Gradings on the octonions and the Albert algebra

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Gradings on Lie algebras have been extensively used since the beginning of Lie theory:

- the Cartan grading on a complex semisimple Lie algebra is the $\mathbb{Z}^r$-grading (r being the rank) whose homogeneous components are the root spaces relative to a Cartan subalgebra (which is the zero component),
- symmetric spaces are related to $\mathbb{Z}_2$-gradings,
- Kac–Moody Lie algebras to gradings by a finite cyclic group,
- the theory of Jordan algebras and pairs to 3-gradings on Lie algebras, etc.

In 1989, a systematic study of gradings on Lie algebras was started by Patera and Zassenhaus.

Fine gradings on the classical simple complex Lie algebras, other than D_4, by arbitrary abelian groups were considered by Havlícek, Patera, and Pelantova in 1998.

The arguments there are computational and the problem of classification of fine gradings is not completely settled. The complete classification, up to equivalence, of fine gradings on all classical simple Lie algebras (including D_4) over algebraically closed fields of characteristic zero has been obtained quite recently.

For any abelian group G, the classification of all G-gradings, up to isomorphism, on the classical simple Lie algebras other than D_4 over algebraically closed fields of characteristic different from two has been achieved in 2010 by Bahturin and Kochetov, using methods developed in the last years by a number of authors.

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The gradings on the octonions and on the Albert algebra are instrumental in obtaining the gradings on the exceptional simple Lie algebras.

Definition

G abelian group, $\mathcal{A}$ algebra over a field $\mathbb{F}$.

G-grading on $\mathcal{A}$:

$$\mathcal{A} = \oplus_{g\in G}\mathcal{A}_g,$$

$$\mathcal{A}_g\mathcal{A}_h \subseteq \mathcal{A}_{gh} \qquad \forall g, h \in G.$$

Example: Pauli matrices

$\mathcal{A} = \mathrm{Mat}_n(\mathbb{F})$

$$X = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & \epsilon & 0 & \dots & 0 \\ 0 & 0 & \epsilon^2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \epsilon^{n-1} \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}$$

(ϵ a primitive nth root of 1)

$$X^n = 1 = Y^n, \qquad YX = \epsilon XY$$

$$\mathcal{A} = \oplus_{(\bar{\imath},\bar{\jmath})\in\mathbb{Z}_n\times\mathbb{Z}_n}\mathcal{A}_{(\bar{\imath},\bar{\jmath})}, \qquad \mathcal{A}_{(\bar{\imath},\bar{\jmath})} = \mathbb{F}X^iY^j.$$

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$\mathcal{A}$ becomes a *graded division algebra*.

Basic definitions (Patera-Zassenhaus)

Let $\Gamma : \mathcal{A} = \oplus_{g\in G}\mathcal{A}_g$ be a grading on $\mathcal{A}$ $\quad(\dim_{\mathbb{F}} \mathcal{A} < \infty$, $\mathbb{F} = \bar{\mathbb{F}}$, $\operatorname{char} \mathbb{F} \neq 2)$:

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- The *universal grading group* of Γ is the group $U(\Gamma)$ generated by $\mathrm{Supp}\,\Gamma$ subject to the relations $g_1g_2 = g_3$ if $0 \neq \mathcal{A}_{g_1}\mathcal{A}_{g_2} \subseteq \mathcal{A}_{g_3}$.

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The grading Γ is then a grading too by $U(\Gamma)$.

Basic definitions (Patera-Zassenhaus)

There appear several groups attached to Γ:

- The *automorphism group*

$$\begin{aligned}\mathrm{Aut}(\Gamma) = \{\varphi \in \mathrm{Aut}\,\mathcal{A} :\\ \exists \alpha \in \mathit{Sym}(\mathrm{Supp}\ \Gamma) \text{ s.t. } \varphi(\mathcal{A}_g) \subseteq \mathcal{A}_{\alpha(g)}\ \forall g\}.\end{aligned}$$

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- The quotient $W(\Gamma) = \mathrm{Aut}(\Gamma)/\,\mathrm{Stab}(\Gamma)$ is the *Weyl group* of Γ.

$W(\Gamma)$ acts by automorphisms on $U(\Gamma)$

Each $\varphi \in \mathrm{Aut}(\Gamma)$ determines a self-bijection α of $\mathrm{Supp}\,\Gamma$ that induces an automorphism of the universal grading group $U(\Gamma)$. Then, there appears a natural group homomorphism:

$$\mathrm{Aut}(\Gamma) \to \mathrm{Aut}(U(\Gamma))$$

with kernel $\mathrm{Stab}(\Gamma)$.

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Remark

$\mathrm{Diag}(\Gamma)$ is isomorphic to the group of characters of $U(\Gamma)$.

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Let $\Gamma : \mathcal{A} = \oplus_{g\in G}\mathcal{A}_g$ and $\Gamma' : \mathcal{A} = \oplus_{g'\in G'}\mathcal{A}'_{g'}$ be two gradings on $\mathcal{A}$:

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- Γ is a *refinement* of Γ' if for any $g \in G$ there is a $g' \in G'$ such that $\mathcal{A}_g \subseteq \mathcal{A}_{g'}$.
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  For example, if $\alpha : G \to H$ is a group homomorphism, then $\mathcal{A} = \oplus_{h \in H} \mathcal{A}_h$, with $\mathcal{A}_h = \oplus_{g \in \alpha^{-1}(h)} \mathcal{A}_g$, is a coarsening.

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  If $G = U(\Gamma)$, any coarsening of Γ is obtained in this way.

- Γ is *fine* if it admits no proper refinement.

  Any grading is a coarsening of a fine grading.

Basic definitions (Patera-Zassenhaus)

- Γ and Γ' are *equivalent* if there is an automorphism $\varphi \in \operatorname{Aut}\mathcal{A}$ such that for any $g \in G$ there is a $g' \in G'$ with $\varphi(\mathcal{A}_g) = \mathcal{A}'_{g'}$.

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- Γ and Γ' are *weakly isomorphic* if there is an automorphism $\varphi \in \operatorname{Aut} \mathcal{A}$ and an isomorphism $\alpha : G \to G'$ such that for any $g \in G$ $\varphi(\mathcal{A}_g) = \mathcal{A}'_{\alpha(g)}$.

