COMPOSITION ALGEBRAS AND THEIR GRADINGS

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ABSTRACT. The goal of this course is the introduction of the basic properties of the classical composition algebras (that is, those algebras which are analogous to the real, complex, quaternion or octonion numbers), and how these basic properties are enough to get all the possible gradings on them. Then a new class of (non unital) composition algebras will be defined and studied, the so called *symmetric composition algebras*. Finally, the gradings on these two families of composition algebras will be shown to induce some interesting gradings on the exceptional simple Lie algebras.

1. UNITAL COMPOSITION ALGEBRAS. THE CAYLEY-DICKSON PROCESS.

Composition algebras constitute a generalization of the classical algebras of the real $\mathbb{R}$, complex $\mathbb{C}$, quaternion $\mathbb{H}$ (1843), and octonion numbers $\mathbb{O}$ (1845).

Definition 1.1. A *composition algebra* (over a field $\mathbb{F}$) is a not necessarily associative algebra C, endowed with a nondegenerate quadratic form (the *norm*) $q : C \to \mathbb{F}$ (i.e., the bilinear form $q(x,y) = q(x+y) - q(x) - q(y)$ is nondegenerate) which is multiplicative: $q(xy) = q(x)q(y)$ $\forall x, y \in C$.

The unital composition algebras will be called *Hurwitz algebras*.

Easy consequences:

- $q(xy, xz) = q(x)q(y,z) = q(yx, zx)$ $\forall x, y, z$. (l_x and r_x are similarities of norm $q(x)$.)
- $q(xy, tz) + q(ty, xz) = q(x,t)q(y,z)$ $\forall x, y, z, t$.
  Assume now that C is unital:
- $t = 1 \ \Rightarrow q(xy, z) = q\big(y, (q(x,1)1 - x)z\big) = q(y, \bar{x}z)$ ($\bar{x} = q(x,1)1 - x$ is an order 2 orthogonal map). That is:
  $$l_x^* = l_{\bar{x}}, \qquad r_x^* = r_{\bar{x}}.$$
  Then $l_x l_{\bar{x}} = r_x r_{\bar{x}} = q(x)id$, and applied to 1 this gives:
  $$\boxed{x^2 - q(x,1)x + q(x)1 = 0, \quad \forall x} \quad \text{(quadratic algebras)}$$
- $q(\overline{xy}, z) = q(xy, \bar{z}) = q(x, \bar{z}\bar{y}) = q(zx, \bar{y}) = q(z, \bar{y}\bar{x})$, so that $\overline{xy} = \bar{y}\bar{x}$. That is, $x \mapsto \bar{x}$ is an involution (the *standard involution*), which satisfies $x\bar{x} = q(x)1 = \bar{x}x$, and $x + \bar{x} = q(x,1)1$ $\forall x$.
- $l_x l_{\bar{x}} = q(x)id \Rightarrow l_x^2 - q(x,1)l_x + q(x)id = 0 \Rightarrow l_x^2 = l_{x^2}$ $(x(xy) = x^2y)$, and in the same vein $(yx)x = yx^2$. That is, $\boxed{\text{Hurwitz algebras are } \textit{alternative}}$.

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Cayley-Dickson doubling process:

Let Q be a subalgebra of a Hurwitz algebra C such that $q|_Q$ is nondegenerate, and let $u \in C$ such that $q(u) \neq 0 = q(u, Q)$. Then $1 \in Q$, so that $q(u, 1) = 0$ and hence $u^2 = -q(u)1$. Then for any $x \in Q$, $q(xu, 1) = q(x, \bar{u}) = -q(x, u) = 0$, so that $\overline{xu} = -xu$. Then:

$$x(yu) = -x(\overline{yu}) = -x(\bar{u}\bar{y}) = u(\bar{x}\bar{y}) = u(\overline{yx}) = -(yx)\bar{u} = (yx)u,$$

$$(yu)x = -\bar{x}(\overline{yu}) = \bar{x}(yu) = (y\bar{x})u,$$

$$(xu)(yu) = -\bar{y}((\overline{xu})u) = \bar{y}((xu)u) = \bar{y}(xu^2) = \alpha\bar{y}x,$$

(for $\alpha = -q(u) \neq 0$).

Thus $Q \oplus Qu$ is a subalgebra of C and $q|_{Q\oplus Qu}$ is nondegenerate.

Conversely, assume that Q is a Hurwitz algebra with norm q and $0 \neq \alpha \in \mathbb{F}$. Consider the vector space $C := Q \oplus Qu$ (this is formal: just the direct sum of two copies of Q), with multiplication:

$$(a + bu)(c + du) = (ac + \alpha\bar{d}b) + (da + b\bar{c})u,$$

and quadratic form

$$q(x + yu) = q(x) - \alpha q(y).$$

Notation: $C = CD(Q, \alpha)$.

Then:

$$\begin{aligned} q\big((a + bu)(c + du)\big) &= q(ac + \alpha\bar{d}b) - \alpha q(da + b\bar{c}),\\ q(a + bu)q(c + du) &= \big(q(a) - \alpha q(b)\big)\big(q(c) - \alpha q(d)\big)\\ &= q(ac) + \alpha^2 q(bd) - \alpha\big(q(da) + q(ba)\big). \end{aligned}$$

and these expressions are equal for any $a, b, c, d \in Q$ if and only if:

$$\begin{aligned} &q(ac, \bar{d}b) = q(da, b\bar{c}) \quad \forall a, b, c, d \in Q\\ &\Leftrightarrow \quad q\big(d(ac), b\big) = q\big((da)c, b\big) \quad \forall a, b, c, d \in Q\\ &\Leftrightarrow \quad d(ac) = (da)c \quad \forall a, c, d \in Q\\ &\Leftrightarrow \quad Q \text{ is associative.} \end{aligned}$$

Theorem 1.2. *Let Q be a Hurwitz algebra with norm q and let $0 \neq \alpha \in \mathbb{F}$. Let $C = CD(Q, \alpha)$ as above. Then:*

(i) *C is a Hurwitz algebra if and only if Q is associative.*

(ii) *C is associative if and only if Q is commutative. (As $x(yu) = (yx)u$.)*

(iii) *C is commutative if and only if $Q = \mathbb{F}1$. (As $xu = u\bar{x}$, so we must have $x = \bar{x}$ for any x.)*

Remark 1.3. $\mathbb{F}$ is a Hurwitz algebra if and only if $\operatorname{char}\mathbb{F} \neq 2$.

Notation: $CD(A, \alpha, \beta) = CD\big(CD(A, \alpha), \beta\big)$.

Generalized Hurwitz Theorem 1.4. *Every Hurwitz algebra over a field $\mathbb{F}$ is isomorphic to one of the following types:*

(i) *The ground field $\mathbb{F}$ if its characteristic is $\neq 2$.*

(ii) *A quadratic commutative and associative separable algebra $K(\mu) = \mathbb{F}1 + \mathbb{F}v$, with $v^2 = v + \mu$ and $4\mu + 1 \neq 0$. Its norm is given by the generic norm.*

(iii) *A* quaternion algebra *$Q(\mu, \beta) = CD\big(K(\mu), \beta\big)$. (These are associative but not commutative.)*

(iv) *A* Cayley algebra $C(\mu,\beta,\gamma)=CD\big(Q(\mu,\beta),\gamma\big)$*. (These are alternative but not associative.)*

In particular, the dimension of any Hurwitz algebras is finite and restricted to 1, 2, 4 *or* 8.

Corollary 1.5. *Two Hurwitz algebras are isomorphic if and only if its norms are isometric.*

Isotropic Hurwitz algebras: Let C be a Cayley algebra such that its norm q represents 0 (*split Cayley algebra*). (This is always the situation if $\mathbb{F}$ is algebraically closed.)

Take $0\neq x\in C$ with $q(x)=0$ and take $y\in C$ with $q(x,\bar y)=1$ ($q(.,.)$ is nondegenerate), then

$$q(xy,1)=q(x,\bar y)=1.$$

Let $e_1=xy$, so $q(e_1)=0$, $q(e_1,1)=1$, and hence $e_1^2=e_1$. Let $e_2=\bar e_1=1-e_1$, so $q(e_2)=0$, $e_2^2=e_2$, $e_1e_2=0=e_2e_1$ and $q(e_1,e_2)=q(e_1,1)=1$.

Then $\boxed{K=\mathbb{F}e_1+\mathbb{F}e_2 \text{ is a composition subalgebra of } C}$.

For any $x\in K^\perp$, $xe_1+\overline{xe_1}=q(xe_1,1)1=q(x,\bar e_1)1=q(x,e_2)1=0$. Hence $xe_1=-\bar e_1\bar x=e_2x$. We get:

$$xe_1=e_2x,\qquad xe_2=e_1x.$$

Also, $x=1x=e_1x+e_2x$, and $e_2(e_1x)=(1-e_1)(e_1x)=((1-e_1)e_1)x=0=e_1(e_2x)$. Therefore,

$$K^\perp=U\oplus V$$

with

$$U=\{x\in C: e_1x=x=xe_2,\ e_2x=0=xe_1\},$$
$$V=\{x\in C: e_2x=x=xe_1,\ e_1x=0=xe_2\}.$$

For any $u\in U$, $q(u)=q(e_1u)=q(e_1)q(u)=0$, and hence U and V are isotropic subspaces of C. Since q is nondegenerate, U and V are paired by q and $\dim U=\dim V=3$.

And for any $u_1,u_2\in U$ and $v\in V$:

$$q(u_1u_2,K)\subseteq q(u_1,Ku_2)\subseteq q(U,U)=0,$$
$$q(u_1u_2,v)=q(u_1u_2,e_2v)=-q(e_2u_2,u_1v)+q(u_1,e_2)q(u_2,v)=0.$$

Hence U^2 is orthogonal to K and V, so it must be contained in V. Also $V^2\subseteq U$. Besides,

$$q(U,UV)\subseteq q(U^2,V)\subseteq q(V,V)=0,$$
$$q(UV,V)\subseteq q(U,V^2)\subseteq q(U,U)=0,$$

so $UV+VU\subseteq K$. Moreover, $q(UV,e_1)\subseteq q(U,e_1V)=0$, so that $UV\subseteq\mathbb{F}e_1$ and $VU\subseteq\mathbb{F}e_2$. More precisely, for $u\in U$ and $v\in V$, $q(uv,e_2)=-q(u,e_2v)=-q(u,v)$, so that $uv=-q(u,v)e_1$, and $vu=-q(u,v)e_2$.

Therefore the decomposition $\boxed{C=K\oplus U\oplus V \text{ is a } \mathbb{Z}_3\text{-grading of } C}$.

For linearly independent elements $u_1,u_2\in U$, let $v\in V$ with $q(u_1,v)\neq 0=q(u_2,v)$, then $(u_1u_2)v=-(u_1v)u_2=q(u_1,v)u_2\neq 0$, so $U^2\neq 0$.

Moreover, the trilinear map

$$\begin{aligned} U\times U\times U &\longrightarrow \mathbb{F}\\ (x,y,z)&\mapsto q(xy,z),\end{aligned}$$

is alternating (for any $x\in U$, $q(x)=0=q(x,1)$, so $x^2=0$ and hence $q(x^2,z)=0$; but $q(xy,y)=-q(x,y^2)=0$ too).

Take a basis $\{u_1, u_2, u_3\}$ of U with $q(u_1u_2, u_3) = 1$ (this is always possible because $q(U^2, U) \neq 0$ since q is nondegenerate). Then $\{v_1 = u_2u_3, v_2 = u_3u_1, v_3 = u_1u_2\}$ is the dual basis in V (relative to q) and the multiplication table is:

	e_1	e_2	u_1	u_2	u_3	v_1	v_2	v_3
e_1	e_1	0	u_1	u_2	u_3	0	0	0
e_2	0	e_2	0	0	0	v_1	v_2	v_3
u_1	0	u_1	0	v_3	$-v_2$	$-e_1$	0	0
u_2	0	u_2	$-v_3$	0	v_1	0	$-e_1$	0
u_3	0	u_3	v_2	$-v_1$	0	0	0	$-e_1$
v_1	v_1	0	$-e_2$	0	0	0	u_3	$-u_2$
v_2	v_2	0	0	$-e_2$	0	$-u_3$	0	u_1
v_3	v_3	0	0	0	$-e_2$	u_2	$-u_1$	0

(For instance, $q(u_1, v_1) = q(u_1, u_2u_3) = 1$, so $u_1v_1 = -e_1$, $v_1u_1 = -e_2$; $q(u_1, v_2) = q(u_1, u_3u_1) = 0$, so $u_1v_2 = 0 = v_2u_1$; $v_1v_2 = v_1(u_3u_1) = -u_3(v_1u_1) = u_3e_2 = u_3$, ...)

Notation: The split Cayley algebra above is denoted by $C(\mathbb{F})$ and the basis considered above is called a canonical basis of $C(\mathbb{F})$.

Theorem 1.6. *Let $n = 2, 4$ or 8. Then there is, up to isomorphism, a unique Hurwitz algebra with isotropic norm of dimension n:*

(i) $\mathbb{F}e_1 + \mathbb{F}e_2$ *in dimension* 2*, which is just the cartesian product of two copies of* $\mathbb{F}$.

(ii) $\mathbb{F}e_1 + \mathbb{F}e_2 + \mathbb{F}u_1 + \mathbb{F}v_1$ *in dimension* 4*, which is isomorphic to* $\mathrm{Mat}_2(\mathbb{F})$*, with the norm given by the determinant.*

(iii) $C(\mathbb{F})$ *in dimension* 8.

What about real Hurwitz algebras?

