Simple modular Lie superalgebras

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- Four infinite families: A_n, B_n, C_n, D_n,
- Five exceptional algebras: E_6, E_7, E_8, F_4, G_2.

E_6,	E_7,	E_8,	F_4,	G_2.
78,	133,	248,	52,	14.

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$F_4 = \mathfrak{der}\,H_3(\mathbb{O})$

$E_6 = \mathfrak{str}_0\,H_3(\mathbb{O})$

(Chevalley-Schafer 1950)

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$$\mathcal{T}(C, J) = \mathfrak{Der}\, C \oplus (C_0 \otimes J_0) \oplus \mathfrak{Der}\, J$$

is a Lie algebra (char $\neq 2, 3$) under a suitable Lie bracket:

$$[a \otimes x, b \otimes y] = \frac{1}{3} tr(xy) D_{a,b} + \Big([a, b] \otimes \big(xy - \frac{1}{3} tr(xy)1\big)\Big) + 2t(ab) d_{x,y}.$$

Freudenthal's Magic Square

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$\mathcal{T}(C, J)$	$H_3(k)$	$H_3(k \times k)$	$H_3(\mathrm{Mat}_2(k))$	$H_3(C(k))$
k	A_1	A_2	C_3	F_4
$k \times k$	A_2	$A_2 \oplus A_2$	A_5	E_6
$\mathrm{Mat}_2(k)$	C_3	A_5	D_6	E_7
$C(k)$	F_4	E_6	E_7	E_8

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(This is valid too in characteristic 3.)

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(This result fails in characteristic 3.)

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$$[e_i, f_j] = \delta_{ij} h_i, \quad [h_i, h_j] = 0, \quad [h_i, e_j] = a_{ij} e_j, \quad [h_i, f_j] = -a_{ij} f_j.$$

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Definition

$\mathfrak{g}(A) = \tilde{\mathfrak{g}}(A)/\mathfrak{j}(A)$ is the *contragredient Lie algebra with Cartan matrix A*.

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- *A one-parametric family of 10-dimensional algebras $L(\epsilon)$ ($\epsilon \neq 0$) first considered by Kostrikin ($L(-1) = \mathfrak{sp}_4$) in characteristic 3:*
  $L(\epsilon) = \big(\mathfrak{sl}_2 \oplus \mathfrak{sl}_2\big) \oplus \big((2) \otimes (2)\big)$.

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  $L(\epsilon) = \big(\mathfrak{sl}_2 \oplus \mathfrak{sl}_2\big) \oplus \big((2) \otimes (2)\big)$.
- *A 29-dimensional algebra discovered by Brown in 1982:*
  $\mathfrak{br}_{29} = \big(\mathfrak{sl}_2 \oplus L(1)\big) \oplus \big((2) \otimes (8)\big)$.

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for any $x, y, z \in \mathfrak{g}_{\bar{0}} \cup \mathfrak{g}_{\bar{1}}$.

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Models of the exceptional simple Lie superalgebras

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It makes sense to replace Jordan algebras by *Jordan superalgebras* in Tits construction:

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Only a few simple Jordan superalgebras work.

A Supermagic Rectangle

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$\mathcal{T}(C,J)$	$H_3(k)$	$H_3(k\times k)$	$H_3(\mathrm{Mat}_2(k))$	$H_3(C(k))$	$J(V)$	D_t	K_{10}
k	A_1	A_2	C_3	F_4	A_1	$B(0,1)$	$B(0,1)\oplus B(0,1)$
$k\times k$	A_2	$A_2\oplus A_2$	A_5	E_6	$B(0,1)$	$A(1,0)$	$C(3)$
$\mathrm{Mat}_2(k)$	C_3	A_5	D_6	E_7	$B(1,1)$	$D(2,1;t)$	$F(4)$
$C(k)$	F_4	E_6	E_7	E_8	$G(3)$	$F(4)$ $(t=2)$	$\mathcal{T}(C(k),K_{10})$ (char 5)

Composition superalgebras

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Theorem (E., Okubo, Shestakov)

In characteristic 3*, and only in this characteristic, besides the classical composition algebras (of dimension* 1, 2, 4 *or* 8*), there appear exactly two composition superalgebras* $C = C_{\bar{0}} \oplus C_{\bar{1}}$*:*

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- $B(1,2)$ *of dimension* $3 = 1 + 2$.
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These superalgebras can be plugged in the more symmetric construction of Freudenthal's Magic Square:

$$\mathfrak{g}(C, C') = \bigl(\mathfrak{tri}(C) \oplus \mathfrak{tri}(C')\bigr) \oplus \bigl(\oplus_{i=0}^{2} \iota_i(C \otimes C')\bigr).$$

