A few exceptional algebras

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Outline

1. Kac's 10-dimensional Jordan superalgebra and an exceptional simple modular Lie superalgebra
2. Composition superalgebras and more exceptional simple Lie superalgebras
3. From algebras to superalgebras via tensor categories

A few exceptional algebras

1. Kac's 10-dimensional Jordan superalgebra and an exceptional simple modular Lie superalgebra

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Classification of the simple Jordan superalgebras

V. Kac obtained in 1977 the classification of the simple finite-dimensional simple Jordan algebras, over algebraically closed fields of characteristic zero, from his classification of simple Lie superalgebras.

V. Kac
Classification of simple $\mathbb{Z}$-graded Lie superalgebras and simple Jordan superalgebras.
Communications in Algebra **5** (1977), no. 13, 1375-1400.

The exceptional Jordan superalgebra of Kac: K_{10}

There appeared an exceptional Jordan superalgebra of dimension $10 = 6 + 4$: K_{10}, related to the simple exceptional Lie superalgebra $F(4)$.

K_{10} is a simple Jordan superalgebra over any field of characteristic $\neq 2, 3$.

McCrimmon showed in 2005 that K_{10} is even more exceptional in characteristic 5:

> Over a field of characteristic 5, K_{10} satisfies the super-version of the Cayley-Hamilton equation of degree 3.

Tits construction (1966)

- $\mathcal{C}$ a composition algebra (analogue of real numbers, complex numbers, quaternions and octonions),
- $\mathcal{J}$ a central simple Jordan algebra satisfying the Cayley-Hamilton equation of degree 3,

then

$$\mathcal{T}(\mathcal{C}, \mathcal{J}) = \mathfrak{der}(\mathcal{C}) \oplus (\mathcal{C}_0 \otimes \mathcal{J}_0) \oplus \mathfrak{der}(\mathcal{J})$$

is a Lie algebra ($\mathrm{char} \neq 2, 3$) under a suitable Lie bracket:

$$[a \otimes x, b \otimes y] = \frac{1}{3} tr(xy) D_{a,b} + \Big([a, b] \otimes \big(xy - \frac{1}{3} tr(xy) 1\big)\Big) + 2t(ab) d_{x,y}.$$

Freudenthal Magic Square

$\mathcal{T}(\mathcal{C},\mathcal{J})$	$H_3(\mathbb{F})$	$H_3(\mathcal{K})$	$H_3(\mathcal{Q})$	$H_3(\mathcal{C})$
$\mathbb{F}$	A_1	A_2	C_3	F_4
$\mathcal{K}$	A_2	$A_2 \oplus A_2$	A_5	E_6
$\mathcal{Q}$	C_3	A_5	D_6	E_7
$\mathcal{C}$	F_4	E_6	E_7	E_8

($\mathcal{K}$ is a quadratic étale algebra, $\mathcal{Q}$ a quaternion algebra, and $\mathcal{C}$ a Cayley algebra.)

An exceptional Lie superalgebra in char. 5 (E. 2007)

In characteristic 5, the Jordan superalgebra K_{10} can be plugged in Tits construction:

$$\mathcal{T}(\mathcal{C}, K_{10}) = \mathfrak{der}(\mathcal{C}) \oplus (\mathcal{C}_0 \otimes (K_{10})_0) \oplus \mathfrak{der}(K_{10})$$

to get an exceptional simple Lie superalgebra of dimension $87 = 55 + 32$: $\mathfrak{el}(5; 5)$.