If Q is a real Hurwitz algebra which is not split (q does not represent 0) then q is positive definite, the norm of $CD(Q, \alpha)$ is positive definite if and only if $\alpha < 0$, and in this case (change u to $\frac{1}{\sqrt{-\alpha}}u$) $CD(Q, \alpha) = CD(Q, -1)$. Thus the list of real Hurwitz algebras is:

- the split ones: $\mathbb{R} \oplus \mathbb{R}$, $\mathrm{Mat}_2(\mathbb{R})$, $C(\mathbb{R})$,
- the "division" ones: $\mathbb{R}$, $\mathbb{C} = CD(\mathbb{R}, -1)$, $\mathbb{H} = CD(\mathbb{C}, -1)$, and $\mathbb{O} = CD(\mathbb{H}, -1)$.

There are many good references that cover the material in this section. Let us mention, for instance, [KMRT98, Chapter VIII] or [ZSSS82, Chapter 2].

2. Symmetric composition algebras. Triality.

Definition 2.1. A composition algebra $(S, *, q)$ is said to be a *symmetric composition algebra* if $l_x^* = r_x$ for any $x \in S$ (that is, $q(x * y, z) = q(x, y * z)$ for any $x, y, z \in S$).

Theorem 2.2. *Let $(S, *, q)$ be a composition algebra. The following conditions are equivalent:*

(a) *$(S, *, q)$ is symmetric.*
(b) *For any $x, y \in S$, $(x * y) * x = x * (y * x) = q(x)y$.*

Proof. If $(S, *, q)$ is symmetric, then for any $x, y, z \in S$,

$$q\big((x * y) * x, z\big) = q(x * y, x * z) = q(x)q(y, z) = q\big(q(x)y, z\big)$$

whence (b), since q is nondegenerate. Conversely, take $x, y, z \in S$ with $q(y) \neq 0$, so that l_y and r_y are bijective, and hence there is an element $z' \in S$ with $z = z' * y$. Then:

$$q(x * y, z) = q(x * y, z' * y) = q(x, z')q(y) = q\big(x, y * (z' * y)\big) = q(x, y * z).$$

This proves (a) assuming $q(y) \neq 0$, but any isotropic element is the sum of two non isotropic elements, so (a) follows. $\square$

Remark 2.3.

- Condition (b) above implies that $((x * y) * x) * (x * y) = q(x * y)x$, but also $((x * y) * x) * (x * y) = q(x)y * (x * y) = q(x)q(y)x$, so that condition (b) already forces the quadratic form q to be multiplicative.
- Let $(S, *, q)$ be a symmetric composition algebra. Take an element $a \in S$ with $q(a) \neq 0$ and define a new multiplication and nondegenerate quadratic form on S by means of

  $$x \bullet y = (a * x) * (y * a), \qquad \tilde{q}(x) = q(x)q(a)^2.$$

  Then $(S, \bullet, \tilde{q})$ is again a composition algebra. Consider the element $e = \frac{1}{q(a)^2} a * a$. Then

  $$e \bullet x = (a * e) * (x * a) = \frac{1}{q(a)^2}(a * (a * a)) * (x * a) = \frac{1}{q(a)} a * (x * a) = x,$$

  and $x \bullet e = x$ too for any $x \in S$. Hence $(S, \bullet, \tilde{q})$ is a Hurwitz algebra. Therefore the dimension of any symmetric composition algebra is restricted to 1, 2, 4 or 8. (And note the the only symmetric composition algebra of dimension 1 is, up to isomorphism, the ground field.)

Examples 2.4. (Okubo 1978 [Oku78])

- **Para-Hurwitz algebras:** Let C be a Hurwitz algebra with norm q and consider the composition algebra $(C, \bullet, q)$ with the new product given by

  $$x \bullet y = \bar{x}\bar{y}.$$

  Then $q(x \bullet y, z) = q(\bar{x}\bar{y}, z) = q(\bar{x}, zy) = q(x, \overline{zy}) = q(x, y \bullet z)$, for any x, y, z, so that $(C, \bullet, q)$ is a symmetric composition algebra. (Note that $1 \bullet x = x \bullet 1 = \bar{x} = q(x, 1)1 - x$ $\forall x$: 1 is the *para-unit* of $(C, \bullet, q)$.)
- **Okubo algebras:** Assume $\operatorname{char} \mathbb{F} \neq 3$ (the case of $\operatorname{char} \mathbb{F} = 3$ requires a different definition), and let $\omega \in \mathbb{F}$ be a primitive cubic root of 1. Let A

be a central simple associative algebra of degree 3 with trace tr, and let $S = A_0 = \{x \in A : \operatorname{tr}(x) = 0\}$ with multiplication and norm given by:

$$x * y = \omega xy - \omega^2 yx - \frac{\omega - \omega^2}{3}\operatorname{tr}(xy)1,$$

$$q(x) = -\frac{1}{2}\operatorname{tr}(x^2), \quad \text{(it is valid in characteristic 2!)}$$

Then, for any $x, y \in S$:

$$\begin{aligned}(x * y) * x &= \omega(x * y)x - \omega^2 x(x * y) - \frac{\omega - \omega^2}{3}\operatorname{tr}\big((x * y)x\big)1\\ &= \omega^2 xyx - yx^2 - \frac{\omega^2 - 1}{3}\operatorname{tr}(xy)x - x^2y + \omega xyx + \frac{1 - \omega}{3}\operatorname{tr}(xy)x\\ &\qquad - \frac{\omega - \omega^2}{3}\operatorname{tr}\Big((\omega - \omega^2)x^2y\Big)1 \quad (\operatorname{tr}(x) = 0)\\ &= -\big(x^2y + yx^2 + xyx\big) + \operatorname{tr}(xy)x + \operatorname{tr}(x^2y)1 \quad ((\omega - \omega^2)^2 = -3).\end{aligned}$$

But if $\operatorname{tr}(x) = 0$, then $x^3 - \frac{1}{2}\operatorname{tr}(x^2)x - \det(x)1 = 0$, so

$$x^2y + yx^2 + xyx - \Big(\operatorname{tr}(xy)x + \frac{1}{2}\operatorname{tr}(x^2)y\Big) \in \mathbb{F}1.$$

Since $(x * y) * x \in A_0$, we have $(x * y) * x = -\frac{1}{2}\operatorname{tr}(x^2)y = x * (y * x)$.

Therefore $(S, *, q)$ is a symmetric composition algebra.

In case $\omega \notin \mathbb{F}$, take $\mathbb{K} = \mathbb{F}[\omega]$ and a central simple associative algebra A of degree 3 over $\mathbb{K}$ endowed with a $\mathbb{K}/\mathbb{F}$-involution of second kind J. Then take $S = K(A, J)_0 = \{x \in A_0 : J(x) = -x\}$ (this is a $\mathbb{F}$-subspace) and use the same formulae above to define the multiplication and the norm.

For instance, for $\mathbb{F} = \mathbb{R}$, take $A = \mathrm{Mat}_3(\mathbb{C})$, $S = \mathfrak{su}_3 = \{x \in \mathrm{Mat}_3(\mathbb{C}) : \operatorname{tr}(x) = 0,\ \overline{x^T} = -x\}$

Remark 2.5. Given an Okubo algebra, note that for any $x, y \in S$,

$$\begin{aligned}x * y &= \omega xy - \omega^2 yx - \frac{\omega - \omega^2}{3}\operatorname{tr}(xy)1,\\ y * x &= \omega yx - \omega^2 xy - \frac{\omega - \omega^2}{3}\operatorname{tr}(xy)1,\end{aligned}$$

so that

$$\omega x * y + \omega^2 y * x = (\omega^2 - \omega)xy - (\omega + \omega^2)\frac{\omega - \omega^2}{3}\operatorname{tr}(xy)1,$$

and

$$xy = \frac{\omega}{\omega^2 - \omega}x * y + \frac{\omega^2}{\omega^2 - \omega}y * x + \frac{1}{3}q(x, y)1,$$

and the product in A is determined by the product in the Okubo algebra.

Classification ($\operatorname{char}\mathbb{F} \neq 3$):

We can go in the reverse direction of Okubo's construction. Given a symmetric composition algebra $(S, *, q)$ over a field containing ω, define the algebra $A = \mathbb{F}1 \oplus S$ with multiplication determined by the formula

$$xy = \frac{\omega}{\omega^2 - \omega}x * y + \frac{\omega^2}{\omega^2 - \omega}y * x + \frac{1}{3}q(x, y)1,$$

for any $x, y \in S$. Then A is a separable alternative algebra of degree 3.

In case $\omega \notin \mathbb{F}$, then we must consider $A = \mathbb{F}[\omega]1 \oplus \big(\mathbb{F}[\omega] \otimes S\big)$, with the same formula for the product. In $\mathbb{F}[\omega]$ we have the Galois automorphism $\omega^\tau = \omega^2$. Then the conditions $J(1) = 1$ and $J(s) = -s$ for any $s \in S$ induce a $\mathbb{F}[\omega]/\mathbb{F}$-involution of the second kind in A.

Theorem 2.6. *(*$\operatorname{char}\mathbb{F} \neq 3$*)*

- $\boxed{\omega \in \mathbb{F}}$**:** *The symmetric composition algebras of dimension* ≥ 2 *are, up to isomorphism, the algebras* $(A_0, *, q)$ *for* A *a separable alternative algebra of degree* 3.

  Two such symmetric composition algebras are isomorphic if and only if so are the corresponding alternative algebras.
- $\boxed{\omega \notin \mathbb{F}}$**:** *The symmetric composition algebras of dimension* ≥ 2 *are, up to isomorphism, the algebras* $\big(K(A; J)_0, *, q\big)$ *for* A *a separable alternative algebra of degree* 3 *over* $\mathbb{K} = \mathbb{F}[\omega]$, *and* J *a* $\mathbb{K}/\mathbb{F}$*-involution of the second kind.*

  Two such symmetric composition algebras are isomorphic if and only if so are the corresponding alternative algebras, as algebras with involution.

Possibilities for such algebras A**:** Let $\mathbb{K} = \mathbb{F}[\omega]$, so that $\mathbb{K} = \mathbb{F}$ if $\omega \in \mathbb{F}$.

- $A = \mathbb{K} \times C$, with $\deg C = 2$ ($\Rightarrow C$ is a Hurwitz algebra!), then $(A_0, *, q)$ is isomorphic to the para-Hurwitz algebra attached to C if $\mathbb{K} = \mathbb{F}$, and $\big(K(A, J)_0, *, q\big)$ to the one attached to $\hat{C} = \{x \in C : J(x) = \bar{x}\}$ if $\mathbb{K} \neq \mathbb{F}$.
- A is a central simple associative algebra of degree 3, and hence $(A_0, *, q)$ or $\big(K(A, J)_0, *, q\big)$ is an Okubo algebra.
- $A = \mathbb{K} \otimes_{\mathbb{F}} L$, for a cubic field extension L of $\mathbb{F}$ (if $\omega \notin \mathbb{F}$ $L = \{x \in A : J(x) = x\}$) and $\dim_{\mathbb{F}} S = 2$.

Remark 2.7. The classification in characteristic 3 follows a different path to arrive at a similar result: any symmetric composition algebra is either para-Hurwitz or "Okubo", with a few exceptions in dimension 2.

Remark 2.8. Assume that $(S, *, q)$ is a two-dimensional symmetric composition algebra (in any characteristic).

If there is an element $a \in S$ such that $q(a) \neq 0$ and $a * a \in \mathbb{F}a$, then we may scale a and get an element $e \in S$ such that $e * e = e$ (so that $q(e) = 1$). Then S is the para-Hurwitz algebra attached to the Hurwitz algebra defined over S with the multiplication

$$x \cdot y = (e * x) * (y * e),$$

with unity $1 = e$.

Otherwise, take $a \in S$ with $q(a) = 1$ (this is always possible). Then $a * a \notin \mathbb{F}a$, so that $S = \mathbb{F}a \oplus \mathbb{F}(a * a)$, and the multiplication is completely determined by the scalar $\alpha = q(a, a * a)$:

$$\begin{aligned} a * (a * a) &= (a * a) * a = q(a)a = a, \\ (a * a) * (a * a) &= -((a * a) * a) * a + q(a, a * a)a = a * a - \alpha a. \end{aligned}$$

Triality:

Assume $\operatorname{char}\mathbb{F} \neq 2$, and let $(S, *, q)$ be a symmetric composition algebra. Consider the associated orthogonal Lie algebra

$$\mathfrak{so}(S, q) = \{d \in \operatorname{End}_{\mathbb{F}}(S) : q\big(d(x), y\big) + q\big(x, d(y)\big) = 0 \ \forall x, y \in S\}.$$

The *triality Lie algebra* of $(S, *, q)$ is defined as the following Lie subalgebra of $\mathfrak{so}(S, q)^3$ (with componentwise bracket):

$$\mathfrak{tri}(S, *, q) = \{(d_0, d_1, d_2) \in \mathfrak{so}(S, q)^3 : d_0(x * y) = d_1(x) * y + x * d_2(y) \ \forall x, y, z \in S\}.$$

Note that the condition $d_0(x * y) = d_1(x) * y + x * d_2(y)$ for any $x, y \in S$ is equivalent to the condition

$$q\big(x * y, d_0(z)\big) + q\big(d_1(x) * y, z\big) + q\big(x * d_2(y), z\big) = 0,$$

for any $x, y, z \in S$. But $q(x * y, z) = q(y * z, x) = q(z * x, y)$. Therefore, the linear map:

$$\begin{aligned} \theta : \mathfrak{tri}(S, *, q) &\longrightarrow \mathfrak{tri}(S, *, q) \\ (d_0, d_1, d_2) &\mapsto (d_2, d_0, d_1), \end{aligned}$$

is an automorphism of the Lie algebra $\mathfrak{tri}(S, *, q)$.

Theorem 2.9. *Let $(S, *, q)$ be an eight-dimensional symmetric composition algebra over a field of characteristic $\neq 2$. Then:*

(i) ***Principle of Local Triality:*** *The projection map:*

$$\begin{aligned} \pi_0 : \mathfrak{tri}(S, *, q) &\longrightarrow \mathfrak{so}(S, q) \\ (d_0, d_1, d_2) &\mapsto d_0 \end{aligned}$$

is an isomorphism of Lie algebras.