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- For $G = G'$, Γ and Γ' are *isomorphic* if there is an automorphism $\varphi \in \operatorname{Aut} \mathcal{A}$ such that $\varphi(\mathcal{A}_g) = \mathcal{A}'_g$ for any $g \in G$.

Cayley-Dickson process:

$$\begin{aligned}
\mathbb{K} &= \mathbb{F} \oplus \mathbb{F}\,\mathbf{i}, && \mathbf{i}^2 = -1,\\
\mathbb{H} &= \mathbb{K} \oplus \mathbb{K}\,\mathbf{j}, && \mathbf{j}^2 = -1,\\
\mathbb{O} &= \mathbb{H} \oplus \mathbb{H}\,\mathbf{l}, && \mathbf{l}^2 = -1,
\end{aligned}$$

$\mathbb{O}$ is $\mathbb{Z}_2^3$-graded with

$$\deg(\mathbf{i}) = (\bar{1},\bar{0},\bar{0}), \quad \deg(\mathbf{j}) = (\bar{0},\bar{1},\bar{0}), \quad \deg(\mathbf{l}) = (\bar{0},\bar{0},\bar{1}).$$

Cartan grading on the Octonions

$\mathbb{O}$ contains canonical bases:

$$\mathcal{B} = \{e_1, e_2, u_1, u_2, u_3, v_1, v_2, v_3\}$$

with

$$n(e_1, e_2) = n(u_i, v_i) = 1, \quad \text{otherwise } 0.$$

$$e_1^2 = e_1, \quad e_2^2 = e_2,$$
$$e_1 u_i = u_i e_2 = u_i, \quad e_2 v_i = v_i e_1 = v_i, \quad (i = 1, 2, 3)$$
$$u_i v_i = -e_1, \quad v_i u_i = -e_2, \quad (i = 1, 2, 3)$$
$$u_i u_{i+1} = -u_{i+1} u_i = v_{i+2}, \ v_i v_{i+1} = -v_{i+1} v_i = u_{i+2}, \ \text{(indices modulo 3)}$$

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The *Cartan grading* is the $\mathbb{Z}^2$-grading determined by:

$$\deg u_1 = -\deg v_1 = (1, 0), \quad \deg u_2 = -\deg v_2 = (0, 1).$$

Fine gradings on the Octonions

Theorem (E. 1998)

Up to equivalence, the fine gradings on $\mathbb{O}$ are

- *the Cartan grading (weight space decomposition relative to a Cartan subalgebra of $\mathfrak{g}_2 = \mathfrak{Der}(\mathbb{O})$), and*
- *the $\mathbb{Z}_2^3$-grading given by the Cayley-Dickson doubling process.*

Fine gradings on the Octonions

Sketch of proof:

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- The Cayley-Hamilton equation: $x^2 - n(x,1)x + n(x)1 = 0$, implies that the norm has a well behavior relative to the grading:

$$n(\mathbb{O}_g) = 0 \text{ unless } g^2 = e, \quad n(\mathbb{O}_g, \mathbb{O}_h) = 0 \text{ unless } gh = e.$$

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- If there is a $g \in \operatorname{Supp} \Gamma$ with either order > 2 or $\dim \mathbb{O}_g \geq 2$, there are elements $x \in \mathbb{O}_g$, $y \in \mathbb{O}_{g^{-1}}$ with $n(x) = 0 = n(y)$, $n(x,y) = 1$. Then $e_1 = x\bar{y}$ and $e_2 = y\bar{x}$ are orthogonal primitive idempotents in $\mathbb{O}_e$, and one uses the corresponding Peirce decomposition to check that, up to equivalence, our grading is the Cartan grading.

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- Otherwise $\dim \mathbb{O}_g = 1$ and $g^2 = e$ for any $g \in \mathrm{Supp}\,\Gamma$. We get the $\mathbb{Z}_2^3$-grading.

$\mathbb{Z}_2^3$-grading: Octonions as a twisted group algebra

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Theorem (Albuquerque-Majid 1999)

The octonion algebra is the twisted group algebra

$$\mathbb{O} = \mathbb{F}_\sigma[\mathbb{Z}_2^3],$$

where

$$e^\alpha e^\beta = \sigma(\alpha, \beta) e^{\alpha+\beta}$$

for $\alpha, \beta \in \mathbb{Z}_2^3$*, with*

$$\sigma(\alpha, \beta) = (-1)^{\psi(\alpha,\beta)},$$

$$\psi(\alpha, \beta) = \beta_1\alpha_2\alpha_3 + \alpha_1\beta_2\alpha_3 + \alpha_1\alpha_2\beta_3 + \sum_{i\leq j} \alpha_i\beta_j.$$

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This allows to consider the algebra of octonions as an "associative algebra in a suitable category".

Cartan grading: Weyl group

Let S be the vector subspace spanned by $(1,1,1)$ in $\mathbb{R}^3$ and consider the two-dimensional real vector space $E = \mathbb{R}^3/S$. Take the elements

$$\epsilon_1 = (1,0,0) + S,\ \epsilon_2 = (0,1,0) + S,\ \epsilon_3 = (0,0,1) + S.$$

The subgroup $G = \mathbb{Z}\epsilon_1 + \mathbb{Z}\epsilon_2 + \mathbb{Z}\epsilon_3$ is isomorphic to $\mathbb{Z}^2$, and we may think of the Cartan grading Γ on the octonions $\mathbb{O}$ as the grading in which

$$\begin{aligned} &\deg(e_1) = 0 = \deg(e_2),\\ &\deg(u_i) = \epsilon_i = -\deg(v_i),\ i = 1,2,3. \end{aligned}$$

Then $\mathrm{Supp}\ \Gamma = \{0\} \cup \{\pm\epsilon_i \mid i = 1,2,3\}$ and G is the universal group. The set

$$\Phi := \Big(\mathrm{Supp}\ \Gamma \cup \{\alpha + \beta \mid \alpha, \beta \in \mathrm{Supp}\ \Gamma, \alpha \neq \pm\beta\}\Big) \setminus \{0\}$$

is the root system of type G_2.

