Codes, S-structures, and exceptional Lie algebras

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Codes: There is a well-known connection between binary linear codes and root lattices. In particular, the E_8 root lattice is obtained from the extended Hamming $[8, 4, 4]$ binary linear code, and the E_7 root lattice from the simplex $[7, 3, 4]$ binary linear code, dual to Hamming's $[7, 4, 3]$ code.

Recently, a new class of commutative nonassociative algebras, called **code algebras** have been defined. These algebras contain a family of orthogonal idempotents and a nice 'Peirce decomposition' relative to this family.

Code algebras are inspired by some axiomatic approaches to Vertex Operator Algebras.

S-**structures:** Vinberg has introduced recently the notion of S-structure in a Lie algebra, as an extension of the notion of grading by an abelian group.

Given an S-structure in a Lie algebra, the **isotypic decomposition** relative to the action of the reductive group S provides a description of the Lie algebra in terms of a nonassociative system (algebra, pair, triple system, ...) that **coordinatizes** the Lie algebra.

Something similar happens for root graded Lie algebras, a subject initiated by Berman and Moody.

Exceptional Lie algebras: Killing-Cartan classification of the finite-dimensional simple Lie algebras over $\mathbb{C}$ includes four families of **classical** Lie algebras, and five **exceptional Lie algebras**: G_2, F_4, E_6, E_7, E_8.

With the exception of G_2, these algebras can be constructed in a unified way by means of a couple of symmetric composition algebras and their triality Lie algebras. A further refinement provides a description of F_4, E_7, and E_8 in terms of very simple components: copies of the 3-dimensional simple Lie algebra and of its 2-dimensional simple representation.

It turns out that these descriptions can be recast in terms of **optimal short SL_2^n-structures** in the corresponding Lie algebras.

The coordinate algebras that appear are quite close to code algebras. Not surprisingly, for E_7 and E_8 the codes involved are the simplex and the extended Hamming binary linear codes.

Outline

1. Codes and root lattices
2. S-structures
3. Exceptional Lie algebras

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Root lattices

Let $\Gamma \subset \mathbb{R}^n$ be an even lattice, i.e., a lattice such that $\mathbf{x}^{\bullet 2} \in 2\mathbb{Z}$ for all $\mathbf{x} \in \Gamma$.
The **roots** of Γ are the elements $\mathbf{x} \in \Gamma$ such that $\mathbf{x}^{\bullet 2} = 2$.
The even lattice Γ is said to be a **root lattice** if its set of roots spans Γ.
Every root lattice Γ is the orthogonal direct sum of the irreducible root lattices corresponding to the simply laced Dynkin diagrams.

Given a binary linear code $\mathbf{C} \subseteq \mathbb{F}_2^n$, consider the reduction modulo 2 map

$$\rho : \mathbb{Z}^n \longrightarrow (\mathbb{Z}/2)^n = \mathbb{F}_2^n .$$

This is a group homomorphism and $\Gamma_{\mathbf{C}} := \frac{1}{\sqrt{2}}\rho^{-1}(\mathbf{C})$ is a lattice in $\mathbb{R}^n$.

Codes and root lattices

Theorem

Let $\Gamma \subset \mathbb{R}^n$ be an irreducible root lattice. Then the following statements are equivalent:

(i) $\Gamma = \Gamma_{\mathbf{C}}$ *for a binary linear code* $\mathbf{C} \subseteq \mathbb{F}_2^n$.

(ii) Γ *contains* n *pairwise orthogonal roots.*

(iii) $nA_1 = A_1 \oplus \cdots \oplus A_1$ *is a sublattice of* Γ.

(iv) $-1 \in W(\Gamma)$.

(v) $2\Gamma^* \subseteq \Gamma$, *where* Γ^* *is the dual lattice* $\Gamma^* := \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} \bullet \mathbf{y} \in \mathbb{Z}\ \forall \mathbf{y} \in \Gamma\}$.

(vi) Γ *is of type* A_1, D_{2n} $(n \geq 2)$, E_7 *or* E_8.

