

Graded-simple algebras and twisted loop algebras



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Problem

How to reduce the study of **graded-simple** algebras to the study of **graded and simple** algebras?

The answer is given by an extension and deformation of the construction of **loop algebras** used in the setting of affine Kac-Moody Lie algebras:

$$\mathfrak{g} = \bigoplus_{\bar{r} \in \mathbb{Z}/n} \mathfrak{g}_{\bar{r}} \longrightarrow L(\mathfrak{g}) = \bigoplus_{r \in \mathbb{Z}} \mathfrak{g}_{\bar{r}} \otimes t^r \subseteq \mathfrak{g} \otimes \mathbb{F}[t, t^{-1}].$$

The graded-central-simple algebras with [split centroid](#) were shown, by Allison, Berman, Faulkner and Pianzola, to be isomorphic to [loop algebras](#) of algebras graded by a quotient group that are central simple as ungraded algebras.

This is a very important reduction, as the graded-central-simple algebras may fail to be nice as ungraded algebras; for instance, they may fail to be simple or semisimple.

It will be shown here [how to remove the restriction of the centroid being split](#), at the expense of allowing certain [deformations of the loop algebra construction](#). These deformations will be based on a symmetric 2-cocycle on the grading group with values in the multiplicative group of the ground field.

Outline

- 1 Group gradings
- 2 Loop algebras
- 3 Technical stuff: extensions and cocycles
- 4 Cocycle twists
- 5 Graded-central-simple algebras
- 6 What is this good for?

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- 2 Loop algebras
- 3 Technical stuff: extensions and cocycles
- 4 Cocycle twists
- 5 Graded-central-simple algebras
- 6 What is this good for?

Definition

Let \mathcal{A} be an algebra (over a field \mathbb{F}) and let G be an *abelian* group.

- A **G -grading** on \mathcal{A} is a vector space decomposition $\Gamma : \mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g$ such that $\mathcal{A}_g \mathcal{A}_h \subseteq \mathcal{A}_{gh}$ for any $g, h \in G$.
- The nonzero elements in \mathcal{A}_g are said to be **homogeneous of degree g** .
- The **support** of Γ is the set $\{g \in G \mid \mathcal{A}_g \neq 0\}$.

Group gradings

Example: group algebra

Given an abelian group G , the group algebra $\mathbb{F}G$ is endowed with a natural G -grading:

$$\mathbb{F}G = \bigoplus_{g \in G} \mathbb{F}g.$$

This is an example of a **graded-field** (commutative graded algebra where all homogeneous elements have an inverse).

Simple algebras

Let \mathcal{B} be an algebra over \mathbb{F} :

- \mathcal{B} is **simple** if it has no proper ideals and $\mathcal{B}^2 \neq 0$.
In other words, \mathcal{B} is simple if it is simple as a module for its **multiplication algebra** $\text{Mult}(\mathcal{B})$.
- The **centroid** of \mathcal{B} is the centralizer of $\text{Mult}(\mathcal{B})$ in $\text{End}_{\mathbb{F}}(\mathcal{B})$:
$$C(\mathcal{B}) := \{f \in \text{End}_{\mathbb{F}}(\mathcal{B}) : f(xy) = f(x)y = xf(y) \forall x, y \in \mathcal{B}\}.$$

 $C(\mathcal{B})$ is commutative if $\mathcal{B}^2 = \mathcal{B}$, and it is a field (an extension field of \mathbb{F}) if \mathcal{B} is simple.
- \mathcal{B} is **central simple** if it is simple and **central**: $C(\mathcal{B}) = \mathbb{F}\text{id}$.

Any simple algebra is a central simple algebra, when considered as an algebra over its centroid.

Graded-simple algebras

Let $\mathcal{B} = \bigoplus_{g \in G} \mathcal{B}_g$ be a G -graded algebra:

- \mathcal{B} is **graded-simple** if it has no proper graded ideals and $\mathcal{B}^2 \neq 0$.