Cartan grading: Weyl group

Identifying the Weyl group $W(\Gamma)$ with a subgroup of $\mathrm{Aut}(G)$, and this with a subgroup of $GL(E)$, we have:

$$\begin{aligned} W(\Gamma) &\subset \{\mu \in \mathrm{Aut}(G) \mid \mu(\mathrm{Supp}\ \Gamma) = \mathrm{Supp}\ \Gamma\} \\ &\subset \{\mu \in GL(E) \mid \mu(\Phi) = \Phi\} =: \mathrm{Aut}\,\Phi. \end{aligned}$$

The latter group is the automorphism group of the root system Φ, which coincides with its Weyl group.

Theorem

Let Γ be the Cartan grading on the octonions. Identify $\mathrm{Supp}\ \Gamma \setminus \{0\}$ with the short roots in the root system Φ of type G_2. Then $W(\Gamma) = \mathrm{Aut}\,\Phi$.

$\mathbb{Z}_2^3$-grading: Weyl group

Theorem

Let Γ be the $\mathbb{Z}_2^3$-grading on the octonions induced by the Cayley-Dickson doubling process. Then $W(\Gamma) = \mathrm{Aut}(\mathbb{Z}_2^3) \cong GL_3(2)$.

$\mathbb{Z}_2^3$-grading: Weyl group

Theorem

Let Γ be the $\mathbb{Z}_2^3$-grading on the octonions induced by the Cayley-Dickson doubling process. Then $W(\Gamma) = \mathrm{Aut}(\mathbb{Z}_2^3) \cong GL_3(2)$.

Remark

As any $\varphi \in \mathrm{Stab}(\Gamma)$ multiplies each of the elements $\mathbf{i}$, $\mathbf{j}$, $\mathbf{l}$ by either 1 or -1, we see that $\mathrm{Stab}(\Gamma) = \mathrm{Diag}(\Gamma)$ is isomorphic to $\mathbb{Z}_2^3$. Therefore, the group $\mathrm{Aut}(\Gamma)$ is a (non-split) extension of $\mathbb{Z}_2^3$ by $W(\Gamma) \cong GL_3(2)$.

Albert algebra

$$\mathbb{A} = H_3(\mathbb{O}, *) = \left\{ \begin{pmatrix} \alpha_1 & \bar{a}_3 & a_2 \\ a_3 & \alpha_2 & \bar{a}_1 \\ \bar{a}_2 & a_1 & \alpha_3 \end{pmatrix} \mid \alpha_1, \alpha_2, \alpha_3 \in \mathbb{F},\ a_1, a_2, a_3 \in \mathbb{O} \right\}$$

$$= \mathbb{F}E_1 \oplus \mathbb{F}E_2 \oplus \mathbb{F}E_3 \oplus \iota_1(\mathbb{O}) \oplus \iota_2(\mathbb{O}) \oplus \iota_3(\mathbb{O}),$$

where

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad E_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\iota_1(a) = 2\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \bar{a} \\ 0 & a & 0 \end{pmatrix}, \quad \iota_2(a) = 2\begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ \bar{a} & 0 & 0 \end{pmatrix}, \quad \iota_3(a) = 2\begin{pmatrix} 0 & \bar{a} & 0 \\ a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

Albert algebra

The multiplication in $\mathbb{A}$ is given by $X \circ Y = \frac{1}{2}(XY + YX)$.

Then E_i are orthogonal idempotents with $E_1 + E_2 + E_3 = 1$. The rest of the products are as follows:

$$E_i \circ \iota_i(a) = 0, \quad E_{i+1} \circ \iota_i(a) = \frac{1}{2}\iota_i(a) = E_{i+2} \circ \iota_i(a),$$

$$\iota_i(a) \circ \iota_{i+1}(b) = \iota_{i+2}(a \bullet b), \quad \iota_i(a) \circ \iota_i(b) = 2n(a, b)(E_{i+1} + E_{i+2}),$$

for any $a, b \in \mathbb{O}$, with $i = 1, 2, 3$ taken modulo 3, where $a \bullet b = \bar{a}\bar{b}$.

Cartan grading

Consider the following elements in $\mathbb{Z}^4 = \mathbb{Z}^2 \times \mathbb{Z}^2$:

$$\begin{aligned} a_1 &= (1,0,0,0), & a_2 &= (0,1,0,0), & a_3 &= (-1,-1,0,0),\\ g_1 &= (0,0,1,0), & g_2 &= (0,0,0,1), & g_3 &= (0,0,-1,-1). \end{aligned}$$

Then $a_1 + a_2 + a_3 = 0 = g_1 + g_2 + g_3$. Take a "good basis" of the octonions. The assignment $\deg e_1 = \deg e_2 = 0, \quad \deg u_i = g_i = -\deg v_i$ gives the Cartan grading on $\mathbb{O}$.

Now, the *Cartan grading* on $\mathbb{A}$ is given by the assignment

$$\begin{aligned} &\deg E_i = 0,\\ &\deg \iota_i(e_1) = a_i = -\deg \iota_i(e_2),\\ &\deg \iota_i(u_i) = g_i = -\deg \iota_i(v_i),\\ &\deg \iota_i(u_{i+1}) = a_{i+2} + g_{i+1} = -\deg \iota_i(v_{i+1}),\\ &\deg \iota_i(u_{i+2}) = -a_{i+1} + g_{i+2} = -\deg \iota_i(v_{i+2}). \end{aligned}$$

Cartan grading: Weyl group

The universal group of the Cartan grading is $\mathbb{Z}^4$, which is contained in $E = \mathbb{R}^4$. Consider the following elements of $\mathbb{Z}^4$:

$$\begin{aligned}
\epsilon_0 &= \deg \iota_1(e_1) = a_1 = (1,0,0,0),\\
\epsilon_1 &= \deg \iota_1(u_1) = g_1 = (0,0,1,0),\\
\epsilon_2 &= \deg \iota_1(u_2) = a_3 + g_2 = (-1,-1,0,1),\\
\epsilon_3 &= \deg \iota_1(u_3) = -a_2 + g_3 = (0,-1,-1,-1).
\end{aligned}$$

Note that the ϵ_i's, $0 \leq i \leq 3$, are linearly independent, but do not form a basis of $\mathbb{Z}^4$. We have for instance $\deg \iota_2(e_1) = a_2 = \frac{1}{2}(-\epsilon_0 - \epsilon_1 - \epsilon_2 - \epsilon_3)$ and $\deg \iota_3(e_1) = \frac{1}{2}(-\epsilon_0 + \epsilon_1 + \epsilon_2 + \epsilon_3)$.