Supermagic Square (char 3, Cunha-E. 2007)

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$\mathfrak{g}(C, C')$	k	$k \times k$	$\mathrm{Mat}_2(k)$	$C(k)$	$B(1,2)$	$B(4,2)$
k	A_1	$\tilde{A}_2$	C_3	F_4	$6\|8$	$21\|14$
$k \times k$		$\tilde{A}_2 \oplus \tilde{A}_2$	$\tilde{A}_5$	$\tilde{E}_6$	$11\|14$	$35\|20$
$\mathrm{Mat}_2(k)$			D_6	E_7	$24\|26$	$66\|32$
$C(k)$				E_8	$55\|50$	$133\|56$
$B(1,2)$					$21\|16$	$36\|40$
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Notation: $\mathfrak{g}(n, m)$ will denote the superalgebra $\mathfrak{g}(C, C')$, with $\dim C = n$, $\dim C' = m$.

Lie superalgebras in the Supermagic Square

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$\mathfrak{g}(C, C')$	$B(1,2)$	$B(4,2)$
k	$\mathfrak{psl}_{2,2}$	$\mathfrak{sp}_6 \oplus (14)$
$k \times k$	$(\mathfrak{sl}_2 \oplus \mathfrak{pgl}_3) \oplus ((2) \otimes \mathfrak{psl}_3)$	$\mathfrak{pgl}_6 \oplus (20)$
$\mathrm{Mat}_2(k)$	$(\mathfrak{sl}_2 \oplus \mathfrak{sp}_6) \oplus ((2) \otimes (13))$	$\mathfrak{so}_{12} \oplus spin_{12}$
$C(k)$	$(\mathfrak{sl}_2 \oplus \mathfrak{f}_4) \oplus ((2) \otimes (25))$	$\mathfrak{e}_7 \oplus (56)$
$B(1,2)$	$\mathfrak{so}_7 \oplus 2spin_7$	$\mathfrak{sp}_8 \oplus (40)$
$B(4,2)$	$\mathfrak{sp}_8 \oplus (40)$	$\mathfrak{so}_{13} \oplus spin_{13}$

Lie superalgebras in the Supermagic Square

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Only $\mathfrak{g}(1,3) \simeq \mathfrak{psl}_{2,2}$ has a counterpart in Kac's classification in characteristic 0. The other Lie superalgebras in the Supermagic Square, or their derived algebras, are new simple Lie superalgebras, specific of characteristic 3.

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The simple Lie superalgebra $\mathfrak{g}(2,3)' = [\mathfrak{g}(2,3), \mathfrak{g}(2,3)]$ can be obtained too through another variation of Tits construction.

Simple modular Lie superalgebras with a Cartan matrix

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The finite dimensional modular Lie superalgebras with indecomposable symmetrizable Cartan matrices (or *contragredient Lie superalgebras*) over algebraically closed fields have been classified by Bouarroudj, Grozman and Leites, under some extra technical hypotheses.

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For characteristic $p \geq 3$, apart from the Lie superalgebras obtained as the analogues of the Lie superalgebras in the classification in characteristic 0, by reducing the Cartan matrices modulo p, there are only the following exceptions:

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2. Two exceptions in characteristic 5: $\mathfrak{br}(2;5)$ and $\mathfrak{el}(5;5)$. (Dimensions $10|12$ and $55|32$.)

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1. The family of exceptions given by the Lie superalgebras in the Supermagic Square in characteristic 3.

2. Two exceptions in characteristic 5: $\mathfrak{br}(2;5)$ and $\mathfrak{el}(5;5)$. (Dimensions $10|12$ and $55|32$.)

3. Another two exceptions in characteristic 3, similar to the ones in characteristic 5: $\mathfrak{br}(2;3)$ and $\mathfrak{el}(5;3)$. (Dimensions $10|8$ and $39|32$.)

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The superalgebra $\mathfrak{el}(5;3)$ lives (as a natural maximal subalgebra) in the Lie superalgebra $\mathfrak{g}(3,8)$ of the Supermagic Square as follows:

$$\mathfrak{el}(5;3)_{\bar{0}} = \mathfrak{sl}_2 \oplus \mathfrak{so}_9 \leq \mathfrak{sl}_2 \oplus \mathfrak{f}_4 = \mathfrak{g}(3,8)_{\bar{0}}$$

$$\mathfrak{el}(5;3)_{\bar{1}} = (2) \otimes \mathit{spin}_9 \leq \mathfrak{g}(3,8)_{\bar{1}}.$$

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The last exception $\mathfrak{br}(2;5)$ satisfies:

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That's all. Thanks