The Albert algebra as a twisted group algebra

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Nonassociative Algebra in Action:
Past, Present, and Future Perspectives

A conference in honor of Professor Kevin McCrimmon

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This means that the Albert algebra $\mathbb{A}$ may be described as the group algebra $\mathbb{F}[\mathbb{Z}_3^3]$, with "twisted" multiplication

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for a suitable map $\sigma : \mathbb{Z}_3^3 \times \mathbb{Z}_3^3 \to \mathbb{F}$.

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In this talk, the classification of fine gradings on the Albert algebra will be reviewed, and it will be shown that σ above can be chosen in a very symmetrical and simple form, which appeared for the first time in work of Griess (1990).

Definition

G abelian group, $\mathcal{A}$ algebra over a field $\mathbb{F}$.

G-grading on $\mathcal{A}$:

$$\mathcal{A} = \oplus_{g\in G}\mathcal{A}_g,$$

$$\mathcal{A}_g\mathcal{A}_h \subseteq \mathcal{A}_{gh} \qquad \forall g, h \in G.$$

Example: Pauli matrices

$\mathcal{A} = \mathrm{Mat}_n(\mathbb{F})$

$$X = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & \epsilon & 0 & \dots & 0 \\ 0 & 0 & \epsilon^2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \epsilon^{n-1} \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}$$

(ϵ a primitive nth root of 1)

$$X^n = 1 = Y^n, \qquad YX = \epsilon XY$$

$$\mathcal{A} = \oplus_{(\bar{\imath},\bar{\jmath})\in\mathbb{Z}_n\times\mathbb{Z}_n}\mathcal{A}_{(\bar{\imath},\bar{\jmath})}, \qquad \mathcal{A}_{(\bar{\imath},\bar{\jmath})} = \mathbb{F}X^iY^j.$$

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$\mathcal{A}$ becomes a *graded division algebra*.

Example: octonions

Cayley-Dickson process:

$$\begin{aligned} \mathbb{K} &= \mathbb{F} \oplus \mathbb{F}i, & i^2 &= -1, \\ \mathbb{H} &= \mathbb{K} \oplus \mathbb{K}j, & j^2 &= -1, \\ \mathbb{O} &= \mathbb{H} \oplus \mathbb{H}l, & l^2 &= -1, \end{aligned}$$

$\mathbb{O}$ is $\mathbb{Z}_2^3$-graded with

$$\deg(i) = (\bar{1}, \bar{0}, \bar{0}), \quad \deg(j) = (\bar{0}, \bar{1}, \bar{0}), \quad \deg(l) = (\bar{0}, \bar{0}, \bar{1}).$$

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- **Cartan grading:** $\mathfrak{g} = \mathfrak{h} \oplus \left(\oplus_{\alpha\in\Phi}\mathfrak{g}_\alpha\right)$
  (root space decomposition of a semisimple complex Lie algebra).
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- **Jordan systems** $\leftrightarrow$ **$\mathbb{Z}$-gradings**

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I. Kantor: “There are no Jordan algebras, there are only Lie algebras.”

K. McCrimmon: “Of course, this can be turned around: nine times out of ten, when you open up a Lie algebra you find a Jordan algebra inside which makes it tick.”

Conventions

In what follows:

- $\mathbb{F}$ will denote an algebraically closed ground field, $\operatorname{char}\mathbb{F} \neq 2$.
- The dimension of the algebras considered will always be finite.

Basic definitions

Let $\Gamma : \mathcal{A} = \oplus_{g\in G}\mathcal{A}_g$ and $\Gamma' : \mathcal{A} = \oplus_{g'\in G'}\mathcal{A}'_{g'}$ be two gradings on $\mathcal{A}$:

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- Γ is a *refinement* of Γ' if for any $g \in G$ there is a $g' \in G'$ such that $\mathcal{A}_g \subseteq \mathcal{A}_{g'}$.
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  Then Γ' is a *coarsening* of Γ.
- Γ is *fine* if it admits no proper refinement.
- Γ and Γ' are *equivalent* if there is an automorphism $\varphi \in \operatorname{Aut}\mathcal{A}$ such that for any $g \in G$ there is a $g' \in G'$ with $\varphi(\mathcal{A}_g) = \mathcal{A}'_{g'}$.