Simplex $[7, 3, 4]$-code and E_7

The **simplex** $[7, 3, 4]$ binary linear code $\mathbf{C}$ is the dual of the Hamming $[7, 4, 3]$-code.

A generator matrix is :

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

The corresponding root lattice is E_7.

Extended Hamming $[8,4,4]$-code and E_8

The Hamming $[7,4,3]$ binary linear code is defined on $\mathbb{F}_2^7$ by the parity check relations:

$$\begin{aligned} c_1 + c_3 + c_5 + c_7 &= 0 \\ c_2 + c_3 + c_6 + c_7 &= 0 \\ c_4 + c_5 + c_6 + c_7 &= 0 \end{aligned}$$

We add one extra dimension and the extra global parity check

$$c_0 + c_1 + c_2 + c_3 + c_4 + c_5 + c_6 + c_7 = 0$$

to get the *extended Hamming* $[8,4,4]$ *binary linear code* with generator matrix:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

The corresponding root lattice is E_8.

Code algebras

Definition (Castillo-Ramírez et al.)

Let $\mathbf{C} \subseteq \mathbb{F}_2^n$ be a binary linear code. A *code algebra* based on $\mathbf{C}$ is a commutative algebra over a field $\mathbb{F}$, endowed with a basis

$$\{t_i \mid i = 1, \ldots, n\} \cup \{e^{\mathbf{c}} \mid \mathbf{c} \in \mathbf{C} \setminus \{\mathbf{0}, \mathbf{1}\}\}$$

that satisfies the following relations:

$$t_i t_j = \begin{cases} t_i & \text{if } i = j, \\ 0 & \text{otherwise,} \end{cases} \qquad t_i e^{\mathbf{c}} \in \mathbb{F} e^{\mathbf{c}},$$

$$e^{\mathbf{c}} e^{\mathbf{d}} \in \mathbb{F} e^{\mathbf{c}+\mathbf{d}}, \text{ for } \mathbf{c} \neq \mathbf{d}, \mathbf{1} - \mathbf{d},$$

$$(e^{\mathbf{c}})^2 \in \sum_{i \in \mathrm{supp}(\mathbf{c})} \mathbb{F} t_i, \qquad e^{\mathbf{c}} e^{\mathbf{1}-\mathbf{c}} = 0.$$

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Gradings

Given a finitely generated abelian group G and a finite-dimensional nonassociative algebra $\mathcal{A}$ (over $\mathbb{C}$), a ***G*-grading** on $\mathcal{A}$ is a vector space decomposition

$$\mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g$$

with $\mathcal{A}_{g_1}\mathcal{A}_{g_2} \subseteq \mathcal{A}_{g_1 g_2}$ for any $g_1, g_2 \in G$.

This is equivalent to a homomorphism

$$\widehat{G} \to \mathrm{Aut}(\mathcal{A})$$

where $\mathcal{A}_g = \{x \in \mathcal{A} \mid \chi \cdot x = \chi(g)x \quad \forall \chi \in \widehat{G}\}$.

Remark

Over arbitrary fields we must replace $\widehat{G}$ by the corresponding diagonalizable group scheme, and $\mathrm{Aut}(\mathcal{A})$ by $\mathbf{Aut}(\mathcal{A})$.

S-structures

Definition (Vinberg 2017)

Given a reductive algebraic group S and a finite-dimensional nonassociative algebra $\mathcal{A}$, an S-structure in a nonassociative algebra $\mathcal{A}$ is a homomorphism $\Phi : \mathsf{S} \to \mathrm{Aut}(\mathcal{A})$.

In this case, we get the corresponding **isotypic decomposition**

$$\mathcal{A} = \bigoplus_{V \in \mathrm{Irr}(\mathsf{S})} \mathcal{A}_V,$$

where $\mathcal{A}_V$ is the sum of the irreducible submodules of $\mathcal{A}$ isomorphic to V.

Vinberg thinks of S-structures as nonabelian gradings, in the sense that the abelian group $\widehat{G}$ is replaced by S.

