shintani lifting

$$\theta_{k,Np,\chi,\mathfrak{d}}(f_\alpha) = \theta_{k,Np,\chi,\mathfrak{d}}(f) - \beta\theta_{k,Np,\chi,\mathfrak{d}}(V_pf)$$

of the ordinary p-stabilization of f with $\theta_{k,N,\chi,\mathfrak{d}}(f)$. Notice that the lifting $\theta_{k,Np,\chi,\mathfrak{d}}(f_\alpha)$ occurs at level Np.

Let $n \geq 1$ be an integer such that $\epsilon(-1)^kn \equiv 0, 1 \pmod 4$. By definition of the $\mathfrak{d}$-th Shintani lifting, the n-th Fourier coefficient of $\theta_{k,N,\chi,\mathfrak{d}}(f)$ is given by the expression

$$\begin{aligned} a_n(\theta_{k,N,\chi,\mathfrak{d}}(f)) &= C(k,\chi,\mathfrak{d}) \sum_{0<t\mid N} \mu(t)\chi_{\mathfrak{d}}\bar{\chi}_0(t)t^{-k-1}r_{k,Nt,\chi}(f;\mathfrak{d},\epsilon(-1)^knt^2) = \\ &= C(k,\chi,\mathfrak{d})r_{k,N,\chi}(f;\mathfrak{d},\epsilon(-1)^kn), \end{aligned}$$

where the second equality is due to the fact that f is new (see equation (12)). Similarly, if $\breve{f} \in S_{2k}(Np, \chi^2)$, then the n-th Fourier coefficient of $\theta_{k,Np,\chi,\mathfrak{d}}(\breve{f})$ is given by

$$a_n(\theta_{k,Np,\chi,\mathfrak{d}}(\breve{f})) = C(k,\chi,\mathfrak{d}) \sum_{0<t\mid Np} \mu(t)\chi_{\mathfrak{d}}\bar{\chi}_0(t)t^{-k-1}r_{k,Npt,\chi}(\breve{f};\mathfrak{d},\epsilon(-1)^knt^2).$$

Using that p does not divide N, that $\mu(pt) = -\mu(t)$, and that $\chi_{\mathfrak{d}}\bar{\chi}_0(pt) = \chi_{\mathfrak{d}}\bar{\chi}_0(p)\chi_{\mathfrak{d}}\bar{\chi}_0(t)$, we may rewrite the sum in the previous expression as

$$\sum_{0<t\mid N} \mu(t)\chi_{\mathfrak{d}}\bar{\chi}_0(t)t^{-k-1}\left(r_{k,Npt,\chi}(\breve{f};\mathfrak{d},\epsilon(-1)^knt^2) - \chi_{\mathfrak{d}}\bar{\chi}_0(p)p^{-k-1}r_{k,Np^2t,\chi}(\breve{f};\mathfrak{d},\epsilon(-1)^knp^2t^2)\right).$$

Our choice of $\mathfrak{d}$ implies that $\chi_{\mathfrak{d}}(p) = 0$, so that we actually have

$$a_n(\theta_{k,Np,\chi,\mathfrak{d}}(\breve{f})) = C(k,\chi,\mathfrak{d}) \sum_{0<t\mid N} \mu(t)\chi_{\mathfrak{d}}\bar{\chi}_0(t)t^{-k-1}r_{k,Npt,\chi}(\breve{f};\mathfrak{d},\epsilon(-1)^knt^2). \tag{21}$$

We are interested in the cases $\breve{f} = f$ and $\breve{f} = V_pf$. In the following lemmas, we will see that since $f \in S_{2k}(N, \chi^2)$ is new, all the terms in (21) corresponding to non-trivial divisors of N vanish, and the remaining term can be related to the $\mathfrak{d}$-th Shintani lifting of f at level N. This will lead to the precise comparison between the coefficients of $\theta_{k,Np,\chi,\mathfrak{d}}(f_\alpha)$ and $\theta_{k,N,\chi,\mathfrak{d}}(f)$ stated in Proposition 3.13 below. We start dealing with the case $\breve{f} = f$.

Lemma 3.9. *With the above notation and assumptions, the $\mathfrak{d}$-th Shintani lifting of f at level Np satisfies*

$$a_n(\theta_{k,Np,\chi,\mathfrak{d}}(f)) = C(k,\chi,\mathfrak{d}) \cdot r_{k,Np,\chi}(f;\mathfrak{d},\epsilon(-1)^kn).$$

Proof. In view of (21), it suffices to show that $r_{k,Npt,\chi}(f;\mathfrak{d},\epsilon(-1)^knt^2) = 0$ for all divisors t of N with $t > 1$. From (8) (with N_1p in place of N_1) and (7) we have

$$r_{k,Npt,\chi}(f;\mathfrak{d},\epsilon(-1)^knt^2) = i_{Npt}\pi^{-1}\binom{2k-2}{k-1}^{-1}2^{2k-2}(|\mathfrak{d}|nt^2)^{k-1/2} \cdot \langle f, V_tf^{k,\bar{\chi}_0}_{N_0,N_1p/t}(-;\mathfrak{d},\epsilon(-1)^kn)\rangle,$$

and we observe that

$$\langle f, V_tf^{k,\bar{\chi}_0}_{N_0,N_1p/t}(-;\mathfrak{d},\epsilon(-1)^kn)\rangle = \langle f, \mathrm{tr}^{Np}_N(V_tf^{k,\bar{\chi}_0}_{N_0,N_1p/t}(-;\mathfrak{d},\epsilon(-1)^kn))\rangle,$$

where $\mathrm{tr}_N^{Np} : S_{2k}(Np,\chi^2) \to S_{2k}(N,\chi^2)$ is the usual trace operator. Furthermore, one can check that

$$\mathrm{tr}_N^{Np}(V_t f^{k,\bar\chi_0}_{N_0,N_1p/t}(-;\mathfrak{d},\epsilon(-1)^k n)) = V_t(\mathrm{tr}_{N/t}^{Np/t} f^{k,\bar\chi_0}_{N_0,N_1p/t}(-;\mathfrak{d},\epsilon(-1)^k n)),$$

and hence the Petersson product on the right hand side of the above identity vanishes. Hence, the terms $r_{k,Npt,\chi}(f;\mathfrak{d},\epsilon(-1)^k nt^2)$ vanish when $t>1$ as we wanted to prove. □

Thanks to the above lemma, it suffices to compute $r_{k,Np,\chi}(f;\mathfrak{d},\epsilon(-1)^k n)$ in order to determine $a_n(\theta_{k,Np,\chi,\mathfrak{d}}(f))$. Notice that the coefficient $a_n(\theta_{k,Np,\chi,\mathfrak{d}}(f))$ is forced to vanish unless n satisfies $\epsilon(-1)^k n \equiv 0,1 \pmod 4$, because $\theta_{k,Np,\chi,\mathfrak{d}}(f)$ belongs to Kohnen's plus subspace. Let $\mathfrak{d}'$ be a discriminant with $\mathfrak{d}\mathfrak{d}'>0$ (note that this is equivalent to $\epsilon(-1)^k\mathfrak{d}'>0$, and that therefore one has $\epsilon(-1)^k|\mathfrak{d}'| = \mathfrak{d}'$).

Lemma 3.10. *With the above notation, suppose that* $\gcd(N_1,\mathfrak{d}')=1$*. Then we have*

$$a_{|\mathfrak{d}'|}(\theta_{k,Np,\chi,\mathfrak{d}}(f)) = a_{|\mathfrak{d}'|}(\theta_{k,N,\chi,\mathfrak{d}}(f)).$$

Proof. By the previous lemma, we know that $a_{|\mathfrak{d}'|}(\theta_{k,Np,\chi,\mathfrak{d}}(f)) = C(k,\chi,\mathfrak{d}) r_{k,Np,\chi}(f;\mathfrak{d},\mathfrak{d}')$, so it suffices to compute

$$r_{k,Np,\chi}(f;\mathfrak{d},\mathfrak{d}') = \sum_{Q\in\mathcal{L}_{Np}(N_0^2\mathfrak{d}\mathfrak{d}')/\Gamma_0(Np)} \omega_{\mathfrak{d}}(Q) I_{k,\chi}(f,Q).$$

Suppose first that $p\nmid\mathfrak{d}'$, so that p divides exactly $N_0^2\mathfrak{d}\mathfrak{d}'$. Notice that our assumptions on $\mathfrak{d}$ and $\mathfrak{d}'$ imply that (†) is satisfied for $\Delta=\mathfrak{d}\mathfrak{d}'$. By our discussion in the previous paragraph, and more precisely by Proposition 3.6, we can use the identity map $[aNp,b,c]\mapsto[(ap)N,b,c]$ to induce a bijection from $\mathcal{L}_{Np}(N_0^2\mathfrak{d}\mathfrak{d}')/\Gamma_0(Np)$ to $\mathcal{L}_N(N_0^2\mathfrak{d}\mathfrak{d}')/\Gamma_0(N)$. Therefore, we have

$$r_{k,Np,\chi}(f;\mathfrak{d},\mathfrak{d}') = \sum_{Q\in\mathcal{L}_N(N_0^2\mathfrak{d}\mathfrak{d}')/\Gamma_0(N)} \omega_{\mathfrak{d}}(Q) I_{k,\chi}(f,Q) = r_{k,N,\chi}(f;\mathfrak{d},\mathfrak{d}'),$$

and it follows that $a_{|\mathfrak{d}'|}(\theta_{k,Np,\chi,\mathfrak{d}}(f)) = a_{|\mathfrak{d}'|}(\theta_{k,N,\chi,\mathfrak{d}}(f))$.

Suppose now that $p\mid\mathfrak{d}'$, hence p^2 divides $\mathfrak{d}\mathfrak{d}'$. In this case, we use the decomposition

$$\mathcal{L}_{Np}(N_0^2\mathfrak{d}\mathfrak{d}')/\Gamma_0(Np) = \mathcal{L}^p_{Np}(N_0^2\mathfrak{d}\mathfrak{d}')/\Gamma_0(Np) \sqcup \mathcal{L}^{(p)}_{Np}(N_0^2\mathfrak{d}\mathfrak{d}')/\Gamma_0(Np)$$

already introduced in the previous paragraph. Since $\omega_{\mathfrak{d}}(Q)=0$ for all forms $Q\in\mathcal{L}^p_{Np}(N_0^2\mathfrak{d}\mathfrak{d}')$, we see that the sum over the first set does not contribute to $r_{k,Np,\chi}(f;\mathfrak{d},\mathfrak{d}')$. Therefore,

$$r_{k,Np,\chi}(f;\mathfrak{d},\mathfrak{d}') = \sum_{Q\in\mathcal{L}^{(p),a}_{Np}(N_0^2\mathfrak{d}\mathfrak{d}')/\Gamma_0(Np)} \omega_{\mathfrak{d}}(Q) I_{k,\chi}(f,Q) + \sum_{Q\in\mathcal{L}^{(p),c}_{Np}(N_0^2\mathfrak{d}\mathfrak{d}')/\Gamma_0(Np)} \omega_{\mathfrak{d}}(Q) I_{k,\chi}(f,Q)$$

By Proposition 3.7, the second sum equals

$$\sum_{Q\in\mathcal{L}^{(p)}_N(N_0^2\mathfrak{d}\mathfrak{d}')/\Gamma_0(N)} \omega_{\mathfrak{d}}(\tau^{-1}(Q)) I_{k,\chi}(f,\tau^{-1}(Q)) = 0,$$

because $\omega_{\mathfrak{d}}(\tau^{-1}(Q)) = \chi_{\mathfrak{d}'}(p)\omega_{\mathfrak{d}}(Q) = 0$ for all $Q\in\mathcal{L}^{(p)}_N(N_0^2\mathfrak{d}\mathfrak{d}')$. Hence the above expression simplifies to

$$r_{k,Np,\chi}(f;\mathfrak{d},\mathfrak{d}') = \sum_{Q\in\mathcal{L}^{(p),a}_{Np}(N_0^2\mathfrak{d}\mathfrak{d}')/\Gamma_0(Np)} \omega_{\mathfrak{d}}(Q) I_{k,\chi}(f,Q). \tag{22}$$

Again by Proposition 3.7, we deduce that

$$r_{k,Np,\chi}(f;\mathfrak{d},\mathfrak{d}') = \sum_{Q\in\mathcal{L}^{(p)}_N(N_0^2\mathfrak{d}\mathfrak{d}')/\Gamma_0(N)} \omega_{\mathfrak{d}}(Q) I_{k,\chi}(f,Q).$$

This sum equals $r_{k,N,\chi}(f;\mathfrak{d},\mathfrak{d}')$, since we can replace $\mathcal{L}^{(p)}_N(N_0^2\mathfrak{d}\mathfrak{d}')$ by $\mathcal{L}_N(N_0^2\mathfrak{d}\mathfrak{d}')$ using again that $\omega_{\mathfrak{d}}(Q)=0$ for all $Q\in\mathcal{L}^p_N(N_0^2\mathfrak{d}\mathfrak{d}')$, which implies the result. □

We proceed now in a similar fashion with V_pf. First, let $n\geq 1$ be an arbitrary positive integer with $\epsilon(-1)^k n\equiv 0,1 \pmod 4$. We have the following observation.

Lemma 3.11. *With the above notation, we have that*

$$a_n(\theta_{k,Np,\chi,\mathfrak{d}}(V_pf)) = C(k,\chi,\mathfrak{d})\cdot r_{k,Np,\chi}(V_pf;\mathfrak{d},\epsilon(-1)^k n).$$

Proof. The argument is similar to the one for the case $\breve{f} = f$. We must show now that for all $t > 1$ the terms $r_{k,Npt,\chi}(V_pf; \mathfrak{d}, \epsilon(-1)^k nt^2)$ vanish. One checks that these are multiples of $\langle V_pf, V_t f^{k,\bar\chi_0}_{N_0,N_1p/t}(-;\mathfrak{d},\epsilon(-1)^k n)\rangle$, and this Petersson product vanishes if $t > 1$ because the second form is old at t while the first one is new at t. □

As above, we focus on fundamental discriminants $\mathfrak{d}'$ with $\mathfrak{d}\mathfrak{d}' > 0$, and study the $|\mathfrak{d}'|$-th Fourier coefficient of $\theta_{k,Np,\chi,\mathfrak{d}}(V_pf)$.

Lemma 3.12. *With the above notation, suppose that* $\gcd(N_1, \mathfrak{d}') = 1$. *Then we have*

$$a_{|\mathfrak{d}'|}(\theta_{k,Np,\chi,\mathfrak{d}}(V_pf)) = \chi_{\mathfrak{d}'}\bar\chi_0(p)p^{-k}a_{|\mathfrak{d}'|}(\theta_{k,Np,\chi,\mathfrak{d}}(f)).$$

Proof. By the previous lemma, we know that $a_{|\mathfrak{d}'|}(\theta_{k,Np,\chi,\mathfrak{d}}(V_pf)) = C(k,\chi,\mathfrak{d})r_{k,Np,\chi}(V_pf;\mathfrak{d},\mathfrak{d}')$, thus we have to compute

$$r_{k,Np,\chi}(V_pf;\mathfrak{d},\mathfrak{d}') = \sum_{Q\in\mathcal{L}_{Np}(N_0^2\mathfrak{d}\mathfrak{d}')/\Gamma_0(Np)} \omega_{\mathfrak{d}}(Q)I_{k,\chi}(V_pf,Q).$$

Suppose first that $p \nmid \mathfrak{d}'$. From Proposition 3.6 the set $\mathcal{L}_{Np}(N_0^2\mathfrak{d}\mathfrak{d}')/\Gamma_0(Np)$ is in bijection with $\mathcal{L}_N(N_0^2\mathfrak{d}\mathfrak{d}')/\Gamma_0(N)$, but in contrast with the case of $\breve{f} = f$, we may now use the map $\tau : [aNp, b, c] \mapsto [aN, b, pc]$ to induce that bijection. Thus we have

$$r_{k,Np,\chi}(V_pf;\mathfrak{d},\mathfrak{d}') = \sum_{Q\in\mathcal{L}_N(N_0^2\mathfrak{d}\mathfrak{d}')/\Gamma_0(N)} \omega_{\mathfrak{d}}(\tau^{-1}(Q))I_{k,\chi}(V_pf,\tau^{-1}(Q)).$$

Now, for $Q \in \mathcal{L}_N(N_0^2\mathfrak{d}\mathfrak{d}')$, using (3) one sees that $\omega_{\mathfrak{d}}(\tau^{-1}(Q)) = \chi_{\mathfrak{d}'}(p)\omega_{\mathfrak{d}}(Q)$, and one also has

$$I_{k,\chi}(V_pf,\tau^{-1}(Q)) = \bar\chi_0(p)p^{-k}I_{k,\chi}(f,Q).$$

Indeed, we have $\chi_0(\tau^{-1}Q) = \bar\chi_0(p)\chi_0(Q)$, and therefore if $Q = [aN, b, c]$ we find

$$\begin{aligned}
I_{k,\chi}(V_pf,\tau^{-1}(Q)) &= \chi_0(\tau^{-1}(Q))\int_{C_{\tau^{-1}(Q)}} f(pz)(paNz^2+bz+c/p)^{k-1}dz = \\
&= \bar\chi_0(p)p^{1-k}\int_{p^{-1}C_Q} f(pz)(aN(pz)^2+b(pz)+c)^{k-1}dz = \\
&= \bar\chi_0(p)p^{-k}\int_{C_Q} f(z)(aNz^2+bz+c)^{k-1}dz = \bar\chi_0(p)p^{-k}I_{k,\chi}(f,Q).
\end{aligned}$$

Therefore, we get

$$r_{k,Np,\chi}(V_pf;\mathfrak{d},\mathfrak{d}') = \chi_{\mathfrak{d}'}\bar\chi_0(p)p^{-k}r_{k,N,\chi}(f;\mathfrak{d},\mathfrak{d}')$$

and the claim follows from this.