Theorem (E. 1998)

Up to equivalence, the fine gradings on $\mathbb{O}$ are

- *the Cartan grading (weight space decomposition relative to a Cartan subalgebra of $\mathfrak{g}_2 = \mathfrak{Der}(\mathbb{O})$), and*
- *the $\mathbb{Z}_2^3$-grading given by the Cayley-Dickson doubling process.*

Sketch of proof:

- The Cayley-Hamilton equation: $x^2 - n(x,1)x + n(x)1 = 0$, implies that the norm has a well behavior relative to the grading:

$$n(\mathbb{O}_g) = 0 \text{ unless } g^2 = e, \quad n(\mathbb{O}_g, \mathbb{O}_h) = 0 \text{ unless } gh = e.$$

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- If there is a $g \in G$ in the support with either $g \neq e \neq g^2$ or $\dim \mathbb{O}_g \geq 2$, there are elements $x \in \mathbb{O}_g$, $y \in \mathbb{O}_{g^{-1}}$ with $n(x) = 0 = n(y)$, $n(x,y) = 1$. Then $e_1 = x\bar{y}$ and $e_2 = y\bar{x}$ are orthogonal primitive idempotents in $\mathbb{O}_e$, and one uses the corresponding Peirce decomposition to check that, up to equivalence, our grading is the Cartan grading.

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- Otherwise $\dim \mathbb{O}_g = 1$ and $g^2 = e$ for any g in the support of the grading, and we get the $\mathbb{Z}_2^3$-grading.

Cartan grading on the Octonions

$\mathbb{O}$ contains canonical bases:

$$\mathcal{B} = \{e_1, e_2, u_1, u_2, u_3, v_1, v_2, v_3\}$$

with

$$n(e_1, e_2) = n(u_i, v_i) = 1, \quad \text{otherwise } 0.$$

$$e_1^2 = e_1, \quad e_2^2 = e_2,$$
$$e_1 u_i = u_i e_2 = u_i, \quad e_2 v_i = v_i e_1 = v_i, \quad (i = 1, 2, 3)$$
$$u_i v_i = -e_1, \quad v_i u_i = -e_2, \quad (i = 1, 2, 3)$$
$$u_i u_{i+1} = -u_{i+1} u_i = v_{i+2}, \; v_i v_{i+1} = -v_{i+1} v_i = u_{i+2}, \; \text{(indices modulo 3)}$$

otherwise 0.

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otherwise 0.

The Cartan grading is determined by:

$$\deg u_1 = -\deg v_1 = (1, 0), \quad \deg u_2 = -\deg v_2 = (0, 1),$$

Octonions as a twisted group algebra

Theorem (Albuquerque-Majid 1999)

The octonion algebra is the twisted group algebra

$$\mathbb{O} = \mathbb{F}_\sigma[\mathbb{Z}_2^3],$$

with

$$\sigma(\alpha, \beta) = (-1)^{\psi(\alpha,\beta)},$$

$$\psi(\alpha, \beta) = \beta_1\alpha_2\alpha_3 + \alpha_1\beta_2\alpha_3 + \alpha_1\alpha_2\beta_3 + \sum_{i\leq j} \alpha_i\beta_j.$$

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This allows to consider the algebra of octonions as an "associative algebra in a suitable category".