Cartan grading: Weyl group

The supports of the Cartan grading Γ on each of the subspaces $\iota_i(\mathbb{O})$ are:

$$
\begin{aligned}
\mathrm{Supp}\ \iota_1(\mathbb{O}) &= \{\pm\epsilon_i \mid 0 \leq i \leq 3\},\\
\mathrm{Supp}\ \iota_2(\mathbb{O}) &= \mathrm{Supp}\ \iota_1(\mathbb{O})(\iota_3(e_1) + \iota_3(e_2))\\
&= \{\frac{1}{2}(\pm\epsilon_0 \pm \epsilon_1 \pm \epsilon_2 \pm \epsilon_3) \mid \text{even number of} + \text{signs}\},\\
\mathrm{Supp}\ \iota_3(\mathbb{O}) &= \mathrm{Supp}\ \iota_1(\mathbb{O})(\iota_2(e_1) + \iota_2(e_2))\\
&= \{\frac{1}{2}(\pm\epsilon_0 \pm \epsilon_1 \pm \epsilon_2 \pm \epsilon_3) \mid \text{odd number of} + \text{signs}\}.
\end{aligned}
$$

Cartan grading: Weyl group

$$\begin{aligned}\Phi &:= \Big(\mathrm{Supp}\ \Gamma \cup \{\alpha+\beta \mid \alpha,\beta \in \mathrm{Supp}\ \iota_1(\mathbb{O}),\ \alpha \neq \pm\beta\}\Big) \setminus \{0\}\\ &= \mathrm{Supp}\ \iota_1(\mathbb{O}) \cup \mathrm{Supp}\ \iota_2(\mathbb{O}) \cup \mathrm{Supp}\ \iota_3(\mathbb{O}) \cup \{\pm\epsilon_i \pm \epsilon_j \mid 0 \leq i \neq j \leq 3\},\end{aligned}$$

is the root system of type F_4. (Note that the ϵ_i's, $i = 0, 1, 2, 3$, form an orthogonal basis of E relative to the unique (up to scalar) inner product that is invariant under the Weyl group of Φ.)

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is the root system of type F_4. (Note that the ϵ_i's, $i = 0, 1, 2, 3$, form an orthogonal basis of E relative to the unique (up to scalar) inner product that is invariant under the Weyl group of Φ.)

Identifying the Weyl group $W(\Gamma)$ with a subgroup of $\mathrm{Aut}(\mathbb{Z}^4)$, and this with a subgroup of $GL(E)$, we have:

Theorem

Let Γ be the Cartan grading on the Albert algebra. Identify $\mathrm{Supp}\ \Gamma \setminus \{0\}$ with the short roots in the root system Φ of type F_4. Then $W(\Gamma) = \mathrm{Aut}\,\Phi$.

$\mathbb{Z}_2^5$-grading

$\mathbb{A}$ is naturally $\mathbb{Z}_2^2$-graded with

$$\mathbb{A}_{(\bar{0},\bar{0})} = \mathbb{F}E_1 + \mathbb{F}E_2 + \mathbb{F}E_3,$$
$$\mathbb{A}_{(\bar{1},\bar{0})} = \iota_1(\mathbb{O}), \qquad \mathbb{A}_{(\bar{0},\bar{1})} = \iota_2(\mathbb{O}), \qquad \mathbb{A}_{(\bar{1},\bar{1})} = \iota_3(\mathbb{O}).$$

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This $\mathbb{Z}_2^2$-grading may be combined with the fine $\mathbb{Z}_2^3$-grading on $\mathbb{O}$ to obtain a fine $\mathbb{Z}_2^5$-grading:

$$\deg E_i = (\bar 0,\bar 0,\bar 0,\bar 0,\bar 0),\ i = 1,2,3,$$
$$\deg \iota_1(x) = (\bar 1,\bar 0,\deg x),\ \deg \iota_2(x) = (\bar 0,\bar 1,\deg x),\ \deg \iota_3(x) = (\bar 1,\bar 1,\deg x).$$

$\mathbb{Z}_2^5$-grading: Weyl group

Write $\mathbb{Z}_2^5 = \mathbb{Z}_2 a \oplus \mathbb{Z}_2 b \oplus \mathbb{Z}_2 c_1 \oplus \mathbb{Z}_2 c_2 \oplus \mathbb{Z}_2 c_3$. Then the $\mathbb{Z}_2^5$-grading Γ is defined by setting

$$\deg \iota_1(1) = a, \quad \deg \iota_2(1) = b,$$
$$\deg \iota_3(\mathbf{i}) = a + b + c_1, \quad \deg \iota_3(\mathbf{j}) = a + b + c_2, \quad \deg \iota_3(\mathbf{l}) = a + b + c_3.$$

Theorem

Let Γ be the $\mathbb{Z}_2^5$-grading on the Albert algebra. Let $T = \oplus_{i=1}^3 \mathbb{Z}_2 c_i$. Then

$$W(\Gamma) = \{\mu \in \mathrm{Aut}(\mathbb{Z}_2^5) : \mu(T) = T\}.$$

$\mathbb{Z}_2^5$-grading: Weyl group

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Remark

Any $\psi \in \mathrm{Stab}(\Gamma)$ fixes E_i and multiplies $\iota_i(\mathbf{i}), \iota_i(\mathbf{j}), \iota_i(\mathbf{l})$, $i = 1, 2, 3$, by either 1 or -1. Hence $\mathrm{Stab}(\Gamma) = \mathrm{Diag}(\Gamma)$ is isomorphic to $\mathbb{Z}_2^5$.