Suppose now that $p \mid \mathfrak{d}'$. Since $\chi_{\mathfrak{d}'}(p) = 0$, we must prove that $r_{k,Np,\chi}(V_pf;\mathfrak{d},\mathfrak{d}') = 0$. Notice that p^2 divides exactly $N_0^2\mathfrak{d}\mathfrak{d}'$, so we can use again the decomposition

$$\mathcal{L}_{Np}(N_0^2\mathfrak{d}\mathfrak{d}')/\Gamma_0(Np) = \mathcal{L}^p_{Np}(N_0^2\mathfrak{d}\mathfrak{d}')/\Gamma_0(Np) \sqcup \mathcal{L}^{(p)}_{Np}(N_0^2\mathfrak{d}\mathfrak{d}')/\Gamma_0(Np).$$

Observe that the sum over the first subset does not contribute to $r_{k,Np,\chi}(V_pf;\mathfrak{d},\mathfrak{d}')$. Using Proposition 3.7 as in Lemma 3.10, we obtain (compare with (22))

$$r_{k,Np,\chi}(V_pf;\mathfrak{d},\mathfrak{d}') = \sum_{Q\in\mathcal{L}^{(p),a}_{Np}(N_0^2\mathfrak{d}\mathfrak{d}')/\Gamma_0(Np)} \omega_{\mathfrak{d}}(Q)I_{k,\chi}(V_pf,Q).$$

We claim that this sum vanishes, and the lemma will follow. Indeed, let $Q = [aNp, b, c] \in \mathcal{L}^{(p),a}_{Np}(N_0^2\mathfrak{d}\mathfrak{d}')$ be any representative of one of the classes appearing in the sum. By definition of this set, we know that $p \mid a$, hence

$$Q = Q' \circ \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$$

for $Q' = [\frac{a}{p}N, \frac{b}{p}, c] \in \mathcal{L}_N(N_0^2\mathfrak{d}\mathfrak{d}'/p^2)$. By equation (5), we deduce that $\omega_{\mathfrak{d}}(Q) = 0$. □

Lemmas 3.10 and 3.12 prove the following precise relation between coefficients of $\theta_{k,Np,\chi,\mathfrak{d}}(f_\alpha)$ and $\theta_{k,N,\chi,\mathfrak{d}}(f)$:

Proposition 3.13. *Let N, k, χ, $f \in S_{2k}^{new}(N, \chi^2)$, and $\mathfrak{d}$ be as above. Let $\mathfrak{d}'$ be a discriminant such that*

i) $\mathfrak{d}\mathfrak{d}' > 0$,
ii) $\gcd(N_1, \mathfrak{d}') = 1$.

Then, one has

$$a_{|\mathfrak{d}'|}(\theta_{k,Np,\chi,\mathfrak{d}}(f_\alpha)) = (1 - \beta\chi_{\mathfrak{d}'}\bar{\chi}_0(p)p^{-k})a_{|\mathfrak{d}'|}(\theta_{k,N,\chi,\mathfrak{d}}(f)).$$

Remark 3.14. *By Lemma 3.10, we have $a_{|\mathfrak{d}'|}(\theta_{k,N,\chi,\mathfrak{d}}(f)) = a_{|\mathfrak{d}'|}(\theta_{k,Np,\chi,\mathfrak{d}}(f))$, hence we can rewrite the identity in Proposition 3.13 as*

$$a_{|\mathfrak{d}'|}(\theta_{k,Np,\chi,\mathfrak{d}}(f_\alpha)) = (1 - \beta\chi_{\mathfrak{d}'}\bar{\chi}_0(p)p^{-k})a_{|\mathfrak{d}'|}(\theta_{k,Np,\chi,\mathfrak{d}}(f)).$$

In this form, this formula holds true also when $f \in S_{2k}^{new}(Np, \chi^2)$ is new of level Np (hence its ordinary p-stabilization is f itself, and $\beta = 0$).

Remark 3.15. *The above proposition should be read as complementary to* [Mak17, Proposition 2.10], *as he imposes the condition* $\gcd(Np, \mathfrak{d}) = 1$ *whereas we choose $\mathfrak{d}$ such that* $\gcd(N, \mathfrak{d}) = 1$ *and* $\mathfrak{d} \equiv 0 \pmod p$.

4. Hida theory and modular symbols

As a preparation for the next section, we now review some basic material of Hida theory and modular symbols, and set the conventions that will be used later. In most of the discussion, we follow closely the approach in [GS93, Ste94].

4.1. **Hida theory.** Let $\mathcal{O}$ be a finite extension of $\mathbf{Z}_p$ and let $\Gamma := 1 + p\mathbf{Z}_p$. We write $\Lambda = \Lambda_{\mathcal{O}} = \mathcal{O}[[\Gamma]]$ for the usual Iwasawa algebra and consider the space:

$$\mathcal{X}(\Lambda) := \mathrm{Hom}_{\mathcal{O}-\mathrm{cont}}(\Lambda, \bar{\mathbf{Q}}_p).$$

Elements $\mathbf{a} \in \Lambda$ can be seen as functions on $\mathcal{X}(\Lambda)$ through evaluation at $\mathbf{a}$, i.e. by setting $\mathbf{a}(\kappa) := \kappa(\mathbf{a})$. The set $\mathcal{X}(\Lambda)$ is endowed with the analytic structure induced from the natural identification

$$\mathcal{X}(\Lambda) \simeq \mathrm{Hom}_{\mathrm{cont}}(\Gamma, \bar{\mathbf{Q}}_p^\times) \tag{23}$$

between $\mathcal{X}(\Lambda)$ and the group of continuous characters $\kappa : \Gamma \to \bar{\mathbf{Q}}_p^\times$. A character $\kappa : \Gamma \to \bar{\mathbf{Q}}_p$ is called *arithmetic* if there exists an integer $r \geq 0$ such that $\kappa(t) = t^r$ for all t *sufficiently close* to 1 in Γ. A point $\kappa \in \mathcal{X}(\Lambda)$ is said to be arithmetic if the associated character of Γ under (23) is arithmetic. We refer to the integer r as the *weight* of κ.

If $\mathcal{R}$ is a finite flat Λ-algebra then we write

$$\mathcal{X}(\mathcal{R}) := \mathrm{Hom}_{\mathcal{O}-\mathrm{cont}}(\mathcal{R}, \bar{\mathbf{Q}}_p)$$

for the set of continuous homomorphisms from $\mathcal{R}$ to $\bar{\mathbf{Q}}_p$, to which we also refer as 'points of $\mathcal{R}$'. The restriction to Λ (via the structure morphism $\Lambda \to \mathcal{R}$) induces a surjective finite-to-one map

$$\pi : \mathcal{X}(\mathcal{R}) \longrightarrow \mathcal{X}(\Lambda).$$

One can define analytic charts around all points κ of $\mathcal{X}(\mathcal{R})$ which are unramified over Λ, by building sections S_κ of the map π, so that $\mathcal{X}(\mathcal{R})$ inherits the structure of rigid analytic cover of $\mathcal{X}(\Lambda)$. A function $f : \mathcal{U} \subseteq \mathcal{X}(\mathcal{R}) \to \bar{\mathbf{Q}}_p$ defined on an analytic neighborhood of κ is analytic if so is $f \circ S_\kappa$. The evaluation at an element $\mathbf{a} \in \mathcal{R}$ yields a function $\mathbf{a} : \mathcal{X}(\mathcal{R}) \to \bar{\mathbf{Q}}_p$, $\mathbf{a}(\kappa) := \kappa(\mathbf{a})$, which is analytic at every unramified point of $\mathcal{X}(\mathcal{R})$. A point $\kappa \in \mathcal{X}(\mathcal{R})$ is said to be arithmetic if the point $\pi(\kappa) \in \mathcal{X}(\Lambda)$ is arithmetic. We will write $\mathcal{X}^{\mathrm{arith}}(\mathcal{R}) \subseteq \mathcal{X}(\mathcal{R})$ for the subset of arithmetic points in $\mathcal{X}(\mathcal{R})$.

If N is a positive integer, with $p \nmid N$, consider the Λ-algebra

$$\Lambda_N := \mathcal{O}[[\mathbf{Z}_{p,N}^\times]] \simeq \Lambda[\Delta_{Np}]$$

associated with the completed group ring on $\mathbf{Z}_{p,N}^\times := \varprojlim(\mathbf{Z}/Np^m\mathbf{Z})^\times$. Under the natural isomorphisms

$$\mathbf{Z}_{p,N}^\times \simeq \mathbf{Z}_p^\times \times \Delta_N \simeq \Gamma \times \Delta_{Np}, \quad \text{where } \Delta_M := (\mathbf{Z}/M\mathbf{Z})^\times,$$

we will write $t_p \in \mathbf{Z}_p^\times$ and $t_N \in \Delta_N$ for the projections of $t \in \mathbf{Z}_{p,N}^\times$ in $\mathbf{Z}_p^\times$ and $\Delta_N^\times$. We will further write $\langle t_p \rangle \in \Gamma$ for the projection of t_p in Γ. Observe that $t_p = \langle t_p \rangle \omega(t_p)$, where ω denotes the Teichmuller character. Notice also that the analytic space $\mathcal{X}(\Lambda_N)$ is naturally isomorphic to a product of $\varphi(Np)$ copies of $\mathcal{X}(\Lambda)$, with the components being in one-to-one correspondence with the $\bar{\mathbf{Q}}_p$-valued characters of Δ_{Np}.

If $\mathcal{R}$ is a Λ_N-algebra, $r \geq 0$ is an integer, and ψ is a finite order character of $\mathbf{Z}_{p,N}^\times$, we say that $\kappa \in \mathcal{X}^{\mathrm{arith}}(\mathcal{R})$ has signature (r, ψ) if its restriction to Λ_N is of the form $\kappa([t]) = \psi(t)\langle t_p \rangle^r$ for all $t \in \mathbf{Z}_{p,N}^\times$. Notice that any finite order character ψ as before can be uniquely written as $\psi = \chi\omega^i\varepsilon$, where χ is a Dirichlet character modulo N, and ε is a finite order character of Γ. We will refer to $\chi\omega^i$ (resp. ε) as the *tame* (resp. *wild*) part of ψ. Using this decomposition and restricting ourselves to *tame* characters, an arithmetic point κ has signature $(r, \chi\omega^i)$ if

$$\kappa([t]) = \chi(t_N)\omega^i(t_p)\langle t_p \rangle^r = \chi(t_N)\omega^{i-r}(t_p)t_p^r$$

for all $t \in \mathbf{Z}_{p,N}^\times$. If $t \in \mathbf{Z}_{p,N}^\times$ we will simply write $\langle t \rangle, \omega(t)$ and $\chi(t)$ to denote $\langle t_p \rangle, \omega(t_p)$ and $\chi(t_N)$ respectively.

Fix a prime $p \geq 5$, and an integer $N \geq 1$ such that $p \nmid N$. If $r \geq 0$ and $m \geq 1$ are integers, recall that an eigenform $f \in S_{r+2}(\Gamma_1(Np^m), \bar{\mathbf{Q}}_p)$ is said to be *ordinary* (at p) if the eigenvalue a_p of T_p acting on f is a p-adic unit. If f is also normalized ($a_1 = 1$) and the prime-to-p part of the conductor is N, one then says that f is a *p-stabilized newform* of tame conductor N. One can check that if f is an ordinary p-stabilized newform of tame conductor N, then either f is already a newform, or f is related to a newform g of conductor N by the so-called process of ordinary p-stabilization (and in this case the level of f is Np). In the second case, if α and β denote the roots of the p-th Hecke polynomial for g, labelled so that α is the unit root and β is the non-unit root, then one has $f(z) = g(z) - \beta g(pz)$. We write $S_{r+2}^{\mathrm{ord}}(\Gamma_1(Np^m), \bar{\mathbf{Q}}_p)$ for the set of ordinary p-stabilized newforms in $S_{r+2}(\Gamma_1(Np^m), \bar{\mathbf{Q}}_p)$.

Consider the abstract Hecke algebra over Λ_N obtainend by considering all the Hecke operators, where group-like elements of Λ_N act as diamond operators. In his work, Hida studied the action of $\mathcal{H}$ on certain spaces of modular forms and defined a Λ_N-algebra $\mathcal{R}_N$, interpolating all ordinary p-stabilised newforms of *tame level N*. This algebra comes equipped with a natural morphism $h : \mathcal{H} \to \mathcal{R}_N$. Writing $\mathbf{a}_n := h(T_n) \in \mathcal{R}_N$ for the image of the Hecke operators, one defines the *universal ordinary p-stabilized newform of tame level N* to be

$$\mathbf{f}_N := \sum_{n \geq 1} \mathbf{a}_n q^n \quad \in \mathcal{R}_N[[q]].$$

Then we have the following:

Theorem 4.1 (Hida theory)**.** *The Λ_N-algebra $\mathcal{R}_N$ is reduced, finite and flat over Λ, and it is unramified at every arithmetic point. The map $\mathcal{X}(\mathcal{R}_N) \to \bar{\mathbf{Q}}_p[[q]], \kappa \mapsto \mathbf{f}_N(\kappa)$ defines a bijection:*

$$\left\{ \begin{array}{c} \kappa \in \mathcal{X}^{\mathrm{arith}}(\mathcal{R}_N) \\ \textit{of signature } (r, \psi) \end{array} \right\} \stackrel{1:1}{\longleftrightarrow} \left\{ \begin{array}{c} \textit{ordinary } p\textit{-stabilized newforms of tame level } N, \\ \textit{weight } r+2 \textit{ and nebentype character } \psi\omega^{-r}. \end{array} \right\}$$

Assume from now on that $\mathcal{O}$ contains the values of the characters of Δ_{Np}. The localization at maximal ideals of $\mathcal{R}_N$ yields natural decompositions

$$\mathcal{R}_N \stackrel{\sim}{\longrightarrow} \bigoplus_{\mathfrak{m}} \mathcal{R}_{\mathfrak{m}}, \qquad \bigsqcup_{\mathfrak{m}} \mathcal{X}(\mathcal{R}_{\mathfrak{m}}) \stackrel{\sim}{\longrightarrow} \mathcal{X}(\mathcal{R}_N),$$

where each of the localizations $\mathcal{R}_{\mathfrak{m}} := (\mathcal{R}_N)_{\mathfrak{m}}$ is a finite flat integral domain extension of Λ. The universal family can be seen as a collection $\mathbf{f}_N = (\mathbf{f}_{\mathfrak{m}})_{\mathfrak{m}}$, where $\mathbf{f}_{\mathfrak{m}}$ denotes the image of $\mathbf{f}_N$ under the natural morphism $\mathcal{R}_N \to \mathcal{R}_{\mathfrak{m}}$ induced by the above decomposition. For each maximal ideal $\mathfrak{m}$, we have the following commutative diagram of Λ_N-algebras, where $\Lambda_{\mathfrak{m}} \simeq \Lambda$ is the localization of Λ_N at $\mathfrak{m} \cap \Lambda_N$:

$$\begin{array}{ccc} \mathcal{R}_N & \longrightarrow & \mathcal{R}_{\mathfrak{m}} \\ \uparrow & & \uparrow \\ \Lambda_N & \longrightarrow & \Lambda_{\mathfrak{m}} \end{array}$$

The maximal ideals $\mathfrak{m}$ are in bijection with the characters of Δ_{Np}: if $\mathfrak{m}$ corresponds to the character $\chi\omega^i$, identifying $\Lambda_{\mathfrak{m}} \simeq \Lambda$, the morphism

$$\mathrm{loc}_{\mathfrak{m}} : \Lambda_N \longrightarrow \Lambda_{\mathfrak{m}} \simeq \Lambda$$

in the bottom row of the above diagram is determined by the fact that

$$\mathrm{loc}_{\mathfrak{m}}([t]) = \chi(t)\omega^i(t)[\langle t\rangle] \qquad \text{for all } t \in \mathbf{Z}_{p,N}^{\times}.$$

In particular, all the arithmetic points in $\mathcal{X}(\mathcal{R}_{\mathfrak{m}})$ have the same character in their signature.