Albert algebra

$$\mathbb{A} = H_3(\mathbb{O}, *) = \left\{ \begin{pmatrix} \alpha_1 & \bar{a}_3 & a_2 \\ a_3 & \alpha_2 & \bar{a}_1 \\ \bar{a}_2 & a_1 & \alpha_3 \end{pmatrix} \mid \alpha_1, \alpha_2, \alpha_3 \in \mathbb{F},\ a_1, a_2, a_3 \in \mathbb{O} \right\}$$

$$= \mathbb{F}E_1 \oplus \mathbb{F}E_2 \oplus \mathbb{F}E_3 \oplus \iota_1(\mathbb{O}) \oplus \iota_2(\mathbb{O}) \oplus \iota_3(\mathbb{O}),$$

where

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad E_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\iota_1(a) = 2\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \bar{a} \\ 0 & a & 0 \end{pmatrix}, \quad \iota_2(a) = 2\begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ \bar{a} & 0 & 0 \end{pmatrix}, \quad \iota_3(a) = 2\begin{pmatrix} 0 & \bar{a} & 0 \\ a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

The multiplication in $\mathbb{A}$ is given by $X \circ Y = \frac{1}{2}(XY + YX)$.

Then E_i are orthogonal idempotents with $E_1 + E_2 + E_3 = 1$. The rest of the products are as follows:

$$E_i \circ \iota_i(a) = 0, \quad E_{i+1} \circ \iota_i(a) = \frac{1}{2}\iota_i(a) = E_{i+2} \circ \iota_i(a),$$

$$\iota_i(a) \circ \iota_{i+1}(b) = \iota_{i+2}(\bar{a}\bar{b}), \quad \iota_i(a) \circ \iota_i(b) = 2n(a, b)(E_{i+1} + E_{i+2}),$$

for any $a, b \in \mathbb{O}$, with $i = 1, 2, 3$ taken modulo 3.

$\mathbb{Z}_2^5$-grading:

$\mathbb{A}$ is naturally $\mathbb{Z}_2^2$-graded with

$$\mathbb{A}_{(\bar{0},\bar{0})} = \mathbb{F}E_1 + \mathbb{F}E_2 + \mathbb{F}E_3,$$
$$\mathbb{A}_{(\bar{1},\bar{0})} = \iota_1(\mathbb{O}), \qquad \mathbb{A}_{(\bar{0},\bar{1})} = \iota_2(\mathbb{O}), \qquad \mathbb{A}_{(\bar{1},\bar{1})} = \iota_3(\mathbb{O}).$$

Gradings on $\mathbb{A}$

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This $\mathbb{Z}_2^2$-grading may be combined with the fine $\mathbb{Z}_2^3$-grading on $\mathbb{O}$ to obtain a fine $\mathbb{Z}_2^5$-grading:

$$\deg E_i = (\bar{0},\bar{0},\bar{0},\bar{0},\bar{0}),\ \ i = 1,2,3,$$
$$\deg \iota_1(x) = (\bar{1},\bar{0},\deg x),\ \ \deg \iota_2(x) = (\bar{0},\bar{1},\deg x),\ \ \deg \iota_3(x) = (\bar{1},\bar{1},\deg x).$$

$\mathbb{Z}\times\mathbb{Z}_2^3$-grading:

Take an element $\mathbf{i}\in\mathbb{F}$ with $\mathbf{i}^2=-1$ and consider the following elements in $\mathbb{A}$:

$$
\begin{aligned}
&E=E_1,\ \widetilde{E}=1-E=E_2+E_3,\\
&\nu(a)=\mathbf{i}\iota_1(a)\quad\text{for all}\quad a\in\mathbb{O}_0,\\
&\nu_\pm(x)=\iota_2(x)\pm\mathbf{i}\iota_3(\bar{x})\quad\text{for all}\quad x\in\mathbb{O},\\
&S^\pm=E_3-E_2\pm\frac{\mathbf{i}}{2}\iota_1(1).
\end{aligned}
$$

$\mathbb{A}$ is then 5-graded:

$$\mathbb{A}=\mathbb{A}_{-2}\oplus\mathbb{A}_{-1}\oplus\mathbb{A}_0\oplus\mathbb{A}_1\oplus\mathbb{A}_2,$$

with $\mathbb{A}_{\pm 2}=\mathbb{F}S^\pm$, $\mathbb{A}_{\pm 1}=\nu_\pm(\mathbb{O})$, and $\mathbb{A}_0=\mathbb{F}E\oplus\Big(\mathbb{F}\widetilde{E}\oplus\nu(\mathbb{O}_0)\Big)$.