Take an element $\mathbf{i} \in \mathbb{F}$ with $\mathbf{i}^2 = -1$ and consider the following elements in $\mathbb{A}$:

$$\begin{aligned} &E = E_1,\ \widetilde{E} = 1 - E = E_2 + E_3, \\ &\nu(a) = \mathbf{i}\iota_1(a) \quad \text{for all} \quad a \in \mathbb{O}_0, \\ &\nu_\pm(x) = \iota_2(x) \pm \mathbf{i}\iota_3(\bar{x}) \quad \text{for all} \quad x \in \mathbb{O}, \\ &S^\pm = E_3 - E_2 \pm \frac{\mathbf{i}}{2}\iota_1(1). \end{aligned}$$

$\mathbb{A}$ is then 5-graded:

$$\mathbb{A} = \mathbb{A}_{-2} \oplus \mathbb{A}_{-1} \oplus \mathbb{A}_0 \oplus \mathbb{A}_1 \oplus \mathbb{A}_2,$$

with $\mathbb{A}_{\pm 2} = \mathbb{F}S^\pm$, $\mathbb{A}_{\pm 1} = \nu_\pm(\mathbb{O})$, and $\mathbb{A}_0 = \mathbb{F}E \oplus \Big(\mathbb{F}\widetilde{E} \oplus \nu(\mathbb{O}_0)\Big)$.

$\mathbb{Z} \times \mathbb{Z}_2^3$-grading

The $\mathbb{Z}_2^3$-grading on $\mathbb{O}$ combines with this $\mathbb{Z}$-grading

$$\mathbb{A} = \mathbb{F}S^- \oplus \nu^-(\mathbb{O}) \oplus \mathbb{A}_0 \oplus \nu^+(\mathbb{O}) \oplus \mathbb{F}S^+$$

to give a fine $\mathbb{Z} \times \mathbb{Z}_2^3$-grading as follows:

$$\begin{aligned}
\deg S^{\pm} &= (\pm 2, \bar{0}, \bar{0}, \bar{0}),\\
\deg \nu_{\pm}(x) &= (\pm 1, \deg x),\\
\deg E = 0 &= \deg \widetilde{E},\\
\deg \nu(a) &= (0, \deg a),
\end{aligned}$$

for homogeneous elements $x \in \mathbb{O}$ and $a \in \mathbb{O}_0$.

$\mathbb{Z} \times \mathbb{Z}_2^3$-grading: Weyl group

Theorem

Let Γ be the $\mathbb{Z} \times \mathbb{Z}_2^3$-grading on the Albert algebra. Then

$$W(\Gamma) = \mathrm{Aut}(\mathbb{Z} \times \mathbb{Z}_2^3).$$

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Theorem

Let Γ be the $\mathbb{Z} \times \mathbb{Z}_2^3$-grading on the Albert algebra. Then

$$W(\Gamma) = \mathrm{Aut}(\mathbb{Z} \times \mathbb{Z}_2^3).$$

Remark

One can show that $\mathrm{Stab}(\Gamma) = \mathrm{Diag}(\Gamma)$, which is isomorphic to $\mathbb{F}^\times \times \mathbb{Z}_2^3$.

$\mathbb{Z}_3^3$-grading

Consider the order three automorphism τ of $\mathbb{O}$:

$$\tau(e_i) = e_i,\ i = 1, 2, \quad \tau(u_j) = u_{j+1},\ \tau(v_j) = v_{j+1},\ j = 1, 2, 3,$$

and define a new multiplication on $\mathbb{O}$:

$$x * y = \tau(\bar{x})\tau^2(\bar{y}).$$

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and define a new multiplication on $\mathbb{O}$:

$$x * y = \tau(\bar{x})\tau^2(\bar{y}).$$

This is the *Okubo algebra*, which is $\mathbb{Z}_3^2$-graded by setting

$$\deg e_1 = (\bar{1}, \bar{0}) \quad \text{and} \quad \deg u_1 = (\bar{0}, \bar{1}).$$

$\mathbb{Z}_3^3$-grading

	e_1	e_2	u_1	v_1	u_2	v_2	u_3	v_3
e_1	e_2	0	0	$-v_3$	0	$-v_1$	0	$-v_2$
e_2	0	e_1	$-u_3$	0	$-u_1$	0	$-u_2$	0
u_1	$-u_2$	0	v_1	0	$-v_3$	0	0	$-e_1$
v_1	0	$-v_2$	0	u_1	0	$-u_3$	$-e_2$	0
u_2	$-u_3$	0	0	$-e_1$	v_2	0	$-v_1$	0
v_2	0	$-v_3$	$-e_2$	0	0	u_2	0	$-u_1$
u_3	$-u_1$	0	$-v_2$	0	0	$-e_1$	v_3	0
v_3	0	$-v_1$	0	$-u_2$	$-e_2$	0	0	u_3

Multiplication table of the Okubo algebra

$\mathbb{Z}_3^3$-grading

If the characteristic of the ground field is $\neq 3$, then the Okubo algebra $(\mathbb{O}, *)$ is isomorphic to $(\mathfrak{sl}_3(\mathbb{F}), *)$, with

$$x * y = \omega xy - \omega^2 yx - \frac{\omega - \omega^2}{3} \operatorname{tr}(xy)1,$$

for a primitive cubic root of unity ω.

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for a primitive cubic root of unity ω.

The $\mathbb{Z}_3^2$-grading on the Okubo algebra is the restriction of the $\mathbb{Z}_3^2$-grading on $\mathrm{Mat}_3(\mathbb{F})$ induced by the Pauli matrices.

$\mathbb{Z}_3^3$-grading

Define $\tilde{\iota}_i(x) = \iota_i(\tau^i(x))$ for all $i = 1, 2, 3$ and $x \in \mathbb{O}$. Then the multiplication in the Albert algebra

$$\mathbb{A} = \oplus_{i=1}^{3}\big(\mathbb{F}E_i \oplus \tilde{\iota}_i(\mathbb{O})\big)$$

becomes:

$$\begin{aligned}
&E_i^{\circ 2} = E_i, \quad E_i \circ E_{i+1} = 0,\\
&E_i \circ \tilde{\iota}_i(x) = 0, \quad E_{i+1} \circ \tilde{\iota}_i(x) = \frac{1}{2}\tilde{\iota}_i(x) = E_{i+2} \circ \tilde{\iota}_i(x),\\
&\tilde{\iota}_i(x) \circ \tilde{\iota}_{i+1}(y) = \tilde{\iota}_{i+2}(x * y), \quad \tilde{\iota}_i(x) \circ \tilde{\iota}_i(y) = 2n(x, y)(E_{i+1} + E_{i+2}),
\end{aligned}$$

for $i = 1, 2, 3$ and $x, y \in \mathbb{O}$.