(ii) *For any $x, y \in S$, the triple*

$$t_{x,y} = \Bigl(\sigma_{x,y} = q(x, .)y - q(y, .)x, \frac{1}{2}q(x, y)id - r_x l_y, \frac{1}{2}q(x, y)id - l_x r_y\Bigr)$$

belongs to $\mathfrak{tri}(S, *, q)$, and $\mathfrak{tri}(S, *, q)$ is spanned by these elements. Moreover, for any $a, b, x, y \in S$:

$$[t_{a,b}, t_{x,y}] = t_{\sigma_{a,b}(x),y} + t_{x,\sigma_{a,b}(y)}.$$

Proof. Let us first check that $t_{x,y} \in \mathfrak{tri}(S, *, q)$:

$$\begin{aligned} \sigma_{x,y}(u * v) &= q(x, u * v)y - q(y, u * v)x \\ r_x l_y(u) * v &= \bigl((y * u) * x\bigr) * v = -(v * x) * (y * u) + q(y * u, v)x, \\ u * l_x r_y(v) &= u * \bigl(x * (v * y)\bigr) = -u * \bigl(y * (v * x)\bigr) + q(x, y)u * v \\ &= (v * x) * (y * u) + q(u, v * x)y + q(x, y)u * v, \end{aligned}$$

and hence

$$\sigma_{x,y}(u * v) - \Bigl(\frac{1}{2}q(x, y)id - r_x l_y\Bigr)(u) * v - u * \Bigl(\frac{1}{2}q(x, y)id - l_x r_y\Bigr)(v) = 0.$$

Also $\sigma_{x,y} \in \mathfrak{so}(S, q)$ and $\Bigl(\frac{1}{2}q(x, y)id - r_x l_y\Bigr)^* = \frac{1}{2}q(x, y)id - r_y l_x$ (adjoint relative to the norm q), but $r_x l_x = q(x)id$, so $r_x l_y + r_y l_x = q(x, y)id$ and hence $\Bigl(\frac{1}{2}q(x, y)id - r_x l_y\Bigr)^* = -\Bigl(\frac{1}{2}q(x, y)id - r_x l_y\Bigr)$, so that $\frac{1}{2}q(x, y)id - r_x l_y \in \mathfrak{so}(S, q)$, and $\frac{1}{2}q(x, y)id - l_x r_y \in \mathfrak{so}(S, q)$ too. Therefore, $t_{x,y} \in \mathfrak{tri}(S, *, q)$.

Since the Lie algebra $\mathfrak{so}(S, q)$ is spanned by the $\sigma_{x,y}$'s, it is clear that the projection π_0 is surjective (and hence so are π_1 and π_2). Consider an element (d_0, d_1, d_2) in $\ker \pi_0$. Hence $d_0 = 0$ and $d_1(x) * y + x * d_2(y) = 0$ for any $x, y \in S$. But since π_1 is onto, the subspace $\{d_1 \in \mathfrak{so}(S, q) : \exists d_2 \in \mathfrak{so}(S, q)\ (0, d_1, d_2) \in \mathfrak{tri}(S, *, q)\}$ is an ideal of the simple Lie algebra $\mathfrak{so}(S, q)$. Hence either $\ker \pi_0 = 0$ or for any $d \in \mathfrak{so}(S, q)$ there is another element $d' \in \mathfrak{so}(S, q)$ such that $d(x) * y + x * d'(y) = 0$ for any $x, y \in S$. This is impossible: take $d = \sigma_{a,b}$ for linearly independent elements $a, b \in S$ and take x orthogonal to a, b and not isotropic. Then $d(x) = 0$, so we would get $x * d'(y) = 0$ for any $y \in S$. This forces $d' = 0$ since l_x is a bijection, and we get a contradiction. Therefore, π_0 is an isomorphism.

Finally the formula $[t_{a,b}, t_{x,y}] = t_{\sigma_{a,b}(x),y} + t_{x,\sigma_{a,b}(y)}$ follows from the "same" formula for the σ's and the fact that π_0 is an isomorphism. □

For the results in this section one may consult [EM93] or [KMRT98, Chapter VIII].

3. Gradings on composition algebras.

3.1. **Generalities on gradings.**

Let A be an algebra (not necessarily associative) over our ground field $\mathbb{F}$, a *grading* on A is a decomposition

$$\Gamma : A = \oplus_{s\in S} A_s$$

of A into a direct sum of subspaces such that for any $s_1, s_2 \in S$ there exists a $s_3 \in S$ with $A_{s_1}A_{s_2} \subseteq A_{s_3}$.

Then:

- The *type* of a grading Γ on a finite dimensional algebra A is the sequence of numbers $(n_1, n_2, \dots, n_r)$ where n_i denotes the number of homogeneous spaces of dimension i, $i = 1, \dots, r$, $n_r \neq 0$. (Thus $\dim A = \sum_{i=1}^{r} in_i$.)
- Two gradings $\Gamma : A = \oplus_{s\in S} A_s$ and $\Gamma' : A' = \oplus_{s'\in S'} A'_{s'}$ are said to be *equivalent* if there is an isomorphism $\psi : A \to A'$ such that for any $s \in S$ there is a $s' \in S'$ with $\psi(A_s) = A'_{s'}$.
- Let Γ and Γ' be two gradings on A. The grading Γ is said to be a *refinement* of Γ' (or Γ' a *coarsening* of Γ) if for any $s \in S$ there is an index $s' \in S'$ such that $A_s \subseteq A_{s'}$. In other words, any homogeneous space in Γ' is a (direct) sum of some homogeneous spaces in Γ. A grading is called *fine* if it admits no proper refinement.
- The grading Γ is said to be a *group grading* if there is a group G containing S such that $A_{s_1}A_{s_2} \subseteq A_{s_1s_2}$ (multiplication of indices in the group G) for any $s_1, s_2 \in S$. Then we can write
$$\Gamma : A = \oplus_{g\in G} A_g.$$
The subset $\{g \in G : A_g \neq 0\}$ is called the *support* of the grading and denoted by $\operatorname{Supp}\Gamma$ (or $\operatorname{Supp} A$ if the context is clear). If the group G is abelian the grading is said to be an *abelian group grading*.
A group grading (respectively abelian group grading) is said to be *fine* if it admits no proper refinement in the class of group gradings (respectively abelian group gradings).
- Given a grading $\Gamma : A = \oplus_{s\in S} A_s$, one may consider the abelian group G generated by $\{s \in S : A_s \neq 0\}$ subject only to the relations $s_1 + s_2 = s_3$ if $0 \neq A_{s_1}A_{s_2} \subseteq A_{s_3}$. Then A is graded over G (or G-graded): $A = \oplus_{g\in G} A_g$, where A_g is the sum of the homogeneous components A_s such that the class of s in G is g. (Note that if Γ is already an abelian group grading there is at most one such homogeneous component.)
This group G has the following property: given any group grading $A = \oplus_{h\in H} A_h$ for an abelian group H which is a coarsening of Γ, then there exists a unique homomorphism of groups $f : G \to H$ such that $A_h = \oplus_{g\in f^{-1}(h)} A_g$. The group G is called the *universal grading group of* Γ. The universal grading groups of two equivalent gradings are isomorphic.
- Two abelian group gradings $\Gamma : A = \oplus_{g\in G} A_g$, and $\Gamma' : A' = \oplus_{g'\in G'} A'_h$ are said to be *isomorphic* if there is a group isomorphism $\varphi : G \to G'$ and an algebra isomorphism $\psi : A \to A'$ such that for any $g \in G$, $\psi(A_g) = A'_{\varphi(g)}$.
It is clear that isomorphic gradings are equivalent, but the converse does

not hold. However, two equivalent abelian group gradings are isomorphic when considered as gradings over their universal grading groups.

We will restrict ourselves most of the time to abelian group gradings, and hence additive notation will be used quite often.

3.2. **Gradings on Hurwitz algebras.**

Let $C = \oplus_{g\in G} C_g$ be a group grading of a Hurwitz algebra C, and assume, without loss of generality, that $\operatorname{Supp} C$ generates G. For any $x \in C$, $x^2 - q(x,1)x + q(x)1 = 0$. Always $1 \in C_e$, and hence if $x \in C_g$, with $g \neq e$:

- $q(x,1) = 0$ so that $\bar{C}_h = C_h$ for any $h \in G$,
- $q(x) = 0$ unless $g^2 = e$.

Take now $x \in C_g$, $y \in C_h$, then $q(x,y) = q(x\bar{y},1) = 0$ unless $gh = e$. But then for $g \neq h^{-1}$, $0 = q(x\bar{y},1)1 = x\bar{y} + y\bar{x}$, so that either $C_gC_h = 0 = C_hC_g$, or $gh = hg$.

Thus, if $g,h \in G$, with $g \neq h$ and $C_g \neq 0 \neq C_h$, $q(C_g + C_{g^{-1}}) \neq 0$ (q is nondegenerate), so that $(C_g + C_{g^{-1}})C_h \neq 0$, and hence either

- $C_gC_h \neq 0$, and then $gh = hg$, or
- $C_{g^{-1}}C_h \neq 0$, and then $g^{-1}h = hg^{-1}$, so $gh = hg$ too.

We conclude that G is abelian. In what follows, additive notation for G will be used.

Examples 3.1.

(1) **Gradings induced by the Cayley-Dickson doubling process:**
- If $C = CD(Q,\alpha) = Q \oplus Qu$, this is a $\mathbb{Z}_2$-grading: $C_{\bar{0}} = Q$, $C_{\bar{1}} = Qu$.
- If, moreover, $Q = CD(K,\beta) = K \oplus Kv$, then $C = K \oplus Kv \oplus Ku \oplus (Kv)u$ is a $\mathbb{Z}_2^2$-grading.
- Finally, if $K = CD(\mathbb{F},\gamma) = \mathbb{F}1 \oplus \mathbb{F}w$, then C is $\mathbb{Z}_2^3$-graded.

(2) **Cartan grading:** Take a canonical basis $\{e_1, e_2, u_1, u_2, u_3, v_1, v_2, v_3\}$ of the split Cayley algebra. Then C is $\mathbb{Z}^2$-graded with

$$\begin{aligned}
&C_{(0,0)} = \mathbb{F}e_1 \oplus \mathbb{F}e_2,\\
&C_{(1,0)} = \mathbb{F}u_1, \quad C_{(-1,0)} = \mathbb{F}v_1,\\
&C_{(0,1)} = \mathbb{F}u_2, \quad C_{(0,-1)} = \mathbb{F}v_2,\\
&C_{(1,1)} = \mathbb{F}v_3, \quad C_{(-1,-1)} = \mathbb{F}u_3.
\end{aligned}$$

The groups $\mathbb{Z}_2^r$ ($r = 1,2,3$) in the gradings induced by the Cayley-Dickson doubling process, and $\mathbb{Z}^2$ in the Cartan grading, are the universal grading groups.

Remark 3.2. The Cartan grading is fine as a group grading, but it is not so as a general grading, because the decomposition $C = \mathbb{F}e_1 \oplus \mathbb{F}e_2 \oplus \mathbb{F}u_1 \oplus \mathbb{F}u_2 \oplus \mathbb{F}u_3 \oplus \mathbb{F}v_1 \oplus \mathbb{F}v_2 \oplus \mathbb{F}v_3$ is a proper refinement. This refinement is not even a semigroup grading (because $(u_1u_2)u_3 = -e_2$ and $u_1(u_2u_3) = -e_1$ are in different homogeneous subspaces).

Theorem 3.3. *Any proper (abelian group) grading of a Cayley algebra is, up to equivalence, either a grading induced by the Cayley-Dickson doubling process or it is a coarsening of the Cartan grading of the split Cayley algebra.*

Proof. Let $C = \oplus_{g\in G} C_g$ be a grading of the Cayley algebra C and assume that $\operatorname{Supp} C$ generates G. Then C_0 is a composition subalgebra of C.

First case: Assume that G is 2-elementary. Then take $0 \neq g_1 \in G$ with $C_{g_1} \neq 0$. The restriction $q|_{C_{g_1}}$ is nondegenerate so we may take an element $u \in C_{g_1}$ with $q(u) \neq 0$, so that $C_{g_1} = C_0u$ and $C_0 \oplus C_{g_1} = C_0 \oplus C_0u = CD(C_0,\alpha)$ with $\alpha = -q(u)$.

This is a composition subalgebra of C, and hence either $C = C_0 \oplus C_{g_1}$ and $G = \mathbb{Z}_2$, or there is another element $g_2 \in G \setminus \{0, g_1\}$ with $C_{g_2} \neq 0$. Again take $v \in C_{g_2}$ with $q(v) \neq 0$ and we get $C_0 \oplus C_{g_1} \oplus C_{g_2} \oplus C_{g_1+g_2} = (C_0 \oplus C_{g_1}) \oplus (C_0 \oplus C_{g_1})v = CD(C_0 \oplus C_{g_1}, \beta) = CD(C_0, \alpha, \beta)$, which is a $\mathbb{Z}_2^2$-graded composition subalgebra of C. Again, either this is the whole C or we can repeat once more the process to get $C = CD(C_0, \alpha, \beta, \gamma)$ $\mathbb{Z}_2^3$-graded (and $\dim C_0 = 1$).

Second case: Assume that G is not 2-elementary, so there exists $g \in G$ with $C_g \neq 0$ and the order of g is > 2. Then $q(C_g) = 0$, so q is isotropic and hence C is the split Cayley algebra. Take elements $x \in C_g$, $y \in C_{-g}$ with $q(x, y) = -1$ (q is nondegenerate). That is, $q(xy, 1) = q(x, \bar{y}) = -q(x, y) = 1$.