$\mathbb{Z} \times \mathbb{Z}_2^3$-grading:

The $\mathbb{Z}_2^3$-grading on $\mathbb{O}$ combines with this $\mathbb{Z}$-grading

$$\mathbb{A} = \mathbb{F}S^- \oplus \nu^-(\mathbb{O}) \oplus \mathbb{A}_0 \oplus \nu^+(\mathbb{O}) \oplus \mathbb{F}S^+$$

to give a fine $\mathbb{Z} \times \mathbb{Z}_2^3$-grading as follows:

$$\begin{aligned}
\deg S^{\pm} &= (\pm 2, \bar{0}, \bar{0}, \bar{0}),\\
\deg \nu_{\pm}(x) &= (\pm 1, \deg x),\\
\deg E &= 0 = \deg \widetilde{E},\\
\deg \nu(a) &= (0, \deg a),
\end{aligned}$$

for homogeneous elements $x \in \mathbb{O}$ and $a \in \mathbb{O}_0$.

$\mathbb{Z}_3^3$-grading (char $\mathbb{F} \neq 3$):

Consider the order three automorphism τ of $\mathbb{O}$:

$$\tau(e_i) = e_i,\ i = 1, 2, \quad \tau(u_j) = u_{j+1},\ \tau(v_j) = v_{j+1},\ j = 1, 2, 3,$$

and define a new multiplication on $\mathbb{O}$:

$$x * y = \tau(\bar{x})\tau^2(\bar{y}).$$

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$$x * y = \tau(\bar{x})\tau^2(\bar{y}).$$

This is the *Okubo algebra*, which is $\mathbb{Z}_3^2$-graded by setting

$$\deg e_1 = (\bar{1}, \bar{0}) \quad \text{and} \quad \deg u_1 = (\bar{0}, \bar{1}).$$

$\mathbb{Z}_3^3$-grading (char $\mathbb{F} \neq 3$):

Define $\tilde{\iota}_i(x) = \iota_i(\tau^i(x))$ for all $i = 1, 2, 3$ and $x \in \mathbb{O}$. Then the multiplication in the Albert algebra

$$\mathcal{A} = \oplus_{i=1}^3 \big(\mathbb{F}E_i \oplus \tilde{\iota}_i(\mathbb{O})\big)$$

becomes:

$$E_i^{\circ 2} = E_i, \quad E_i \circ E_{i+1} = 0,$$
$$E_i \circ \tilde{\iota}_i(x) = 0, \quad E_{i+1} \circ \tilde{\iota}_i(x) = \frac{1}{2}\tilde{\iota}_i(x) = E_{i+2} \circ \tilde{\iota}_i(x),$$
$$\tilde{\iota}_i(x) \circ \tilde{\iota}_{i+1}(y) = \tilde{\iota}_{i+2}(x * y), \quad \tilde{\iota}_i(x) \circ \tilde{\iota}_i(y) = 2n(x, y)(E_{i+1} + E_{i+2}),$$

for $i = 1, 2, 3$ and $x, y \in \mathbb{O}$.

$\mathbb{Z}_3^3$-grading (char $\mathbb{F} \neq 3$):

Assume now char $\mathbb{F} \neq 3$. Then the $\mathbb{Z}_3^2$-grading on the Okubo algebra is determined by two commuting order 3 automorphisms $\varphi_1, \varphi_2 \in \mathrm{Aut}(\mathbb{O}, *)$:

$$\begin{aligned} \varphi_1(e_1) &= \omega e_1, & \varphi_1(u_1) &= u_1, \\ \varphi_2(e_1) &= e_1, & \varphi_2(u_1) &= \omega u_1, \end{aligned}$$

where ω is a primitive third root of unity in $\mathbb{F}$.