Its centroid *inherits* a G -grading:

$$C(\mathcal{B})_g := \{f \in C(\mathcal{B}) : f(\mathcal{B}_h) \subseteq \mathcal{B}_{gh} \forall h \in G\}.$$

- \mathcal{B} is **graded-central** if $C(\mathcal{B})_e = \mathbb{F}\text{id}$.
- \mathcal{B} is **graded-central-simple** if it is graded-simple and graded-central.

Graded-simple algebras

Let $\mathcal{B} = \bigoplus_{g \in G} \mathcal{B}_g$ be a graded-simple algebra, then:

- $C(\mathcal{B})$ is a graded-field.
- \mathcal{B} is simple (ungraded) if and only if $C(\mathcal{B})$ is a field.
- $\mathbb{K} = C(\mathcal{B})_e$ is a field, and \mathcal{B} is graded-central-simple considered as an algebra over \mathbb{K} .
- If \mathcal{B} is graded-simple, and H is the support of the induced grading on $C(\mathcal{B})$, then H is a subgroup of G .
- If \mathcal{B} is graded-central-simple, its centroid $C(\mathcal{B})$ is said to be **split** if it is isomorphic, as a graded algebra, to the group algebra $\mathbb{F}H$: $C(\mathcal{B}) \simeq_G \mathbb{F}H$.

- 1 Group gradings
- 2 Loop algebras**
- 3 Technical stuff: extensions and cocycles
- 4 Cocycle twists
- 5 Graded-central-simple algebras
- 6 What is this good for?

Definition

Given an abelian group G , a subgroup H , the canonical projection

$$\pi : G \rightarrow \bar{G} = G/H, \quad g \mapsto \pi(g) = \bar{g},$$

and an algebra \mathcal{A} graded by \bar{G} : $\mathcal{A} = \bigoplus_{\bar{g} \in \bar{G}} \mathcal{A}_{\bar{g}}$, the **loop algebra** $L_{\pi}(\mathcal{A})$ is the G -graded algebra

$$L_{\pi}(\mathcal{A}) := \bigoplus_{g \in G} \mathcal{A}_{\bar{g}} \otimes g$$

which is a subalgebra of the tensor product $\mathcal{A} \otimes_{\mathbb{F}} \mathbb{F}G$.

L_{π} is right-adjoint to the natural functor from G -graded to \bar{G} -graded algebras.

Theorem (Allison, Berman, Faulkner, and Pianzola)

Let G be an abelian group, H a subgroup of G , and $\pi : G \rightarrow \overline{G} = G/H$ the canonical projection.

1. If \mathcal{A} is a central simple algebra graded by \overline{G} , then the loop algebra $L_\pi(\mathcal{A})$ is a G -graded-central-simple algebra, and the map

$$\begin{aligned} \mathbb{F}H &\longrightarrow C(L_\pi(\mathcal{A})) \\ h &\mapsto (x \otimes g \mapsto x \otimes hg) \end{aligned}$$

for $g \in G$, $x \in \mathcal{A}_{\pi(g)}$, is an isomorphism of G -graded algebras. (Hence $C(L_\pi(\mathcal{A})) \simeq_G \mathbb{F}H$.)

Theorem (continued)

2. Conversely, if \mathcal{B} is a G -graded-central-simple algebra *with split centroid*: $C(\mathcal{B}) \simeq_G \mathbb{F}H$, then there exists a central simple and \overline{G} -graded algebra \mathcal{A} such that $\mathcal{B} \simeq_G L_\pi(\mathcal{A})$.

This central simple algebra \mathcal{A} may be obtained as $\mathcal{B}/\mathcal{J}\mathcal{B}$, where \mathcal{J} is the augmentation ideal of $C(\mathcal{B}) \simeq_G \mathbb{F}H$.

Theorem (continued)

3. If \mathcal{A} and \mathcal{A}' are central simple and \overline{G} -graded algebras, then $L_\pi(\mathcal{A}) \simeq_G L_\pi(\mathcal{A}')$ if and only if there is a character $\chi \in \text{Hom}(H, \mathbb{F}^\times)$ such that $\mathcal{A}' \simeq_{\overline{G}} \mathcal{A}_\chi$.