Our considerations on isotropic Hurwitz algebras show that $e_1 = xy$ satisfies $e_1^2 = e_1$, $q(e_1) = 0$, $\bar{e}_1 = 1 - e_1 =: e_2$. Therefore $\mathbb{F}e_1 \oplus \mathbb{F}e_2$ is a composition subalgebra of C_0 and hence the subspaces $U = \{x \in C : e_1 x = x = x e_2\}$ and $V = \{x \in C : e_2 x = x = x e_1\}$ are graded subspaces of C and we may choose a basis $\{u_1, u_2, u_3\}$ of U consisting of homogeneous elements and such that $q(u_1 u_2, u_3) = 1$. With $v_1 = u_2 u_3$, $v_2 = u_3 u_1$ and $v_3 = u_1 u_2$ we get a canonical basis $\{e_1, e_2, u_1, u_2, u_3, v_1, v_2, v_3\}$ of C formed by homogeneous elements and such that $\deg(e_1) = \deg(e_2) = 0$. Let $g_i = \deg(u_i)$, $i = 1, 2, 3$. From $u_i v_i = -e_1$ we conclude that $\deg(v_i) = -g_i$, and from $v_1 = u_2 u_3$ we conclude that $g_1 + g_2 + g_3 = 0$. The grading is a coarsening of the Cartan grading. □

Remark 3.4. The number of non-equivalent gradings induced by the Cayley-Dickson doubling process depends on the ground field. Actually, the number of non equivalent $\mathbb{Z}_2$-gradings coincides with the number of isomorphism classes of quaternion subalgebras Q of the Cayley algebra.

For an algebraically closed ground field $\mathbb{F}$ this is one. Over $\mathbb{R}$ there are two non isomorphic Cayley algebras, the classical division algebra of the octonions $\mathbb{O} = C(-1, -1, -1)$ and the split Cayley algebra $\mathbb{O}_s = C(1, 1, 1)$. Any quaternion subalgebra of $\mathbb{O}$ is isomorphic to $\mathbb{H} = Q(-1, -1)$, while $\mathbb{O}_s$ contains quaternion subalgebras isomorphic to either $\mathbb{H}$ and $\mathrm{Mat}_2(\mathbb{F})$.

On the other hand, for p, q prime numbers congruent to 3 modulo 4, it is easy to check that the quaternion subalgebras $Q_p = CD\big(\mathbb{Q}(i), p\big)$ and $Q_q = CD\big(\mathbb{Q}(i), q\big)$ are not isomorphic. Consider the division algebra $Q = CD\big(\mathbb{Q}(i), -1\big)$. The split Cayley algebra over $\mathbb{Q}$ is isomorphic to $C = CD(Q, 1)$, and by the classical *Four Squares Theorem*, $Q^\perp$ contains elements whose norm is $-p$ for any prime number p. Therefore C contains a quaternion subalgebra isomorphic to Q_p for any prime number p, and hence the split Cayley algebra over $\mathbb{Q}$ is endowed with infinitely many non-equivalent $\mathbb{Z}_2$-gradings.

Over an algebraically closed field there is a unique $\mathbb{Z}_2^r$-grading, up to equivalence, for any $r = 1, 2, 3$. Over $\mathbb{R}$, $\mathbb{O}$ is endowed with a unique $\mathbb{Z}_2^r$-grading ($r = 1, 2, 3$) up to equivalence, while $\mathbb{O}_s$ is endowed with two non equivalent $\mathbb{Z}_2$ and $\mathbb{Z}_2^2$-gradings, but a unique $\mathbb{Z}_2^3$-grading. □

Up to symmetry, any coarsening of the Cartan grading is obtained as follows (recall $g_i = \deg(u_i)$, $i = 1, 2, 3$):

$\boxed{g_1 = 0}$**:** Then we obtain a "3-grading" over $\mathbb{Z}$: $C = C_{-1} \oplus C_0 \oplus C_1$, with $C_0 = \langle e_1, e_2, u_1, v_1 \rangle$, $C_1 = \langle u_2, v_3 \rangle$, $C_{-1} = \langle u_3, v_2 \rangle$. Its proper coarsenings are all "2-elementary".

$\boxed{g_1 = g_2}$**:** Here we obtain a "5-grading" over $\mathbb{Z}$, with $C_{-2} = \mathbb{F}u_3$, $C_{-1} = \langle v_1, v_2 \rangle$, $C_0 = \langle e_1, e_2 \rangle$, $C_1 = \langle u_1, u_2 \rangle$ and $C_2 = \mathbb{F}v_3$, which has two proper coarsenings which are not 2-elementary:

$\boxed{g_1 = g_2 = g_3}$: This gives a $\mathbb{Z}_3$-grading: $C_{\bar 0} = \langle e_1, e_2\rangle$, $C_{\bar 1} = U$, $C_{\bar 2} = V$.

$\boxed{g_3 = -g_3}$: This gives a $\mathbb{Z}_4$-grading.

$\boxed{g_1 = -g_1}$: Here we get a $\mathbb{Z} \times \mathbb{Z}_2$-grading

$$\begin{array}{ccccccccccccc} C = & C_{(0,\bar 0)} & \oplus & C_{(1,\bar 0)} & \oplus & C_{(-1,\bar 0)} & \oplus & C_{(0,\bar 1)} & \oplus & C_{(-1,\bar 1)} & \oplus & C_{(1,\bar 1)} \\ & \shortparallel & & \shortparallel & & \shortparallel & & \shortparallel & & \shortparallel & & \shortparallel \\ & \langle e_1, e_2\rangle & & \mathbb{F}u_2 & & \mathbb{F}v_2 & & \langle u_1, v_1\rangle & & \mathbb{F}u_3 & & \mathbb{F}v_3 \end{array}$$

Any of its coarsenings is a coarsening of the previous gradings.

$\boxed{g_1 = -g_2}$: In this case $g_3 = 0$, and this is equivalent to the grading obtained with $g_1 = 0$.

Theorem 3.5. *Up to equivalence, the (abelian group) gradings of the split Cayley algebra are:*

(i) *The $\mathbb{Z}_2^r$-gradings induced by the Cayley-Dickson doubling process.*

(ii) *The Cartan grading over $\mathbb{Z}^2$.*

(iii) *The 3-grading: $C_0 = span\{e_1, e_2, u_3, v_3\}$, $C_1 = span\{u_1, v_2\}$, $C_{-1} = span\{u_2, v_1\}$.*

(iv) *The 5-grading: $C_0 = span\{e_1, e_2\}$, $C_1 = span\{u_1, u_2\}$, $C_2 = span\{v_3\}$, $C_{-1} = span\{v_1, v_2\}$, $C_{-2} = span\{u_3\}$.*

(v) *The $\mathbb{Z}_3$-grading: $C_{\bar 0} = span\{e_1, e_2\}$, $C_{\bar 1} = U$, $C_{\bar 2} = V$.*

(vi) *The $\mathbb{Z}_4$-grading: $C_{\bar 0} = span\{e_1, e_2\}$, $C_{\bar 1} = span\{u_1, u_2\}$, $C_{\bar 2} = span\{u_3, v_3\}$, $C_{\bar 3} = span\{v_1, v_2\}$.*

(vii) *The $\mathbb{Z} \times \mathbb{Z}_2$-grading.*

Remark 3.6. The gradings on quaternion algebras are obtained in a similar but simpler way. Any (abelian group) grading is either induced by the Cayley-Dickson doubling process ($\mathbb{Z}_2^r$-grading for $0 \le r \le 2$) or it is the Cartan grading of $\mathrm{Mat}_2(\mathbb{F})$.

3.3. **Gradings on symmetric composition algebras.**

Let $S = \oplus_{g\in G} S_g$ be a group grading of the symmetric composition algebra $(S, *, q)$ and assume that $\mathrm{Supp}\, S$ generates G. Take nonzero homogeneous elements $x \in S_a$, $y \in S_b$ and $z \in S_c$. Then

$$(x * y) * z + (z * y) * x = q(x, z)y,$$

so $q(S_a, S_c) = 0$ unless $abc = b$ or $cba = b$. With $b = a$ we get $q(S_a, S_c) = 0$ unless $c = a^{-1}$. With $c = a^{-1}$, since q is nondegenerate we may take x and z with $q(x, z) = 1$, and hence either $aba^{-1} = b$ or $a^{-1}ba = b$. In any case $ab = ba$. Hence again the grading group must be abelian and additive notation will be used.

Proposition 3.7. *Let $(S, *, q)$ be a para-Hurwitz algebra of dimension 4 or 8, so that $x * y = \bar x \cdot \bar y$ for a Hurwitz product. Then the group gradings on $(S, *, q)$ and on the Hurwitz algebra $(S, \cdot, q)$ coincide.*

Proof. We know that given any grading $S = \oplus_{g\in G} S_g$ of the Hurwitz algebra $(S, \cdot, q)$, $\bar S_g = S_g$ for any g, and hence this is a grading too of $(S, *, q)$. Conversely, let $S = \oplus_{g\in G} S_g$ be a grading of $(S, *, q)$. Then

$$\begin{aligned} K &= \{x \in S : x * y = y * x \ \forall y \in S\} \\ &= \{x \in S : \bar x \cdot y = y \cdot \bar x \ \forall y \in S\} = \mathbb{F}1, \end{aligned}$$

because the dimension is at least 4. Thus $\mathbb{F}1$ is a graded subspace of $(S,*,q)$ and as $1*1=1$, it follows that $1\in S_0$. But then it is clear that $\bar{S}_g=S_g$ for any $g\in G$ (because $q(S_g,1)=0$ unless $g=0$) and the grading is a grading of the Hurwitz algebra. □

Therefore it is enough to study the gradings of the Okubo algebras. (And of the two-dimensional symmetric composition algebras, but this is quite easy: one gets either the trivial grading or a $\mathbb{Z}_2$-grading of a para-Hurwitz algebra or some $\mathbb{Z}_3$-gradings.)

To do this, let us first check what the possible gradings on the central simple associative algebras of degree 3 look like.

For $R=\mathrm{Mat}_3(\mathbb{F})$, there is the *Cartan grading* over $\mathbb{Z}^2$, with $\deg(E_{21})=(1,0)=-\deg(E_{12})$, $\deg(E_{32})=(0,1)=-\deg(E_{23})$. This is a fine grading and induces the Cartan grading on the Lie algebra $\mathfrak{sl}_3(\mathbb{F})$.

If $\operatorname{char}\mathbb{F}\neq 3$ and $\mathbb{F}$ contains the cubic roots of 1, then given any central simple degree three associative algebra R there are nonzero scalars α,β such that

$$R=\operatorname{alg}\left\langle x,y : x^3=\alpha,\ y^3=\beta,\ yx=\omega xy\right\rangle.$$

Think for example in $R=\mathrm{Mat}_3(\mathbb{F})$, and x and y given by:

$$x=\begin{pmatrix}1&0&0\\0&\omega&0\\0&0&\omega^2\end{pmatrix},\qquad y=\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}.$$

Then R is naturally $\mathbb{Z}_3^2$-graded with $\deg(x)=(\bar{1},\bar{0})$, $\deg(y)=(\bar{0},\bar{1})$. This is a fine grading too and it is called a *division grading*, as R becomes a graded division algebra (any nonzero homogeneous element is invertible).

Proposition 3.8. *Any proper (abelian group) grading of a central simple associative algebra of degree* 3 *is, up to equivalence, either a coarsening of the Cartan grading of* $\mathrm{Mat}_3(\mathbb{F})$ *or a division grading over* $\mathbb{Z}_3$ *or* $\mathbb{Z}_3^2$. *(In the latter case* $\operatorname{char}\mathbb{F}\neq 3$ *and* $\mathbb{F}$ *contains the cubic roots of* 1.*)*

Proof. Let $R=\oplus_{g\in G}R_g$ be such a grading, and let I be a minimal graded left ideal of R. I^2 is not 0 since R is simple, so there is an homogeneous element $0\neq x\in I$ such that $Ix\neq 0$ and, by minimality of I, we have $I=Ix$. Hence there is an element $e\in I_0$ such that $x=ex$. Again by minimality, $I\cap\{r\in R : rx=0\}=0$ holds, so $e^2-e=0$ and $I=Re$.

By the natural graded version of Schur's Lemma, the minimality of I forces the endomorphism ring $D=\mathrm{End}_R(I)$ to be a G-graded division algebra. (The action of the elements of D on I will be considered on the right.) Moreover, the map $D=\mathrm{End}_R(I)\to eRe$, $\varphi\mapsto e\varphi=e^2\varphi=e(e\varphi)\in eI=eRe$ is a G-graded isomorphism.

Now the map $R\to\mathrm{End}_{eRe}(I)$: $r\mapsto\varphi_r(: x\mapsto rx)$, is an isomorphism, as the image φ_R equals $\varphi_{IR}=\varphi_I\varphi_R$, which is a left ideal of $\mathrm{End}_{eRe}(I)$ containing the identity element $1=\varphi_1$, and hence it is the whole $\mathrm{End}_{eRe}(I)$. Also, $D\simeq eRe$ is a central simple algebra and R is a free right D-module, so by dimension count we get that either $D=\mathbb{F}1$ or $D=R$.

In the first case ($D=\mathbb{F}1$) take an homogeneous basis $\{e,x,y\}$ of I, $x\in I_{g_1}$, $y\in I_{g_2}$. Then the grading on $R=\mathrm{End}_D(I)=\mathrm{End}_{\mathbb{F}}(I)$ is induced by the grading on I. By means of the chosen basis of I, R can be identified to $\mathrm{Mat}_3(\mathbb{F})$ and its grading is a coarsening of the Cartan grading.