$\mathbb{Z}_3^3$-grading (char $\mathbb{F} \neq 3$):

The commuting order 3 automorphisms φ_1, φ_2 of $(\mathbb{O}, *)$ extend to commuting order 3 automorphisms of $\mathbb{A}$:

$$\varphi_j(E_i) = E_i, \quad \varphi_j\big(\tilde{\iota}_i(x)\big) = \tilde{\iota}_i(\varphi_j(x)).$$

On the other hand, the linear map $\varphi_3 \in \mathrm{End}(\mathcal{A})$ defined by

$$\varphi_3(E_i) = E_{i+1}, \quad \varphi_3\big(\tilde{\iota}_i(x)\big) = \tilde{\iota}_{i+1}(x),$$

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The subgroup of $\mathrm{Aut}(\mathbb{A})$ generated by $\varphi_1, \varphi_2, \varphi_3$ is isomorphic to $\mathbb{Z}_3^3$ and induces a $\mathbb{Z}_3^3$-grading on $\mathbb{A}$.

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All the homogeneous components have dimension 1.

Theorem (Draper–Martín-González 2009 (char $= 0$), E.–Kochetov 2010)

Up to equivalence, the fine gradings of the Albert algebra are:

1. *The Cartan grading (weight space decomposition relative to a Cartan subalgebra of $\mathfrak{f}_4 = \mathfrak{Der}(\mathbb{A})$).*
2. *The $\mathbb{Z}_2^5$-grading obtained by combining the natural $\mathbb{Z}_2^2$-grading on 3×3 hermitian matrices with the fine grading over $\mathbb{Z}_2^3$ of $\mathbb{O}$.*
3. *The $\mathbb{Z} \times \mathbb{Z}_2^3$-grading obtained by combining a 5-grading and the $\mathbb{Z}_2^3$-grading on $\mathbb{O}$.*
4. *The $\mathbb{Z}_3^3$-grading with $\dim \mathbb{A}_g = 1 \ \forall g$ (char $\mathbb{F} \neq 3$).*

The fine gradings on $\mathfrak{f}_4 = \mathfrak{Der}(\mathbb{A})$ are the ones induced by these.

Gradings on $\mathbb{A}$

Theorem (Draper–Martín-González 2009 (char = 0), E.–Kochetov 2010)

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The fine gradings on $\mathfrak{f}_4 = \mathfrak{Der}(\mathbb{A})$ *are the ones induced by these.*

All the gradings up to isomorphism on $\mathbb{A}$ and $\mathfrak{f}_4$ have been classified too (E.–Kochetov).

In order to present the Albert algebra as a twisted group algebra:

$$\mathbb{A} = \mathbb{F}_\sigma[\mathbb{Z}_3^3],$$

in a nice way, i.e., with a simple expression for σ, it is important to look at the fine $\mathbb{Z}_3^3$-grading on $\mathbb{A}$ in a suitable way.

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Tits construction of the Albert algebra provides such a way.

Tits construction

Let $\mathcal{R} = \mathsf{Mat}_3(\mathbb{F})$. Then

$$\mathbb{A} = \mathcal{R}_0 \oplus \mathcal{R}_1 \oplus \mathcal{R}_2,$$

with $\mathcal{R}_0$, $\mathcal{R}_1$, $\mathcal{R}_2$ copies of $\mathcal{R}$.

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The product in $\mathbb{A}$ satisfies $\mathcal{R}_i \circ \mathcal{R}_j \subseteq \mathcal{R}_{i+j}$ (mod 3) and it is given by:

$\circ$	a'_0	b'_1	c'_2
a_0	$(a \circ a')_0$	$(\bar{a}b')_1$	$(c'\bar{a})_2$
b_1	$(\bar{a}'b)_1$	$(b \times b')_2$	$(\overline{bc'})_2$
c_2	$(c\bar{a}')_2$	$(\overline{b'c})_0$	$(c \times c')_1$

where

- $a \circ a' = \frac{1}{2}(aa' + a'a)$,
- $a \times b = a \circ b - \frac{1}{2}(\mathrm{tr}(a)b + \mathrm{tr}(b)a) + \frac{1}{2}\big(\mathrm{tr}(a)\,\mathrm{tr}(b) - \mathrm{tr}(ab)\big)1$,
- $\bar{a} = a \times 1 = \frac{1}{2}\big(\mathrm{tr}(a)1 - a\big)$.