Very short SL_2-structures

A nontrivial SL_2-structure Φ in a Lie algebra $\mathfrak{g}$ is called **very short** if the representation Φ decomposes into 1- and 3-dimensional irreducible representations.

In a semisimple Lie algebra $\mathfrak{g}$, a very short SL_2-structure gives rise to an isotypic decomposition of the form

$$\mathfrak{g} = \left(\mathfrak{sl}_2 \otimes \mathcal{J}\right) \oplus \mathfrak{der}(\mathcal{J})$$

for a semisimple Jordan algebra $\mathcal{J}$.

Short SL_3-structures

A nontrivial SL_3-structure in a simple Lie algebra $\mathfrak{g}$ is called **short** if the representation Φ decomposes into the adjoint representation of SL_3 and 1- and 3-dimensional irreducible representations.

In this case, the Lie algebra $\mathfrak{g}$ can also be described in terms of a cubic Jordan algebra $\mathcal{J}$:

$$\mathfrak{g} = \mathfrak{sl}_3 \oplus (V \otimes \mathcal{J}) \oplus (V^* \otimes \mathcal{J}) \oplus \mathfrak{str}_0(\mathcal{J}).$$

SL_2^n-structures

Definition

Let $\mathfrak{g}$ be a simple Lie algebra, and let $n \in \mathbb{N}$.

- An SL_2^n-structure $\Phi : \mathrm{SL}_2^n \longrightarrow \mathrm{Aut}(\mathfrak{g})$ is called **short** if the representation Φ decomposes into the adjoint representation of SL_2^n, irreducible representations formed by tensor products of the 2-dimensional natural representations of some of the copies of SL_2 (without repetitions), and 1-dimensional representations.
- A short SL_2^n-structure is said to be **optimal** if $n = \mathrm{rank}(\mathfrak{g})$.

Short SL_2^n-structures

Given an SL_2^n-structure in a Lie algebra $\mathfrak{g}$, let V_i be the 2-dimensional irreducible representation for the i^{th} factor in SL_2^n.

Given any $\mathbf{c} \in \mathbb{F}_2^n$, denote by $V^{\mathbf{c}}$ the SL_2^n-module obtained as the tensor product of the V_i's with $i \in \mathrm{supp}(\mathbf{c})$.

Thus, for example, with $n = 8$ and $\mathbf{c} = (1,0,0,1,0,1,1,0)$,

$$V^{\mathbf{c}} = V_1 \otimes V_4 \otimes V_6 \otimes V_7.$$

In particular, $V^{\mathbf{0}} = \mathbb{F}$ is the 1-dimensional trivial representation.

Short SL_2^n-structures

Isotypic decomposition

If $\Phi : \mathrm{SL}_2^n \longrightarrow \mathrm{Aut}(\mathfrak{g})$ is a short SL_2^n-structure in $\mathfrak{g}$, the isotypic decomposition of $\mathfrak{g}$ is of the form:

$$\mathfrak{g} = \mathfrak{sl}_2^n \oplus \Big(\bigoplus_{\mathbf{c} \in \mathbb{F}_2^n \setminus \{\mathbf{0}\}} (V^{\mathbf{c}} \otimes \mathcal{A}^{\mathbf{c}}) \Big) \oplus \mathfrak{c}$$

where

- the subalgebra $\mathfrak{sl}_2^n$ (the image of $\mathrm{d}\Phi$) is the adjoint representation of SL_2^n,
- the $\mathcal{A}^{\mathbf{c}}$'s are vector spaces whose dimension indicates the multiplicity of $V^{\mathbf{c}}$, and
- $\mathfrak{c}$ is the sum of the 1-dimensional representations, so that $\mathfrak{c}$ is the centralizer in $\mathfrak{g}$ of the subalgebra $\mathfrak{sl}_2^n$ and, as such, it is a subalgebra of $\mathfrak{g}$.

Short SL_2^n-structures

For each $i = 1, \dots, n$, let $\{e_i, f_i, h_i\}$ be a standard basis of the i^{th} copy of $\mathfrak{sl}_2$: $[h_i, e_i] = 2e_i$, $[h_i, f_i] = -2f_i$, $[e_i, f_i] = h_i$.