In the second case (R a graded division algebra), R_0 is a finite dimensional division algebra over $\mathbb{F}$ contained strictly in R. By dimension count either $R_0=\mathbb{F}1$, and then for any $g\in G$ with $R_g\neq 0$, $\dim R_g=1$, or $R_0=L$ is a cubic field extension of $\mathbb{F}$. If $R_0 = \mathbb{F}1$ we obtain easily that $\operatorname{Supp} R$ is a subgroup of G of order 9, which is not cyclic as R is not generated by a single element since it is not commutative. Hence $\operatorname{Supp} R = \mathbb{Z}_3^2$ and this is the universal grading group. Take nonzero elements $x \in R_{(\bar{1},\bar{0})}$, $y \in R_{(\bar{0},\bar{1})}$. Then $0 \neq x^3, y^3 \in R_0 = \mathbb{F}1$, so there are nonzero scalars α, β with $x^3 = \alpha$, $y^3 = \beta$. Besides, $yxy^{-1} \in R_{(\bar{1},\bar{0})} = \mathbb{F}x$, x and y do not commute, and the inner automorphism induced by y has order 3. It follows that $\mathbb{F}$ contains the cubic roots of 1 and (permuting x and y if necessary) $yxy^{-1} = \omega y$, thus getting a division grading as above.

If $R_0 = L$ is a cubic field extension of the ground field, then $\operatorname{Supp} R$ is a subgroup isomorphic to $\mathbb{Z}_3$. Note that for $x \in R_{\bar{1}}$, $1, x, x^2$ are linearly independent and $0 \neq x^3 \in \mathbb{F}1$ by the Cayley-Hamilton equation.. The automorphism $y \mapsto xyx^{-1}$ of $R_{\bar{0}} = L$ is nontrivial, so L is a Galois extension of $\mathbb{F}$. If the characteristic of $\mathbb{F}$ is $\neq 3$ and $\omega \in \mathbb{F}$, then there is an element $0 \neq y \in L$ with $xyx^{-1} = \omega^2 y$, so $yx = \omega xy$, $y^3 \in \mathbb{F}1$, and this grading is a coarsening of a $\mathbb{Z}_3^2$-grading. $\square$

Theorem 3.9. *Let $\mathbb{F}$ be a field of characteristic $\neq 3$ containing the cubic roots of 1. Then any (abelian group) grading of an Okubo algebra over $\mathbb{F}$ is a coarsening of either a $\mathbb{Z}^2$-grading or of a $\mathbb{Z}_3^2$-grading.*

Proof. Let $(S, *, q)$ be an Okubo algebra over $\mathbb{F}$ and $S = \oplus_{g\in G} S_g$ be a grading over the abelian group G. Let $A = \mathbb{F}1 \oplus S$ be the central simple associative algebra of degree 3 with multiplication determined by

$$xy = \frac{\omega}{\omega^2 - \omega} x * y + \frac{\omega^2}{\omega^2 - \omega} y * x + \frac{1}{3} q(x, y)1,$$

for any $x, y \in S$. Then since $q(S_g, S_h) = 0$ unless $g + h = 0$, the grading on S induces a grading on A, which is a coarsening of either the Cartan grading over $\mathbb{Z}^2$ of $\operatorname{Mat}_3(\mathbb{F})$, or a $\mathbb{Z}_3^2$-grading on either $\operatorname{Mat}_3(\mathbb{F})$ or a central division algebra of degree 3. $\square$

Remark 3.10. The group gradings on Okubo algebras have been completely determined over arbitrary fields, but the methods needed are different.

What do these $\mathbb{Z}^2$ and $\mathbb{Z}_3^2$-gradings look like?

$\mathbb{Z}^2$-grading: The type of this grading on $\operatorname{Mat}_3(\mathbb{F})$ ($\omega \in \mathbb{F}$) is $(6, 0, 1)$, so its type on S is $(6, 1)$, with $\dim S_0 = 2$ and $\dim S_g \leq 1$ for $g \neq 0$. Take $0 \neq g \in \mathbb{Z}^2$ with $S_g \neq 0 = S_{2g}$. Then $S_0 \oplus S_g \oplus S_{-g}$ is a para-quaternion subalgebra S with "para-unit" $e \in S_0$. Consider the Hurwitz algebra $(S, \cdot, q)$ with multiplication

$$x \cdot y = (e * x) * (y * e),$$

and unity e.

Lemma 3.11. *The map $\tau : S \to S$, such that $\tau(x) = q(x, e)e - x * e$ is an order 3 automorphism of both $(S, *)$ and $(S, \cdot)$.*

Proof. Define $\bar{x} = q(x, e)e - x$, then $\tau(x) = \bar{x} * e = \overline{x * e}$ ($q(x * e, e) = q(x, e * e) = q(x, e)$), so that $\tau(x) = r_e(\bar{x}) = \overline{r_e(x)}$, and hence $\tau^3(x) = r_e^3(\bar{x})$. But

$$((x*e)*e)*e = -(e*e)*(x*e) + q(x*e, e)e = -e*(x*e) + q(x, e)e = -x + q(e, x)e = \bar{x}.$$

Therefore, $\tau^3 = id$, and $\tau \neq id$, because otherwise e would be a "para-unit" of $(S, *, q)$ and this would force this algebra to be para-Hurwitz. Also $\tau^2(x) = (x *$

$e) * e = q(e,x)e - x * e = l_e(\bar{x}) = \overline{l_e(x)}$. Now,

$$\begin{aligned}\tau(x) * \tau(y) &= (q(e,x)e - x * e) * (q(e,y)e - y * e)\\ &= q(e,x)q(e,y)e - q(e,x)y - q(e,y)(x * e) * e + (x * e) * (y * e)\\ &= q(e,x)q(e,y)e - q(e,x)y - q(e,y)\Big(q(e,x)e - e * x\Big)\\ &\qquad + \Big(q(x * e, e)y - e * (y * (x * e))\Big)\\ &= q(e,y)e * x - e * (y * (x * e))\\ &= e * (e * (x * y)) = q(e, x * y)e - (x * y) * e = \tau(x * y),\end{aligned}$$

and hence τ is an automorphism of $(S, *, q)$. Since $\tau(e) = e$, it follows that τ is an automorphism too of $(S, \cdot, q)$. □

Note that the restriction of τ to the subalgebra $S_0 \oplus S_g \oplus S_{-g}$ is the identity, that all the homogeneous subspaces are invariant under τ and that for any $x, y \in S$ $x * y = (e * (x * e)) * ((e * y) * e) = (x * e) \cdot (e * y) = \tau(\bar{x}) \cdot \tau^2(\bar{y})$. That is,

$$x * y = \tau(\bar{x}) \cdot \tau^2(\bar{y}), \tag{3.12}$$

for any $x, y \in S$.

The automorphism τ being of order 3, it induces a $\mathbb{Z}_3$-grading of the Cayley algebra $(S, \cdot, q)$ with $\dim S_{\bar{0}} = 4$. There is just one possibility for such a grading (which is a $\mathbb{Z}$-grading too). It follows that there exists a canonical basis $\{e_1, e_2, u_1, u_2, u_3, v_1, v_2, v_3\}$ with $S_{\bar{0}} = \operatorname{span}\{e_1, e_2, u_1, v_1\}$, $S_{\bar{1}} = \operatorname{span}\{u_2, v_3\}$ and $S_{\bar{2}} = \operatorname{span}\{u_3, v_2\}$ (that is, $\tau|_{S_{\bar{0}}} = id$, $\tau|_{S_{\bar{1}}} = \omega id$ and $\tau|_{S_{\bar{2}}} = \omega^2 id$) and such that the $\mathbb{Z}^2$-grading is given by the canonical $\mathbb{Z}^2$-grading on the Hurwitz algebra $(S, \cdot, q)$ relative to this basis, with the product given by (3.12). The grading is thus expressed in terms of the Cartan grading of the split Cayley algebra.

$\mathbb{Z}_3^2$-grading: Here the type of this grading on the central simple associative algebra A is (9), and hence the type in S is (8) with $S_0 = 0$ and $\dim S_g = 1$ for any $g \neq 0$. As before, the associative algebra A appears as a crossed product

$$A = \operatorname{alg}\left\langle x, y : x^3 = \alpha,\ y^3 = \beta,\ yx = \omega xy \right\rangle.$$

In this situation $S = A_0 = \operatorname{span}\left\{x^i y^j : 0 \le i, j \le 2,\ (i,j) \neq (0,0)\right\}$ and $S_{(\bar{\imath},\bar{\jmath})} = \mathbb{F}x^i y^j$ for any $(\bar{0}, \bar{0}) \neq (\bar{\imath}, \bar{\jmath}) \in \mathbb{Z}_2^3$.

The material in this section is taken from [Eld98] and [Eld09b].

4. Gradings on the central simple Lie algebras of type G_2.

From now on, only abelian group gradings will be considered, and these will be referred to just as gradings.

4.1. Gradings and affine group schemes of automorphisms.

Let H be an abelian group, and let A be a finite dimensional nonassociative algebra over our ground field $\mathbb{F}$. Then, with $\mathbb{F}H$ the group algebra of H, one has equivalences:

$$\begin{aligned}H\text{-grading on } A \ &\leftrightarrow\ \rho : A \to A \otimes \mathbb{F}H \text{ structure of } \mathbb{F}H\text{-comodule algebra}\\ &\leftrightarrow\ \text{morphism of affine group schemes } \hat{\rho} : H_{\text{diag}} \to \mathbf{Aut}\, A\end{aligned}$$

where a grading $A = \oplus_{h\in H} A_h$ corresponds to the comodule algebra structure given by $\rho : a_h \mapsto a_h \otimes h$, for any $a_h \in A_h$, and conversely, such a structure ρ gives the H-grading where $A_h = \{x \in A : \rho(x) = x \otimes h\}$. Besides, H_{diag} denotes the affine group scheme whose Hopf algebra is $\mathbb{F}H$ (so that its $\mathbb{F}$-points are precisely the group homomorphisms (characters) $H \to \mathbb{F}^\times$), and given a comodule structure ρ as before, the attached morphism $\hat\rho$ is given by

$$\begin{aligned}\hat\rho_R : H_{\text{diag}}(R) = \operatorname{Hom}_{\text{alg}}(\mathbb{F}H, R) &\mapsto \operatorname{Aut}(A \otimes R),\\ \varphi\quad &\mapsto \hat\rho_R(\varphi) : a_h \otimes r \mapsto a_h \otimes \varphi(h)r.\end{aligned}$$

(Note that $\hat\rho_R(\varphi) = (I \otimes \text{mult}_R) \circ (I \otimes \varphi \otimes I) \circ (\rho \otimes I)$ where I denotes the identity map. If more precision is needed, we will use I_R to denote the identity map on the algebra R.)

And conversely, given any morphism $\hat\rho : H_{\text{diag}} \to \mathbf{Aut}\, A$, one gets the automorphism $\hat\rho_{\mathbb{F}H}(I) \in \operatorname{Aut}(A \otimes \mathbb{F}H)$, which restricts to $\rho : A \simeq A \otimes 1 \to A \otimes \mathbb{F}H$.

Remark 4.1. The affine group scheme $\mathbf{Aut}\, A$ takes any (unital commutative) algebra R over $\mathbb{F}$ to the group of automorphisms $\operatorname{Aut}(A\otimes R)$ (automorphisms as an algebra over R), and any algebra homomorphism $\varphi : R \to S$ to the group homomorphism $\mathbf{Aut}\, A(\varphi) : \operatorname{Aut}(A \otimes R) \to \operatorname{Aut}(A \otimes S)$, $\psi_R \mapsto \psi_S$, where ψ_S is the automorphism of $A \otimes S$ such that the following diagram:

$$\begin{array}{ccc} A\otimes R & \xrightarrow{\psi_R} & A\otimes R \\ {\scriptstyle I\otimes\varphi}\downarrow & & \downarrow{\scriptstyle I\otimes\varphi} \\ A\otimes S & \xrightarrow[\psi_S]{} & A\otimes S \end{array}$$

is commutative. □

Let $\varphi : G \to H$ be a homomorphism of abelian groups, let $\Gamma : A = \oplus_{g\in G} A_g$ be a grading of a finite dimensional algebra A, and let $\bar\Gamma : A = \oplus_{h\in H} A_h$ be the associated coarsening: $A_h = \oplus_{g\in\varphi^{-1}(h)} A_g$ for any $h \in H$. Denote by ρ_Γ and $\rho_{\bar\Gamma}$ the associated comodule maps, and by $\hat\rho_\Gamma$ and $\hat\rho_{\bar\Gamma}$ the corresponding morphisms $G_{\text{diag}} \to \mathbf{Aut}\, A$ and $H_{\text{diag}} \to \mathbf{Aut}\, A$.

The group homomorphism φ induces an algebra homomorphism $\mathbb{F}G \to \mathbb{F}H$ which will be denoted by φ too, and this induces a morphism of affine group schemes $\varphi^* : H_{\text{diag}} \to G_{\text{diag}}$.