Finally, by the Weierstrass preparation theorem, each of the power series $\mathbf{f}_{\mathfrak{m}}$ is uniquely determined locally by its values $\mathbf{f}_{\mathfrak{m}}(\kappa)$ at arithmetic points κ having trivial wild character. We define the set of *classical points* in $\mathcal{X}(\mathcal{R}_{\mathfrak{m}})$ as the set

$$\mathcal{X}^{\mathrm{cl}}(\mathcal{R}_{\mathfrak{m}}) := \{\kappa \in \mathcal{X}^{\mathrm{arith}}(\mathcal{R}_{\mathfrak{m}}) : \exists k \in \mathbf{Z} \text{ such that } \kappa([t]) = t^k, \forall t \in \Gamma\}.$$

In fact, $\mathbf{f}_{\mathfrak{m}}$ is also uniquely determined by the values $\mathbf{f}_{\mathfrak{m}}(\kappa)$ when restricting to classical points κ whose images in $\mathcal{X}^{\mathrm{cl}}(\Lambda)$ have big enough weights and are contained in a single residue class[4] modulo $p-1$. This justifies the following definition:

Definition 4.2. *A* Hida family *of tame level N and tame character χ modulo N is a quadruple $(\mathcal{R}_{\mathbf{f}}, \mathcal{U}_{\mathbf{f}}, \mathcal{U}_{\mathbf{f}}^{\mathrm{cl}}, \mathbf{f})$ where:*

- *$\mathcal{R}_{\mathbf{f}}$ is a finite flat integral domain extension of Λ;*
- *$\mathcal{U}_{\mathbf{f}} \subset \mathcal{X}(\mathcal{R}_{\mathbf{f}})$ is an open subset for the rigid analytic topology;*
- *$\mathcal{U}_{\mathbf{f}}^{\mathrm{cl}} \subset \mathcal{U}_{\mathbf{f}} \cap \mathcal{X}^{\mathrm{cl}}(\mathcal{R}_{\mathbf{f}})$ is a dense subset of $\mathcal{U}_{\mathbf{f}}$ whose weights are contained in a single residue class modulo $p-1$;*
- *$\mathbf{f} \in \mathcal{R}_{\mathbf{f}}[[q]]$ is a power series in q with coefficients in $\mathcal{R}_{\mathbf{f}}$, such that for all $\kappa \in \mathcal{U}_{\mathbf{f}}^{\mathrm{cl}}$ of weight $r > 0$,*
$$\mathbf{f}(\kappa) \in S_{r+2}^{\mathrm{ord}}(Np, \chi)$$
is the ordinary p-stabilization of a newform $f_\kappa \in S_{r+2}(N, \chi)$.

Let $(\mathcal{R}_{\mathbf{f}}, \mathcal{U}_{\mathbf{f}}, \mathcal{U}_{\mathbf{f}}^{\mathrm{cl}}, \mathbf{f})$ be a Hida family as in the definition. By the universal property of $\mathcal{R}_N$, there is a unique Λ-algebra homomorphism $\mathcal{R}_N \to \mathcal{R}_{\mathbf{f}}$, which gives $\mathcal{R}_{\mathbf{f}}$ the structure of Λ_N-algebra. This fits in a commutative diagram

$$\begin{array}{ccc} \mathcal{R}_N & \longrightarrow & \mathcal{R}_{\mathbf{f}} \\ \uparrow & & \uparrow \\ \Lambda_N & \xrightarrow{\mathrm{loc}_{\mathbf{f}}} & \Lambda, \end{array}$$

where the vertical arrows are the structure maps of $\mathcal{R}_N$ and $\mathcal{R}_{\mathbf{f}}$ as Λ_N- and Λ-algebras, respectively, and the morphism $\mathrm{loc}_{\mathbf{f}}$ is determined by the property that

$$\mathrm{loc}_{\mathbf{f}}([t]) = \chi(t)\omega^{r_0}(t)[\langle t\rangle] \qquad \text{for all } t \in \mathbf{Z}_{p,N}^{\times},$$

where r_0 is an integer (only defined modulo $p-1$) determined by the quadruple $(\mathcal{R}_{\mathbf{f}}, \mathcal{U}_{\mathbf{f}}, \mathcal{U}_{\mathbf{f}}^{\mathrm{cl}}, \mathbf{f})$. In particular, every classical point $\kappa \in \mathcal{U}_{\mathbf{f}}^{\mathrm{cl}}$ has weight r congruent to r_0 modulo $p-1$.

Remark 4.3. *When $r = 0$, the form $\mathbf{f}(\kappa)$ can be either old or new at p. Only in the first case, $\mathbf{f}(\kappa)$ would be the p-stabilization of a weight 2 newform of level N.*

4.2. **Modular symbols.** Let $M \geq 1$ be an integer and let $\Gamma_0(M)$. Let $\Delta^0 := \mathrm{Div}^0(\mathbf{P}^1(\mathbf{Q}))$ be the group of degree 0 divisors on the set of rational cusps of Poincaré's upper half plane $\mathfrak{H}$. The congruence group $\Gamma_0(M)$ acts by linear fractional transformations on Δ^0. Let V be a right $\mathbf{Z}[1/6][\Gamma_0(M)]$-module. There is a natural right action of $\Gamma_0(M)$ on the set $\mathrm{Hom}(\Delta^0, V)$ of additive homomorphisms from Δ^0 to V, defined by the rule

$$(\Phi|\gamma)(D) := \Phi(\gamma D)|\gamma, \qquad \gamma \in \Gamma_0(M),\ D \in \Delta^0.$$

The group of V-valued modular symbols over $\Gamma_0(M)$ is the set of $\Phi : \Delta^0 \to V$ such that $\Phi|\gamma = \Phi$ for all $\gamma \in \Gamma_0(M)$, i.e.:

$$\mathrm{Symb}_{\Gamma_0(M)}(V) := \mathrm{Hom}_{\Gamma_0(M)}(\Delta^0, V) = \mathrm{Hom}(\Delta^0, V)^{\Gamma_0(M)}.$$

[4] The finite-to-1 map $\mathcal{X}(\mathcal{R}_{\mathfrak{m}}) \to \mathcal{X}(\Lambda)$ is unramified at arithmetic points. In particular, using the analytic sections S_κ mentioned above one can identify the weight space, locally analytically around κ, with $\mathcal{X}(\Lambda)$. The coefficients of $\mathbf{f}_{\mathfrak{m}} \circ S_\kappa$ can then be identified with power series in one variable. By the Weierstrass Preparation theorem, each of them is uniquely determined by its values at infinitely many points.

Let now $G_0(M)$ denote the semigroup of two-by-two matrices $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_2(\mathbf{Z})$ such that $\gcd(a, M) = 1$ and $c \equiv 0 \pmod{M}$. If the action of $\Gamma_0(M)$ on V extends to an action of $G_0(M)$, then $\mathrm{Symb}_{\Gamma_0(M)}(V)$ inherits a natural action of Hecke operators.

The element $\iota = \mathrm{diag}(1, -1) \in G_0(M)$ induces an involution on $\mathrm{Symb}_{\Gamma_0(M)}(V)$. Notice that ι acts by $\Phi \mapsto \Phi|\iota$, with $(\Phi|\iota)(D) = \Phi(\iota D)|\iota$. Under the assumption that V is a $\mathbf{Z}[1/2]$-module, the action of ι decomposes the modular symbol Φ as $\Phi = \Phi^+ + \Phi^-$, where $\Phi^\pm := \frac{1}{2}(\Phi \pm \iota\Phi) \in \mathrm{Symb}_{\Gamma_0(M)}(V)^\pm$ are such that $\iota\Phi^\pm = \pm\Phi^\pm$, i.e.:

$$\mathrm{Symb}_{\Gamma_0(M)}(V) = \mathrm{Symb}_{\Gamma_0(M)}(V)^+ \oplus \mathrm{Symb}_{\Gamma_0(M)}(V)^-.$$

Let us now focus our discussion on some special choices of V. Fix an integer $r \geq 0$ and a commutative ring R in which $2r!$ is invertible (e.g. of characteristic zero). Let $\mathrm{Sym}^r(R^2)$ denote the R-module of homogeneous 'divided powers polynomials' of degree r in X, Y over R generated by the monomials

$$\frac{X^n}{n!}\frac{Y^{r-n}}{(r-n)!} \quad \text{with } 0 \leq n \leq r.$$

Similarly, let $\mathrm{Sym}^r(R^2)^*$ be the R-module of homogeneous polynomials of degree r in X, Y over R, generated by the monomials X^nY^{r-n} with $0 \leq n \leq r$. Both $\mathrm{Sym}^r(R^2)$ and $\mathrm{Sym}^r(R^2)^*$ are equipped with a natural action of $M_2(R)$, by the rule

$$(F|\gamma)(X, Y) := F((X, Y)^t\gamma)$$

where $\gamma \mapsto {}^t\gamma$ denotes the usual transpose.

Let us write $\langle \cdot, \cdot \rangle_r$ for the unique perfect pairing

$$\langle \cdot, \cdot \rangle_r : \mathrm{Sym}^r(R^2) \times \mathrm{Sym}^r(R^2)^* \longrightarrow R$$

satisfying

$$\left\langle \frac{X^iY^{r-i}}{i!(r-i)!}, X^{r-j}Y^j \right\rangle_r = (-1)^j\delta_{ij},$$

where δ_{ij} is the usual Kronecker's delta function. This pairing satisfies in addition the properties

$$\left\langle \frac{(aY - bX)^r}{r!}, P(X, Y) \right\rangle_r = P(a, b) \;\; \forall\, a, b \in R, \quad \langle P_1|\gamma, P_2|\gamma\rangle_r = \det(\gamma)^r\langle P_1, P_2\rangle.$$

From now on, we will write $L_r(R)$ (resp. $L_r^*(R)$) for the $R[\Gamma_0(M)]$-module $\mathrm{Sym}^r(R^2)$ (resp. $\mathrm{Sym}^r(R^2)^*$) equipped with the action of $\Gamma_0(M)$ induced by the above described action. If in addition χ is an R-valued Dirichlet character modulo M, we denote by $L_{r,\chi}(R)$ (resp. $L_{r,\chi}^*(R)$) the same underlying $R[\Gamma_0(M)]$-module as $L_r(R)$ (resp. $L_r^*(R)$) but with the action of $\Gamma_0(M)$ twisted by χ^{-1} (resp. χ). That is, for an element

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(M)$$

one has

$$(F|\gamma)(X, Y) := \begin{cases} \chi(a) \cdot F((X, Y)^t\gamma) & \text{if } F \in L_{r,\chi}(R), \\ \chi(d) \cdot F((X, Y)^t\gamma) & \text{if } F \in L_{r,\chi}^*(R). \end{cases}$$

If $f \in S_{r+2}(M, \chi)$ is a cusp form of weight $r + 2$, level $\Gamma_0(M)$, and Nebentype character χ, then the $L_{r,\chi}(\mathbf{C})$-valued differential form

$$\omega_f := \frac{1}{r!}f(\tau)(\tau Y - X)^r d\tau$$

on $\mathfrak{H}$ satisfies $\gamma^*\omega_f|\gamma = \omega_f$ for all $\gamma \in \Gamma_0(M)$. The additive map $\psi_f : \Delta^0 \to L_{r,\chi}(\mathbf{C})$ induced by

$$\{c_2\} - \{c_1\} \longmapsto \int_{c_1}^{c_2} \omega_f$$

yields an $L_{r,\chi}(\mathbf{C})$-valued modular symbol over $\Gamma_0(M)$ (where the integral is along the oriented geodesic in $\mathfrak{H}$ from c_1 to c_2). We then have the following:

Theorem 4.4. *For each choice of sign $\pm$, the map $f \mapsto \psi_f$ yields a Hecke-equivariant inclusion*

$$S_{r+2}(M, \chi) \hookrightarrow \mathrm{Symb}_{\Gamma_0(M)}(L_{r,\chi}(\mathbf{C}))^\pm. \tag{24}$$

Proof. This is a combination of the Eichler–Shimura isomorphism, the Manin–Drinfeld principle and the Ash–Stevens isomorphism (see [AS86, Proposition 4.2]), which composed give us maps

$$S_{r+2}(M,\chi) \xrightarrow[\text{ES}]{\sim} H^1_{\text{par}}(\Gamma_0(M), L_{r,\chi}(\mathbf{C}))^{\pm} \underset{\text{DM}}{\hookrightarrow} H^1_c(\Gamma_0(M), L_{r,\chi}(\mathbf{C}))^{\pm} \xrightarrow[\text{AS}]{\sim} \text{Symb}_{\Gamma_0(M)}(L_{r,\chi}(\mathbf{C}))^{\pm}.$$

We want to stress that the first isomorphism comes from integration and it does not respect algebraicity. If f is defined over a subring R of $\mathbf{C}$ its image is not necessarily in $H^1_{\text{par}}(\Gamma_0(M), L_{r,\chi}(R))^{\pm}$. Also, the inclusion provided by the Manin–Drinfeld principle arises by taking a section of the natural projection map $H^1_c(\Gamma_0(M), L_{r,\chi}(\mathbf{C})) \to H^1_{\text{par}}(\Gamma_0(M), L_{r,\chi}(\mathbf{C}))$. Such a section is defined over any characteristic zero field, but it does not need to descend to any subring R of $\mathbf{C}$. $\square$

A modular symbol $\varphi \in \text{Symb}_{\Gamma_0(M)}(L_{r,\chi}(\mathbf{C}))$ is said to be defined over a subring R of $\mathbf{C}$ if it takes values in $L_{r,\chi}(R)$, i.e. if it lies in the image of the natural map

$$\text{Symb}_{\Gamma_0(M)}(L_{r,\chi}(R)) \hookrightarrow \text{Symb}_{\Gamma_0(M)}(L_{r,\chi}(\mathbf{C})).$$

If $f \in S_{r+2}(M,\chi)$ is a Hecke eigenform and $\mathcal{O}_f$ denotes the ring of integers of its Hecke field, it is well-known that there exist two complex numbers $\Omega_f^{\pm} \in \mathbf{C}^{\times}$ such that the normalized modular symbols

$$\varphi_f^{\pm} := \frac{1}{\Omega_f^{\pm}} \cdot \psi_f^{\pm} \in \text{Symb}_{\Gamma_0(M)}(L_{r,\chi}(\mathbf{C}))^{\pm}, \tag{25}$$

one for each choice of sign, are defined over $\mathcal{O}_f$ (cf. [Man73]).

4.3. **p-adic interpolation of modular symbols.** The modular symbols associated with modular forms as recalled above can be p-adically interpolated, giving rise to 'Λ-adic modular symbols'. We now recall this construction, mainly due to Greenberg and Stevens [GS93].

Let $N \geq 1$ and p be as before, and let $\mathbf{f}_N$ be the universal p-stabilized ordinary newform of tame conductor N as defined in section 4.1. Recall that we can see $\mathbf{f}_N$ via its q-expansion

$$\mathbf{f}_N = \sum_{n\geq 1} \mathbf{a}_n q^n \in \mathcal{R}_N[[q]],$$

where $\mathcal{R}_N$ is the universal ordinary p-adic Hecke algebra of tame conductor N. For each arithmetic point $\kappa \in \mathcal{X}^{\text{arith}}(\mathcal{R}_N)$, we may fix complex periods $\Omega^{\pm}_{\mathbf{f}_N(\kappa)} \in \mathbf{C}^{\times}$ so that the normalized cohomology classes

$$\varphi^{\pm}_{\mathbf{f}_N(\kappa)} := \frac{1}{\Omega^{\pm}_{\mathbf{f}_N(\kappa)}} \psi_{\mathbf{f}_N(\kappa)} \tag{26}$$

are defined over the ring of integers $\mathcal{O}_{\mathbf{f}_N(\kappa)}$ of the Hecke field of $\mathbf{f}_N(\kappa)$ (cf. (25)). Following [GS93], we recall how the collection of cohomology classes $\varphi_\kappa^{\pm}$, as κ varies on the arithmetic points of $\mathcal{R}_N$, can be put together into Λ-adic cohomology classes (or modular symbols) $\Phi^{\pm}$, recovering the classes $\varphi_\kappa^{\pm}$ under specialization maps.