A short SL_2^n-structure in the simple Lie algebra $\mathfrak{g}$ is given then by a subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}_2^n$, such that the eigenvalues of the adjoint map $\operatorname{ad} h_i$ are ± 2 with multiplicity 1, and ± 1 and 0, because the eigenvalues of $\operatorname{ad} h_i$ on V_i are ± 1.

The subspace $\mathbb{F}h_1 \oplus \cdots \oplus \mathbb{F}h_n$ is a toral subalgebra of $\mathfrak{g}$, and hence contained in a Cartan subalgebra

$$\mathfrak{h} = \mathbb{F}h_1 \oplus \cdots \oplus \mathbb{F}h_n \oplus (\mathfrak{h} \cap \mathfrak{c}).$$

The linear map $\alpha_i : \mathfrak{h} \to \mathbb{F}$ given by

$$\alpha_i(h_i) = 2, \quad \alpha_i(h_j) = 0 \text{ if } i \neq j, \quad \alpha_i(\mathfrak{h} \cap \mathfrak{c}) = 0,$$

is a root of $\mathfrak{h}$ with root space $\mathfrak{g}_{\alpha_i} = \mathbb{F}e_i$.

Short SL_2^n-structures

Theorem

Let $\Phi : \mathrm{SL}_2^n \longrightarrow \mathrm{Aut}(\mathfrak{g})$ *be a short* SL_2^n*-structure in the simple Lie algebra* $\mathfrak{g}$*. Let* $\mathfrak{h}$ *be the Cartan subalgebra and* $\alpha_1, \dots, \alpha_n$ *be the roots above. Then* $\{\alpha_1, \dots, \alpha_n\}$ *is a set of pairwise orthogonal long roots.*

Conversely, if $\mathfrak{h}$ *is a Cartan subalgebra of the simple Lie algebra* $\mathfrak{g}$ *with associated root system* R*, and if* $\{\alpha_1, \dots, \alpha_n\}$ *is a set of pairwise orthogonal long roots in* R*, then*

$$\mathfrak{s}_i = \mathfrak{g}_{\alpha_i} \oplus \mathfrak{g}_{-\alpha_i} \oplus [\mathfrak{g}_{\alpha_i}, \mathfrak{g}_{-\alpha_i}]$$

is a Lie subalgebra isomorphic to $\mathfrak{sl}_2$*,* $[\mathfrak{s}_i, \mathfrak{s}_j] = 0$ *for* $i \neq j$*, and the embedding*

$$\mathfrak{sl}_2^n \simeq \mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_n \hookrightarrow \mathfrak{g}$$

integrates to a short SL_2^n*-structure.*

Optimal short SL_2^n-structures

Corollary

Let $\mathfrak{g}$ be a simple Lie algebra. Then $\mathfrak{g}$ admits an optimal short SL_2^n-structure if and only if $\mathfrak{g}$ is of type A_1, B_{2n} $(n \geq 2)$, C_n $(n \geq 2)$, D_{2n} $(n \geq 2)$, E_7, E_8, or F_4.

Any two optimal short SL_2^n-structures of $\mathfrak{g}$ are conjugate by an automorphism.

Given an optimal short SL_2^n-structure in a simple Lie algebra $\mathfrak{g}$, we get $\mathfrak{c} = 0$, and $\dim \mathcal{A}^{\mathbf{c}} \leq 1$ for any $\mathbf{c} \in \mathbb{F}_2^n \setminus \{\mathbf{0}\}$, because the multiplicity of any root is 1.

The isotypic decomposition above may be rewritten as:

$$\mathfrak{g} = \Big(\bigoplus_{i=1}^{n}(\mathfrak{sl}(V_i) \otimes \mathbb{F}t_i)\Big) \oplus \Big(\bigoplus_{\mathbf{c}\in\mathbf{S}}(V^{\mathbf{c}} \otimes \mathbb{F}e^{\mathbf{c}})\Big)$$

for some subset $\mathbf{S} \subseteq \mathbb{F}_2^n \setminus \{\mathbf{0}\}$.