$\mathbb{Z}_3^3$-grading

Assume now char $\mathbb{F} \neq 3$. Then the $\mathbb{Z}_3^2$-grading on the Okubo algebra is determined by two commuting order 3 automorphisms $\varphi_1, \varphi_2 \in \mathrm{Aut}(\mathbb{O}, *)$:

$$\begin{aligned} \varphi_1(e_1) &= \omega e_1, \qquad & \varphi_1(u_1) &= u_1, \\ \varphi_2(e_1) &= e_1, \qquad & \varphi_2(u_1) &= \omega u_1, \end{aligned}$$

where ω is a primitive cubic root of unity in $\mathbb{F}$.

$\mathbb{Z}_3^3$-grading

The commuting order 3 automorphisms φ_1, φ_2 of $(\mathbb{O}, *)$ extend to commuting order 3 automorphisms of $\mathbb{A}$:

$$\varphi_j(E_i) = E_i, \quad \varphi_j\bigl(\tilde{\iota}_i(x)\bigr) = \tilde{\iota}_i(\varphi_j(x)).$$

On the other hand, the linear map $\varphi_3 \in \mathrm{End}(\mathcal{A})$ defined by

$$\varphi_3(E_i) = E_{i+1}, \quad \varphi_3\bigl(\tilde{\iota}_i(x)\bigr) = \tilde{\iota}_{i+1}(x),$$

is another order 3 automorphism, which commutes with φ_1 and φ_2.

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The subgroup of $\mathrm{Aut}(\mathbb{A})$ generated by $\varphi_1, \varphi_2, \varphi_3$ is isomorphic to $\mathbb{Z}_3^3$ and induces a $\mathbb{Z}_3^3$-grading on $\mathbb{A}$.

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The subgroup of $\mathrm{Aut}(\mathbb{A})$ generated by $\varphi_1, \varphi_2, \varphi_3$ is isomorphic to $\mathbb{Z}_3^3$ and induces a $\mathbb{Z}_3^3$-grading on $\mathbb{A}$.

All the homogeneous components have dimension 1.

$\mathbb{Z}_3^3$-grading: Weyl group

The $\mathbb{Z}_3^3$-grading is determined by

$$\begin{aligned} \deg\left(\sum_{i=1}^3 \tilde{\iota}_i(e_1)\right) &= (\bar{1},\bar{0},\bar{0}),\\ \deg\left(\sum_{i=1}^3 \tilde{\iota}_i(u_1)\right) &= (\bar{0},\bar{1},\bar{0}),\\ \deg\left(\sum_{i=1}^3 \omega^{-i}E_i\right) &= (\bar{0},\bar{0},\bar{1}), \end{aligned}$$

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Theorem

Let Γ be the $\mathbb{Z}_3^3$-grading on the Albert algebra. Then $W(\Gamma)$ is the commutator subgroup of $\mathrm{Aut}(\mathbb{Z}_3^3)$, *i.e.,*

$$W(\Gamma) \cong SL_3(3).$$

Why $SL_3(3)$ and not $GL_3(3)$?

$\mathbb{Z}_3^3$-grading: Weyl group

Why $SL_3(3)$ and not $GL_3(3)$?

Consider the $\mathbb{Z}_3^3$-grading Γ^- determined by

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\deg\left(\sum_{i=1}^3 \tilde{\iota}_i(e_1)\right) &= (\bar{0},\bar{1},\bar{0}),\\
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\end{aligned}$$

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\deg\left(\textstyle\sum_{i=1}^3 \omega^{-i} E_i\right) &= (\bar{0},\bar{0},\bar{1}),
\end{aligned}$$

Then, for $X_1 \in \mathbb{A}_{(\bar{1},\bar{0},\bar{0})}$, $X_2 \in \mathbb{A}_{(\bar{0},\bar{1},\bar{0})}$, $X_3 \in \mathbb{A}_{(\bar{0},\bar{0},\bar{1})}$, we have:

$$(X_1 \circ X_2) \circ X_3 = \begin{cases} \omega X_1 \circ (X_2 \circ X_3), & \text{for } \Gamma,\\ \omega^{-1} X_1 \circ (X_2 \circ X_3), & \text{for } \Gamma^-. \end{cases}$$

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Hence Γ and Γ^- are equivalent, but NOT isomorphic, gradings on $\mathbb{A}$ by $\mathbb{Z}_3^3$.

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\end{aligned}$$

Then, for $X_1 \in \mathbb{A}_{(\bar{1},\bar{0},\bar{0})}$, $X_2 \in \mathbb{A}_{(\bar{0},\bar{1},\bar{0})}$, $X_3 \in \mathbb{A}_{(\bar{0},\bar{0},\bar{1})}$, we have:

$$(X_1 \circ X_2) \circ X_3 = \begin{cases} \omega X_1 \circ (X_2 \circ X_3), & \text{for } \Gamma,\\ \omega^{-1} X_1 \circ (X_2 \circ X_3), & \text{for } \Gamma^-. \end{cases}$$

Hence Γ and Γ^- are equivalent, but NOT isomorphic, gradings on $\mathbb{A}$ by $\mathbb{Z}_3^3$. Besides, any fine $\mathbb{Z}_3^3$-grading on $\mathbb{A}$ is isomorphic to either Γ or Γ^-, so $W(\Gamma)$ has index two in $\mathrm{Aut}(U(\Gamma)) \cong GL_3(3)$.

$\mathbb{Z}_3^3$-grading and the Tits construction

Let $\mathcal{R} = \mathrm{Mat}_3(\mathbb{F})$. Then

$$\mathbb{A} = \mathcal{R}_0 \oplus \mathcal{R}_1 \oplus \mathcal{R}_2,$$

with $\mathcal{R}_0$, $\mathcal{R}_1$, $\mathcal{R}_2$ copies of $\mathcal{R}$.