Consider the subset $(\mathbf{Z}_p^2)'$ of *primitive* vectors in $\mathbf{Z}_p^2$, meaning the subset of vectors which do not lie in $(p\mathbf{Z}_p)^2$, and let $\mathbf{D} = \text{Meas}((\mathbf{Z}_p^2)')$ denote the group of $\mathbf{Z}_p$-valued measures on $(\mathbf{Z}_p^2)'$. Namely, if $\text{Cont}((\mathbf{Z}_p^2)')$ denotes the space of continuous $\mathbf{Z}_p$-valued functions on $(\mathbf{Z}_p^2)'$, then $\mathbf{D}$ is the space of continuous $\mathbf{Z}_p$-valued functionals on $\text{Cont}((\mathbf{Z}_p^2)')$. One can also see $\mathbf{D}$ as a direct summand of $\text{Meas}(\mathbf{Z}_p^2)$, via restriction of continuous functions on $\mathbf{Z}_p^2$ to $(\mathbf{Z}_p^2)'$. Following standard conventions of measure theory, if $\mu \in \mathbf{D}$ we will write

$$\int_U f d\mu$$

to denote $\mu(f \cdot \mathbf{1}_U)$ for any $f \in \text{Cont}((\mathbf{Z}_p^2)')$ and any compact open subset $U \subseteq (\mathbf{Z}_p^2)'$.

There are various natural actions on $\mathbf{D}$. On the one hand, the scalar action of $\mathbf{Z}_p^{\times}$ on $(\mathbf{Z}_p^2)'$ induces a natural action of $\mathbf{Z}_p[[\mathbf{Z}_p^{\times}]]$ on $\mathbf{D}$. On the other hand, viewing elements of $(\mathbf{Z}_p^2)'$ as row vectors, we let $\Gamma_0(N)$ act on $(\mathbf{Z}_p^2)'$ by multiplication on the right. This action induces a natural (right) action on $\mathbf{D}$, which is characterized by the fact that, for all $\mu \in \mathbf{D}$, $\gamma \in \Gamma_0(N)$, and $f \in \text{Cont}((\mathbf{Z}_p^2)')$, one has

$$\int f(x,y) d(\mu|\gamma)(x,y) = \int f((x,y)^t\gamma^{-1}) d\mu(x,y).$$

The two actions just described clearly commute one with each other, hence we can view $\mathbf{D}$ as a $\mathbf{Z}_p[[\mathbf{Z}_p^\times]][\Gamma_0(N)]$-module.

If $\mathcal{R}$ is a Λ_N-algebra, we define the $\mathcal{R}[\Gamma_0(N)]$-module

$$\mathbf{D}_{\mathcal{R}} := \mathbf{D} \otimes_{\mathbf{Z}_p[[\mathbf{Z}_p^\times]]} \mathcal{R},$$

where $\Gamma_0(N)$ acts through the rule

$$(\mu \otimes \lambda)|\gamma := \mu|\gamma \otimes [a]_N \lambda,$$

for $\mu \in \mathbf{D}$, $\lambda \in \mathcal{R}$, and $\gamma \in \Gamma_0(N)$, where a is the upper-left entry of γ and $[a]_N \in \Delta_N = (\mathbf{Z}/N\mathbf{Z})^\times \subseteq \Lambda_N \to \mathcal{R}$ is the image in $\mathcal{R}$ of the group-like element of a modulo N. To lighten the notation, we will write $\mathbf{D}_N = \mathbf{D}_{\Lambda_N}$.

If $\kappa \in \mathcal{X}^{\mathrm{arith}}(\mathcal{R}_N)$ is an arithmetic point of signature (r, ψ), factor $\psi = \psi_N \psi_p$ as a product of two Dirichlet characters, with ψ_N defined modulo N and ψ_p defined modulo a power of p. Then one has a map

$$\mathrm{sp}_\kappa : \mathbf{D}_{\mathcal{R}_N} \longrightarrow L_{r,\psi\omega^{-r}}(R_\kappa),$$

where $R_\kappa = \kappa(\mathcal{R}_N)$, defined by

$$\mathrm{sp}_\kappa(\mu \otimes \alpha) := \kappa(\alpha) \cdot \int_{\mathbf{Z}_p \times \mathbf{Z}_p^\times} \psi_p \omega^{-r}(y) \cdot \frac{(xY - yX)^r}{r!} d\mu(x, y) \tag{27}$$

for $\mu \in \mathbf{D}$ and $\alpha \in \mathcal{R}_N$. This is a $\Gamma_0(Np^m)$-homomorphism if ψ is defined modulo Np^m, and hence it yields a map on modular symbols

$$\mathrm{sp}_{\kappa,*} : \mathrm{Symb}_{\Gamma_0(N)}(\mathbf{D}_{\mathcal{R}_N}) \longrightarrow \mathrm{Symb}_{\Gamma_0(Np^m)}(L_{r,\psi\omega^{-r}}(R_\kappa)).$$

In addition, the specialization map $\mathrm{sp}_{\kappa,*}$ is Hecke equivariant. If $\Phi \in \mathrm{Symb}_{\Gamma_0(N)}(\mathbf{D}_{\mathcal{R}_N})$ and $\kappa \in \mathcal{X}^{\mathrm{arith}}(\mathcal{R}_N)$, we abbreviate

$$\Phi_\kappa := \mathrm{sp}_{\kappa,*}(\Phi)$$

for the weight κ specialization of Φ.

Theorem 4.5. *Fix an arithmetic point $\kappa_0 \in \mathcal{X}^{\mathrm{arith}}(\mathcal{R}_N)$. There is a modular symbol $\Phi \in \mathrm{Symb}_{\Gamma_0(N)}(\mathbf{D}_{\mathcal{R}_N})$ and a choice of p-adic periods $\Omega_\kappa \in R_\kappa$, one for each $\kappa \in \mathcal{X}^{\mathrm{arith}}(\mathcal{R}_N)$, satisfying the following properties:*

i) $\Omega_{\kappa_0} \neq 0$;
ii) $\Phi_\kappa = \Omega_\kappa \cdot \varphi^-_{\mathbf{f}_N(\kappa)}$ *for every arithmetic point* $\kappa \in \mathcal{X}^{\mathrm{arith}}(\mathcal{R}_N)$.

Proof. The proof is essentially a consequence of [GS93, Theorem 5.13], as explained in [Ste94, Theorem 5.5]. We recall it here for convenience of the reader. Let (r_0, ψ_0) be the signature of κ_0, and let m_0 be the smallest positive integer such that ψ_0 is defined modulo Np^{m_0}. Then the modular symbol

$$\varphi^-_{\mathbf{f}_N(\kappa_0)} \in \mathrm{Symb}_{\Gamma_0(Np^{m_0})}(L_{r_0,\psi_0\omega^{-r_0}}(R_{\kappa_0}))$$

is a Hecke eigenclass. If $\mathcal{R}_{(\kappa_0)}$ denotes the localization of $\mathcal{R}_N$ at κ_0, and $h : \mathcal{H} \to \mathcal{R}_{(\kappa_0)}$ is the canonical map, then [GS93, Theorem 5.13] tells us that the $\mathcal{R}_{(\kappa_0)}$-module of h-eigenclasses in the space $\mathrm{Symb}_{\Gamma_0(N)}(\mathbf{D}_{\mathcal{R}_{(\kappa_0)}})^-$ is free of rank one, and it is generated by an element Ψ whose image under $\mathrm{sp}_{\kappa_0,*}$ equals $\varphi^-_{\mathbf{f}_N(\kappa_0)}$. One can choose an element $\alpha \in \mathcal{R}_N$ such that $\alpha(\kappa_0) \neq 0$ and $\alpha\Psi$ is everywhere regular. Then $\Phi = \alpha\Psi$ is the desired modular symbol. Indeed, by weak multiplicity one, the weight κ specialization Φ_κ is a multiple of $\varphi^-_{\mathbf{f}_N(\kappa)}$ for each arithmetic point κ, thus one can choose periods $\Omega_\kappa \in R_\kappa$ verifying conditions i) and ii). □

Let now $(\mathcal{R}_{\mathbf{f}}, \mathcal{U}_{\mathbf{f}}, \mathcal{U}_{\mathbf{f}}^{\mathrm{cl}}, \mathbf{f})$ be a Hida family of tame level N and tame character χ modulo N, as in Definition 4.2. Recall that $\mathbf{f}$ is the image of the universal family $\mathbf{f}_N$ under a unique morphism $\mathcal{R}_N \to \mathcal{R}_{\mathbf{f}}$. We shall consider the $\mathcal{R}_N[\Gamma_0]$-module $\mathbf{D}_{\mathbf{f}} := \mathbf{D}_{\mathcal{R}_{\mathbf{f}}}$, together with the natural map

$$\mathbf{D}_{\mathcal{R}_N} = \mathbf{D} \otimes_{\mathbf{Z}_p[[\mathbf{Z}_p^\times]]} \mathcal{R}_N \longrightarrow \mathbf{D}_{\mathbf{f}} = \mathbf{D} \otimes_{\mathbf{Z}_p[[\mathbf{Z}_p^\times]]} \mathcal{R}_{\mathbf{f}}. \tag{28}$$

If $\kappa \in \mathcal{U}_{\mathbf{f}}^{\mathrm{cl}}$ has weight $r \geq 0$, so that $\mathbf{f}(\kappa) \in S^{\mathrm{ord}}_{r+2}(Np, \chi)$, then observe that the specialization map becomes

$$\mathrm{sp}_\kappa(\mu \otimes \alpha) = \kappa(\alpha) \cdot \int_{\mathbf{Z}_p \times \mathbf{Z}_p^\times} \frac{(xY - yX)^r}{r!} d\mu(x, y)$$

for $\mu \in \mathbf{D}$, $\alpha \in \mathcal{R}_N$ (compare with (27)). Also, sp_κ factors through factors through (28), and hence the specialization map on modular symbols

$$\mathrm{sp}_{\kappa,*} : \mathrm{Symb}_{\Gamma_0(N)}(\mathbf{D}_{\mathcal{R}_N}) \longrightarrow \mathrm{Symb}_{\Gamma_0(Np)}(L_{r,\chi}(R_\kappa))$$

factors through a map, that we still denote $\mathrm{sp}_{\kappa,*}$,

$$\mathrm{sp}_{\kappa,*} : \mathrm{Symb}_{\Gamma_0(N)}(\mathbf{D}_{\mathbf{f}}) \longrightarrow \mathrm{Symb}_{\Gamma_0(Np)}(L_{r,\chi}(R_\kappa)).$$

For simplicity, we may write $\psi^{\pm}_{\mathbf{f}(\kappa)}$ for the $\pm$-components of the modular symbol associated with $\mathbf{f}(\kappa)$, $\Omega^{\pm}_{\mathbf{f}(\kappa)} \in \mathbf{C}^\times$ for the corresponding complex periods, and $\varphi^{\pm}_{\mathbf{f}(\kappa)} := \psi^{\pm}_{\mathbf{f}(\kappa)}/\Omega^{\pm}_{\mathbf{f}(\kappa)}$ as in (26). Then, the above theorem yields the following immediate consequence.

Corollary 4.6. *Let $\kappa_0 \in \mathcal{U}^{\mathrm{cl}}_{\mathbf{f}} \subseteq \mathcal{X}^{\mathrm{cl}}(\mathcal{R}_{\mathbf{f}})$ be a classical point. There exists a modular symbol $\Phi_{\mathbf{f}} \in \mathrm{Symb}_{\Gamma_0(N)}(\mathbf{D}_{\mathbf{f}})$, and a choice of p-adic periods $\Omega_\kappa \in R_\kappa$, one for each $\kappa \in \mathcal{U}^{\mathrm{cl}}_{\mathbf{f}}$, such that:*
- i) $\Omega_{\kappa_0} \neq 0$;
- ii) $\Phi_{\mathbf{f},\kappa} = \Omega_\kappa \cdot \varphi^{-}_{\mathbf{f}(\kappa)}$ *for all* $\kappa \in \mathcal{U}_{\mathbf{f}}$.

5. The Λ-adic $\mathfrak{d}$-th Shintani lifting

This section is devoted to the construction of the so-called 'Λ-adic $\mathfrak{d}$-th Shintani lifting', which interpolates the $\mathfrak{d}$-th Shintani liftings of modular forms in p-adic families. To provide this construction, first we explain in 5.1 a cohomological interpretation of the classical $\mathfrak{d}$-th Shintani lifting described in Section 2, which is better suited for the p-adic interpolation and yields also an algebraicity statement. After that, in Section 5.2 we refine Stevens' approach in [Ste94] to define the Λ-adic $\mathfrak{d}$-th Shintani lifting $\Theta_{\mathfrak{d}}(\mathbf{f})$ of a Hida family $\mathbf{f}$ (cf. Equation (30)). The interpolation property is precisely stated in Theorem 5.9. The main result of Section 5.3 is the Λ-adic version of Kohnen's formula given in Theorem 5.13. As an application of this, Corollary 5.15 gives a mild generalization of Kohnen's classical formula. Finally, in Section 5.4 we study the evaluation of the Λ-adic Kohnen formula at classical points outside the interpolation region.

5.1. **Cohomological $\mathfrak{d}$-th Shintani lifting and integrality.** Let $M \geq 1$ be an odd integer, and χ be a Dirichlet character modulo M (later we will be interested in $M = N$ or Np). Let χ_0 be the primitive character associated with M, M_0 be its conductor, and set $M_1 = M/M_0$, $\epsilon = \chi(-1)$. Let $k \geq 1$ be an integer, and fix a fundamental discriminant $\mathfrak{d}$ with $\gcd(M_0, \mathfrak{d}) = 1$ and $\epsilon(-1)^k\mathfrak{d} > 0$. We explain the cohomological interpretation of the $\mathfrak{d}$-th Shintani lifting from $S_{2k}(M,\chi^2)$ to $S^+_{k+1/2}(M,\chi)$.

If $f \in S_{2k}(M,\chi^2)$, recall that the definition of its $\mathfrak{d}$-th Shintani lifting $\theta_{k,M,\mathfrak{d}}(f)$ involves certain integrals $I_{k,\chi}(f,Q)$ associated with integral binary quadratic forms Q of discriminant divisible by $|\mathfrak{d}|$ (see (10) and (9)). Namely, if $m \geq 1$ is an integer such that $\epsilon(-1)^k m \equiv 0, 1 \pmod 4$, then the m-th Fourier coefficient of $\theta_{k,M,\chi,\mathfrak{d}}(f)$ involves the computation of integrals

$$I_{k,\chi}(f,Q) = \chi_0(Q)\int_{C_Q} f(z)Q(z,1)^{k-1}dz$$

for integral binary quadratic forms Q in $\mathcal{L}_{Mt}(|\mathfrak{d}|mt^2)$ (modulo $\Gamma_0(Mt)$-equivalence), where t is a positive divisor of M. If Q is such a quadratic form, we let

$$D_Q := \partial C_Q = \{\omega'_Q\} - \{\omega_Q\} \in \Delta^0$$

be the degree zero divisor given by the boundary of the geodesic path C_Q.

Definition 5.1. *Let R be a $\mathbf{Z}[1/6]$-algebra containing the values of χ, $\varphi \in \mathrm{Symb}_{\Gamma_0(M)}(L_{2k-2,\chi^2}(R))$ be a modular symbol, and $t > 0$ be a divisor of M. For each integral binary quadratic form $Q \in \mathcal{L}_{Mt}$ with positive discriminant, we put*

$$J_{k,\chi}(\varphi,Q) := \chi_0(Q)\langle\varphi(D_Q), Q^{k-1}\rangle \in R,$$

where here $\langle\,\rangle = \langle\,\rangle_{2k-2}$ is the pairing on modular symbols as defined in the previous section. For each integer $m \geq 1$ with $\epsilon(-1)^k m \equiv 0, 1 \pmod 4$ we also define

$$s_{k,Mt,\chi}(\varphi,\mathfrak{d};\epsilon(-1)^k m) := \sum_{Q\in\mathcal{L}_{Mt}(|\mathfrak{d}|m)/\Gamma_0(Mt)} \omega_{\mathfrak{d}}(Q)\cdot J_{k,\chi}(\varphi,Q).$$

In the above definition, $\varphi(D_Q)$ stands for the value at D_Q of any cocycle representing φ. One can check that $J_{k,\chi}(\varphi, Q)$ does not depend on the choice of such representative for φ, and that it depends on $Q \in \mathcal{L}_{Mt}$ only up to $\Gamma_0(Mt)$-equivalence. When χ is trivial, we will write J_k instead of $J_{k,\chi}$, and similarly $s_{k,Mt}$ instead of $s_{k,Mt,\chi}$. With the help of the quantities $s_{k,Mt,\chi}(\varphi, \mathfrak{d}; \epsilon(-1)^k m)$, we can now define the cohomological $\mathfrak{d}$-th Shintani lifting as follows (compare with the classical definition in (10)).