Another look at the $\mathbb{Z}_3^3$-grading on $\mathbb{A}$

Assume $\operatorname{char}\mathbb{F} \neq 3$. Take Pauli matrices in $\mathcal{R}$:

$$x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

where ω, ω^2 are the primitive cubic roots of 1, which satisfy

$$x^3 = 1 = y^3, \quad yx = \omega xy.$$

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These Pauli matrices give a grading over $\mathbb{Z}_3^2$ on $\mathcal{R}$, with

$$\mathcal{R}_{(\alpha_1,\alpha_2)} = \mathbb{F}x^{\alpha_1}y^{\alpha_2}.$$

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This grading combines with the $\mathbb{Z}_3$-grading on $\mathbb{A}$ induced by Tits construction, to give the unique, up to equivalence, fine grading over $\mathbb{Z}_3^3$ of the Albert algebra.

Multiplication in Tits construction

For $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{Z}_3^3$ consider the element

$$Z^\alpha := (x^{\alpha_1} y^{\alpha_2})_{\alpha_3} \in \mathcal{R}_{\alpha_3} \subseteq \mathbb{A}.$$

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Then, for any $\alpha, \beta \in \mathbb{Z}_3^3$:

$$Z^\alpha \circ Z^\beta = \begin{cases} \omega^{\tilde{\psi}(\alpha,\beta)} Z^{\alpha+\beta} & \text{if } \dim_{\mathbb{Z}_3}(\mathbb{Z}_3\alpha + \mathbb{Z}_3\beta) \leq 1, \\ -\frac{1}{2}\omega^{\tilde{\psi}(\alpha,\beta)} Z^{\alpha+\beta} & \text{otherwise,} \end{cases}$$

where

$$\tilde{\psi}(\alpha, \beta) = (\alpha_2\beta_1 - \alpha_1\beta_2)(\alpha_3 - \beta_3) - (\alpha_1\beta_2 + \alpha_2\beta_1).$$

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Consider now the elements (Racine 1990, unpublished)

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$$W^{\alpha} \circ W^{\beta} = \omega^{-\alpha_1\alpha_2-\beta_1\beta_2} Z^{\alpha} \circ Z^{\beta}$$

$$= \begin{cases} \omega^{\tilde{\psi}(\alpha,\beta)-(\alpha_1\alpha_2+\beta_1\beta_2)} Z^{\alpha+\beta} & \text{if } \dim_{\mathbb{Z}_3}(\mathbb{Z}_3\alpha + \mathbb{Z}_3\beta) \leq 1, \\ -\frac{1}{2}\omega^{\tilde{\psi}(\alpha,\beta)-(\alpha_1\alpha_2+\beta_1\beta_2)} Z^{\alpha+\beta} & \text{otherwise,} \end{cases}$$

$$= \begin{cases} \omega^{\tilde{\psi}(\alpha,\beta)+(\alpha_1\beta_2+\alpha_2\beta_1)} W^{\alpha+\beta} & \text{if } \dim_{\mathbb{Z}_3}(\mathbb{Z}_3\alpha + \mathbb{Z}_3\beta) \leq 1, \\ -\frac{1}{2}\omega^{\tilde{\psi}(\alpha,\beta)+(\alpha_1\beta_2+\alpha_2\beta_1)} W^{\alpha+\beta} & \text{otherwise.} \end{cases}$$

The Albert algebra as a twisted group algebra

Theorem (Griess 1990)

The Albert algebra is, up to isomorphism, the twisted group algebra

$$\mathbb{A} = \mathbb{F}_\sigma[\mathbb{Z}_3^3],$$

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That's all. Thanks