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with $\mathcal{R}_0$, $\mathcal{R}_1$, $\mathcal{R}_2$ copies of $\mathcal{R}$.
The product in $\mathbb{A}$ satisfies $\mathcal{R}_i \circ \mathcal{R}_j \subseteq \mathcal{R}_{i+j}$ (mod 3) and it is given by:

$\circ$	a'_0	b'_1	c'_2
a_0	$(a \circ a')_0$	$(\bar{a}b')_1$	$(c'\bar{a})_2$
b_1	$(\bar{a}'b)_1$	$(b \times b')_2$	$(\overline{bc'})_2$
c_2	$(c\bar{a}')_2$	$(\overline{b'c})_0$	$(c \times c')_1$

where

- $a \circ a' = \frac{1}{2}(aa' + a'a)$,
- $a \times b = a \circ b - \frac{1}{2}(\mathrm{tr}(a)b + \mathrm{tr}(b)a) + \frac{1}{2}\big(\mathrm{tr}(a)\,\mathrm{tr}(b) - \mathrm{tr}(ab)\big)1$,
- $\bar{a} = a \times 1 = \frac{1}{2}\big(\mathrm{tr}(a)1 - a\big)$.

$\mathbb{Z}_3^3$-grading and the Tits construction

Assume $\operatorname{char}\mathbb{F} \neq 3$. Take Pauli matrices in $\mathcal{R}$:

$$x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

where ω, ω^2 are the primitive cubic roots of 1, which satisfy

$$x^3 = 1 = y^3, \quad yx = \omega xy.$$

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These Pauli matrices give a grading over $\mathbb{Z}_3^2$ on $\mathcal{R}$, with

$$\mathcal{R}_{(\alpha_1,\alpha_2)} = \mathbb{F}x^{\alpha_1}y^{\alpha_2}.$$

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This grading combines with the $\mathbb{Z}_3$-grading on $\mathbb{A}$ induced by Tits construction, to give the unique, up to equivalence, fine grading over $\mathbb{Z}_3^3$ of the Albert algebra.

$\mathbb{Z}_3^3$-grading and the Tits construction

For $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{Z}_3^3$ consider the element

$$Z^\alpha := (x^{\alpha_1} y^{\alpha_2})_{\alpha_3} \in \mathcal{R}_{\alpha_3} \subseteq \mathbb{A}.$$

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Then, for any $\alpha, \beta \in \mathbb{Z}_3^3$:

$$Z^\alpha \circ Z^\beta = \begin{cases} \omega^{\tilde{\psi}(\alpha,\beta)} Z^{\alpha+\beta} & \text{if } \dim_{\mathbb{Z}_3}(\mathbb{Z}_3\alpha + \mathbb{Z}_3\beta) \leq 1, \\ -\frac{1}{2}\omega^{\tilde{\psi}(\alpha,\beta)} Z^{\alpha+\beta} & \text{otherwise,} \end{cases}$$

where

$$\tilde{\psi}(\alpha, \beta) = (\alpha_2\beta_1 - \alpha_1\beta_2)(\alpha_3 - \beta_3) - (\alpha_1\beta_2 + \alpha_2\beta_1).$$

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$$= \begin{cases} \omega^{\tilde{\psi}(\alpha,\beta)-(\alpha_1\alpha_2+\beta_1\beta_2)} Z^{\alpha+\beta} & \text{if } \dim_{\mathbb{Z}_3}(\mathbb{Z}_3\alpha + \mathbb{Z}_3\beta) \leq 1, \\ -\frac{1}{2}\omega^{\tilde{\psi}(\alpha,\beta)-(\alpha_1\alpha_2+\beta_1\beta_2)} Z^{\alpha+\beta} & \text{otherwise,} \end{cases}$$

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The Albert algebra as a twisted group algebra

Theorem (Griess 1990)

The Albert algebra is, up to isomorphism, the twisted group algebra

$$\mathbb{A} = \mathbb{F}_\sigma[\mathbb{Z}_3^3],$$

with

$$\sigma(\alpha,\beta) = \begin{cases} \omega^{\psi(\alpha,\beta)} & \text{if } \dim_{\mathbb{Z}_3}(\mathbb{Z}_3\alpha + \mathbb{Z}_3\beta) \leq 1, \\ -\frac{1}{2}\omega^{\psi(\alpha,\beta)} & \text{otherwise,} \end{cases}$$

where

$$\psi(\alpha,\beta) = (\alpha_2\beta_1 - \alpha_1\beta_2)(\alpha_3 - \beta_3).$$

Fine gradings on the Albert algebra

Theorem (Draper–Martín-González 2009 (char $= 0$), E.–Kochetov 2010)

Up to equivalence, the fine gradings of the Albert algebra are:

1. *The Cartan grading (weight space decomposition relative to a Cartan subalgebra of $\mathfrak{f}_4 = \mathfrak{Der}(\mathbb{A})$).*
2. *The $\mathbb{Z}_2^5$-grading obtained by combining the natural $\mathbb{Z}_2^2$-grading on 3×3 hermitian matrices with the fine grading over $\mathbb{Z}_2^3$ of $\mathbb{O}$.*
3. *The $\mathbb{Z} \times \mathbb{Z}_2^3$-grading obtained by combining a 5-grading and the $\mathbb{Z}_2^3$-grading on $\mathbb{O}$.*
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All the gradings up to isomorphism on $\mathbb{A}$ have been classified too (E.–Kochetov).