Definition 5.2. *Let R be a $\mathbf{Z}[1/6]$-algebra containing the values of χ. Define an R-linear map*

$$\Theta_{k,M,\chi,\mathfrak{d}} : \mathrm{Symb}_{\Gamma_0(M)}(L_{2k-2}(R)) \to R[[q]]$$

by setting

$$\Theta_{k,M,\chi,\mathfrak{d}}(\varphi) := C(k,\chi,\mathfrak{d}) \sum_{\substack{m \geq 1,\\ \epsilon(-1)^k m \equiv 0,1 (4)}} \left(\sum_{0<t|M_1} \mu(t)\chi_{\mathfrak{d}}\bar{\chi}_0(t) t^{-k-1} s_{k,Mt,\chi}(\varphi, \mathfrak{d}; \epsilon(-1)^k m t^2) \right) q^m.$$

Proposition 5.3. *Let the notation be as before, and R be a $\mathbf{Z}[1/6]$-algebra.*

i) *For every $\varphi \in \mathrm{Symb}_{\Gamma_0(M)}(L_{2k-2,\chi^2}(R))$, one has $\Theta_{k,M,\chi,\mathfrak{d}}(\varphi|\iota) = -\Theta_{k,M,\chi,\mathfrak{d}}(\varphi)$.*
ii) *Let $f \in S_{2k}(M,\chi^2)$, and let $\psi_f \in \mathrm{Symb}_{\Gamma_0(M)}(L_{2k-2,\chi^2}(\mathbf{C}))$ be its associated modular symbol as above. Then $\Theta_{k,M,\chi,\mathfrak{d}}(\psi_f) = \Theta_{k,M,\chi,\mathfrak{d}}(\psi_f^-) = \theta_{k,M,\chi,\mathfrak{d}}(f)$.*
iii) *If $R = K$ is a field of characteristic zero, and $\varphi \in \mathrm{Symb}_{\Gamma_0(M)}(L_{2k-2,\chi^2}(K))$ belongs to the image of the inclusion (24), then $\Theta_{k,M,\chi,\mathfrak{d}}(\varphi) \in S^+_{k+1/2}(M,\chi;K)$.*

Proof. First of all, one easily checks from the definitions that $\iota D_Q = -D_{Q\circ\iota}$. This implies that, for an arbitrary integral binary quadratic form Q,

$$\begin{aligned} J_{k,\chi}(\varphi|\iota, Q) &= \chi_0(Q)\langle \varphi(\iota D_Q)|\iota, Q^{k-1}\rangle = \chi_0(Q)\langle \varphi(\iota D_Q), (Q|\iota)^{k-1}\rangle = \\ &= -\chi_0(Q\circ\iota)\langle \varphi(D_{Q\circ\iota}), (Q\circ\iota)^{k-1}\rangle = -J_{k,\chi}(\varphi, Q\circ\iota), \end{aligned}$$

where in the third equality we use that $Q\circ\iota = Q|\iota$ and that $\chi_0(Q) = \chi_0(Q\circ\iota)$. Statement i) follows from this by the definition of $\Theta_{k,M,\chi,\mathfrak{d}}$ (noticing that also $\omega_{\mathfrak{d}}(Q) = \omega_{\mathfrak{d}}(Q\circ\iota)$, since Q and $Q\circ\iota$ represent the same integers, and that $Q \mapsto Q\circ\iota$ gives bijections on each of the sets $\mathcal{L}_{Mt}(|\mathfrak{d}|m)/\Gamma_0(Mt)$).

Next, we observe that

$$\langle (\tau Y - X)^{2k-2}, Q^{k-1}\rangle = (2k-2)! \cdot Q(\tau,1)^{k-1}.$$

This implies that

$$I_{k,\chi}(f,Q) = \frac{\chi_0(Q)}{(2k-2)!}\int_{C_Q} f(\tau)\langle(\tau Y - X)^{2k-2}, Q^{k-1}\rangle d\tau = \chi_0(Q)\langle \psi_f(D_Q), Q^{k-1}\rangle = J_{k,\chi}(\psi_f, Q),$$

and hence one easily sees that $\Theta_{k,M,\chi,\mathfrak{d}}(\psi_f) = \theta_{k,M,\chi,\mathfrak{d}}(f)$. Furthermore, decomposing $\psi_f = \psi_f^+ + \psi_f^-$ into its $+$ and $-$ components, part i) tells us that $\Theta_{k,M,\chi,\mathfrak{d}}(\psi_f^+) = 0$, and hence $\Theta_{k,M,\chi,\mathfrak{d}}(\psi_f) = \Theta_{k,M,\chi,\mathfrak{d}}(\psi_f^-)$, thereby completing the proof of ii). Finally, statement iii) follows already from the proof of ii). ☐

A cohomology class (or modular symbol) $\Phi \in H^1_c(\Gamma_0(M), L_{2k-2}(\mathbf{C}))$ is said to be *defined over* a subring R of $\mathbf{C}$ if Φ lies in the image of the natural map

$$H^1_c(\Gamma_0(M), L_{2k-2}(R)) \longrightarrow H^1_c(\Gamma_0(M), L_{2k-2}(\mathbf{C})).$$

Equivalently, Φ is defined over R if the corresponding modular symbol (under Ash–Stevens isomorphism) takes values in $L_{2k-2}(R)$.

If $f \in S_{2k}(M,\chi^2)$, recall from (25) that there exist complex numbers $\Omega_f^{\pm} \in \mathbf{C}^\times$ such that the normalized modular symbols

$$\varphi_f^{\pm} := \frac{1}{\Omega_f^{\pm}} \cdot \psi_f^{\pm},$$

are defined over the ring of integers $\mathcal{O}_f$ of the Hecke field of f. We fix once and for all periods $\Omega_f^\pm \in \mathbf{C}^\times$ with this algebraicity property, and then define

$$\theta^{\mathrm{alg}}_{k,M,\chi,\mathfrak{d}}(f) := \frac{1}{\Omega_f^-}\cdot \theta_{k,M,\chi,\mathfrak{d}}(f). \tag{29}$$

Theorem 5.4. *Let $f \in S_{2k}(M,\chi^2)$ be a Hecke eigenform, and let ψ_f and φ_f be as above. Then*

$$\theta^{\mathrm{alg}}_{k,M,\chi,\mathfrak{d}}(f) = \Theta_{k,M,\chi,\mathfrak{d}}(\varphi_f^-) \in S^+_{k+1/2}(M,\chi;\mathcal{O}_f).$$

Proof. The equality $\theta^{\mathrm{alg}}_{k,M,\chi,\mathfrak{d}}(f) = \Theta_{k,M,\chi,\mathfrak{d}}(\varphi_f^-)$ follows from the previous proposition and the definition of φ_f^- and $\theta^{\mathrm{alg}}_{k,M,\chi,\mathfrak{d}}(f)$. Now, since φ_f^- is defined over $\mathcal{O}_f$, the values $\chi_0(Q)\langle\varphi_f^-(D_Q), Q^{k-1}\rangle$ belong to $\mathcal{O}_f$ for all Q, hence also the values $s_{k,Mt,\chi}(\varphi_f^-,\mathfrak{d};(-1)^k mt^2)$ belong to $\mathcal{O}_f$ for all m, t. As a consequence, all the coefficients of $\Theta_{k,M,\chi,\mathfrak{d}}(\varphi_f^-)$ lie in $\mathcal{O}_f$ and the theorem is proved. □

As a direct consequence of Proposition 3.13, and of the cohomological $\mathfrak{d}$-th Shintani lifting introduced in this section, we have the following statement.

Corollary 5.5. *Let $k \geq 1$ be an integer, $N \geq 1$ be an odd integer, χ be a Dirichlet character modulo N, and $p > 2$ be a prime with $p \nmid N$. Let $f \in S_{2k}(N,\chi^2)$ be a normalized newform ordinary at p, and let $f_\alpha \in S_{2k}(Np,\chi^2)$ be its ordinary p-stabilization. Let $\mathfrak{d}$ be a fundamental discriminant with $\epsilon(-1)^k\mathfrak{d} > 0$, $\gcd(\mathfrak{d},N) = 1$ and $\mathfrak{d} \equiv 0 \pmod p$, and let $\mathfrak{d}'$ be another fundamental discriminant with $\mathfrak{d}\mathfrak{d}' > 0$ and $\gcd(\mathfrak{d}',N) = 1$. Then*

$$a_{|\mathfrak{d}'|}(\Theta_{k,Np,\chi,\mathfrak{d}}(\psi_{f_\alpha})) = (1-\beta\chi_{\mathfrak{d}'}\bar\chi_0(p)p^{-k})a_{|\mathfrak{d}'|}(\theta_{k,N,\chi,\mathfrak{d}}(f)),$$

and hence

$$\frac{1}{\Omega_f^-}\cdot a_{|\mathfrak{d}'|}(\Theta_{k,Np,\chi,\mathfrak{d}}(\psi_{f_\alpha})) = (1-\beta\chi_{\mathfrak{d}'}\bar\chi_0(p)p^{-k})a_{|\mathfrak{d}'|}(\theta^{\mathrm{alg}}_{k,N,\chi,\mathfrak{d}}(f))$$

Proof. Indeed, we have

$$a_{|\mathfrak{d}'|}(\Theta_{k,Np,\chi,\mathfrak{d}}(\psi_{f_\alpha})) = a_{|\mathfrak{d}'|}(\theta_{k,Np,\chi,\mathfrak{d}}(f_\alpha)) = (1-\beta\chi_{\mathfrak{d}'}\bar\chi_0(p)p^{-k})a_{|\mathfrak{d}'|}(\theta_{k,N,\chi,\mathfrak{d}}(f)),$$

where the first equality follows from part ii) of Proposition 5.3, and the second one from Proposition 3.13. This proves the first identity. The second one follows immediately from the first by the definition of $\theta^{\mathrm{alg}}_{k,N,\chi,\mathfrak{d}}$. □

5.2. **The $\mathfrak{d}$-th Shintani lifting of a Hida family.** Recall the universal ordinary p-adic Hecke algebra $\mathcal{R}_N$ of tame conductor N, over $\Lambda_N = \mathcal{O}[[\mathbf{Z}^\times_{p,N}]]$. We introduce now the universal ordinary metaplectic p-adic Hecke algebra of tame conductor N, defined as the Λ_N-algebra

$$\widetilde{\mathcal{R}}_N := \mathcal{R}_N \otimes_{\Lambda_N,\sigma} \Lambda_N,$$

where the tensor product is taken with respect to the $\mathcal{O}$-algebra homomorphism $\sigma : \Lambda_N \to \Lambda_N$ associated to the group homomorphism $t \mapsto t^2$ on $\mathbf{Z}^\times_{p,N}$. We note that $\widetilde{\mathcal{R}}_N$ is seen as a Λ_N-algebra with the structure morphism $\lambda \mapsto 1\otimes\lambda$, $\lambda \in \Lambda_N$. Also, we observe that the ring homomorphism

$$\mathcal{R}_N \longmapsto \widetilde{\mathcal{R}}_N, \quad \alpha \longmapsto \alpha\otimes 1$$

is *not* a homomorphism of Λ_N-algebras, but only of $\mathcal{O}$-algebras. Indeed, the map induced by pullback on weight spaces

$$\mathcal{X}(\widetilde{\mathcal{R}}_N) \longrightarrow \mathcal{X}(\mathcal{R}_N)$$

sends arithmetic points of signature (r,ψ) to arithmetic points of signature $(2r,\psi^2)$. In particular, for any quadratic character ε, the arithmetic points of signature $(r,\varepsilon\psi)$ are still sent to arithmetic points of signature $(2r,\psi^2)$.

Fix now a Hida family $(\mathcal{R}_\mathbf{f},\mathcal{U}_\mathbf{f},\mathcal{U}_\mathbf{f}^{\mathrm{cl}},\mathbf{f})$ of tame level N and tame character χ^2 modulo N. As usual, we write χ_0 be the primitive character associated with χ, N_0 for its conductor and $N_1 = N/N_0$. We assume that $\gcd(N_0,N_1) = 1$ and write $\epsilon = \chi(-1)$. By definition of Hida family, there exists an integer k_0 (only determined modulo $(p-1)/2$) such that every classical point $\kappa \in \mathcal{U}_\mathbf{f}^{\mathrm{cl}}$ has weight $2k-2$ with $2k \equiv 2k_0 \pmod{p-1}$. We define

$$\widetilde{\mathcal{R}}_\mathbf{f} := \mathcal{R}_\mathbf{f} \otimes_{\Lambda,\sigma} \Lambda,$$

where $\sigma : \Lambda \to \Lambda$ is the $\mathcal{O}$-algebra isomorphism induced by $t \mapsto t^2$ on $1 + p\mathbf{Z}_p$. In particular, notice that $\widetilde{\mathcal{R}}_{\mathbf{f}}$ is isomorphic to $\mathcal{R}_{\mathbf{f}}$ as $\mathcal{O}$-algebras. We equip $\widetilde{\mathcal{R}}_{\mathbf{f}}$ with the structure of Λ-algebra via the map $\lambda \mapsto 1 \otimes \lambda$. The natural homomorphism

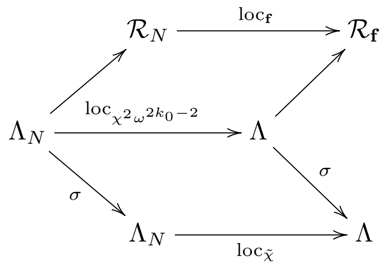
$$\mathcal{R}_{\mathbf{f}} \longrightarrow \widetilde{\mathcal{R}}_{\mathbf{f}}, \quad \alpha \longmapsto \alpha \otimes 1$$

is an isomorphism of $\mathcal{O}$-algebras, but it is *not* even a homomorphism of Λ-algebras. Indeed, similarly as above this is reflected in the fact that the induced map

$$\pi : \mathcal{X}(\widetilde{\mathcal{R}}_{\mathbf{f}}) \longmapsto \mathcal{X}(\mathcal{R}_{\mathbf{f}})$$

on weight spaces doubles the weights.

For any choice of square root $\tilde{\chi}$ of $\chi^2\omega^{2k_0-2}$ in $\widehat{\Delta}_{Np}$ we have a Λ_N-algebra structure for $\widetilde{\mathcal{R}}_{\mathbf{f}}$. This also uniquely determines the $\widetilde{\mathcal{R}}_N$-algebra structure because of the following diagram:



Since $p \nmid N$, we have a natural decomposition $\widehat{\Delta}_{Np} \simeq \widehat{\Delta}_N \times \widehat{\Delta}_p$, hence $\tilde{\chi}$ is uniquely determined by the choice of a square root of χ^2 in $\widehat{\Delta}_N$ and of ω^{2k_0-2} in $\widehat{\Delta}_p$. We choose r_0 to be one of the two solutions of the congruence $2x \equiv 2k_0 \pmod{p-1}$ and consider the Λ_N-structure of $\widetilde{\mathcal{R}}_{\mathbf{f}}$ induced by the choice $\tilde{\chi} := \chi\omega^{r_0-1}$.

Let $\widetilde{\mathcal{U}}_{\mathbf{f}} := \pi^{-1}(\mathcal{U}_{\mathbf{f}})$ and $\widetilde{\mathcal{U}}_{\mathbf{f}}^{\mathrm{cl}} := \pi^{-1}(\mathcal{U}_{\mathbf{f}}^{\mathrm{cl}})$. Notice that not all the weights of classical points in $\widetilde{\mathcal{U}}_{\mathbf{f}}^{\mathrm{cl}}$ are contained in a single residue class modulo $p-1$. Indeed, a point $\tilde{\kappa} \in \widetilde{\mathcal{U}}_{\mathbf{f}}^{\mathrm{cl}}$ has weight $k-1$ for some integer k such that $2k \equiv 2k_0 \pmod{p-1}$. Therefore either $k \equiv r_0 \pmod{p-1}$ or $k \equiv r_0 + \frac{p-1}{2} \pmod{p-1}$, yielding a partition of $\widetilde{\mathcal{U}}_{\mathbf{f}}^{\mathrm{cl}}$ as a union of two sets. In the following discussion we write $\widetilde{\mathcal{U}}_{\mathbf{f}}^{\mathrm{cl}}(r_0)$ for the subset of classical points in $\widetilde{\mathcal{U}}_{\mathbf{f}}^{\mathrm{cl}}$ whose weights are of the form $k-1$ with $k \equiv r_0 \pmod{p-1}$.