The algebra \mathcal{A}_χ is defined on the same vector space as \mathcal{A} , but with new multiplication

$$x \cdot_\chi y = \chi\left(s(\overline{g}_1)s(\overline{g}_2)s(\overline{g}_1\overline{g}_2)^{-1}\right)xy$$

for $\overline{g}_1, \overline{g}_2 \in \overline{G}$, $x \in \mathcal{A}_{\overline{g}_1}$, $y \in \mathcal{A}_{\overline{g}_2}$, where $s : \overline{G} \rightarrow G$ is an arbitrary section of the canonical projection $\pi : G \rightarrow \overline{G}$.

- 1 Group gradings
- 2 Loop algebras
- 3 Technical stuff: extensions and cocycles**
- 4 Cocycle twists
- 5 Graded-central-simple algebras
- 6 What is this good for?

$\text{Ext}(A, B)$ and extensions

Given two abelian groups A, B , the abelian group $\text{Ext}(A, B)$ is the set of equivalence classes of extensions of A by B (in the category of abelian groups): $1 \rightarrow B \rightarrow E \rightarrow A \rightarrow 1$.

Two extensions

$$1 \longrightarrow B \longrightarrow E \longrightarrow A \longrightarrow 1 \quad \text{and} \quad 1 \longrightarrow B \longrightarrow E' \longrightarrow A \longrightarrow 1$$

are equivalent if there is a homomorphism $\varphi : E \rightarrow E'$, necessarily bijective, such that the diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & B & \longrightarrow & E & \longrightarrow & A & \longrightarrow & 1 \\ & & \parallel & & \downarrow \varphi & & \parallel & & \\ 1 & \longrightarrow & B & \longrightarrow & E' & \longrightarrow & A & \longrightarrow & 1 \end{array}$$

is commutative.

$\text{Ext}(A, B)$ and extensions

For a homomorphism $f : A' \rightarrow A$, the natural homomorphism

$$\begin{aligned} f^* : \text{Ext}(A, B) &\longrightarrow \text{Ext}(A', B) \\ [\xi] &\mapsto [\xi f] \end{aligned}$$

is obtained by means of the commutative diagram:

$$\begin{array}{ccccccccc} \xi f : 1 & \longrightarrow & B & \xrightarrow{j} & \tilde{E} & \xrightarrow{\tilde{p}_2} & A' & \longrightarrow & 1 \\ & & \parallel & & \downarrow \tilde{p}_1 & & \downarrow f & & \\ \xi : 1 & \longrightarrow & B & \xrightarrow{i} & E & \xrightarrow{p} & A & \longrightarrow & 1 \end{array}$$

where \tilde{E} is the pull-back of p and f .

$$\text{Ext}(A, B) \simeq H_{\text{sym}}^2(A, B)$$

On the other hand, the set of equivalence classes of central extensions, in the category of groups, of the group A by the abelian group B :

$$1 \longrightarrow B \xrightarrow{i} E \xrightarrow{p} A \longrightarrow 1$$

with $i(B)$ central in E , can be identified with the second cohomology group $H^2(A, B) = Z^2(A, B)/B^2(A, B)$, where

$$Z^2(A, B) = \{ \sigma : A \times A \rightarrow B \mid \\ \sigma(a_1, a_2)\sigma(a_1a_2, a_3) = \sigma(a_1, a_2a_3)\sigma(a_2, a_3) \quad \forall a_1, a_2, a_3 \in A \}$$

is the set of **2-cocycles**, and $B^2(A, B) = \{ d\gamma \mid \gamma : A \rightarrow B \text{ a map} \}$ is the set of **2-coboundaries**:

$$d\gamma(a_1, a_2) = \gamma(a_1)\gamma(a_2)\gamma(a_1a_2)^{-1}.$$

The element in $H^2(A, B)$ that corresponds to the central extension

$$\xi : 1 \longrightarrow B \xrightarrow{i} E \xrightarrow{p} A \longrightarrow 1$$

is obtained by fixing a section $s : A \rightarrow E$ of p . Then for any $a_1, a_2 \in A$, there is a unique element $\sigma(a_1, a_2) \in B$ such that

$$i(\sigma(a_1, a_2)) = s(a_1)s(a_2)s(a_1a_2)^{-1},$$

and $\sigma : A \times A \rightarrow B$ is a 2-cocycle whose equivalence class $[\sigma]$ is the element in $H^2(A, B)$ that corresponds to the equivalence class of ξ .