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Composition superalgebras

Definition

A superalgebra $\mathcal{C} = \mathcal{C}_{\bar{0}} \oplus \mathcal{C}_{\bar{1}}$, endowed with a regular quadratic superform $\mathsf{q} = (\mathsf{q}_{\bar{0}}, \mathsf{b})$, called the *norm*, is said to be a **composition superalgebra** in case

$$\mathsf{q}_{\bar{0}}(x_{\bar{0}}y_{\bar{0}}) = \mathsf{q}_{\bar{0}}(x_{\bar{0}})q_{\bar{0}}(y_{\bar{0}}),$$

$$\mathsf{b}(x_{\bar{0}}y, x_{\bar{0}}z) = \mathsf{q}_{\bar{0}}(x_{\bar{0}})\mathsf{b}(y, z) = \mathsf{b}(yx_{\bar{0}}, zx_{\bar{0}}),$$

$$\mathsf{b}(xy, zt) + (-1)^{|x||y|+|x||z|+|y||z|}\mathsf{b}(zy, xt) = (-1)^{|y||z|}\mathsf{b}(x, z)\mathsf{b}(y, t),$$

All the composition (super)algebras will be assumed to be unital.

Composition superalgebras: examples (Shestakov)

$$B(1,2) = \mathbb{F}1 \oplus V,$$

$\operatorname{char}\mathbb{F} = 3$, V a two-dimensional vector space with a nonzero alternating bilinear form $\langle .|.\rangle$, with

$$1x = x1 = x, \quad uv = \langle u|v\rangle 1, \qquad \mathsf{q}_{\bar{0}}(1) = 1, \quad \mathsf{b}(u,v) = \langle u|v\rangle,$$

is a composition superalgebra.

Composition superalgebras: examples (Shestakov)

$$B(4,2) = \mathrm{End}_{\mathbb{F}}(V) \oplus V,$$

$\mathbb{F}$ and V as before, $\mathrm{End}_{\mathbb{F}}(V)$ is equipped with the symplectic involution $f \mapsto \bar{f}$, ($\langle f(u)|v\rangle = \langle u|\bar{f}(v)\rangle$), the multiplication is given by:

- the usual multiplication (composition of maps) in $\mathrm{End}_{\mathbb{F}}(V)$,
- $v \cdot f = f(v) = \bar{f} \cdot v$ for any $f \in \mathrm{End}_k(V)$ and $v \in V$,
- $u \cdot v = \langle .|u\rangle v \ \big(w \mapsto \langle w|u\rangle v\big) \in \mathrm{End}_{\mathbb{F}}(V)$ for any $u, v \in V$,

and with quadratic superform

$$\mathsf{q}_{\bar{0}}(f) = \det f, \quad \mathsf{b}(u,v) = \langle u|v\rangle,$$

is a composition superalgebra.

Composition superalgebras: classification

Theorem

A composition superalgebra is either:

- *a composition algebra,*
- *a $\mathbb{Z}_2$-graded composition algebra in characteristic 2,*
- *isomorphic to either $B(1,2)$ or $B(4,2)$ in characteristic 3.*

A. Elduque and S. Okubo,
Composition superalgebras.
Communications in Algebra **30** (2002), no. 11, 5447–5471.

Symmetric construction of Freudenthal Magic Square

Let $\mathcal{C}$ be a composition algebra over a field $\mathbb{F}$ of characteristic not 2.

Its triality Lie algebra is

$$\mathfrak{tri}(\mathcal{C}) := \{(d_0, d_1, d_2) \in \mathfrak{so}(\mathcal{C})^3 \mid \\ d_0(x \bullet y) = d_1(x) \bullet y + x \bullet d_2(y)\ \forall x, y \in \mathcal{C}\}$$

with $x \bullet y = \overline{x}\,\overline{y}$. ($\overline{x}$ is the canonical involution.)

This is a Lie algebra with componentwise Lie bracket, and the cyclic permutation

$$\theta : (d_0, d_1, d_2) \mapsto (d_2, d_0, d_1)$$

is an automorphism (triality automorphism).