Lemma 5.6. *Let $(\mathcal{R}_{\mathbf{f}}, \mathcal{U}_{\mathbf{f}}, \mathcal{U}_{\mathbf{f}}^{\mathrm{cl}}, \mathbf{f})$ be a Hida family and keep the notation as in the above discussion. For each $Q \in \mathcal{L}_{Np}^{(p)}$ with positive discriminant divisible by p, there is a unique $\mathcal{R}_{\mathbf{f}}$-homomorphism*

$$J_{Q,\mathbf{f}}^{r_0} : \mathbf{D}_{\mathbf{f}} \longrightarrow \widetilde{\mathcal{R}}_{\mathbf{f}}$$

such that for all $\tilde{\kappa} \in \widetilde{\mathcal{U}}_{\mathbf{f}}^{\mathrm{cl}}(r_0)$ of weight $k-1$ one has

$$\tilde{\kappa}(J_{Q,\mathbf{f}}^{r_0}(\mu)) = \chi_0(Q)\langle \mathrm{sp}_{\pi(\tilde{\kappa})}(\mu), Q^{k-1}\rangle \quad \textit{for all } \mu \in \mathbf{D}_{\mathbf{f}}.$$

Proof. Fix a quadratic form Q as in the statement. The uniqueness of J_Q is clear because $\widetilde{\mathcal{U}}_{\mathbf{f}}^{\mathrm{cl}}(r_0)$ is dense in $\widetilde{\mathcal{U}}_{\mathbf{f}}$. To prove the existence, we adapt the arguments in the proof of [Ste94, Lemma 6.1].

Recall that there is a canonical isomorphism $\mathbf{Z}_p[[\mathbf{Z}_p^\times]] \simeq \mathrm{Meas}(\mathbf{Z}_p^\times)$, $j \mapsto dj$. Using this, we first define a map

$$j_Q : \mathbf{D} \longrightarrow \mathbf{Z}_p[[\mathbf{Z}_p^\times]], \quad \nu \mapsto j_Q(\nu),$$

where $j_Q(\nu)$ is determined by requiring that

$$\int_{\mathbf{Z}_p^\times} f\, dj_Q(\nu) = \int_{\mathbf{Z}_p \times \mathbf{Z}_p^\times} f(Q(x,y))\, d\nu(x,y)$$

for all continuous functions $f : \mathbf{Z}_p^\times \to \mathbf{Z}_p$. Observe that since $Q \in \mathcal{L}_{Np}^{(p)}$ has discriminant divisible by p, we have $Q(x,y) \in \mathbf{Z}_p^\times$ for all $(x,y) \in \mathbf{Z}_p \times \mathbf{Z}_p^\times$, so this is well-defined. We notice that j_Q is a $\mathbf{Z}_p$-linear map, and that $j_Q([t]\nu) = [t^2] \cdot j_Q(\nu)$ for $t \in \mathbf{Z}_p^\times$. We compose the map j_Q with the map $\mathrm{loc}_{r_0} := \mathrm{loc}_{\tilde{\chi}} |_{\Lambda_1}$, i.e.

$$\begin{aligned} \mathrm{loc}_{r_0} : \Lambda_1 &\to \Lambda \\ [x] &\mapsto \omega^{r_0-1}(x)[\langle x\rangle] \quad \text{for } x \in \mathbf{Z}_p^\times. \end{aligned}$$

In this way we get a map

$$j_{Q,\mathbf{f}} : \mathbf{D} \longrightarrow \Lambda, \quad \nu \longmapsto j_{Q,\mathbf{f}}(\nu) = \mathrm{loc}_{r_0}(j_Q(\nu)).$$

which, in terms of measures, is characterized by requiring that

$$\int_\Gamma \varphi \, dj_{Q,\mathbf{f}}(\nu) = \int_{\mathbf{Z}_p^\times} \varphi^\dagger \, dj_Q(\nu)$$

for all continuous functions $\varphi : \Gamma \to \mathbf{Z}_p$, where $\varphi^\dagger(x) = \omega^{r_0-1}(x)\varphi(\langle x\rangle)$ for $x \in \mathbf{Z}_p^\times$. Observe that $j_{Q,\mathbf{f}}([t]\nu) = \mathrm{loc}_{r_0}([t]^2 j_Q(\nu))$ for all $t \in \mathbf{Z}_p^\times$, $\nu \in \mathbf{D}$. The map $j_{Q,\mathbf{f}}$ extends by $\mathcal{R}_\mathbf{f}$-linearity to a unique $\mathcal{R}_\mathbf{f}$-linear map

$$J_{Q,\mathbf{f}} : \mathbf{D}_\mathbf{f} = \mathbf{D} \otimes_{\mathbf{Z}_p[[\mathbf{Z}_p^\times]]} \mathcal{R}_\mathbf{f} \longrightarrow \widetilde{\mathcal{R}}_\mathbf{f} = \mathcal{R}_\mathbf{f} \otimes_{\Lambda,\sigma} \Lambda$$

such that

$$J_{Q,\mathbf{f}}(\nu \otimes \alpha) = \chi_0(Q) \cdot \alpha \otimes_\sigma j_{Q,\mathbf{f}}(\nu) \quad \text{for } \nu \in \mathbf{D}, \alpha \in \mathcal{R}_\mathbf{f}.$$

Now, let $\mu = \nu \otimes \alpha \in \mathbf{D}_\mathbf{f} = \mathbf{D} \otimes \mathcal{R}_\mathbf{f}$, and $\tilde\kappa \in \widetilde{\mathcal{U}}_\mathbf{f}^{\mathrm{cl}}(r_0)$ be a classical point of weight $k-1$, and let $\kappa = \pi(\tilde\kappa) \in \mathcal{U}_\mathbf{f}^{\mathrm{cl}}$. Then we have

$$\begin{aligned}
\tilde\kappa(J_{Q,\mathbf{f}}^{r_0}(\mu)) &= \chi_0(Q) \cdot \tilde\kappa(\alpha \otimes 1)\tilde\kappa(1 \otimes j_{Q,\mathbf{f}}(\nu)) = \chi_0(Q)\kappa(\alpha) \cdot \int_\Gamma \tilde\kappa \, dj_{Q,\mathbf{f}}(\nu) = \\
&= \chi_0(Q)\kappa(\alpha) \cdot \int_{\mathbf{Z}_p^\times} \tilde\kappa^\dagger \, dj_Q(\nu) = \chi_0(Q)\kappa(\alpha) \cdot \int_{\mathbf{Z}_p \times \mathbf{Z}_p^\times} \omega^{r_0-1}(Q(x,y))\tilde\kappa(\langle Q(x,y)\rangle) d\nu(x,y) = \\
&= \chi_0(Q)\kappa(\alpha) \cdot \int_{\mathbf{Z}_p \times \mathbf{Z}_p^\times} \omega(Q(x,y))^{r_0-k} Q(x,y)^{k-1} d\nu(x,y) = \\
&= \chi_0(Q)\kappa(\alpha) \cdot \int_{\mathbf{Z}_p \times \mathbf{Z}_p^\times} \left\langle \frac{(xY-yX)^{2k-2}}{(2k-2)!}, Q^{k-1} \right\rangle d\nu(x,y) = \\
&= \chi_0(Q) \left\langle \kappa(\alpha) \cdot \int_{\mathbf{Z}_p \times \mathbf{Z}_p^\times} \frac{(xY-yX)^{2k-2}}{(2k-2)!} d\nu(x,y), Q^{k-1} \right\rangle = \chi_0(Q)\langle \mathrm{sp}_\kappa(\mu), Q^{k-1}\rangle,
\end{aligned}$$

as we wanted to prove. □

Definition 5.7. *Let $\Phi_\mathbf{f} \in \mathrm{Symb}_{\Gamma_0(N)}(\mathbf{D}_\mathbf{f})$ be the Λ-adic modular symbol attached to* $\mathbf{f}$ *as in Corollary 4.6. For each $Q \in \mathcal{L}_{Np}^{(p)}$ with positive discriminant divisible by p, we define*

$$J^{r_0}(\mathbf{f}; Q) := J_{Q,\mathbf{f}}^{r_0}(\Phi_\mathbf{f}(D_Q)) \in \widetilde{\mathcal{R}}_\mathbf{f}.$$

An easy computation shows that $J^{r_0}(\mathbf{f}; Q)$ depends only on the $\Gamma_0(Np)$-equivalence class of Q. The notation suggests that $J^{r_0}(\mathbf{f}; -)$ depends only on the Hida family $\mathbf{f}$ (and on the choice of r_0), but the definition clearly shows that it depends on the modular symbol $\Phi_\mathbf{f}$, which is only determined up to the choice of p-adic periods as explained in Corollary 4.6. Despite of this, we prefer to write $J^{r_0}(\mathbf{f}; -)$ instead of $J^{r_0}(\Phi_\mathbf{f}; -)$.

Now let $\mathfrak{d}$ be a fundamental discriminant *divisible by p* such that $\gcd(N_0, \mathfrak{d}) = 1$.

Definition 5.8. *If $m \geq 1$ is an integer, we define*

$$s_N^{r_0}(\mathbf{f}; \mathfrak{d}, \chi_\mathfrak{d}(-1)m) := \sum_{Q \in \mathcal{L}_{Np}^{(p)}(|\mathfrak{d}|m)/\Gamma(Np)} \omega_\mathfrak{d}(Q) \cdot J^{r_0}(\mathbf{f}; Q) \in \widetilde{\mathcal{R}}_\mathbf{f}.$$

Observe that the function $s_N^{r_0}(\mathbf{f}; \mathfrak{d}, \chi_\mathfrak{d}(-1)m)$ is identically zero if $\chi_\mathfrak{d}(-1)m \not\equiv 0, 1 \pmod 4$, since in this case $|\mathfrak{d}|m = \mathfrak{d}\chi_\mathfrak{d}(-1)m$ is not a discriminant. And when $\chi_\mathfrak{d}(-1)m \equiv 0, 1 \pmod 4$, the function $s_N^{r_0}(\mathbf{f}; \mathfrak{d}, \chi_\mathfrak{d}(-1)m)$ on $\mathcal{X}(\widetilde{\mathcal{R}}_\mathbf{f})$ is key to interpolate the m-th Fourier coefficients of the $\mathfrak{d}$-th Shintani liftings of the specializations of $\mathbf{f}$. Indeed, we define the Λ-adic $\mathfrak{d}$-th Shintani lifting of $\mathbf{f}$ as the power series with coefficients in $\widetilde{\mathcal{R}}_\mathbf{f}$ given by

$$\Theta_\mathfrak{d}^{r_0}(\mathbf{f}) := \sum_{m \geq 1} \left(\sum_{0 < t | N_1} \mu(t)\chi_\mathfrak{d}\bar\chi_0(t) t^{-2} \omega(t)^{r_0-1} [\langle t\rangle]^{-1} s_N^{r_0}(\mathbf{f}; \mathfrak{d}, \chi_\mathfrak{d}(-1)mt^2) \right) q^m \in \widetilde{\mathcal{R}}_\mathbf{f}[[q]], \tag{30}$$

where $[\langle t\rangle] \in \Lambda$ is the group-like element associated with $\langle t\rangle \in \Gamma$ (notice that $p \nmid N_1$). Observe that $\Theta_\mathfrak{d}^{r_0}(\mathbf{f})$ vanishes identically if $\epsilon(-1)^{r_0}\mathfrak{d} < 0$ because the quantities $s_N^{r_0}(\mathbf{f}; \mathfrak{d}, \chi_\mathfrak{d}(-1)mt^2)$ are all zero (compare with Remark 2.2). Note also that the definition of $\Theta_{\mathfrak{d}}^{r_0}(\mathbf{f})$ depends on our choice of r_0, as indicated by the notation (in fact, note that Lemma 5.6 and Definitions 5.7 and 5.8 do depend on the choice of r_0). Hence, for each solution of the congruence $2x \equiv 2k_0 \pmod{p-1}$ we have a Λ-adic $\mathfrak{d}$-th Shintani lifting of $\mathbf{f}$, and they are different liftings. The interpolation property for $\Theta_{\mathfrak{d}}^{r_0}(\mathbf{f})$ then reads as follows:

Theorem 5.9. *Keep the notation notation as above. For every classical point $\tilde\kappa \in \widetilde{\mathcal{U}}_{\mathbf{f}}^{\rm cl}(r_0)$ of weight $k-1$, we have*

$$\Theta_{\mathfrak{d}}^{r_0}(\mathbf{f})(\tilde\kappa) = C(k,\chi,\mathfrak{d})^{-1}\Theta_{k,Np,\mathfrak{d}}(\Phi_{\mathbf{f},\kappa}) = \Omega_\kappa\cdot C(k,\chi,\mathfrak{d})^{-1}\theta_{k,Np,\chi,\mathfrak{d}}^{\rm alg}(\mathbf{f}(\kappa)),$$

where $\kappa = \pi(\tilde\kappa)\in\mathcal{U}_{\mathbf{f}}^{\rm cl}$, and $C(k,\chi,\mathfrak{d})$ is the constant defined in (11).

Proof. The proof follows from the above construction of the Λ-adic $\mathfrak{d}$-th Shintani lifting. First suppose that $\epsilon(-1)^{r_0}\mathfrak{d} < 0$. In this case, $\epsilon(-1)^k\mathfrak{d} < 0$ as well for all points $\tilde\kappa$ as in the statement, and hence both sides of the stated equality are zero. We assume for the rest of the proof that $\epsilon(-1)^{r_0}\mathfrak{d} > 0$. Then, if $\tilde\kappa$ is as in the statement we have $\chi_{\mathfrak{d}}(-1) = \epsilon(-1)^k$. If $m\geq 1$ is an integer with $\epsilon(-1)^k m \not\equiv 0,1 \pmod 4$, then $s_N^{r_0}(\mathbf{f};\mathfrak{d},\epsilon(-1)^k m) = 0$, since $|\mathfrak{d}|m$ is not a discriminant. For the remaining positive integers, we can apply Lemma 5.6 to find

$$\begin{aligned}\tilde\kappa(s_N^{r_0}(\mathbf{f},\mathfrak{d};\chi_{\mathfrak{d}}(-1)m)) &= \sum_{Q\in\mathcal{L}_{Np}^{(p)}(|\mathfrak{d}|m)/\Gamma(Np)}\omega_{\mathfrak{d}}(Q)\cdot\tilde\kappa(J^{r_0}(\mathbf{f};Q)) =\\ &= \sum_{Q\in\mathcal{L}_{Np}^{(p)}(|\mathfrak{d}|m)/\Gamma(Np)}\omega_{\mathfrak{d}}(Q)\cdot\langle \mathrm{sp}_\kappa(\Phi_{\mathbf{f}}(D_Q), Q^{k-1})\rangle =\\ &= \sum_{Q\in\mathcal{L}_{Np}(|\mathfrak{d}|m)/\Gamma(Np)}\omega_{\mathfrak{d}}(Q)\cdot J_k(\Phi_{\mathbf{f},\kappa},Q) =\\ &= s_{k,Np,\chi}(\Phi_{\mathbf{f},\kappa},\mathfrak{d};\epsilon(-1)^k m).\end{aligned}$$

In the last line, the presence of $\omega_{\mathfrak{d}}$ makes trivial the contribution of non-p-primitive forms (because $p\mid\mathfrak{d}$). It then easily follows by the definition of $\Theta_{\mathfrak{d}}^{r_0}$ that

$$\Theta_{\mathfrak{d}}^{r_0}(\mathbf{f})(\tilde\kappa) = C(k,\chi,\mathfrak{d})^{-1}\Theta_{k,Np,\chi,\mathfrak{d}}(\Phi_{\mathbf{f},\kappa}),$$

observing that

$$\tilde\kappa(t^{-2}\omega(t)^{r_0-1}[\langle t\rangle]^{-1}) = t^{-2}\omega(t)^{r_0-k}t^{-k+1} = t^{-k-1}.$$

The second equality in the statement is now deduced using that $\Phi_{\mathbf{f},\kappa} = \Omega_\kappa\cdot\varphi_{\mathbf{f}(\kappa)}^-$, by Theorem 4.5, and that $\Theta_{k,Np,\chi,\mathfrak{d}}(\varphi_{\mathbf{f}(\kappa)}^-) = \theta_{k,Np,\chi,\mathfrak{d}}^{\rm alg}(\mathbf{f}(\kappa))$, by Theorem 5.4. ☐

Remark 5.10. *As we have pointed out above, it may happen that $\Theta_{\mathfrak{d}}^{r_0}(\mathbf{f})$ vanishes identically. However, suppose that there is at least one classical point $\tilde\kappa\in\widetilde{\mathcal{U}}_{\mathbf{f}}^{\rm cl}(r_0)$, say of weight $k-1$, such that the classical $\mathfrak{d}$-th Shintani lifting $\theta_{k,Np,\mathfrak{d}}(\mathbf{f}(\kappa))$ is non-zero, where $\kappa=\pi(\tilde\kappa)$. This implies that $\epsilon(-1)^k\mathfrak{d}>0$, and hence also $\epsilon(-1)^{r_0}\mathfrak{d}>0$. Then, by virtue of Corollary 4.6 we can choose the p-adic periods so that $\Omega_\kappa\neq 0$, and the interpolation property of the previous theorem ensures that $\Theta_{\mathfrak{d}}^{r_0}(\mathbf{f})$ does not vanish. Hence, as soon as the classical $\mathfrak{d}$-th Shintani lifting is not vanishing on $\mathbf{f}(\kappa)$ for some $\kappa\in\pi(\widetilde{\mathcal{U}}_{\mathbf{f}}^{\rm cl}(r_0))$, one can construct a non-zero Λ-adic $\mathfrak{d}$-th Shintani lifting of $\mathbf{f}$, $\Theta_{\mathfrak{d}}^{r_0}(\mathbf{f})$. When doing this choice, we can say that $\Theta_{\mathfrak{d}}^{r_0}(\mathbf{f})$ is 'centered at κ'.*