Moreover, for A and B abelian, ξ is an abelian extension if and only if σ is symmetric.

$$\text{Ext}(A, B) \simeq H_{\text{sym}}^2(A, B)$$

Denote by $Z_{\text{sym}}^2(A, B)$ the subgroup of $Z^2(A, B)$ of the symmetric 2-cocycles, and note that $B^2(A, B)$ is contained in $Z_{\text{sym}}^2(A, B)$. Then, for A and B abelian, $\text{Ext}(A, B)$ can be identified with the quotient

$$H_{\text{sym}}^2(A, B) = Z_{\text{sym}}^2(A, B)/B^2(A, B).$$

The map $f^* : \text{Ext}(A, B) \rightarrow \text{Ext}(A', B)$ becomes:

$$\begin{aligned} f^* : H_{\text{sym}}^2(A, B) &\longrightarrow H_{\text{sym}}^2(A', B) \\ [\sigma] &\mapsto [\sigma \circ (f \times f)] \end{aligned}$$

A long exact sequence

Given an abelian group G , a subgroup H , and the associated quotient group $\bar{G} = G/H$, consider the corresponding short exact sequence

$$\zeta : 1 \longrightarrow H \xhookrightarrow{\iota} G \xrightarrow{\pi} G/H \longrightarrow 1.$$

For any abelian group F , this induces a long exact sequence:

$$\begin{aligned} 1 \rightarrow \operatorname{Hom}(G/H, F) &\xrightarrow{\pi^*} \operatorname{Hom}(G, F) \xrightarrow{\iota^*} \operatorname{Hom}(H, F) \\ &\xrightarrow{\delta} H_{\operatorname{sym}}^2(G/H, F) \xrightarrow{\pi^*} H_{\operatorname{sym}}^2(G, F) \xrightarrow{\iota^*} H_{\operatorname{sym}}^2(H, F) \rightarrow 1 \end{aligned}$$

the *connecting homomorphism* $\delta : \operatorname{Hom}(H, F) \rightarrow H_{\operatorname{sym}}^2(G/H, F)$ being given by

$$\delta(f) = [f \circ \sigma],$$

with $\sigma(\bar{g}_1, \bar{g}_2) = s(\bar{g}_1)s(\bar{g}_2)s(\bar{g}_1\bar{g}_2)^{-1}$.

A long exact sequence

Proposition

Let G and F be abelian groups, H a subgroup of G , and $\tau' : H \times H \rightarrow F$ a symmetric 2-cocycle. Then there is a symmetric 2-cocycle $\tau \in Z_{\text{sym}}^2(G, F)$ that extends τ' (i.e., $\tau' = \tau|_{H \times H}$).

Proof:

$\iota^* : H_{\text{sym}}^2(G, F) \rightarrow H_{\text{sym}}^2(H, F)$ surjective

$\Rightarrow \exists \tilde{\tau} \in Z_{\text{sym}}^2(G, F)$ such that $[\tilde{\tau}|_{H \times H}] = \iota^*([\tilde{\tau}]) = [\tau']$,

$\Rightarrow \exists \gamma : H \rightarrow F$ such that $\tau' = (\tilde{\tau}|_{H \times H})(d\gamma)$.

$$\tau'(h_1, h_2) = \tilde{\tau}(h_1, h_2)\gamma(h_1)\gamma(h_2)\gamma(h_1 h_2)^{-1}.$$

Extend γ to a map $\tilde{\gamma} : G \rightarrow F$. Then $\tau = \tilde{\tau}(d\tilde{\gamma})$ satisfies $\tau|_{H \times H} = \tau'$. □

- 1 Group gradings
- 2 Loop algebras
- 3 Technical stuff: extensions and cocycles
- 4 Cocycle twists**
- 5 Graded-central-simple algebras
- 6 What is this good for?