Lemma

Let V be a 2-dimensional vector space endowed with a nonzero skew-symmetric bilinear form $\langle . \mid . \rangle$. Then:

- $\mathrm{Hom}_{\mathrm{SL}(V)}(V \otimes V, \mathbb{F})$ *is spanned by* $\langle . \mid . \rangle$.
- $\mathrm{Hom}_{\mathrm{SL}(V)}(\mathfrak{sl}(V) \otimes V, V)$ *is spanned by the natural action of* $\mathfrak{sl}(V)$ *on* V.
- $\mathrm{Hom}_{\mathrm{SL}(V)}(V \otimes V, \mathfrak{sl}(V))$ *is spanned by the map* $u \otimes v \mapsto \left(s_{u,v} : w \mapsto \frac{1}{2}\big(\langle w \mid u \rangle v + \langle w \mid v \rangle u\big)\right)$.
- $\mathrm{Hom}_{\mathrm{SL}(V)}(\mathfrak{sl}(V) \otimes \mathfrak{sl}(V), V) = 0 = \mathrm{Hom}_{\mathrm{SL}(V)}(V \otimes V, V)$.

Hence given $\mathbf{c} \neq \mathbf{d} \in \mathbb{F}_2^n \setminus \{\mathbf{0}\}$, there is a unique, up to scalars, nonzero bilinear map

$$\varphi_{\mathbf{c},\mathbf{d}} : V^{\mathbf{c}} \times V^{\mathbf{d}} \longrightarrow V^{\mathbf{c}+\mathbf{d}}$$

invariant under the action of $\mathrm{SL}_2^n = \mathrm{SL}(V_1) \times \cdots \times \mathrm{SL}(V_n)$, and this is given by contraction on the 'common indices':

$$\varphi_{(1,1,1,0),(1,0,1,1)}\big(u_1 \otimes u_2 \otimes u_3, v_1 \otimes v_3 \otimes v_4\big) = \langle u_1 \mid v_1\rangle\langle u_3 \mid v_3\rangle u_2 \otimes v_4,$$

for $u_i, v_i \in V_i$, $1 \leq i \leq 4$.

In the same vein, for $\mathbf{c} \in \mathbb{F}_2^n \setminus \{\mathbf{0}\}$ with 1 in the i^{th} position, there is a unique, up to scalars, nonzero bilinear map

$$\varphi^i_{\mathbf{c},\mathbf{c}} : V^{\mathbf{c}} \times V^{\mathbf{c}} \longrightarrow \mathfrak{sl}(V^i),$$

invariant under the action of $\mathrm{SL}_2^n = \mathrm{SL}(V_1) \times \cdots \times \mathrm{SL}(V_n)$, given by contraction on the indices different from i and using $s_{u,v}$'s:

$$\varphi^2_{(1,1,1,0),(1,1,1,0)}\big(u_1 \otimes u_2 \otimes u_3, v_1 \otimes v_2 \otimes v_3\big) = \langle u_1 \mid v_1\rangle\langle u_3 \mid v_3\rangle s_{u_2,v_2}.$$

Consider the vector space

$$\mathcal{C} = \mathbb{F}t_1 \oplus \cdots \oplus \mathbb{F}t_n \oplus \Big(\bigoplus_{\mathbf{c}\in\mathbf{S}} \mathbb{F}e^{\mathbf{c}}\Big).$$

The invariance of the Lie bracket of $\mathfrak{g}$ under the action of $\mathsf{SL}_2^n = \mathsf{SL}(V_1) \times \cdots \times \mathsf{SL}(V_n)$ induces a bilinear multiplication on $\mathcal{C}$ with:

$$t_i^2 = t_i,\ i = 1, \ldots, n; \quad t_i t_j = 0 \text{ for } i \neq j,$$

$$t_i e^{\mathbf{c}} = e^{\mathbf{c}} t_i = \begin{cases} e^{\mathbf{c}} & \text{if } i \in \mathrm{supp}(\mathbf{c}), \\ 0 & \text{otherwise}, \end{cases}$$

$$e^{\mathbf{c}} e^{\mathbf{d}} \in \mathbb{F}e^{\mathbf{c}+\mathbf{d}} \quad \text{for } \mathbf{c} \neq \mathbf{d} \text{ in } \mathbf{S},$$

$$e^{\mathbf{c}} e^{\mathbf{c}} \in \sum_{i\in\mathrm{supp}(\mathbf{c})} \mathbb{F}t_i, \quad \text{for } \mathbf{c} \in \mathbf{S}.$$