Symmetric construction of Freudenthal magic square

The vector space

$$\mathfrak{g}(\mathcal{C},\mathcal{C}') = \big(\mathfrak{tri}(\mathcal{C}) \oplus \mathfrak{tri}(\mathcal{C}')\big) \oplus \Big(\oplus_{i=0}^{2} \iota_i(\mathcal{C} \otimes \mathcal{C}')\Big),$$

where $\mathcal{C}$ and $\mathcal{C}'$ are composition algebras and $\iota_i(\mathcal{C} \otimes \mathcal{C}')$ is just a copy of $\mathcal{C} \otimes \mathcal{C}'$ $(i = 0, 1, 2)$, becomes a Lie algebra with:

- the Lie bracket in $\mathfrak{tri}(\mathcal{C}) \oplus \mathfrak{tri}(\mathcal{C}')$,
- $[(d_0, d_1, d_2), \iota_i(x \otimes x')] = \iota_i\big(d_i(x) \otimes x'\big)$,
- $[(d'_0, d'_1, d'_2), \iota_i(x \otimes x')] = \iota_i\big(x \otimes d'_i(x')\big)$,

Symmetric construction of Freudenthal magic square

- $[\iota_i(x \otimes x'), \iota_{i+1}(y \otimes y')] = \iota_{i+2}\big((x \bullet y) \otimes (x' \bullet y')\big)$,
- $[\iota_i(x \otimes x'), \iota_i(y \otimes y')] = \mathsf{q}'(x', y')\theta^i(t_{x,y}) + \mathsf{q}(x, y)\theta'^i(t'_{x',y'}) \in \mathfrak{tri}(\mathcal{C}) \oplus \mathfrak{tri}(\mathcal{C}')$,
  with $t_{x,y} := \Big(s_{x,y}, \frac{1}{2}\big(r_y l_x - r_x l_y\big), \frac{1}{2}\big(l_y r_x - l_x r_y\big)\Big)$, and
  $s_{x,y} : z \mapsto q(x,z)y - q(y,z)x$, $l_x : z \mapsto x \bullet z$, and $r_x : z \mapsto z \bullet x$.

Freudenthal magic square

$\dim \mathcal{C}'$

$\dim \mathcal{C}$

$\mathfrak{g}(\mathcal{C}, \mathcal{C}')$	1	2	4	8
1	A_1	A_2	C_3	F_4
2	A_2	$A_2 \oplus A_2$	A_5	E_6
4	C_3	A_5	D_6	E_7
8	F_4	E_6	E_7	E_8

(Vinberg, Allison-Faulkner, Barton-Sudbery, Landsberg-Manivel, E.)

Extended Freudenthal magic square in characteristic 3

The previous symmetric construction of Freudenthal magic square works if the composition algebras are replaced by composition superalgebras:

$\mathfrak{g}(\mathcal{C},\mathcal{C}')$	$\mathbb{F}$	$\mathcal{K}$	$\mathcal{Q}$	$\mathcal{C}$	$B(1,2)$	$B(4,2)$
$\mathbb{F}$	A_1	$\tilde{A}_2$	C_3	F_4	$6\vert 8$	$21\vert 14$
$\mathcal{K}$		$\tilde{A}_2 \oplus \tilde{A}_2$	$\tilde{A}_5$	$\tilde{E}_6$	$11\vert 14$	$35\vert 20$
$\mathcal{Q}$			D_6	E_7	$24\vert 26$	$66\vert 32$
$\mathcal{C}$				E_8	$55\vert 50$	$133\vert 56$
$B(1,2)$					$21\vert 16$	$36\vert 40$
$B(4,2)$						$78\vert 64$

(Cunha-E. 2007)

Lie superalgebras in the extended magic square

	$B(1,2)$	$B(4,2)$
$\mathbb{F}$	$\mathfrak{psl}_{2,2}$	$\mathfrak{sp}_6 \oplus (14)$
$\mathcal{K}$	$(\mathfrak{sl}_2 \oplus \mathfrak{pgl}_3) \oplus ((2) \otimes \mathfrak{psl}_3)$	$\mathfrak{pgl}_6 \oplus (20)$
$\mathcal{Q}$	$(\mathfrak{sl}_2 \oplus \mathfrak{sp}_6) \oplus ((2) \otimes (13))$	$\mathfrak{so}_{12} \oplus spin_{12}$
$\mathcal{C}$	$(\mathfrak{sl}_2 \oplus \mathfrak{f}_4) \oplus ((2) \otimes (25))$	$\mathfrak{e}_7 \oplus (56)$
$B(1,2)$	$\mathfrak{so}_7 \oplus 2spin_7$	$\mathfrak{sp}_8 \oplus (40)$
$B(4,2)$	$\mathfrak{sp}_8 \oplus (40)$	$\mathfrak{so}_{13} \oplus spin_{13}$

In this way one obtains a whole bunch of new simple exceptional modular simple Lie superalgebras in characteristic 3.