Theorem 4.2. *With the hypotheses above, the equality $\hat\rho_{\bar\Gamma} = \hat\rho_\Gamma \circ \varphi^*$ holds.*

Proof. From the definition of $\bar\Gamma$, the diagram

$$\begin{array}{ccc} A\otimes \mathbb{F}G & \xrightarrow{(\hat\rho_\Gamma)_{\mathbb{F}G}(I_{\mathbb{F}G})} & A\otimes \mathbb{F}G \\ {\scriptstyle I\otimes\varphi}\downarrow & & \downarrow{\scriptstyle I\otimes\varphi} \\ A\otimes \mathbb{F}H & \xrightarrow[(\hat\rho_{\bar\Gamma})_{\mathbb{F}H}(I_{\mathbb{F}H})]{} & A\otimes \mathbb{F}H. \end{array}$$

is commutative, and this amounts to the condition

$$(\hat\rho_{\bar\Gamma})_{\mathbb{F}H}(I_{\mathbb{F}H}) = \mathbf{Aut}\, A(\varphi)\big((\hat\rho_\Gamma)_{\mathbb{F}G}(I_{\mathbb{F}G})\big).$$

But $\hat\rho_\Gamma$ is a natural transformation, so we have the commutative diagram

$$\begin{array}{ccc} G_{\text{diag}}(\mathbb{F}G) & \xrightarrow{(\hat\rho_\Gamma)_{\mathbb{F}G}} & \operatorname{Aut}(A\otimes \mathbb{F}G) \\ {\scriptstyle G_{\text{diag}}(\varphi)}\downarrow & & \downarrow{\scriptstyle \mathbf{Aut}\, A(\varphi)} \\ G_{\text{diag}}(\mathbb{F}H) & \xrightarrow[(\hat\rho_\Gamma)_{\mathbb{F}H}]{} & \operatorname{Aut}(A\otimes \mathbb{F}H), \end{array}$$

and we get

$$\big(\hat\rho_\Gamma\circ\varphi^*\big)_{\mathbb{F}H}(I_{\mathbb{F}H}) = (\hat\rho_\Gamma)_{\mathbb{F}H}(\varphi) = (\hat\rho_\Gamma)_{\mathbb{F}H}\circ G_{\mathrm{diag}}(\varphi)(I_{\mathbb{F}G}) = \mathbf{Aut}\,A(\varphi)\big((\hat\rho_\Gamma)_{\mathbb{F}G}(I_{\mathbb{F}G})\big).$$

That is,

$$\big(\hat\rho_\Gamma\circ\varphi^*\big)(I_{\mathbb{F}H}) = \mathbf{Aut}\,A(\varphi)\big((\hat\rho_\Gamma)_{\mathbb{F}G}(I_{\mathbb{F}G})\big) = (\hat\rho_{\bar\Gamma})_{\mathbb{F}H}(I_{\mathbb{F}H}),$$

and this shows that the comodule maps attached to $\hat\rho_\Gamma\circ\varphi^*$ and $\hat\rho_{\bar\Gamma}$ coincide, so $\hat\rho_\Gamma\circ\varphi^* = \hat\rho_{\bar\Gamma}$, as required. □

To ease the notation, given a grading $\Gamma : A = \oplus_{g\in G}A_g$ as above, with comodule map ρ_Γ and associated morphism $\hat\rho_\Gamma : G_{\mathrm{diag}} \to \mathbf{Aut}\,A$, the algebra automorphism $(\hat\rho_\Gamma)_{\mathbb{F}G}(I_{\mathbb{F}G}) : A\otimes\mathbb{F}G \to A\otimes\mathbb{F}G$ (which is determined by its restriction to $A \simeq A\otimes 1$, and this is nothing else but ρ_Γ), will be denoted too by ρ_Γ.

Assume now that A' is a second finite dimensional nonassociative algebra over $\mathbb{F}$ and that $\Phi : \mathbf{Aut}\,A \to \mathbf{Aut}\,A'$ is a morphism of affine group schemes. Given any G-grading $\Gamma : A = \oplus_{g\in G}A_g$ with associated comodule map ρ_Γ and morphism $\hat\rho_\Gamma$, we get a composite morphism

$$G_{\mathrm{diag}} \xrightarrow{\hat\rho_\Gamma} \mathbf{Aut}\,A \xrightarrow{\Phi} \mathbf{Aut}\,A',$$

which gives a G-grading Γ' on A'. This grading Γ' will be said to be *induced* from Γ through Φ.

Let $\varphi : G \to H$ be a group homomorphism as above, and let $\bar\Gamma$ and $\bar\Gamma'$ be the associated coarsenings in A and A'.

Then the next natural result holds:

Theorem 4.3. *Under the conditions above, $\bar\Gamma'$ is the grading on A' induced from $\bar\Gamma$ through Φ.*

Proof. It is enough to use the previous Proposition:

$$\hat\rho_{\bar\Gamma'} = \hat\rho_{\Gamma'}\circ\varphi^* = \Phi\circ\hat\rho_\Gamma\circ\varphi^* = \Phi\circ\hat\rho_{\bar\Gamma}.$$ □

Definition 4.4. Let $\Gamma : A = \oplus_{g\in G}A_g$ and $\check\Gamma : B = \oplus_{h\in H}B_h$ be isomorphic gradings on the finite dimensional nonassociative algebras A and B. That is, there is an algebra isomorphism $\psi : A \to B$ and a group isomorphism $\varphi : G \to H$ such that for any $g\in G$, $\psi(A_g) = B_{\varphi(g)}$. Then Γ and $\check\Gamma$ are said to be *isomorphic by means of* (ψ,φ).

Any algebra isomorphism $\psi : A \to B$ induces a morphism

$$\psi_* : \mathbf{Aut}\,A \to \mathbf{Aut}\,B$$

such that for any (unital commutative) $\mathbb{F}$-algebra R, the group homomorphism

$$\psi_*(R) : \mathrm{Aut}(A\otimes R) \to \mathrm{Aut}(B\otimes R)$$

is given by

$$\psi_*(R)(f) = (\psi\otimes I)f(\psi^{-1}\otimes I),$$

for any $f\in\mathrm{Aut}(A\otimes R)$.

Theorem 4.5. *Let $\Gamma : A = \oplus_{g\in G}A_g$ and $\check\Gamma : B = \oplus_{h\in H}B_h$ be gradings on the finite dimensional nonassociative algebras A and B. Then Γ and $\check\Gamma$ are isomorphic by means of (ψ,φ) if and only if the diagram*

$$\begin{array}{ccc} G_{\mathrm{diag}} & \xrightarrow{\hat\rho_\Gamma} & \mathbf{Aut}\,A \\ {\scriptstyle\varphi^*}\Big\uparrow & & \Big\downarrow{\scriptstyle\psi_*} \\ H_{\mathrm{diag}} & \xrightarrow[\hat\rho_{\check\Gamma}]{} & \mathbf{Aut}\,B, \end{array}$$

is commutative.

Proof. The gradings Γ and $\check{\Gamma}$ are isomorphic by means of (ψ,φ) if and only if the diagram

$$\begin{tikzcd} A\otimes\mathbb{F}G \arrow[r, "\rho_\Gamma"] \arrow[d, "\psi\otimes\varphi"'] & A\otimes\mathbb{F}G \arrow[d, "\psi\otimes\varphi"] \\ B\otimes\mathbb{F}H \arrow[r, "\rho_{\check{\Gamma}}"'] & B\otimes\mathbb{F}H, \end{tikzcd}$$

is commutative, and this happens if and only if

$$\begin{aligned}\rho_{\check{\Gamma}} &= (\psi\otimes I)(I\otimes\varphi)\rho_\Gamma(I\otimes\varphi^{-1})(\psi^{-1}\otimes I)\\ &= \psi_*\bigl((I\otimes\varphi)\rho_\Gamma(I\otimes\varphi^{-1})\bigr)\\ &= \psi_*\bigl(\rho_{\bar{\Gamma}}\bigr),\end{aligned}$$

where $\bar{\Gamma}$ is the 'coarsening' of Γ associated to φ (see the proof of Theorem 4.2). That is, Γ and $\check{\Gamma}$ are isomorphic by means of (ψ,φ) if and only if (Theorem 4.2) the equation

$$\hat{\rho}_{\check{\Gamma}}(I_{\mathbb{F}H}) = \psi_*\circ\hat{\rho}_{\bar{\Gamma}}(I_{\mathbb{F}H}) = \psi_*\circ\hat{\rho}_\Gamma\circ\varphi^*(I_{\mathbb{F}H})$$

holds.

But any morphism $H_{\text{diag}}\to\mathbf{Aut}\,B$ is determined by the image of $I_{\mathbb{F}H}$, hence this last equation is equivalent to the condition $\hat{\rho}_{\check{\Gamma}} = \psi_*\circ\hat{\rho}_\Gamma\circ\varphi^*$, as required. □

Corollary 4.6. *Let $\Gamma : A = \oplus_{g\in G}A_g$ and $\check{\Gamma} : A = \oplus_{h\in H}A_h$ be two gradings of the finite dimensional nonassociative algebra A which are isomorphic by means of (ψ,φ). Let A' be a second nonassociative algebra and let $\Phi : \mathbf{Aut}\,A\to\mathbf{Aut}\,A'$ be a morphism of affine group schemes. Then the induced gradings Γ' and $\check{\Gamma}'$ on A' through Φ are isomorphic by means of $(\Phi_{\mathbb{F}}(\psi),\varphi)$.*

Proof. Fist note that the diagram

$$\begin{tikzcd} \mathbf{Aut}\,A \arrow[r, "\Phi"] \arrow[d, "\psi_*"'] & \mathbf{Aut}\,A' \arrow[d, "\Phi_{\mathbb{F}}(\psi)_*"] \\ \mathbf{Aut}\,A \arrow[r, "\Phi"'] & \mathbf{Aut}\,A' \end{tikzcd}$$

commutes because for any R, if $\iota:\mathbb{F}\to R$ denotes the inclusion map and since Φ is a morphism of affine group schemes, the following diagram commutes:

$$\begin{tikzcd} \mathrm{Aut}\,A\simeq\mathrm{Aut}(A\otimes\mathbb{F}) \arrow[r, "\Phi_{\mathbb{F}}"] \arrow[d, "\mathbf{Aut}\,A(\iota)"'] & \mathrm{Aut}\,A'\simeq\mathrm{Aut}(A'\otimes\mathbb{F}) \arrow[d, "\mathbf{Aut}\,A'(\iota)"] \\ \mathrm{Aut}(A\otimes R) \arrow[r, "\Phi_R"'] & \mathrm{Aut}(A'\otimes R). \end{tikzcd}$$

Therefore $\Phi_{\mathbb{F}}(\psi)\otimes I_R = \mathbf{Aut}\,A'(\iota)\circ\Phi_{\mathbb{F}}(\psi) = \Phi_R\circ\mathbf{Aut}\,A(\iota)(\psi) = \Phi_R(\psi\otimes I_R)$, and this immediately shows that the previous diagram commutes.

Now we compute easily:

$$\begin{aligned}\hat{\rho}_{\check{\Gamma}'} &= \Phi\circ\hat{\rho}_{\check{\Gamma}}\\ &= \Phi\circ\psi_*\circ\hat{\rho}_\Gamma\circ\varphi^* \quad \text{(Theorem 4.5)}\\ &= \Phi_{\mathbb{F}}(\psi)_*\circ\Phi\circ\hat{\rho}_\Gamma\circ\varphi^*\\ &= \Phi_{\mathbb{F}}(\psi)_*\circ\hat{\rho}_{\Gamma'}\circ\varphi^*,\end{aligned}$$

as required. □

Remark 4.7. Therefore, if $\Phi : \mathbf{Aut}\, A \to \mathbf{Aut}\, A'$ is an isomorphism, the classification of gradings on A' up to isomorphism reduced to the corresponding classification on A.

Also, fine gradings on A correspond to fine gradings on A', *but only if universal grading groups are used!!*

Corollary 4.8. *Let $\Gamma : A = \oplus_{g\in G} A_g$ be a fine grading of the finite dimensional nonassociative algebra A and assume that G is the universal grading group. Let A' be a second finite dimensional nonassociative algebra and let $\Phi : \mathbf{Aut}\, A \to \mathbf{Aut}\, A'$ be an isomorphism of affine group schemes. Then the induced grading Γ' on A' through Φ is also a fine group grading and G is its universal grading group.*

Proof. Let $\check{\Gamma}' : A' = \oplus_{u\in U} A'_u$ be a fine refinement of Γ', with U being its universal grading group. Then there is a group homomorphism $\varphi : U \to G$ such that for any $g \in G$, $A'_g = \oplus_{u\in\varphi^{-1}(g)} A'_u$. Through Φ^{-1} we get a U-grading $\check{\Gamma} : A = \oplus_{u\in U} A_u$. By Theorem 4.3, Γ is the corresponding coarsening of $\check{\Gamma}$, so that for any $g \in G$ we have $A_g = \oplus_{u\in\varphi^{-1}(g)} A_u$. But Γ is fine so for any $g \in \operatorname{Supp}\Gamma$ there exists a unique $u \in \operatorname{Supp}\check{\Gamma}$ such that $\varphi(u) = g$. Since G is the universal grading group of Γ there is a group homomorphism $\psi : G \to U$ such that $\psi(g) = u$, and hence $\varphi \circ \psi = I$.

But then $\check{\Gamma}$ is the coarsening of Γ corresponding to ψ, so that Theorem 4.3 asserts that $\check{\Gamma}'$ is the corresponding coarsening of Γ'. It turns out that $\check{\Gamma}'$ and Γ' are equivalent, so that Γ' is fine and for any $u \in \operatorname{Supp}\check{\Gamma}'$ there is a unique $g \in \operatorname{Supp}\Gamma'$ such that $u = \psi(g)$. Since U is generated by $\operatorname{Supp}\Gamma'$, it follows that ψ is onto, and hence φ is a group isomorphism with inverse ψ, whence the result. $\square$

The condition on G being the universal grading group is essential, as we will check later on with Hurwitz algebras and their derivation algebras. In particular the relationship between gradings on A and A' given by a morphism Φ *does not in general preserve equivalence of gradings.*

4.2. **Gradings on Lie algebras of derivations.**

Given an affine group scheme G, for any (unital commutative) algebra R, $G(R)$ acts by conjugation on

$$\ker\Bigl(G(R(\epsilon)) \to G(R)\Bigr) = \operatorname{Lie}(G) \otimes R,$$

where $R(\epsilon) = R \otimes \mathbb{F}(\epsilon)$, with $\mathbb{F}(\epsilon) = \mathbb{F}1 \oplus \mathbb{F}\epsilon$ and $\epsilon^2 = 0$ (dual numbers), thus giving a linear representation (the adjoint action):

$$\operatorname{Ad} : G \to \mathbf{GL}\bigl(\operatorname{Lie}(G)\bigr).$$

For $G = \mathbf{Aut}\, A$, we have $\operatorname{Lie}(G) = \mathfrak{Der}(A)$ (the Lie algebra of derivations of A) and the image of the adjoint action is contained in the closed subgroup $\mathbf{Aut}\bigl(\mathfrak{Der}(A)\bigr)$.