Definition

Let G be an abelian group, let \mathcal{A} be an algebra over \mathbb{F} endowed with a G -grading: $\mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g$, and let $\tau : G \times G \rightarrow \mathbb{F}^\times$ be a symmetric 2-cocycle.

Define a new multiplication on \mathcal{A} by the formula

$$x * y := \tau(g_1, g_2)xy$$

for $g_1, g_2 \in G$, $x \in \mathcal{A}_{g_1}$, $y \in \mathcal{A}_{g_2}$.

The new algebra thus defined will be called the τ -twist of \mathcal{A} , and will be denoted by \mathcal{A}^τ .

Cocycle twists

Example: graded fields

For any abelian group G and symmetric 2-cocycle $\tau \in Z_{\text{sym}}^2(G, \mathbb{F}^\times)$, the τ -twist $(\mathbb{F}G)^\tau$ of the group algebra $\mathbb{F}G$ is denoted traditionally by $\mathbb{F}^\tau G$ (a twisted group algebra).

Any G -graded-field \mathcal{F} with $\mathcal{F}_e = \mathbb{F}$ is isomorphic to $\mathbb{F}^\tau H$, for some subgroup H of G and some $\tau \in Z_{\text{sym}}^2(H, \mathbb{F}^\times)$.

Cocycle twists

Example: \mathcal{A}_χ

Let G be an abelian group, H a subgroup, and $\overline{G} = G/H$ the corresponding quotient. Let \mathcal{A} be a \overline{G} -graded algebra and let $\chi \in \text{Hom}(H, \mathbb{F}^\times)$ (a character on H).

The algebra \mathcal{A}_χ , considered by Allison et al., defined on \mathcal{A} by

$$x \cdot_\chi y = \chi\left(s(\overline{g}_1)s(\overline{g}_2)s(\overline{g}_1\overline{g}_2)^{-1}\right)xy$$

for a section $s : \overline{G} \rightarrow G$, coincides with the $(\chi \circ \sigma)$ -twist $\mathcal{A}^{\chi \circ \sigma}$. ($\sigma \in Z_{\text{sym}}^2(\overline{G}, H)$ is given by $\sigma(\overline{g}_1, \overline{g}_2) = s(\overline{g}_1)s(\overline{g}_2)s(\overline{g}_1\overline{g}_2)^{-1}$.)

Note that $[\chi \circ \sigma] = \delta(\chi)$, where δ is the connecting homomorphism for $F = \mathbb{F}^\times$.

Cocycle twists

Some properties

Let G be an abelian group and let $\mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g$ be a G -graded algebra over \mathbb{F} .

1. For $\sigma, \tau \in Z_{\text{sym}}^2(G, \mathbb{F}^\times)$, $(\mathcal{A}^\sigma)^\tau = \mathcal{A}^{\sigma\tau}$.
2. If $\tau \in B^2(G, \mathbb{F}^\times)$, then $\mathcal{A}^\tau \simeq_G \mathcal{A}$. More generally, if $\tau_1, \tau_2 \in Z_{\text{sym}}^2(G, \mathbb{F}^\times)$ and $[\tau_1] = [\tau_2]$ in $H_{\text{sym}}^2(G, \mathbb{F}^\times)$, then $\mathcal{A}^{\tau_1} \simeq_G \mathcal{A}^{\tau_2}$. In other words, for $\tau \in Z_{\text{sym}}^2(G, \mathbb{F}^\times)$, **the G -graded isomorphism class of \mathcal{A}^τ depends only on $[\tau] \in H_{\text{sym}}^2(G, \mathbb{F}^\times)$.**
3. If $\overline{\mathbb{F}}$ is an algebraic closure of \mathbb{F} and $\tau \in Z_{\text{sym}}^2(G, \mathbb{F}^\times)$, then $\mathcal{A}^\tau \otimes_{\mathbb{F}} \overline{\mathbb{F}} \simeq_G \mathcal{A} \otimes_{\mathbb{F}} \overline{\mathbb{F}}$. In particular, if \mathcal{A} is an associative, alternative, Lie, linear Jordan, ..., algebra, so is \mathcal{A}^τ .
4. If \mathcal{A} is graded-simple, so is \mathcal{A}^τ , and $C(\mathcal{A}^\tau) \simeq_G C(\mathcal{A})^\tau$.