5.3. **A Λ-adic Kohnen formula.** We keep the notation and assumptions as in the previous paragraph. Associated with a Hida family $(\mathcal{R}_{\mathbf{f}},\mathcal{U}_{\mathbf{f}},\mathcal{U}_{\mathbf{f}}^{\rm cl},\mathbf{f})$ and a fundamental discriminant $\mathfrak{d}$ satisfying the assumptions of Theorem 5.9, let us rewrite (30) as

$$\Theta_{\mathfrak{d}}^{r_0}(\mathbf{f}) = \sum_{m\geq 1}\mathbf{a}_m(\Theta_{\mathfrak{d}}^{r_0}(\mathbf{f}))q^m \in \widetilde{\mathcal{R}}_{\mathbf{f}}[[q]].$$

According to Theorem 5.9, the elements $\mathbf{a}_m(\Theta_{\mathfrak{d}}^{r_0}(\mathbf{f}))\in\widetilde{\mathcal{R}}_{\mathbf{f}}$ interpolate Fourier coefficients of the $\mathfrak{d}$-th Shintani liftings of the classical forms $\mathbf{f}(\kappa)$ on the set of classical points $\pi(\widetilde{\mathcal{U}}_{\mathbf{f}}^{\rm cl}(r_0))\subset\mathcal{U}_{\mathbf{f}}^{\rm cl}$. In particular, we may consider the function

$$\mathbf{a}_{|\mathfrak{d}|}(\Theta_{\mathfrak{d}}^{r_0}(\mathbf{f})) : \mathcal{X}(\widetilde{\mathcal{R}}_{\mathbf{f}}) \longrightarrow \mathbf{C}_p,\quad \tilde\kappa\longmapsto \mathbf{a}_{|\mathfrak{d}|}(\Theta_{\mathfrak{d}}^{r_0}(\mathbf{f}))(\tilde\kappa)$$

given by the $|\mathfrak{d}|$-th Fourier coefficient of $\Theta_{\mathfrak{d}}^{r_0}(\mathbf{f})$.

Proposition 5.11. *With notation and assumptions as above, suppose that* $\gcd(N,\mathfrak{d}) = 1$ *and* $\epsilon(-1)^{r_0}\mathfrak{d} > 0$. *If* $\tilde{\kappa} \in \widetilde{\mathcal{U}}_{\mathbf{f}}^{\mathrm{cl}}(r_0)$ *has weight* $k-1$, $\kappa = \pi(\tilde{\kappa})$, *and* $\mathbf{f}(\kappa) \neq f_\kappa$, *then*

$$\mathbf{a}_{|\mathfrak{d}|}(\Theta_{\mathfrak{d}}^{r_0}(\mathbf{f}))(\tilde{\kappa}) = \Omega_\kappa \cdot \chi_{\mathfrak{d}}(-1) R_{\mathfrak{d}}(f_\kappa)(-1)^k |\mathfrak{d}|^k N_0^k (k-1)! \cdot \frac{L(f_\kappa, \chi_{\mathfrak{d}}\bar{\chi}_0, k)}{(2\pi i)^k \mathfrak{g}(\chi_{\mathfrak{d}}\bar{\chi}_0)\Omega^-_{\mathbf{f}(\kappa)}}, \tag{31}$$

where $R_{\mathfrak{d}}(f_\kappa)$ *as defined in Proposition 2.4.*

Proof. For any $\tilde{\kappa} \in \widetilde{\mathcal{U}}_{\mathbf{f}}^{\mathrm{cl}}(r_0)$ of weight $k-1$, Theorem 5.9 and Proposition 3.13 (see also Remark 3.14) imply that

$$\begin{aligned}\mathbf{a}_{|\mathfrak{d}|}(\Theta_{\mathfrak{d}}^{r_0}(\mathbf{f}))(\tilde{\kappa}) &= \Omega_\kappa \cdot \frac{a_{|\mathfrak{d}|}(\theta_{k,Np,\chi,\mathfrak{d}}(\mathbf{f}(\kappa)))}{C(k,\chi,\mathfrak{d}) \cdot \Omega^-_{\mathbf{f}(\kappa)}} = \Omega_\kappa (1 - \beta\chi_{\mathfrak{d}}\bar{\chi}_0(p)p^{-k}) \cdot \frac{a_{|\mathfrak{d}|}(\theta_{k,Np,\chi,\mathfrak{d}}(f_\kappa))}{C(k,\chi,\mathfrak{d}) \cdot \Omega^-_{\mathbf{f}(\kappa)}} = \\ &= \Omega_\kappa \cdot \frac{a_{|\mathfrak{d}|}(\theta_{k,Np,\chi,\mathfrak{d}}(f_\kappa))}{C(k,\chi,\mathfrak{d}) \cdot \Omega^-_{\mathbf{f}(\kappa)}},\end{aligned} \tag{32}$$

where $\kappa = \pi(\tilde{\kappa})$, and the last equality uses that p divides $\mathfrak{d}$. If $\mathbf{f}(\kappa) \neq f_\kappa$, the result follows by combining (32) with Lemma 3.10 and Corollary 2.5. ☐

Remark 5.12. *Notice that the condition* $\mathbf{f}(\kappa) = f_\kappa$ *can only happen when* $k = 1$. *See the forthcoming Corollary 5.14 for the value at those* κ *(in the case that* χ *is trivial).*

Note that the values

$$L^{\mathrm{alg}}(f_\kappa, \chi_{\mathfrak{d}}\bar{\chi}_0, k) := \frac{L(f_\kappa, \chi_{\mathfrak{d}}\bar{\chi}_0, k)}{(2\pi i)^k \mathfrak{g}(\chi_{\mathfrak{d}}\bar{\chi}_0)\Omega^-_{\mathbf{f}(\kappa)}}$$

in (31) are algebraic, hence the Fourier coefficient $\mathbf{a}_{|\mathfrak{d}|}(\Theta_{\mathfrak{d}}^{r_0}(\mathbf{f}))$ interpolates the 'algebraic parts' of the special values $L(f_\kappa, \chi_{\mathfrak{d}}\bar{\chi}_0, k)$ at classical points $\tilde{\kappa} \in \widetilde{\mathcal{U}}_{\mathbf{f}}(r_0)$. A p-adic L-function interpolating such special values was studied in Greenberg–Stevens [GS93], generalizing Mazur–Tate–Teitelbaum [MTT86]. Associated with $\mathbf{f}$ and a Dirichlet character ψ, they define a two-variable p-adic L-function $\mathcal{L}_p^{\mathrm{GS}}(\mathbf{f},\psi)$ on some local domain $\mathcal{U}_{\mathbf{f}} \times \mathcal{U} \subset \mathcal{X}(\mathcal{R}_{\mathbf{f}}) \times \mathcal{X}(\Lambda)$ satisfying the following interpolation property: for every pair of classical points $(\kappa, j) \in \mathcal{U}_{\mathbf{f}}^{\mathrm{cl}} \times \mathcal{U}^{\mathrm{cl}}$ in the cone defined by $0 < j < \mathrm{wt}(\kappa) + 2$,

$$\mathcal{L}_p^{\mathrm{GS}}(\mathbf{f},\psi)(\kappa, j) = \Omega_\kappa \cdot \mathcal{E}_p(\mathbf{f}(\kappa), \psi, j) \cdot \frac{\mathrm{c}^{j-1}(j-1)!\mathfrak{g}(\psi\omega^{1-j})}{(2\pi i)^j \Omega^{\mathrm{sgn}(\psi)}_{\mathbf{f}(\kappa)}} L(\mathbf{f}(\kappa), \bar{\psi}\omega^{j-1}, j).$$

Here $c = \mathrm{cond}(\psi\omega^{1-j})$, m is the exponent of p in c, $a_p(\kappa)$ is the p-th Fourier coefficient of $\mathbf{f}(\kappa)$, and the Euler-like factor:

$$\mathcal{E}_p(\mathbf{f}(\kappa), \psi, j) = a_p(\kappa)^{-m}\left(1 - \frac{\psi\omega^{1-j}(p)p^{j-1}}{a_p(\kappa)}\right). \tag{33}$$

When restricted to the line (κ, k), where $k = (\mathrm{wt}(\kappa) + 2)/2$, this identity becomes

$$\mathcal{L}_p^{\mathrm{GS}}(\mathbf{f},\psi)(\kappa, k) = \Omega_\kappa \cdot \mathcal{E}_p(\mathbf{f}(\kappa), \psi, k) \cdot \frac{\mathrm{c}^{k-1}(k-1)!\mathfrak{g}(\psi\omega^{1-k})}{(2\pi i)^k \Omega^{\mathrm{sgn}(\psi)}_{\mathbf{f}(\kappa)}} L(\mathbf{f}(\kappa), \bar{\psi}\omega^{k-1}, k).$$

This suggests defining a one-variable p-adic L-function as the restriction of $\mathcal{L}_p^{\mathrm{GS}}(\mathbf{f},\psi)$ to the 'line' (κ, k). More precisely, this one-variable p-adic L-function is the pullback of $\mathcal{L}_p^{\mathrm{GS}}(\mathbf{f},\psi)$ along the map

$$\mathcal{U}_{\mathbf{f}} \xrightarrow{\Delta} \mathcal{U}_{\mathbf{f}} \times \mathcal{U}$$

given by $\kappa \mapsto (\kappa, \mathrm{wt}(\kappa)/2 + 1)$ on $\mathcal{U}_{\mathbf{f}}^{\mathrm{cl}}$. For our purposes, we will rather consider the pullback of $\mathcal{L}_p^{\mathrm{GS}}(\mathbf{f},\psi)$ along

$$\iota : \widetilde{\mathcal{U}}_{\mathbf{f}} \xrightarrow{\pi} \mathcal{U}_{\mathbf{f}} \xrightarrow{\Delta} \mathcal{U}_{\mathbf{f}} \times \mathcal{U},$$

which yields a one-variable p-adic L-function $\tilde{\mathcal{L}}_p^{\mathrm{GS}}(\mathbf{f},\psi)$ on $\widetilde{\mathcal{U}}_{\mathbf{f}}$. Indeed, by construction we see that if $\tilde{\kappa} \in \widetilde{\mathcal{U}}_{\mathbf{f}}^{\mathrm{cl}}(r_0)$ has weight $k-1$ and $\kappa = \pi(\tilde{\kappa})$, then

$$\tilde{\mathcal{L}}_p^{\mathrm{GS}}(\mathbf{f},\psi)(\tilde{\kappa}) = \Omega_\kappa \cdot \mathcal{E}_p(\mathbf{f}(\kappa), \psi, k) \cdot \frac{\mathrm{c}^{k-1}(k-1)!\mathfrak{g}(\psi\omega^{1-k})}{(2\pi i)^k \Omega^{\mathrm{sgn}(\psi)}_{\mathbf{f}(\kappa)}} \cdot L(\mathbf{f}(\kappa), \bar{\psi}\omega^{k-1}, k).$$

Similarly, by pulling back via the map $\pi : \widetilde{\mathcal{U}}_{\mathbf{f}} \to \mathcal{U}_{\mathbf{f}}$ we can define Λ-adic functions $\mathbf{R}_{\mathfrak{d}}$ and $\mathbf{a}_p$ on $\widetilde{\mathcal{U}}_{\mathbf{f}}$ such that

$$\mathbf{R}_{\mathfrak{d}}(\tilde{\kappa}) = R_{\mathfrak{d}}(f_\kappa) \qquad \text{and} \qquad \mathbf{a}_p(\tilde{\kappa}) = a_p(\kappa)$$

for all $\tilde{\kappa} \in \widetilde{\mathcal{U}}_{\mathbf{f}}^{\mathrm{cl}}$ of weight $k-1>1$, with $\kappa = \pi(\tilde{\kappa})$.

The following theorem might be seen as a 'Λ-adic Kohnen formula' in the spirit of the classical formula stated in Corollary 2.5.

Theorem 5.13. *With the above notation, suppose that* $\gcd(N, \mathfrak{d}) = 1$ *and* $\epsilon(-1)^{r_0}\mathfrak{d} > 0$. *Then the equality*

$$\mathbf{a}_{|\mathfrak{d}|}(\Theta_{\mathfrak{d}}^{r_0}(\mathbf{f})) = \chi_{\mathfrak{d}}(-1) \cdot \mathbf{a}_p \cdot \mathbf{R}_{\mathfrak{d}} \cdot \tilde{\mathcal{L}}_p^{\mathrm{GS}}(\mathbf{f}, \chi_{\mathfrak{d}}\chi_0\omega^{r_0-1})$$

holds, as an equality of functions on $\widetilde{\mathcal{U}}_{\mathbf{f}}$.

Proof. It suffices to prove the claimed equality at classical points in $\widetilde{\mathcal{U}}_{\mathbf{f}}^{\mathrm{cl}}(r_0)$ of weight $k-1>1$, since they are dense in $\widetilde{\mathcal{U}}_{\mathbf{f}}$. Let $\tilde{\kappa} \in \widetilde{\mathcal{U}}_{\mathbf{f}}^{\mathrm{cl}}(r_0)$ be a classical point of weight $k-1$, with $k \equiv r_0 \pmod{p-1}$, and let $\kappa = \pi(\tilde{\kappa})$. We consider $\psi = \chi_{\mathfrak{d}}\chi_0\omega^{r_0-1}$. Since $k \equiv r_0 \pmod{p-1}$, we have $\psi\omega^{1-k} = \chi_{\mathfrak{d}}\chi_0$, which has conductor $|\mathfrak{d}|N_0$. Therefore, since $\chi_{\mathfrak{d}}(p) = 0$ the Euler-like factor $\mathcal{E}_p(\mathbf{f}(\kappa), \psi, k)$ is just 1. The fact that $\chi_{\mathfrak{d}}(p) = 0$ also implies that $L(\mathbf{f}(\kappa), \bar{\psi}\omega^{k-1}, k) = L(\mathbf{f}(\kappa), \chi_{\mathfrak{d}}\bar{\chi}_0, k) = L(f_\kappa, \chi_{\mathfrak{d}}\bar{\chi}_0, k)$. Besides, note that since ω is odd and we are assuming $\epsilon(-1)^{r_0}\mathfrak{d} > 0$ we have $\mathrm{sgn}(\chi_{\mathfrak{d}}\chi_0\omega^{r_0-1}) = -1$. We thus obtain that

$$\tilde{\mathcal{L}}_p^{\mathrm{GS}}(\mathbf{f}, \chi_{\mathfrak{d}}\chi_0\omega^{r_0-1})(\tilde{\kappa}) = \Omega_\kappa a_p(\kappa)^{-1} \frac{N_0^{k-1}|\mathfrak{d}|^{k-1}(k-1)!\mathfrak{g}(\chi_{\mathfrak{d}}\chi_0)}{(2\pi i)^k \Omega^-_{\mathbf{f}(\kappa)}} \cdot L(f_\kappa, \chi_{\mathfrak{d}}\bar{\chi}_0, k).$$

Using that

$$\mathfrak{g}(\chi_{\mathfrak{d}}\chi_0) = \chi_{\mathfrak{d}}\chi_0(-1)|\mathfrak{d}|N_0/\mathfrak{g}(\chi_{\mathfrak{d}}\bar{\chi}_0) = (-1)^k|\mathfrak{d}|N_0/\mathfrak{g}(\chi_{\mathfrak{d}}\bar{\chi}_0),$$

we can rewrite the above identity as

$$\tilde{\mathcal{L}}_p^{\mathrm{GS}}(\mathbf{f}, \chi_{\mathfrak{d}}\chi_0\omega^{r_0-1})(\tilde{\kappa}) = \Omega_\kappa a_p(\kappa)^{-1}(-1)^k N_0^k |\mathfrak{d}|^k (k-1)! \cdot \frac{L(f_\kappa, \chi_{\mathfrak{d}}\bar{\chi}_0, k)}{(2\pi i)^k \mathfrak{g}(\chi_{\mathfrak{d}}\bar{\chi}_0)\Omega^-_{\mathbf{f}(\kappa)}}. \tag{34}$$