Gradings and comodule algebras

G-grading $\quad\longleftrightarrow\quad$ comodule algebra over the group algebra $\mathbb{F}G$

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$$\Gamma : \mathcal{A} = \oplus_{g\in G}\mathcal{A}_g \qquad \Rightarrow \qquad \begin{aligned} \rho_\Gamma : \mathcal{A} &\longrightarrow \mathcal{A}\otimes \mathbb{F}G \\ x_g &\mapsto \ x_g \otimes g \end{aligned}$$

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(algebra morphism and comodule structure)

$$\Gamma_\rho : \mathcal{A} = \oplus_{g\in G}\mathcal{A}_g \qquad \Leftarrow \qquad \rho : \mathcal{A}\to\mathcal{A}\otimes\mathbb{F}G$$

$$(\mathcal{A}_g = \{x\in\mathcal{A} : \rho(x) = x\otimes g\})$$

A comodule algebra map

$$\rho : \mathcal{A} \to \mathcal{A} \otimes \mathbb{F}G$$

induces a *generic automorphism* of $\mathbb{F}G$-algebras

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All the information on the grading Γ attached to ρ is contained in this single automorphism!

Gradings and affine group schemes

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Now,

$$\rho_\Gamma : \mathcal{A} \to \mathcal{A}\otimes \mathbb{F}G \quad \Longleftrightarrow \quad \eta_\Gamma : G^D \to \mathbf{Aut}\,\mathcal{A}$$

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(comodule algebra) (morphism of affine group schemes)

For any $\varphi \in G^D(\mathcal{R})$, $\eta_\Gamma(\varphi) \in \mathrm{Aut}_{\mathcal{R}}(\mathcal{A} \otimes \mathcal{R})$ is given by:

$$\eta_\Gamma(\varphi)(x_g \otimes r) = x_g \otimes \varphi(g) r.$$

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and ρ_Γ is recovered as

$$\rho_\Gamma(x) = \eta_\Gamma(id_{\mathbb{F}G})(x\otimes 1) \qquad \Big(\eta_\Gamma(id_{\mathbb{F}G}) \in \mathrm{Aut}_{\mathbb{F}G}(\mathcal{A}\otimes\mathbb{F}G)\Big)$$

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Consider a homomorphism $\Phi : \mathbf{Aut}\,\mathcal{A} \longrightarrow \mathbf{Aut}\,\mathcal{A}'$ of affine group schemes.

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$$\eta_{\Gamma'} = \Phi \circ \eta_\Gamma : G^D \to \mathbf{Aut}\,\mathcal{A}'.$$

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If $\Gamma_1 : \mathcal{A} = \oplus_{g\in G}\mathcal{A}_g$ and $\Gamma_2 : \mathcal{A} = \oplus_{h\in H}\mathcal{A}'_h$ are weakly isomorphic, then the induced gradings Γ'_1 and Γ'_2 on $\mathcal{A}'$ are weakly isomorphic too through the automorphism $\Phi_{\mathbb{F}}(\psi) \in \mathrm{Aut}\,\mathcal{A}'$ and $\varphi : G \to H$.

For $\mathbf{G} = \mathbf{Aut}\,\mathcal{A}$, $\mathrm{Lie}(\mathbf{G}) = \mathfrak{Der}(\mathcal{A})$, so

$$\mathrm{Ad} : \mathbf{Aut}\,\mathcal{A} \to \mathbf{Aut}\big(\mathfrak{Der}(\mathcal{A})\big)$$

is a homomorphism, and any grading $\Gamma : \mathcal{A} = \oplus_{g\in G}\mathcal{A}_g$ induces a grading

$$\Gamma' : \mathfrak{Der}(\mathcal{A}) = \oplus_{g\in G}\,\mathfrak{Der}(\mathcal{A})_g,$$

$$\mathfrak{Der}(\mathcal{A})_g = \{d \in \mathfrak{Der}(\mathcal{A}) : d(\mathcal{A}_h) \subseteq \mathcal{A}_{gh}\ \forall h \in G\}.$$

Gradings on G_2 and F_4

If $\mathbf{Aut}\,\mathcal{A} \cong \mathbf{Aut}\,\mathcal{B}$, then the problem of the classification of fine gradings up to equivalence, and of gradings up to isomorphism, on $\mathcal{A}$ and $\mathcal{B}$ are equivalent.

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If the characteristic of the ground field $\mathbb{F}$ is $\neq 2, 3$, then

$$\mathrm{Ad} : \textbf{Aut}\,\mathbb{O} \to \textbf{Aut}\,\mathfrak{g}_2$$

is an isomorphism, and (assuming just $\mathrm{char}\,\mathbb{F} \neq 2$),

$$\mathrm{Ad} : \textbf{Aut}\,\mathbb{A} \to \textbf{Aut}\,\mathfrak{f}_4$$

is an isomorphism too.

Gradings on G_2

Theorem

Up to equivalence, the fine gradings on $\mathfrak{g}_2$ are

- *the Cartan grading, and*
- *a $\mathbb{Z}_2^3$-grading with $(\mathfrak{g}_2)_0 = 0$ and where $(\mathfrak{g}_2)_g$ is a Cartan subalgebra of $\mathfrak{g}_2$ for any $0 \neq g \in \mathbb{Z}_2^3$.*

Gradings on F_4

Theorem

Up to equivalence, the fine gradings on $\mathfrak{f}_4$ are

- *the Cartan grading,*
- *a grading by $\mathbb{Z}_2^5$, obtained by combining the $\mathbb{Z}_2^2$-grading given by the decomposition $\mathfrak{f}_4 = \mathfrak{d}_4 \oplus natural \oplus spin \oplus \overline{spin}$, with the $\mathbb{Z}_2^3$-grading on the octonions (which is the vector space behind the natural and spin representations of $\mathfrak{d}_4$).*
- *a grading by $\mathbb{Z} \times \mathbb{Z}_2^3$, obtained by looking at $\mathfrak{f}_4$ as the Kantor Lie algebra of a structurable algebra: $\mathfrak{f}_4 = \mathcal{K}(\mathbb{O}, -)$, and combining the natural 5-grading on $\mathcal{K}(\mathbb{O}, -)$ and the $\mathbb{Z}_2^3$-grading on $\mathbb{O}$.*
- *a $\mathbb{Z}_3^3$-grading (only if $\mathrm{char}\,\mathbb{F} \neq 3$), with $(\mathfrak{f}_4)_0 = 0$ and where $(\mathfrak{f}_4)_g \oplus (\mathfrak{f}_4)_{-g}$ is a Cartan subalgebra of $\mathfrak{f}_4$ for any $0 \neq g \in \mathbb{Z}_3^3$.*

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That's all. Thanks