Together with $\mathfrak{el}(5;5)$ these account for most of the exceptional simple contragredient Lie superalgebras in low characteristics ($\neq 2$).

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Verlinde category

Let $\mathbb{F}$ be a field of characteristic p.

The category $\mathsf{Rep}\,\mathsf{C}_p$, whose objects are the finite-dimensional representations of the finite group C_p over $\mathbb{F}$, and whose morphisms are the equivariant homomorphisms, is a **symmetric tensor category**, with the usual tensor product of vector spaces and the braiding given by the usual swap: $X \otimes Y \to Y \otimes X$, $x \otimes y \mapsto y \otimes x$.

A homomorphism $f \in \mathrm{Hom}_{\mathsf{Rep}\,\mathsf{C}_p}(X, Y)$ is said to be **negligible** if for all homomorphisms $g \in \mathrm{Hom}_{\mathsf{Rep}\,\mathsf{C}_p}(Y, X)$, $\mathrm{tr}(fg) = 0$ holds. Denote by $\mathrm{N}(X, Y)$ the subspace of negligible homomorphisms in $\mathrm{Hom}_{\mathsf{Rep}\,\mathsf{C}_p}(X, Y)$.

Verlinde category

Negligible homomorphisms form a tensor ideal and this allows us to define the **semisimplification** of $\mathsf{Rep}\,\mathsf{C}_p$, which is the Verlinde category Ver_p, whose objects are the objects of $\mathsf{Rep}\,\mathsf{C}_p$, but whose morphisms are given by

$$\mathrm{Hom}_{\mathsf{Ver}_p}(X,Y) := \mathrm{Hom}_{\mathsf{Rep}\,\mathsf{C}_p}(X,Y)/\mathrm{N}(X,Y).$$

This is again a symmetric tensor category, with the tensor product in $\mathsf{Rep}\,\mathsf{C}_p$, and the braiding induced by the one in $\mathsf{Rep}\,\mathsf{C}_p$. Moreover, it is semisimple.

Superalgebras and Verlinde category

Theorem

The category sVec *of vector superspaces is equivalent to the full tensor subcategory of* Ver_p *generated by the one-dimensional and* $(p-1)$*-dimensional irreducible objects in* $\mathsf{Rep}\,\mathsf{C}_p$.

Consequence

Any order p automorphism of an algebra $\mathcal{A}$ allows to see $\mathcal{A}$ as an algebra in $\mathsf{Rep}\,\mathsf{C}_p$, which induces an algebra in the categories Ver_p and in sVec.
But an algebra in the category sVec is a superalgebra.
This superalgebra is said to be obtained by semisimplification of $\mathcal{A}$.

From the Albert algebra to Kac's superalgebra

The exceptional split central simple Jordan algebra, or Albert algebra, over a field of characteristic 5 is endowed with a suitable automorphism of order 5.

Theorem (E.-Etingof-Kannan 2024)

Over a field of characteristic 5, Kac's superalgebra K_{10} is obtained by semisimplification of the Albert algebra.

Corollary

The exceptional modular simple Lie superalgebra $\mathfrak{el}(5;5)$ is obtained by semisimplification of the exceptional simple Lie algebra E_8.

From octonions to composition superalgebras

The algebra of (split) octonions over fields of characteristic 3 is endowed with some very specific order 3 automorphisms.

Theorem (Daza-E.-Sayin 2024)

Over a field of characteristic 3, the composition superalgebras $B(1,2)$ and $B(4,2)$ are both obtained by semisimplification of the octonions.

Corollary

The exceptional modular simple Lie superalgebras in the extended Freudenthal magic square are all obtained by semisimplification of the exceptional simple Lie algebra E_8.

Thank you!