Flexible composition algebras

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International Conference in Memory of Hyo Chul Myung
April 25, 2011

1. Hurwitz algebras

2. Symmetric composition algebras

3. Flexible composition algebras

Composition algebras

Definition

A *composition algebra* over a field k is a triple $(C, \cdot, n)$ where

- C is a vector space over k,
- $x \cdot y$ is a bilinear multiplication $C \times C \to C$,
- $n : C \to k$ is a multiplicative nondegenerate quadratic form:
  - its polar $n(x, y) = n(x + y) - n(x) - n(y)$ is nondegenerate,
  - $n(x \cdot y) = n(x)n(y)$ $\forall x, y \in C$.

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The unital composition algebras will be called *Hurwitz algebras*.

Hurwitz algebras

Hurwitz algebras form a class of degree two algebras:

$$x^2 - n(x,1)x + n(x)1 = 0$$

for any x.

They are endowed with an antiautomorphism, the *standard conjugation*:

$$\bar{x} = n(x,1)1 - x,$$

satisfying

$$\bar{\bar{x}} = x, \quad x + \bar{x} = n(x,1)1, \quad x \cdot \bar{x} = \bar{x} \cdot x = n(x)1.$$

Cayley-Dickson doubling process

Let B be an associative Hurwitz algebra with norm n, and let λ be a nonzero scalar in the ground field k. Consider the direct sum of two copies of B:

$$C = B \oplus Bu,$$

with the following multiplication and nondegenerate quadratic form that extend those on B:

$$\begin{aligned}(a + bu)(c + du) &= (ac + \lambda \bar{d} b) + (da + b\bar{c})u,\\ n(a + bu) &= n(a) - \lambda n(b).\end{aligned}$$

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Notation: $CD(A,\mu,\lambda) := CD\big(CD(A,\mu),\lambda\big).$

Generalized Hurwitz Theorem

Theorem

Every Hurwitz algebra over a field k is isomorphic to one of the following:

(i) *The ground field k if its characteristic is $\neq 2$.*

(ii) *A quadratic commutative and associative separable algebra $K(\mu) = k1 + kv$, with $v^2 = v + \mu$ and $4\mu + 1 \neq 0$. The norm is given by its generic norm.*

(iii) *A quaternion algebra $Q(\mu, \beta) = CD(K(\mu), \beta)$. (These four dimensional algebras are associative but not commutative.)*

(iv) *A Cayley algebra $C(\mu, \beta, \gamma) = CD(K(\mu), \beta, \gamma)$. (These eight dimensional algebras are alternative, but not associative.)*

General composition algebras

Corollary

The dimension of any finite-dimensional composition algebra is restricted to 1, 2, 4 *or* 8.

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Proof.

- Take any element a of C with $n(a) \neq 0$. Then the norm of $e = \frac{1}{n(a)}a^2$ is 1.
- Consider the new multiplication on C (Kaplansky's trick):

$$x \cdot y = (R_e^{-1}x)(L_e^{-1}y).$$

- Then $(C, \cdot, n)$ is a Hurwitz algebra with unity $1 = e^2$.

$\square$

The split Hurwitz algebras

There are 4 split (i.e., $\exists x$ s.t. $n(x) = 0$) Hurwitz algebras:

$$k, \qquad k \times k, \qquad \mathrm{Mat}_2(k), \qquad C(k).$$

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Canonical basis of the *split Cayley algebra* $C(k) = CD\big(\mathrm{Mat}_2(k), -1\big)$:

$$\mathcal{B} = \{e_1, e_2, u_1, u_2, u_3, v_1, v_2, v_3\}$$

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$$n(e_1, e_2) = n(u_i, v_i) = 1, \quad \text{(otherwise 0)}$$

$$e_1^2 = e_1, \quad e_2^2 = e_2,$$
$$e_1 u_i = u_i e_2 = u_i, \quad e_2 v_i = v_i e_1 = v_i, \quad (i = 1, 2, 3)$$
$$u_i v_i = -e_1, \quad v_i u_i = -e_2, \quad (i = 1, 2, 3)$$
$$u_i u_{i+1} = -u_{i+1} u_i = v_{i+2}, \; v_i v_{i+1} = -v_{i+1} v_i = u_{i+2}, \; \text{(indices modulo 3)}$$

(otherwise 0).

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Symmetric composition algebras

Definition

A composition algebra $(S, *, n)$ is said to be *symmetric* if the polar form of its norm is associative:

$$n(x * y, z) = n(x, y * z),$$

for any $x, y, z \in S$.

This is equivalent to the condition:

$$(x * y) * x = n(x)y = x * (y * x),$$

for any $x, y \in S$.

Examples

Let C be a Hurwitz algebra with norm n.