The Lie bracket of $\mathfrak{g}$ is completely determined by its coordinate algebra:

$$[x \otimes t_i, y \otimes t_i] = [x, y] \otimes t_i, \quad \text{for } i = 1, \dots, n,\ x, y \in \mathfrak{sl}(V_i),$$

$$[x \otimes t_i, y \otimes t_j] = 0, \quad \text{for } 1 \le i \ne j \le n,\ x \in \mathfrak{sl}(V_i),\ y \in \mathfrak{sl}(V_j),$$

$$[x \otimes t_i, (u_{i_1} \otimes \cdots \otimes u_{i_r}) \otimes e^{\mathbf{c}}] = \begin{cases} 0 & \text{if } i \notin \operatorname{supp}(\mathbf{c}), \\ (u_{i_1} \otimes \cdots \otimes (x u_{i_j}) \otimes \cdots \otimes u_{i_r}) \otimes e^{\mathbf{c}} & \text{if } i = i_j \in \operatorname{supp}(\mathbf{c}), \end{cases}$$

$$[X \otimes e^{\mathbf{c}}, Y \otimes e^{\mathbf{d}}] = \varphi_{\mathbf{c},\mathbf{d}}(X, Y) \otimes e^{\mathbf{c}} e^{\mathbf{d}}, \quad \text{for } \mathbf{c} \ne \mathbf{d} \text{ in } \mathbf{S},$$

$$[X \otimes e^{\mathbf{c}}, Y \otimes e^{\mathbf{c}}] = \sum_{i \in \operatorname{supp}(\mathbf{c})} \mu_i \varphi^i_{\mathbf{c},\mathbf{c}}(X, Y) \otimes t_i, \quad \text{for } \mathbf{c} \in \mathbf{S}.$$

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Optimal short SL_2^7-structure on E_7

$$\mathfrak{e}_7 = \Big(\bigoplus_{i=1}^{7} \mathfrak{sl}(V_i) \otimes \mathbb{F} t_i\Big) \oplus \Big(\bigoplus_{\mathbf{c}\in\mathbf{C}\setminus\{\mathbf{0}\}} (V^{\mathbf{c}} \otimes \mathbb{F} e^{\mathbf{c}})\Big),$$

where $\mathbf{C}$ is the simplex $[7,3,4]$-code.

The coordinate algebra is

$$\mathcal{C} = \Big(\bigoplus_{i=1}^{7} \mathbb{F} t_i\Big) \oplus \Big(\bigoplus_{\mathbf{c}\in\mathbf{C}\setminus\{\mathbf{0}\}} \mathbb{F} e^{\mathbf{c}}\Big),$$

Optimal short SL_2^7-structure on E_7

$$t_i t_j = \delta_{ij} t_i, \quad \text{for } 1 \leq i, j \leq 7,$$

$$t_i e^{\mathbf{c}} = e^{\mathbf{c}} t_i = \begin{cases} e^{\mathbf{c}} & \text{if } i \in \operatorname{supp}(\mathbf{c}), \\ 0 & \text{otherwise,} \end{cases}$$

$$e^{\mathbf{c}} e^{\mathbf{d}} = \epsilon(\mathbf{c}, \mathbf{d}) e^{\mathbf{c}+\mathbf{d}}, \quad \text{for } \mathbf{c} \neq \mathbf{d} \in \mathbf{C} \setminus \{\mathbf{0}\},$$

$$e^{\mathbf{c}} e^{\mathbf{c}} = \epsilon(\mathbf{c}, \mathbf{c}) \sum_{i \in \operatorname{supp}(\mathbf{c})} t_i, \quad \text{for } \mathbf{c} \in \mathbf{C} \setminus \{\mathbf{0}\},$$

where $\epsilon(\mathbf{c}, \mathbf{d}) \in \{\pm 1\}$ is the sign that appears in the multiplication table of $\mathbb{O}$.