Remark 4.9. The differential of the morphism $\operatorname{Ad} : \mathbf{Aut}\, A \to \mathbf{Aut}\bigl(\mathfrak{Der}(A)\bigr)$ is the adjoint map

$$\begin{aligned} \operatorname{ad} : \mathfrak{Der}(A) &\to \mathfrak{Der}(\mathfrak{Der}(A)) \\ X &\mapsto \operatorname{ad}_X : Y \mapsto [X, Y], \end{aligned}$$

because, with $G = \mathbf{Aut}\, A$, for any $X, Y \in \mathfrak{Der}(A)$, $(I+X\epsilon) \in \ker\Bigl(G(\mathbb{F}(\epsilon)) \to G(\mathbb{F})\Bigr)$ and

$$(I + X\epsilon)Y(I + X\epsilon)^{-1} = (I + X\epsilon)Y(I - X\epsilon) = Y + [X, Y]\epsilon = (I + \operatorname{ad}_X \epsilon)Y. \qquad \square$$

Therefore, given an H-grading on A with attached ρ, we get a composite morphism

$$H_{\mathrm{diag}} \xrightarrow{\hat\rho} \mathbf{Aut}\, A \xrightarrow{\mathrm{Ad}} \mathbf{Aut}\big(\mathfrak{Der}(A)\big),$$

which gives a H-grading on the Lie algebra $\mathfrak{Der}(A)$.

Proposition 4.10. *Given an H-grading on A: $A = \oplus_{h\in H} A_h$, the induced H-grading on $\mathfrak{Der}(A)$ is the natural one:*

$$\mathfrak{Der}(A) = \oplus_{h\in H}\mathfrak{Der}(A)_h, \quad \mathfrak{Der}(A)_h = \{d \in \mathfrak{Der}(A) : d(A_g) \subseteq A_{hg}\ \forall g \in H\}.$$

Proof. Let $\rho : A \to A\otimes\mathbb{F}H$ be the comodule algebra structure given by our grading, and recall that we denote too by ρ the automorphism $\hat\rho_{\mathbb{F}H}(I) : A\otimes\mathbb{F}H \to A\otimes\mathbb{F}H$ which extends ρ.

The comodule algebra structure on $\mathfrak{Der}(A)$ attached to the morphism $\mathrm{Ad}\circ\hat\rho$ is the map $\tilde\rho : \mathfrak{Der}(A) \mapsto \mathfrak{Der}(A)\otimes\mathbb{F}H$, given by

$$\tilde\rho(d) = \big(\mathrm{Ad}\circ\hat\rho\big)(I_{\mathbb{F}H})(d\otimes 1) = \hat\rho_{\mathbb{F}H}(I_{\mathbb{F}H})(d\otimes 1)\hat\rho_{\mathbb{F}H}(I_{\mathbb{F}H})^{-1}.$$

Thus for $x \in A_g$ and $d \in \mathfrak{Der}(A)_h$,

$$\begin{aligned}\tilde\rho(d)(x\otimes 1) &= \hat\rho_{\mathbb{F}H}(I_{\mathbb{F}H})(d\otimes 1)(x\otimes g^{-1})\\ &= \hat\rho_{\mathbb{F}H}(I_{\mathbb{F}H})(dx\otimes g^{-1})\\ &= dx\otimes hgg^{-1} = dx\otimes h\\ &= (d\otimes h)(x\otimes 1).\end{aligned}$$

Hence $\tilde\rho(d) = d\otimes h$ for any $d \in \mathfrak{Der}(A)_h$ as required. □

4.3. **Gradings on G_2.**

Let us consider now a Cayley algebra C over our ground field $\mathbb{F}$ and assume that the characteristic of $\mathbb{F}$ is $\neq 2, 3$ in what follows. Denote by $\mathbb{F}_{\mathrm{alg}}$ the algebraic closure of $\mathbb{F}$, and by C_{alg} the Cayley algebra $C\otimes\mathbb{F}_{\mathrm{alg}}$.

Then we have the affine group scheme $G = \mathbf{Aut}\, C$ and the adjoint morphism $\mathrm{Ad} : G \to \mathbf{Aut}\big(\mathfrak{Der}(C)\big)$. Since any derivation of $\mathfrak{Der}(C)$ is inner, its differential ad is an isomorphism.

The affine group scheme $G_{\mathrm{alg}} = \mathbf{Aut}\, C_{\mathrm{alg}}$ contains the algebraic (smooth) matrix group $\mathrm{Aut}\, C_{\mathrm{alg}}$ as a closed subgroup. Its coordinate algebra is $\mathbb{F}_{\mathrm{alg}}[\mathrm{Aut}\, C_{\mathrm{alg}}] = \mathbb{F}_{\mathrm{alg}}[\mathbf{Aut}\, C_{\mathrm{alg}}]_{\mathrm{red}}$ (the quotient of $\mathbb{F}_{\mathrm{alg}}[\mathbf{Aut}\, C_{\mathrm{alg}}]$ by its nilradical) and its Lie algebra is $\mathfrak{Der}(C_{\mathrm{alg}}) = \mathfrak{Der}(C)\otimes\mathbb{F}_{\mathrm{alg}}$, which coincides with the Lie algebra of $\mathbf{Aut}\, C_{\mathrm{alg}}$. Hence

$$\begin{aligned}\dim\mathrm{Lie}(G) \geq \dim G = \dim G_{\mathrm{alg}} &\geq \dim\mathrm{Aut}\, C_{\mathrm{alg}}\\ &= \dim\mathrm{Lie}(G_{\mathrm{alg}}) = \dim\mathfrak{Der}(C) = \dim\mathrm{Lie}(G),\end{aligned}$$

and we conclude that $G = \mathbf{Aut}\, C$ is smooth.

In the same vein, since any derivation of $\mathrm{Lie}(G) = \mathfrak{Der}(C)$ is inner:

$$\begin{aligned}\dim\mathrm{Lie}(G) \geq \dim\mathbf{Aut}(\mathrm{Lie}(G)) &\geq \dim\mathrm{Aut}\,\mathrm{Lie}(G)_{\mathrm{alg}}\\ &= \dim\mathrm{Lie}(G)_{\mathrm{alg}} = \dim\mathrm{Lie}(G),\end{aligned}$$

and $\mathbf{Aut}\big(\mathfrak{Der}(C)\big)$ is smooth too.

Then, since the adjoint map gives the known group isomorphism

$$\mathrm{Ad} : \mathbf{Aut}\, C(\mathbb{F}_{\mathrm{alg}}) = \mathrm{Aut}\, C_{\mathrm{alg}} \to \mathbf{Aut}(\mathfrak{Der}(C))(\mathbb{F}_{\mathrm{alg}}) = \mathrm{Aut}\big(\mathfrak{Der}(C_{\mathrm{alg}})\big),$$

it follows that Ad is an isomorphism of affine group schemes (see [KMRT98, (22.5)]).

The following result is then immediately drawn from this isomorphism.

Theorem 4.11. *The gradings on $\mathfrak{Der}(C)$ are those induced by gradings on C.*

Corollary 4.12. *Any proper grading of $\mathfrak{Der}(C)$ is either a $\mathbb{Z}_2^r$-grading $r = 0, 1, 2$ induced by the Cayley-Dickson doubling process or C is split, so $\mathfrak{Der}(C)$ is the split simple Lie algebra of type G_2, and the grading is a coarsening of the Cartan grading of $\mathfrak{Der}(C)$ (the root spaces relative to a split Cartan subalgebra).*

The Theorem above allows us to get all the (abelian group) gradings on $\mathfrak{Der}(C)$, but one has to be careful: each grading on our list in Theorem 3.5 is obtained as a group grading for many different groups.

Thus, for instance, consider the 3-grading of the split Cayley algebra C in Theorem 3.5(iii): $C_0 = \text{span}\,\{e_1, e_2, u_3, v_3\}$, $C_1 = \text{span}\,\{u_1, v_2\}$, $C_{-1} = \text{span}\,\{u_2, v_1\}$. As a $\mathbb{Z}$-grading it induces a 5-grading on $\mathfrak{Der}(C)$, with $\mathfrak{Der}(C)_2 = \text{span}\,\{D_{u_1,v_2}\} \neq 0$, where $D_{a,b} : c \mapsto [[a, b], c] + 3\big((ac)b - a(cb)\big)$ is the *inner derivation* defined by $a, b \in C$ (the linear span of the inner derivations fills $\mathfrak{Der}(C)$), so it has 5 different nonzero homogeneous components. Its type is $(2, 0, 0, 3)$. However, up to equivalence, this grading of C is also a $\mathbb{Z}_3$-grading, and as such it induces a $\mathbb{Z}_3$-grading on $\mathfrak{Der}(C)$ of type $(0, 0, 0, 1, 2)$.

As a further example, the Cartan grading of the split Cayley algebra C is obtained as a grading by any abelian group G containing two elements g_1, g_2 and g_3 such that $g_1 + g_2 + g_3 = 0$ and the elements $0, g_1, g_2, g_3, -g_1, -g_2, -g_3$ are all different. In particular, it can be obtained as a grading over $\mathbb{Z}_3^2$, with $g_1 = (\bar{1}, \bar{0})$ and $g_2 = (\bar{0}, \bar{1})$. However, the induced $\mathbb{Z}_3^2$-grading on $\mathfrak{Der}(C)$ is not equivalent to the Cartan grading, as some of the nonzero root spaces get together in this grading, and hence it is not fine. (See Corollary 4.8.)

Easy combinatorial arguments give all the gradings on $\mathfrak{Der}(C)$ in terms of the gradings on the Cayley algebra C in Theorem 3.5

Theorem 4.13. *Let C be a Cayley algebra over a field of characteristic $\neq 2, 3$. Up to equivalence, the proper (abelian group) gradings on $\mathfrak{Der}(C)$ are either the $\mathbb{Z}_2^r$-gradings ($r = 1, 2, 3$) induced by the Cayley-Dickson doubling process, or one of the following gradings in the split case:*

(i) *Eleven gradings induced by the Cartan grading on C with universal grading groups: $\mathbb{Z}^2$,$\mathbb{Z}_7$, $\mathbb{Z}_8$, $\mathbb{Z}_9$, $\mathbb{Z}_{10}$, $\mathbb{Z}$, $\mathbb{Z}_6 \times \mathbb{Z}_2$, $\mathbb{Z} \times \mathbb{Z}_2$, $\mathbb{Z}_{12}$, $\mathbb{Z} \times \mathbb{Z}_3$ and $\mathbb{Z}_3^2$.*

(ii) *Three gradings induced by the 3-grading on C with universal grading groups $\mathbb{Z}$, $\mathbb{Z}_3$ and $\mathbb{Z}_4$.*

(iii) *Three gradings induced by the 5-grading on C with universal grading groups $\mathbb{Z}$, $\mathbb{Z}_5$ and $\mathbb{Z}_6$.*

(iv) *The $\mathbb{Z}_3$-grading induced by the $\mathbb{Z}_3$-grading on C.*

(v) *The $\mathbb{Z}_4$-grading induced by the $\mathbb{Z}_4$-grading on C.*

(vi) *Three gradings induced by the $\mathbb{Z} \times \mathbb{Z}_2$-grading on C with universal grading groups $\mathbb{Z} \times \mathbb{Z}_2$, $\mathbb{Z}_3 \times \mathbb{Z}_2$ and $\mathbb{Z}_4 \times \mathbb{Z}_2$.*

In particular, over an algebraically closed field of characteristic $\neq 2, 3$ there are exactly 25 different gradings, up to equivalence, of the central simple Lie algebra of type G_2.

Much of the material in this section is new. It relies on [KMRT98, Chapter VI]. The gradings on the simple Lie algebra of type G_2 (but only over algebraically closed fields of characteristic 0) have been considered in [DM06] and [BT09].

5. The Magic Square of exceptional Lie algebras. Induced gradings.

Throughout this lecture the characteristic of the ground field $\mathbb{F}$ will always be assumed to be $\neq 2,3$.

Given two symmetric composition algebras $(S,*,q)$ and $(S',\star,q')$, consider the vector space:

$$\mathfrak{g}=\mathfrak{g}(S,S')=\bigl(\mathfrak{tri}(S)\oplus\mathfrak{tri}(S')\bigr)\oplus\Bigl(\oplus_{i=0}^{2}\iota_i(S\otimes S')\Bigr),$$

where $\iota_i(S\otimes S')$ is just a copy of $S\otimes S'$ ($i=0,1,2$) and we write $\mathfrak{tri}(S)$, $\mathfrak{tri}(S')$ instead of $\mathfrak{tri}(S,*,q)$ and $\mathfrak{tri}(S',\star,q')$ for short. Define now an anticommutative bracket on $\mathfrak{g}$ by means of:

- the Lie bracket in $\mathfrak{tri}(S)\oplus\mathfrak{tri}(S')$, which thus becomes a Lie subalgebra of $\mathfrak{g}$,
- $[(d_0,d_1,d_2),\iota_i(x\otimes x')]=\iota_i\bigl(d_i(x)\otimes x'\bigr)$,
- $[(d_0',d_1',d_2'),\iota_i(x\otimes x')]=\iota_i\bigl(x\otimes d_i'(x')\bigr)$,
- $[\iota_i(x\otimes x'),\iota_{i+1}(y\otimes y')]=\iota_{i+2}\bigl((x*y)\otimes(x'\star y')\bigr)$ (indices modulo 3),
- $[\iota_i(x\otimes x'),\iota_i(y\otimes y')]=q'(x',y')\theta^i(t_{x,y})+q(x,y)\theta'^i(t'_{x',y'})\in\mathfrak{tri}(S)\oplus\mathfrak{tri}(S')$.