- *Para-Hurwitz algebras (Okubo-Myung 1980)*: Consider the new multiplication on C:

  $$x \bullet y = \bar{x} \cdot \bar{y}.$$

  Then $(C, \bullet, n)$ is a composition algebra, which will be denoted by $\bar{C}$ for short.
  The unity of C becomes a *para-unit* in $\bar{C}$, that is, an element e such that $e \bullet x = x \bullet e = n(e, x)e - x$. If the dimension is at least 4, the para-unit is unique, and it is the unique idempotent that spans the commutative center of the para-Hurwitz algebra.

Examples

- *Petersson algebras (1969)*: Let τ be an automorphism of C with $\tau^3 = 1$, and consider the new multiplication defined on C by means of:
$$x * y = \tau(\bar{x}) \cdot \tau^2(\bar{y}).$$
The algebra $(C, *, n)$ is a symmetric composition algebra, which will be denoted by $\bar{C}_\tau$ for short.

Okubo algebras

Let $\mathcal{B} = \{e_1, e_2, u_1, u_2, u_3, v_1, v_2, v_3\}$ be a canonical basis of $C(k)$. Then the linear map $\tau_{st} : C(k) \to C(k)$ determined by the conditions:

$$\tau_{st}(e_i) = e_i,\ i = 1, 2; \quad \tau_{st}(u_i) = u_{i+1},\ \tau_{st}(v_i) = v_{i+1} \text{ (indices modulo 3)},$$

is clearly an order 3 automorphism of $C(k)$. ("*st*" stands for *standard*.)

Definition

The associated Petersson algebra $P_8(k) = \overline{C(k)}_{\tau_{st}}$ is called the *pseudo-octonion algebra* over the field k.

The forms of $P_8(k)$ are called *Okubo algebras* [E.-Myung 1990].

(This is not the original definition of the pseudo-octonion algebra due to Okubo in 1978.)

Classification

Theorem (Okubo-Osborn 81, E.–Myung 91,93, E.–Pérez-Izquierdo 96, E. 97)

Any symmetric composition algebra is either:

- *a para-Hurwitz algebra,*
- *a form of a two-dimensional para-Hurwitz algebra without idempotent elements (with a precise description),*
- *an Okubo algebra.*

Classification

Moreover:

- If char $k \neq 3$ and $\exists \omega \neq 1 = \omega^3$ in k, then any Okubo algebra is, up to isomorphism, the algebra A_0 of zero trace elements in a central simple degree 3 associative algebra with multiplication

$$x * y = \omega xy - \omega^2 yx - \frac{\omega - \omega^2}{3}\,\mathrm{tr}(xy)1,$$

  and norm $n(x) = -\frac{1}{2}\,\mathrm{tr}(x^2)$.

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  and norm $n(x) = -\frac{1}{2}\operatorname{tr}(x^2)$.

- If char $k \neq 3$ and $\nexists \omega \neq 1 = \omega^3$ in k, then any Okubo algebra is, up to isomorphism, the algebra $S(A, j)_0 = \{x \in A_0 : j(x) = -x\}$, where (A, j) is a central simple degree three associative algebra over $k[\omega]$ and j is a $k[\omega]/k$-involution of second kind, with multiplication and norm as above.

Classification

Finally, if char $k = 3$, for any Okubo algebra there are nonzero scalars $\alpha, \beta \in k$ and a basis such that the multiplication table is:

$*$	$x_{1,0}$	$x_{-1,0}$	$x_{0,1}$	$x_{0,-1}$	$x_{1,1}$	$x_{-1,-1}$	$x_{-1,1}$	$x_{1,-1}$
$x_{1,0}$	$-\alpha x_{-1,0}$	0	0	$x_{1,-1}$	0	$x_{0,-1}$	0	$\alpha x_{-1,-1}$
$x_{-1,0}$	0	$-\alpha^{-1} x_{1,0}$	$x_{-1,1}$	0	$x_{0,1}$	0	$\alpha^{-1} x_{1,1}$	0
$x_{0,1}$	$x_{1,1}$	0	$-\beta x_{0,-1}$	0	$\beta x_{1,-1}$	0	0	$x_{1,0}$
$x_{0,-1}$	0	$x_{-1,-1}$	0	$-\beta^{-1} x_{0,1}$	0	$\beta^{-1} x_{-1,1}$	$x_{-1,0}$	0
$x_{1,1}$	$\alpha x_{-1,1}$	0	0	$x_{1,0}$	$-(\alpha\beta) x_{-1,-1}$	0	$\beta x_{0,-1}$	0
$x_{-1,-1}$	0	$\alpha^{-1} x_{1,-1}$	$x_{-1,0}$	0	0	$-(\alpha\beta)^{-1} x_{1,1}$	0	$\beta^{-1} x_{0,1}$
$x_{-1,1}$	$x_{0,1}$	0	$\beta x_{-1,-1}$	0	0	$\alpha^{-1} x_{1,0}$	$-\alpha^{-1}\beta x_{1,-1}$	0
$x_{1,-1}$	0	$x_{0,-1}$	0	$\beta^{-1} x_{1,1}$	$\alpha x_{-1,0}$	0	0	$-\alpha\beta^{-1} x_{-1,1}$

Okubo algebras with isotropic norm

Remark

Okubo algebras with isotropic norm present this same multiplication table, no matter what the characteristic of the ground field is.