Optimal short SL_2^8-structure on E_8

$$\mathfrak{e}_8 = \Big(\bigoplus_{i=1}^{8} \mathfrak{sl}(V_i) \otimes \mathbb{F} t_i\Big) \oplus \Big(\bigoplus_{\mathbf{c}\in\mathbf{H}\setminus\{\mathbf{0},\mathbf{1}\}} (V^{\mathbf{c}} \otimes \mathbb{F} e^{\mathbf{c}})\Big),$$

where **H** is the extended Hamming $[8,4,4]$-code.

The coordinate algebra is

$$\mathcal{H} = \Big(\bigoplus_{i=1}^{8} \mathbb{F} t_i\Big) \oplus \Big(\bigoplus_{\mathbf{c}\in\mathbf{H}\setminus\{\mathbf{0},\mathbf{1}\}} \mathbb{F} e^{\mathbf{c}}\Big),$$

Optimal short SL_2^8-structure on E_8

$$t_i t_j = \delta_{ij} t_i, \quad \text{for } 1 \le i, j \le 8,$$

$$t_i e^{\mathbf{c}} = e^{\mathbf{c}} t_i = \begin{cases} e^{\mathbf{c}} & \text{if } i \in \operatorname{supp}(\mathbf{c}), \\ 0 & \text{otherwise,} \end{cases}$$

$$e^{\mathbf{c}} e^{\mathbf{d}} = \epsilon(\mathbf{c}, \mathbf{d}) e^{\mathbf{c}+\mathbf{d}}, \quad \text{for } \mathbf{c} \ne \mathbf{d} \in \mathbf{H} \setminus \{\mathbf{0}, \mathbf{1}\},$$

$$e^{\mathbf{c}} e^{\mathbf{c}} = \epsilon(\mathbf{c}, \mathbf{c}) \sum_{i \in \operatorname{supp}(\mathbf{c})} t_i, \quad \text{for } \mathbf{c} \in \mathbf{H} \setminus \{\mathbf{0}, \mathbf{1}\},$$

where $\epsilon(\mathbf{c}, \mathbf{d}) \in \{\pm 1\}$, and these signs are also related to $\mathbb{O}$.

Optimal short SL_2^4-structure on F_4

$$\mathfrak{f}_4 = \Big(\bigoplus_{i=1}^{4} \mathfrak{sl}(V_i) \otimes \mathbb{F} t_i\Big) \oplus \Big(\bigoplus_{\mathbf{c} \in \mathbf{F} \setminus \{\mathbf{0}\}} (V^{\mathbf{c}} \otimes \mathbb{F} e^{\mathbf{c}})\Big),$$

where $\mathbf{F}$ is the binary $[4, 3, 2]$-code

$$\{(c_1, c_2, c_3, c_4) \in \mathbb{F}_2^4 \mid c_1 + c_2 + c_3 + c_4 = 0\}.$$

Concluding remarks

- It is not difficult to get the coordinate algebras of the optimal short SL_2^n-structures on the classical Lie algebras A_1, B_{2n}, C_n, D_{2n}.
- Fine gradings are gradings where the irreducible modules for the corresponding quasitorus appear with low multiplicity, that is, the homogeneous components are small. In this work, the optimal SL_2^n-structures that have been considered satisfy that the multiplicities of the irreducible modules that appear are all equal to 1, so they can be thought of as a sort of **fine S-structures**. The outcome is that some nice descriptions of several simple Lie algebras are obtained, showing an intriguing connection with code algebras.
- The notion of S-structure opens a broad area of research. It also extends the notion of Lie algebras graded by finite root systems, which may be seen as algebras with particular S-structures.

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Thank you!