- 1 Group gradings
- 2 Loop algebras
- 3 Technical stuff: extensions and cocycles
- 4 Cocycle twists
- 5 Graded-central-simple algebras**
- 6 What is this good for?

Twisted loop algebras

Given an abelian group G , a subgroup H , a $\overline{G} = G/H$ -graded algebra \mathcal{A} , and a symmetric 2-cocycle $\tau \in Z_{\text{sym}}^2(G, \mathbb{F}^\times)$, the τ -twist $(L_\pi(\mathcal{A}))^\tau$ will be denoted by $L_\pi^\tau(\mathcal{A})$ and called a **cocycle twisted loop algebra**.

Remark

$L_\pi^\tau(\mathcal{A})$ is the subalgebra $\bigoplus_{g \in G} \mathcal{A}_{\overline{g}} \otimes g$ of the tensor product $\mathcal{A} \otimes_{\mathbb{F}} \mathbb{F}^\tau G$, where $\mathbb{F}^\tau G$ is the twisted group algebra.

Theorem

Let G be an abelian group.

1. Let H be a subgroup of G , \mathcal{A} a central simple and \overline{G} -graded algebra, and let $\tau \in Z_{\text{sym}}^2(G, \mathbb{F}^\times)$. Then $L_\pi^\tau(\mathcal{A})$ is G -graded-central-simple and $C(L_\pi^\tau(\mathcal{A})) \simeq_G \mathbb{F}^{\tau'} H$, where $\tau' = \tau|_{H \times H}$.
2. Conversely, if \mathcal{B} is a G -graded-central-simple algebra, then there is a subgroup H of G , a central simple and $\overline{G} = G/H$ -graded algebra \mathcal{A} , and a symmetric 2-cocycle $\tau \in Z_{\text{sym}}^2(G, \mathbb{F}^\times)$ such that $\mathcal{B} \simeq_G L_\pi^\tau(\mathcal{A})$.

Theorem (continued)

3. For $i = 1, 2$, let H_i be a subgroup of G , \mathcal{A}_i a central simple and G/H_i -graded algebra, $\tau_i \in Z_{\text{sym}}^2(G, \mathbb{F}^\times)$. Denote by $\pi_i : G \rightarrow \overline{G}_i = G/H_i$ the canonical projection, $i = 1, 2$. Then $L_{\pi_1}^{\tau_1}(\mathcal{A}_1) \simeq_G L_{\pi_2}^{\tau_2}(\mathcal{A}_2)$ if and only if the following conditions are satisfied:

- $H_1 = H_2 =: H$, so $\pi_1 = \pi_2 =: \pi : G \rightarrow \overline{G} = G/H$.
- $\iota^*([\tau_1]) = \iota^*([\tau_2])$ in $H_{\text{sym}}^2(H, \mathbb{F}^\times)$, where $\iota : H \hookrightarrow G$ is the inclusion, and
- there is a 2-cocycle $\mu \in Z_{\text{sym}}^2(\overline{G}, \mathbb{F}^\times)$ such that $[\tau_1] = \pi^*([\mu])[\tau_2]$ in $H_{\text{sym}}^2(G, \mathbb{F}^\times)$ and $\mathcal{A}_1^\mu \simeq_{\overline{G}} \mathcal{A}_2$.