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Can these two classes of composition algebras (Hurwitz and symmetric) be characterized in terms of identities?

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Definition

- An algebra is said to be *flexible* in case for any x, y:

$$(xy)x = x(yx).$$

- An algebra is said to be *third power-associative* in case for any x:

$$x^2x = xx^2.$$

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Definition

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- An algebra is said to be *third power-associative* in case for any x:

$$x^2x = xx^2.$$

Flexibility and third power-associativity are some of the weakest identities that can be imposed on a nonassociative algebra. (In particular, any commutative or anticommutative algebra is flexible.)

Flexible composition algebras

Theorem (Okubo 82)

Any finite dimensional flexible composition algebra over a field of characteristic $\neq 2$ *is either a Hurwitz algebra or a symmetric composition algebra.*

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The last joint paper with Hyo Myung (2004) dealt with the problem of simplifying and extending to arbitrary fields Okubo's result.

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- The ground field may be assumed to be algebraically closed (and hence infinite). Then Zariski topology arguments can be used.

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Sketch of proof:

- The ground field may be assumed to be algebraically closed (and hence infinite). Then Zariski topology arguments can be used.
- For any $x \in C$, the dimension of the subalgebra generated by x is at most 2.

  It is enough to prove this with the added condition $n(x) \neq 0$ (Zariski dense). The new multiplication $u \cdot v = (R_x^{-1}u)(L_x^{-1}v)$ is then a Hurwitz algebra with unity x^2 and norm $\frac{1}{n(x)^2}n$.

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  (a) $x^2x = xx^2$ and $(x^2)^2 = (x^2x)x$ for any x, or
  (b) $x^2x = n(x)x = xx^2$ for any x.
- In case (a) $n(x)$ divides $n(x, x^2)$ as polynomial maps, so there is a special element e such that

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- In case (b) for any x, y

$$n(x)n(xy, x) = n(xy, n(x)x) = n(xy, xx^2) = n(x)n(y, x^2),$$

  and hence $n(xy, x) = n(y, x^2) = n(yx, x)$ in a Zariski dense subset, which shows $n([x, y], x) = 0$, or $n([x, y], z]) = n(x, [y, z])$ for any x, y, z.
  It then follows that $n(xy, z) = n(x, yz)$ for any x, y, z in a Zariski dense subset, and hence C is a symmetric composition algebra.

Third power-associative composition algebras

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Any finite-dimensional third power-associative algebra over a field of characteristic $\neq 2, 3$ is either a Hurwitz algebra or a symmetric composition algebra.

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Can this be extended too to arbitrary fields?

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Theorem (E.–Myung 04)

Any finite-dimensional third power-associative composition algebra over a field with $|F| > 2$ *is either a Hurwitz algebra or a symmetric composition algebra.*

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Example

Over $\mathbb{F}_2$, the vector space $\mathbb{F}_2 \times \mathbb{F}_2$, with multiplication

$$(\alpha, \beta)(\gamma, \delta) = (\beta\gamma, \alpha\delta)$$

is a composition algebra with $n\big((\alpha, \beta)\big) = \alpha\beta$. It satisfies $(1, 0)^2 = (0, 1)^2 = (0, 0)$, $(1, 1)^2 = (1, 1)$, so it is trivially third power-associative, but it is not flexible as

$$\big((1, 0)(1, 1)\big)(1, 0) = (1, 0), \quad (1, 0)\big((1, 1)(1, 0)\big) = (0, 0).$$

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- In the latter case for any x, y, z we get

  $$n(xz, x) = n(z, x^2) = n(zx, x) \text{ and } n([x, y], z) = n(x, [y, z]).$$

  Linearization gives $2n(yx, z) = n(xz, y) + n(x, zy)$.
  Add this equation to twice the equation obtained by cyclically permuting x, y, z to get

  $$3n(yx, z) = 3n(y, xz),$$

  and hence this is a symmetric composition algebra in char $\neq 3$.

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- The characteristic 3 case is more involved.

Some concluding remarks

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That's all. Thanks