Theorem 5.1. *With this bracket, $\mathfrak{g}(S,S')$ is a Lie algebra and, if S_r and S_s' denote symmetric composition algebras of dimension r and s, then the Lie algebra $\mathfrak{g}(S_r,S_s')$ is a (semi)simple Lie algebra whose type is given by Freudenthal's Magic Square:*

	S_1	S_2	S_4	S_8
S_1'	A_1	A_2	C_3	F_4
S_2'	A_2	$A_2\oplus A_2$	A_5	E_6
S_4'	C_3	A_5	D_6	E_7
S_8'	F_4	E_6	E_7	E_8

Proof. "Straightforward" (but lengthy). □

The Lie algebra $\mathfrak{g}=\mathfrak{g}(S,S')$ is naturally $\mathbb{Z}_2^2$-graded with

$$\mathfrak{g}_{(\bar0,\bar0)}=\mathfrak{tri}(S)\oplus\mathfrak{tri}(S'),$$
$$\mathfrak{g}_{(\bar1,\bar0)}=\iota_0(S\otimes S'),\quad \mathfrak{g}_{(\bar0,\bar1)}=\iota_1(S\otimes S'),\quad \mathfrak{g}_{(\bar1,\bar1)}=\iota_2(S\otimes S').$$

Now, this $\mathbb{Z}_2^2$-grading can be combined with gradings on S and S' to obtain some nice gradings of the exceptional simple Lie algebras.

Also, the triality automorphisms θ and θ' induce an order 3 automorphism $\Theta\in\operatorname{Aut}\mathfrak{g}$ such that

$$\begin{cases}\Theta|_{\mathfrak{tri}(S)}=\theta,\quad \Theta|_{\mathfrak{tri}(S')}=\theta',\\ \Theta\bigl(\iota_i(x\otimes x')\bigr)=\iota_{i+1}(x\otimes x')\quad\text{(indices modulo 3)}\end{cases}$$

If $\omega\in\mathbb{F}$ this gives a $\mathbb{Z}_3$-grading which can be combined too with the gradings on S and S'.

Examples 5.2.

- The $\mathbb{Z}_2^3$-grading on a Cayley algebra C give a fine grading of the simple Lie algebra $\mathfrak{g}_2=\mathfrak{Der}(C)$, where

$$\mathfrak{g}_2=\oplus_{0\neq\alpha\in\mathbb{Z}_2^3}(\mathfrak{g}_2)_\alpha,$$

and

$\boxed{(\mathfrak{g}_2)_\alpha \text{ is a Cartan subalgebra for any } 0 \neq \alpha \in \mathbb{Z}_2^3!}$

It induces too a $\mathbb{Z}_2^3$-grading on $\mathfrak{d}_4 = \mathfrak{so}(C,q)$ with

$$\mathfrak{d}_4 = \oplus_{0\neq\alpha\in\mathbb{Z}_2^3}(\mathfrak{d}_4)_\alpha,$$

where again

$\boxed{(\mathfrak{d}_4)_\alpha \text{ is a Cartan subalgebra for any } 0 \neq \alpha \in \mathbb{Z}_2^3!}$

But this grading is not fine. It can be refined (if $\omega \in \mathbb{F}$) by means of the triality automorphism θ of $\mathfrak{tri}(\bar{C}) \simeq \mathfrak{so}(C,q)$ to get a fine $\mathbb{Z}_2^3 \times \mathbb{Z}_3$-grading of type $(14,7)$.

- Let $(\mathcal{O},*,q)$ be an Okubo algebra and assume that $\omega \in \mathbb{F}$. A $\mathbb{Z}_3^2$-grading on $\mathcal{O}$, combined with the automorphism Θ, induces a $\mathbb{Z}_3^3$-grading of $\mathfrak{f}_4 = \mathfrak{g}(\mathbb{F},\mathcal{O})$. Again,
$$\mathfrak{f}_4 = \oplus_{0\neq\alpha\in\mathbb{Z}_3^3}(\mathfrak{f}_4)_\alpha,$$
with $\dim(\mathfrak{f}_4)_\alpha = 2$ for any $0 \neq \alpha \in \mathbb{Z}_3^3$, and
$\boxed{(\mathfrak{f}_4)_\alpha \oplus (\mathfrak{f}_4)_{-\alpha} \text{ is a Cartan subalgebra for any } 0 \neq \alpha \in \mathbb{Z}_3^3!}$
This can be extended to a $\mathbb{Z}_3^3$-grading on $\mathfrak{e}_6 = \mathfrak{g}(S_2,\mathcal{O})$ with similar properties: $\dim(\mathfrak{e}_6)_\alpha = 3$ and
$\boxed{(\mathfrak{e}_6)_\alpha \oplus (\mathfrak{e}_6)_{-\alpha} \text{ is a Cartan subalgebra for any } 0 \neq \alpha \in \mathbb{Z}_3^3!}$
- Consider now two $\mathbb{Z}_2^3$-graded para-Cayley algebras $\bar{C}$ and $\bar{C}'$. The natural $\mathbb{Z}_2^2$-grading of $\mathfrak{g}(\bar{C},\bar{C}')$ combined with the $\mathbb{Z}_2^3$-grading on $\bar{C}\otimes\bar{C}'$ induces a $\mathbb{Z}_2^5$-grading:
$$\mathfrak{e}_8 = \oplus_{0\neq\alpha\in\mathbb{Z}_2^5}(\mathfrak{e}_8)_\alpha,$$
such that
$\boxed{(\mathfrak{e}_8)_\alpha \text{ is a Cartan subalgebra for any } 0 \neq \alpha \in \mathbb{Z}_3^3!}$
This is a famous *Dempwolff* decomposition considered by Thompson [Tho76].

Jordan gradings: Alekseevskii [Al74] considered *Jordan subgroups* A of $\operatorname{Aut}\mathfrak{g}$ for the simple complex Lie algebras. Any such group is abelian and:

(i) its normalizer is finite,

(ii) A is a minimal normal subgroup of its normalizer,

(iii) its normalizer is maximal among the normalizers of abelian subgroups satisfying (i) and (ii).

He classified (1974) these groups and gave detailed models of all the possibilities for classical simple Lie algebras. The exceptional cases are:

$\mathfrak{g}$	A	$\dim\mathfrak{g}_\alpha$ $(\alpha\neq 0)$
G_2	$\mathbb{Z}_2^3$	2
F_4	$\mathbb{Z}_3^3$	2
E_8	$\mathbb{Z}_5^3$	2
D_4	$\mathbb{Z}_2^3$	4
E_8	$\mathbb{Z}_2^5$	8
E_6	$\mathbb{Z}_3^3$	3

With the exception of the $\mathbb{Z}_5^3$-grading of E_8, these are precisely the gradings considered in the previous examples.

This exception can be obtained as follows. Let V_1 and V_2 be two vector spaces over $\mathbb{F}$ of dimension 5, and consider the $\mathbb{Z}_5$-graded vector space

$$\mathfrak{g} = \oplus_{i=0}^4 \mathfrak{g}_{\bar{i}},$$

where

$$\begin{aligned}
\mathfrak{g}_{\bar{0}} &= \mathfrak{sl}(V_1) \oplus \mathfrak{sl}(V_2),\\
\mathfrak{g}_{\bar{1}} &= V_1 \otimes \textstyle\bigwedge^2 V_2,\\
\mathfrak{g}_{\bar{2}} &= \textstyle\bigwedge^2 V_1 \otimes \bigwedge^4 V_2,\\
\mathfrak{g}_{\bar{3}} &= \textstyle\bigwedge^3 V_1 \otimes V_2,\\
\mathfrak{g}_{\bar{4}} &= \textstyle\bigwedge^4 V_1 \otimes \bigwedge^3 V_2.
\end{aligned}$$

This is a $\mathbb{Z}_5$-graded Lie algebra, with the natural action of the semisimple algebra $\mathfrak{g}_{\bar{0}}$ on each of the other homogeneous components, and the brackets between elements in different components are given by suitable scalar multiples of the only $\mathfrak{g}_{\bar{0}}$-invariant possibilities. In this way, $\mathfrak{g}$ is the exceptional simple Lie algebra of type E_8. The details of the Lie multiplication have been computed in [Dra05].

Fix bases for V_1 and V_2, let ξ be a primitive fifth root of 1 in $\mathbb{F}$ and consider the endomorphisms b_1 and c_1 of V_1 with coordinate matrices:

$$b_1 \leftrightarrow \begin{pmatrix} 1&0&0&0&0\\ 0&\xi&0&0&0\\ 0&0&\xi^2&0&0\\ 0&0&0&\xi^3&0\\ 0&0&0&0&\xi^4 \end{pmatrix}, \qquad c_1 \leftrightarrow \begin{pmatrix} 0&0&0&0&1\\ 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0 \end{pmatrix},$$

and the endomorphisms of V_2 with coordinate matrices:

$$b_2 \leftrightarrow \begin{pmatrix} 1&0&0&0&0\\ 0&\xi^2&0&0&0\\ 0&0&\xi^4&0&0\\ 0&0&0&\xi&0\\ 0&0&0&0&\xi^3 \end{pmatrix}, \qquad c_2 \leftrightarrow \begin{pmatrix} 0&0&0&0&1\\ 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0 \end{pmatrix}.$$

Then the only $\mathbb{Z}_5^3$-grading of $\mathfrak{g}$ such that $\dim \mathfrak{g}_\alpha = 2$ for any $0 \neq \alpha \in \mathbb{Z}_5^3$ is given by the common eigenspaces of the automorphisms $\sigma_1, \sigma_2, \sigma_3$ such that

$$\begin{aligned}
&\sigma_1(x) = \xi^i x \quad \text{for any } x \in \mathfrak{g}_{\bar{i}} \text{ and } 0 \leq i \leq 4,\\
&\sigma_2|_{\mathfrak{g}_{\bar{1}}} = b_1 \otimes \wedge^2 b_2,\\
&\sigma_3|_{\mathfrak{g}_{\bar{1}}} = c_1 \otimes \wedge^2 c_2.
\end{aligned}$$

Some other related results: Assume $\mathbb{F}$ algebraically closed of characteristic 0.

- **Fine gradings of F_4 [DM09]:**
  - Cartan grading (over $\mathbb{Z}^4$),
  - The $\mathbb{Z}_2^5$-grading on $\mathfrak{f}_4 = \mathfrak{g}(k, \bar{C})$ (C a Cayley algebra) obtained by combining the natural $\mathbb{Z}_2^2$-grading on $\mathfrak{g}(k, \bar{C})$ and the $\mathbb{Z}_2^3$-grading on $\bar{C}$.
  - The $\mathbb{Z}_3^3$-grading on $\mathfrak{f}_4 = \mathfrak{g}(k, \mathcal{O})$ obtained by combining the $\mathbb{Z}_3^2$-grading on $\mathcal{O}$ with the $\mathbb{Z}_3$-grading induced by the automorphism Θ.
  - A $\mathbb{Z}_2^3 \times \mathbb{Z}$-grading: the $\mathbb{Z}_2^2$-grading on $\mathfrak{g}(k, \bar{C})$ can be "unfolded" to a $\mathbb{Z}$-grading compatible with the $\mathbb{Z}_2^3$-grading on $\bar{C}$. (This is related to the fact that C is a *structurable algebra*.)

- **Fine gradings of E_6:**
  Many of the fine gradings here are related to the construction of $\mathfrak{e}_6$ as either $\mathfrak{g}(\bar{K},\bar{C})$ or $\mathfrak{g}(\bar{K},\mathcal{O})$, where $K=\mathbb{F}\times\mathbb{F}$.
- **Fine gradings of D_4 (**[DV08]**,** [DMVpr]**,** [Eldpr]**):**
  Among the **17** fine gradings of D_4, there are 3 of them which have no counterparts for D_n, $n>4$:
  - A $\mathbb{Z}^2\times\mathbb{Z}_3$-grading obtained by combining the $\mathbb{Z}^2$-grading on C and the $\mathbb{Z}_3$-grading given by the triality automorphism.
  - A $\mathbb{Z}_2^3\times\mathbb{Z}_3$-grading obtained by combining the $\mathbb{Z}_2^3$-grading on C and the $\mathbb{Z}_3$-grading given by the triality automorphism.
  - A $\mathbb{Z}_3^3$-grading obtained by combining the $\mathbb{Z}_3^2$- grading on $\mathcal{O}$ and the $\mathbb{Z}_3$-grading given by the triality automorphism.
- **Fine gradings of the exceptional simple Lie superalgebras** [DEMpr]**:**
  - The two fine gradings of $\mathfrak{g}(3)=\big(\mathfrak{sp}(V)\oplus\mathfrak{Der}(C)\big)\oplus\big(V\otimes[C,C]\big)$ $(\dim V=2)$, over $\mathbb{Z}^3=\mathbb{Z}\times\mathbb{Z}^2$ and $\mathbb{Z}\times\mathbb{Z}_2^3$, are induced by the fine gradings of C.
  - Among the **5** fine gradings of $\mathfrak{f}(4)=\big(\mathfrak{sp}(V)\oplus\mathfrak{so}(C)\big)\oplus\big(V\otimes C\big)$, two of them are induced by the fine gradings on C, another two are induced by the two fine gradings of the 10-dimensional Kac superalgebra (a simple exceptional Jordan superalgebra), as $\mathfrak{f}(4)$ is the Lie superalgebra obtained by the Tits-Kantor-Koecher construction from K_{10}: $\mathfrak{f}(4)=\mathcal{TKK}(K_{10})=\big([Q,Q]\otimes K_{10}\big)\oplus\mathfrak{Der}(K_{10})$, combining the $\mathbb{Z}_2^2$ fine grading of the quaternion algebra Q with either the $\mathbb{Z}^2$ or $\mathbb{Z}\times\mathbb{Z}_2$ fine grading of K_{10}. The remaining fine grading (over $\mathbb{Z}_2^3\times\mathbb{Z}_4$) is related to a construction of $\mathfrak{f}(4)$ in terms of two quaternion algebras.
  - Most of the fine gradings of $\mathfrak{d}(2,1;\alpha)$ are related to the fine gradings of the 4-dimensional Jordan superalgebras D_t through the Tits-Kantor-Koecher construction.

It is hoped that the construction $\mathfrak{g}(S,S')$, together with some other constructions of the exceptional simple Lie algebras in terms of nonassociative algebra, will allow nice descriptions of a large portion of the fine gradings on the exceptional Lie algebras E_6,E_7,E_8.

The results in this section are taken from [Eld04], [Eld09a] and [Eld09b].

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