Main result

Sketch of proof:

If \mathcal{B} is a G -graded-central-simple algebra, and H is the support of its centroid, then $C(\mathcal{B})$ is a graded-field, and hence a twisted group algebra:

$$C(\mathcal{B}) \simeq_G \mathbb{F}^{\tau'} H, \text{ for a 2-cocycle } \tau' \in Z_{\text{sym}}^2(H, \mathbb{F}^\times).$$

Then there is a 2-cocycle $\tau \in Z_{\text{sym}}^2(G, \mathbb{F}^\times)$ such that $\tau|_{H \times H} = \tau'$, and

$$C(\mathcal{B}^{\tau^{-1}}) \simeq_G C(\mathcal{B})^{\tau^{-1}} \simeq_G (\mathbb{F}^{\tau'} H)^{\tau^{-1}} = \mathbb{F}H,$$

so $C(\mathcal{B}^{\tau^{-1}})$ is split, and we are in the situation studied by Allison et al. □

- 1 Group gradings
- 2 Loop algebras
- 3 Technical stuff: extensions and cocycles
- 4 Cocycle twists
- 5 Graded-central-simple algebras
- 6 What is this good for?**

Gradings on semisimple Lie algebras

In 1989, Patera and Zassenhaus undertook a systematic study of gradings by abelian groups on finite-dimensional simple Lie algebras over the complex numbers, with fine gradings as the central objects. A key example of fine grading is the root space decomposition of a finite-dimensional semisimple Lie algebra relative to a Cartan subalgebra, but there are many other fine gradings that reflect the symmetries of these algebras.

A description of fine gradings on the classical simple Lie algebras (other than D_4 , which is exceptional in many aspects) over \mathbb{C} followed in 1998 by Havlicek, Patera, and Pelantova. The classification of fine gradings on all finite-dimensional simple Lie algebras over an algebraically closed field has been recently completed through the efforts of many authors.

Gradings on semisimple Lie algebras

Key observation

Let \mathcal{A} be a finite direct sum of simple ideals (for instance, a finite-dimensional semisimple Lie algebra over a field of characteristic 0).

Then, given any G -grading

$$\Gamma : \mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g,$$

\mathcal{A} decomposes as a direct sum of **graded-simple ideals**.

(Joint work with Alejandra S. Córdova-Martínez.)

Graded-division algebras

Division algebras and the Brauer group play a key role in the theory of finite-dimensional associative algebras.

The elements of the Brauer group correspond to the isomorphism classes of central division algebras.

Graded-division algebras are key objects in the graded theory, as any finite-dimensional G -graded-central-simple associative algebra \mathcal{A} is graded-isomorphic to $\text{End}_{\mathcal{D}}(V)$ where \mathcal{D} is a G -graded-central-division algebra and V is a graded right \mathcal{D} -module of finite rank.

Graded-division algebras

However, the definition of a **graded Brauer group** is not evident. For abelian G (for example, $G = \mathbb{Z}/2$ in the case of the Brauer-Wall group), one possibility is to fix a bicharacter $\phi : G \times G \rightarrow \mathbb{F}^\times$ and define central simple algebras and (twisted) tensor products relative to ϕ .

The **loop algebra construction** allows us to consider, not **graded-central-simple algebras**, but **central simple graded algebras**, and consider the **graded Brauer group** $\text{Br}_G(\mathbb{F})$, consisting of the equivalence classes of finite-dimensional associative algebras that are central simple and G -graded.

Graded-division algebras





Given a finite abelian group G , Picco and Platzeck proved in 1970 the existence of a split short exact sequence of abelian groups:

$$1 \longrightarrow \mathrm{Br}(\mathbb{F}) \xrightarrow{\iota} \mathrm{Br}_G(\mathbb{F}) \xrightarrow{\zeta} \mathrm{E}_G(\mathbb{F}) \longrightarrow 1$$

where $\mathrm{E}_G(\mathbb{F})$ is the group of isomorphism classes of **G -Galois extensions** of \mathbb{F} . This allows to classify finite-dimensional G -graded-division algebras in terms of (ungraded) division algebras and G -Galois extensions, and it can also be used to understand the structure of these Galois extensions.

(Joint work with Mikhail Kochetov.)

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Thank you!