Since $k-1>1$, we have $\mathbf{f}(\kappa) \neq f_\kappa$ and we can compare the above with equation (31) to deduce that

$$\mathbf{a}_{|\mathfrak{d}|}(\Theta_{\mathfrak{d}}^{r_0}(\mathbf{f}))(\tilde{\kappa}) = a_p(\kappa) \cdot (-1)^k\epsilon \cdot R_{\mathfrak{d}}(f_\kappa) \cdot \tilde{\mathcal{L}}_p^{\mathrm{GS}}(\mathbf{f}, \chi_{\mathfrak{d}}\chi_0\omega^{r_0-1})(\tilde{\kappa}).$$

To conclude, we may just use that $(-1)^k\epsilon = \chi_{\mathfrak{d}}(-1)$ to get

$$\mathbf{a}_{|\mathfrak{d}|}(\Theta_{\mathfrak{d}}^{r_0}(\mathbf{f}))(\tilde{\kappa}) = \chi_{\mathfrak{d}}(-1) \cdot a_p(\kappa) \cdot R_{\mathfrak{d}}(f_\kappa) \cdot \tilde{\mathcal{L}}_p^{\mathrm{GS}}(\mathbf{f}, \chi_{\mathfrak{d}}\chi_0\omega^{r_0-1})(\tilde{\kappa}).$$

□

Corollary 5.14. *Let* $N \geq 1$ *be a squarefree integer,* p *be an odd prime with* $p \nmid N$, $f \in S_2(Np)$ *be a normalized newform, and* $\mathfrak{d} < 0$ *be a fundamental discriminant divisible by* p *such that* $\gcd(N, \mathfrak{d}) = 1$. *Assume that* $\chi_{\mathfrak{d}}(\ell) = w_\ell$ *for every prime* ℓ *dividing* N. *Let* $\mathbf{f}$ *be the Hida family passing through* f, $\kappa \in \mathcal{U}_{\mathbf{f}}^{\mathrm{cl}}$ *be such that* $\mathbf{f}(\kappa) = f_\kappa = f$, *and* $\tilde{\kappa} \in \widetilde{\mathcal{U}}_{\mathbf{f}}^{\mathrm{cl}}(1)$ *such that* $\pi(\tilde{\kappa}) = \kappa$. *Then*

$$\mathbf{a}_{|\mathfrak{d}|}(\Theta_{\mathfrak{d}}^{1}(\mathbf{f}))(\tilde{\kappa}) = \Omega_\kappa \cdot 2^{\nu(N)}|\mathfrak{d}| \cdot \frac{L(f, \chi_{\mathfrak{d}}, 1)}{(2\pi i)\mathfrak{g}(\chi_{\mathfrak{d}})\Omega_f^-}.$$

Proof. This is an immedate consequence of the above theorem, taking $k = r_0 = 1$ and noticing that $R_{\mathfrak{d}}(f) = 2^{\nu(N)}$ under the hypotheses in the statement. □

By using the interpolation property of the Λ-adic $\mathfrak{d}$-th Shintani lifting, the above corollary can be rewritten in classical terms. This yields a mild generalization of Kohnen's formula in (15), in which the level of the newform and the fundamental discriminant are not relatively prime:

Corollary 5.15. *Let* $N \geq 1$ *be a squarefree integer,* p *be an odd prime with* $p \nmid N$, $f \in S_2(Np)$ *be a normalized newform, and* $\mathfrak{d} < 0$ *be a fundamental discriminant divisible by* p *such that* $\gcd(N, \mathfrak{d}) = 1$. *Assume that* $\chi_{\mathfrak{d}}(\ell) = w_\ell$ *for every prime* ℓ *dividing* N. *Then*

$$a_{|\mathfrak{d}|}(\theta_{1,Np,\mathfrak{d}}(f)) = 2^{1+\nu(N)}|\mathfrak{d}|\frac{L(f, \chi_{\mathfrak{d}}, 1)}{(2\pi i)\mathfrak{g}(\chi_{\mathfrak{d}})}.$$

Proof. It follows from Corollary 5.14, equation (32) (where $C(k, \mathfrak{d}) = 2$) and the fact that the family of p-adic periods can be chosen so that the relevant one is non-zero by Corollary 4.6. □

Remark 5.16. *This formula could have been obtained by adapting the classical computation of Proposition 2.4, choosing a suitable set of representatives for $\mathcal{L}_{Np}(N_0^2\mathfrak{d}^2)/\Gamma_0(Np)$ for the computation of $r_{k,Np,\chi}(f;\mathfrak{d},\mathfrak{d})$ (for this, one needs to use arguments similar to those used in Section 3 to classify integral binary quadratic forms). Instead, the previous corollary shows that such computation can be avoided if one disposes of a $\mathfrak{d}$-th Shintani lifting in p-adic families.*

5.4. **Classical points outside the interpolation region.** Continue to assume that $(\mathcal{R}_{\mathbf{f}},\mathcal{U}_{\mathbf{f}},\mathcal{U}_{\mathbf{f}}^{\mathrm{cl}},\mathbf{f})$ is a Hida family as usual, and let $\mathfrak{d}$ be a fundamental discriminant as in Theorem 5.13 (depending on a choice of r_0). We have seen in the previous paragraph how the $|\mathfrak{d}|$-th Fourier coefficient of $\Theta_{\mathfrak{d}}^{r_0}(\mathbf{f})$ interpolates special values of twisted L-series associated with classical specializations of the Hida family $\mathbf{f}$ on 'half' of the classical points. Namely, for classical points $\tilde\kappa\in\widetilde{\mathcal{U}}_{\mathbf{f}}^{\mathrm{cl}}(r_0)$ of weight $k-1$ the specializations $\mathbf{a}_{|\mathfrak{d}|}(\Theta_{\mathfrak{d}}^{r_0}(\mathbf{f}))(\tilde\kappa)$ interpolate the special values $L(f_{\pi(\tilde\kappa)},\chi_{\mathfrak{d}}\bar\chi_0,k)$ (see Proposition 5.11). This was just an immediate consequence of the fact that $\Theta_{\mathfrak{d}}^{r_0}(\mathbf{f})(\tilde\kappa)$ interpolates the $\mathfrak{d}$-th Shintani liftings of the forms $\mathbf{f}(\pi(\tilde\kappa))$ when varying $\tilde\kappa\in\widetilde{\mathcal{U}}_{\mathbf{f}}^{\mathrm{cl}}(r_0)$, together with the classical relation between Fourier coefficients of Shintani liftings and special L-values.

A natural question is: what about the values of $\mathbf{a}_{|\mathfrak{d}|}(\Theta_{\mathfrak{d}}^{r_0}(\mathbf{f}))$ at classical points $\tilde\kappa\in\widetilde{\mathcal{U}}_{\mathbf{f}}^{\mathrm{cl}}-\widetilde{\mathcal{U}}_{\mathbf{f}}^{\mathrm{cl}}(r_0)$? At those points, we cannot use Theorem 5.9 to relate $\Theta_{\mathfrak{d}}^{r_0}(\mathbf{f})(\tilde\kappa)$ to a classical $\mathfrak{d}$-th Shintani lifting. However, we can still use Theorem 5.13 to evaluate $\mathbf{a}_{|\mathfrak{d}|}(\Theta_{\mathfrak{d}}^{r_0}(\mathbf{f}))$ by evaluating $\tilde{\mathcal{L}}_p^{\mathrm{GS}}(\mathbf{f},\chi_{\mathfrak{d}}\chi_0\omega^{r_0-1})$. Indeed, first notice that if $\tilde\kappa\in\widetilde{\mathcal{U}}_{\mathbf{f}}^{\mathrm{cl}}$ does not belong to $\widetilde{\mathcal{U}}_{\mathbf{f}}^{\mathrm{cl}}(r_0)$, then $\tilde\kappa$ has weight $k-1$ for some integer k such that $k\equiv r_0+(p-1)/2$ modulo $p-1$. Then the character

$$\chi_{\mathfrak{d}}\chi_0\omega^{r_0-1}\omega^{1-k}=\chi_{\mathfrak{d}}\omega^{(p-1)/2}\chi_0=\chi_{\mathfrak{d}/p^*}\chi_0$$

has conductor $\mathrm{c}=|\mathfrak{d}|N_0/p$, where $p^*=(-1)^{(p-1)/2}p$. In particular, $m=0$ in the notation of the previous paragraph. If $\kappa=\pi(\tilde\kappa)$, then

$$\mathcal{E}_p(\mathbf{f}(\kappa),\chi_{\mathfrak{d}}\chi_0\omega^{r_0-1},k)=\left(1-\frac{\chi_{\mathfrak{d}/p^*}\chi_0(p)p^{k-1}}{a_p(\kappa)}\right),$$

and therefore we find

$$\mathbf{a}_{|\mathfrak{d}|}(\Theta_{\mathfrak{d}}^{r_0}(\mathbf{f}))(\tilde\kappa)=a_p(\kappa)\chi_{\mathfrak{d}}(-1)\mathbf{R}_{\mathfrak{d}}(\tilde\kappa)\Omega_\kappa\left(1-\frac{\chi_{\mathfrak{d}/p^*}\chi_0(p)p^{k-1}}{a_p(\kappa)}\right)\times$$
$$\times\frac{\mathrm{c}^{k-1}(k-1)!\mathfrak{g}(\chi_{\mathfrak{d}/p^*}\chi_0)}{(2\pi i)^k\Omega_{\mathbf{f}(\kappa)}^-}\cdot L(\mathbf{f}(\kappa),\chi_{\mathfrak{d}/p^*}\bar\chi_0,k).$$

Now, we have

$$\mathfrak{g}(\chi_{\mathfrak{d}/p^*}\chi_0)=\frac{\chi_{\mathfrak{d}/p^*}\chi_0(-1)|\mathfrak{d}|N_0}{p\mathfrak{g}(\chi_{\mathfrak{d}/p^*}\bar\chi_0)}=\frac{(-1)^k|\mathfrak{d}|N_0}{p\mathfrak{g}(\chi_{\mathfrak{d}/p^*}\bar\chi_0)},$$

where in the last equality we use that $\chi_{\mathfrak{d}/p^*}\chi_0(-1)=(-1)^{(p-1)/2}\chi_{\mathfrak{d}}(-1)\epsilon=(-1)^{r_0+(p-1)/2}=(-1)^k$. Hence we can rewrite the above identity as

$$\mathbf{a}_{|\mathfrak{d}|}(\Theta_{\mathfrak{d}}^{r_0}(\mathbf{f}))(\tilde\kappa)=\frac{a_p(\kappa)}{p^k}\left(1-\frac{\chi_{\mathfrak{d}/p^*}\chi_0(p)p^{k-1}}{a_p(\kappa)}\right)\cdot\mathbf{R}_{\mathfrak{d}}(\tilde\kappa)\Omega_\kappa\cdot\chi_{\mathfrak{d}}(-1)(-1)^k|\mathfrak{d}|^kN_0^k(k-1)!\times$$
$$\times\frac{L(\mathbf{f}(\kappa),\chi_{\mathfrak{d}/p^*}\bar\chi_0,k)}{(2\pi i)^k\mathfrak{g}(\chi_{\mathfrak{d}/p^*}\bar\chi_0)\Omega_{\mathbf{f}(\kappa)}^-}. \tag{35}$$

Remark 5.17. *This identity complements the interpolation formula for $\Theta_{\mathfrak{d}}^{r_0}(\mathbf{f})$ in (31) at classical weights in $\widetilde{\mathcal{U}}_{\mathbf{f}}^{\mathrm{cl}}-\widetilde{\mathcal{U}}_{\mathbf{f}}^{\mathrm{cl}}(r_0)$. In turn, this identity also suggests that the specializations of $\Theta_{\mathfrak{d}}^{r_0}(\mathbf{f})$ at classical points in $\widetilde{\mathcal{U}}_{\mathbf{f}}^{\mathrm{cl}}-\widetilde{\mathcal{U}}_{\mathbf{f}}^{\mathrm{cl}}(r_0)$ would be 'p-adic shadows' of the $\mathfrak{d}$-th Shintani liftings $\theta_{k,Np,\chi\omega^{(p-1)/2},\mathfrak{d}}(\mathbf{f}(\kappa))$. Note that in these $\mathfrak{d}$-th Shintani liftings the conductor of the character $\chi\omega^{(p-1)/2}$ is not relatively prime with $\mathfrak{d}$, and hence the discussion in Section 2 should be reformulated in order to define such liftings.*

The identity in (35) shows also that on the subset of classical points $\widetilde{\mathcal{U}}_{\mathbf{f}}^{\mathrm{cl}}-\widetilde{\mathcal{U}}_{\mathbf{f}}^{\mathrm{cl}}(r_0)$ one can find *exceptional zeroes* (precisely at those points where $\tilde{\mathcal{L}}_p^{\mathrm{GS}}(\mathbf{f},\chi_{\mathfrak{d}}\chi_0\omega^{r_0-1})$ has exceptional zeroes).

Indeed, it is apparent from (35) that for a classical point $\tilde{\kappa} \in \widetilde{\mathcal{U}}_{\mathbf{f}}^{\mathrm{cl}} - \widetilde{\mathcal{U}}_{\mathbf{f}}^{\mathrm{cl}}(r_0)$ of weight $k-1$, $\mathbf{a}_{|\mathfrak{d}|}(\Theta_{\mathfrak{d}}^{r_0}(\mathbf{f})(\tilde{\kappa})$ vanishes whenever

$$\chi_{\mathfrak{d}/p^*}\chi_0(p)p^{k-1} = a_p(\kappa).$$

When this holds, the order of vanishing of the $|\mathfrak{d}|$-th Fourier coefficient $a_{|\mathfrak{d}|}(\Theta_{\mathfrak{d}}^{r_0}(\mathbf{f}))$ at $\tilde{\kappa}$ is at least one more than the order of vanishing of the relevant classical special L-value. This extra vanishing is due to the p-adic interpolation.

To illustrate one of the settings in which exceptional zeroes arise, suppose that $E/\mathbf{Q}$ is an elliptic curve of conductor Np with multiplicative reduction at the prime p (thus $a_p(E) = \pm 1$ according to whether E has split or non-split multiplicative reduction at p, respectively). Let $f \in S_2(Np)$ be the normalized weight 2 newform (with rational coefficients) associated with E by modularity. Let also $\mathbf{f}$ be the Hida family passing through f at a classical point $\kappa \in \mathcal{U}_{\mathbf{f}}^{\mathrm{cl}}$ (thus $\mathbf{f}(\kappa) = f_\kappa = f$), and let $\tilde{\kappa} \in \widetilde{\mathcal{U}}_{\mathbf{f}}^{\mathrm{cl}}(1)$ be such that $\pi(\tilde{\kappa}) = \kappa$.

Let r_0 be an integer congruent to $1+(p-1)/2 = (p+1)/2$ modulo $p-1$, and $\mathfrak{d}$ be a fundamental discriminant divisible by p such that $(-1)^{r_0}\mathfrak{d} > 0$. Associated with this choice, consider the Λ-adic $\mathfrak{d}$-th Shintani lifting $\Theta_{\mathfrak{d}}^{r_0}(\mathbf{f})$. Then, equation (35) reads

$$\begin{aligned}\mathbf{a}_{|\mathfrak{d}|}(\Theta_{\mathfrak{d}}^{r_0}(\mathbf{f}))(\tilde{\kappa}) &= \chi_{\mathfrak{d}}(-1)\mathbf{R}_{\mathfrak{d}}(\tilde{\kappa})\Omega_\kappa(a_p(\kappa) - \chi_{\mathfrak{d}/p^*}(p))\frac{(-1)|\mathfrak{d}|N_0}{p} \cdot \frac{L(f_\kappa, \chi_{\mathfrak{d}/p^*}, 1)}{(2\pi i)\mathfrak{g}(\chi_{\mathfrak{d}/p^*})\Omega_{f_\kappa}^-} = \\ &= (a_p(E) - \chi_{\mathfrak{d}/p^*}(p)) \cdot \mathbf{R}_{\mathfrak{d}}(\tilde{\kappa})\Omega_\kappa \cdot (-\mathfrak{d})N_0 p^{-1} \cdot \frac{L(E, \chi_{\mathfrak{d}/p^*}, 1)}{(2\pi i)\mathfrak{g}(\chi_{\mathfrak{d}/p^*})\Omega_f^-},\end{aligned}$$

and we see that $\mathbf{a}_{|\mathfrak{d}|}(\Theta_{\mathfrak{d}}^{r_0}(\mathbf{f}))$ has an exceptional zero at $\tilde{\kappa}$ when $\chi_{\mathfrak{d}/p^*}(p) = a_p(E)$. We notice that under the assumption that $\chi_{\mathfrak{d}}(q) = w_{q^e}(f)$ for all primes q such that $q^e \mid\mid N$, the quantity $\mathbf{R}_{\mathfrak{d}}(\tilde{\kappa})$ equals $2^{\nu(N)}$, where $\nu(N)$ is the number of primes dividing N.

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