Gradings on simple Lie algebras

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Gradings on Lie algebras have been extensively used since the beginning of Lie theory:

- the Cartan grading on a complex semisimple Lie algebra is the $\mathbb{Z}^r$-grading (r being the rank) whose homogeneous components are the root spaces relative to a Cartan subalgebra (which is the zero component),
- symmetric spaces are related to $\mathbb{Z}_2$-gradings,
- Kac–Moody Lie algebras to gradings by a finite cyclic group,
- the theory of Jordan algebras and pairs to 3-gradings on Lie algebras, etc.

In 1989, a systematic study of gradings on Lie algebras was started by Patera and Zassenhaus.

Fine gradings on the classical simple complex Lie algebras, other than D_4, by arbitrary abelian groups were considered by Havlícek, Patera, and Pelantova in 1998.

The arguments there are computational and the problem of classification of fine gradings is not completely settled. The complete classification, up to equivalence, of fine gradings on all classical simple Lie algebras (including D_4) over algebraically closed fields of characteristic zero has been obtained quite recently.

For any abelian group G, the classification of all G-gradings, up to isomorphism, on the classical simple Lie algebras other than D_4 over algebraically closed fields of characteristic different from two has been achieved in 2010 by Bahturin and Kochetov, using methods developed in the last years by a number of authors.

As to the exceptional simple Lie algebras, the classification of all gradings (up to equivalence) for type G_2 over an algebraically closed field of characteristic 0 was obtained independently by Draper and Martín-González and by Bahturin and Tvalavadze, using the known results on gradings on the Cayley algebras.

Also, the classification of fine gradings (up to equivalence) for type F_4 over an algebraically closed field of characteristic 0 has recently been obtained by Draper and Martín-González.

The method used for F_4 relies on the fact that, under the stated assumptions on the ground field, any abelian group grading on an algebra is the decomposition into common eigenspaces for some diagonalizable subgroup of the automorphism group of the algebra.

It is shown that any such subgroup is contained in the normalizer of a maximal torus of the automorphism group.

Since the automorphism groups of the simple Lie algebra of type F_4 and of the exceptional simple Jordan algebra (the Albert algebra) are isomorphic, the fine gradings on the Albert algebra have been classified as well.

In joint work with M. Kochetov, both gradings up to isomorphism and fine gradings up to equivalence in the simple Lie algebras of types G_2 and F_4 over algebraically closed fields of characteristic different from two have been classified.

This is done by first classifying gradings on the Cayley algebra and on the Albert algebra, and then use automorphism group schemes to transfer the classification to the corresponding Lie algebras.

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The purpose of this mini-course is to review these methods and results.

General definitions and methods

1. Gradings
2. Characteristic 0
3. Gradings and affine group schemes
4. Gradings on matrix algebras

State of the art

5. Classical simple Lie algebras

6. Octonions and G_2

7. The Albert algebra and F_4

8. And now?

1 Gradings

2 Characteristic 0

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Definition

G abelian group, $\mathcal{A}$ algebra over a field $\mathbb{F}$.

G-grading on $\mathcal{A}$:

$$\mathcal{A} = \oplus_{g\in G}\mathcal{A}_g,$$

$$\mathcal{A}_g\mathcal{A}_h \subseteq \mathcal{A}_{gh} \qquad \forall g, h \in G.$$

Example: Pauli matrices

$\mathcal{A} = \mathrm{Mat}_n(\mathbb{F})$

$$X = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & \epsilon & 0 & \dots & 0 \\ 0 & 0 & \epsilon^2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \epsilon^{n-1} \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}$$

(ϵ a primitive nth root of 1)

$$X^n = 1 = Y^n, \qquad YX = \epsilon XY$$

$$\mathcal{A} = \oplus_{(\bar{\imath},\bar{\jmath}) \in \mathbb{Z}_n \times \mathbb{Z}_n} \mathcal{A}_{(\bar{\imath},\bar{\jmath})}, \qquad \mathcal{A}_{(\bar{\imath},\bar{\jmath})} = \mathbb{F} X^i Y^j.$$

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$\mathcal{A}$ becomes a *graded division algebra*.

Example: octonions

Cayley-Dickson process:

$$
\begin{aligned}
\mathbb{K} &= \mathbb{F} \oplus \mathbb{F}i, && i^2 = -1,\\
\mathbb{H} &= \mathbb{K} \oplus \mathbb{K}j, && j^2 = -1,\\
\mathbb{O} &= \mathbb{H} \oplus \mathbb{H}l, && l^2 = -1,
\end{aligned}
$$

$\mathbb{O}$ is $\mathbb{Z}_2^3$-graded with

$$\deg(i) = (\bar{1}, \bar{0}, \bar{0}), \quad \deg(j) = (\bar{0}, \bar{1}, \bar{0}), \quad \deg(l) = (\bar{0}, \bar{0}, \bar{1}).$$

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I. Kantor: “There are no Jordan algebras, there are only Lie algebras.”

K. McCrimmon: “Of course, this can be turned around: nine times out of ten, when you open up a Lie algebra you find a Jordan algebra inside which makes it tick.”

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- **Symmetric spaces** $\leftrightarrow$ $\mathbb{Z}_2$**-gradings:** $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$.

- Finite order automorphisms of simple complex Lie algebras (classified by V. Kac) correspond to gradings over finite cyclic groups:

$$\varphi \in \operatorname{Aut}\mathfrak{g},\ \varphi^n = 1 \quad \leftrightarrow \quad \mathfrak{g} = \oplus_{\bar{\imath}\in\mathbb{Z}_n}\mathfrak{g}_{\bar{\imath}},$$

$$\mathfrak{g}_{\bar{\imath}} = \{x \in \mathfrak{g} : \varphi(x) = \epsilon^i x\}\ (\epsilon = e^{\frac{2\pi i}{n}}).$$

(This is important in the theory of Kac-Moody Lie algebras.)

Conventions

In what follows:

- $\mathbb{F}$ will denote an algebraically closed ground field, $\operatorname{char}\mathbb{F} \neq 2$.
- The dimension of the algebras considered will always be finite.
- The stress will be put more on the methods than on the results. Some of them are quite technical.

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More on Weyl groups in M. Kochetov's talk!!

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Abelian diagonalizable subgroups

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  - $\mathcal{A} = \oplus_{g\in G}\mathcal{A}_g$ is a grading, with

$$\mathcal{A}_g = \{x \in \mathcal{A} : \varphi(x) = g(\varphi)x \ \forall \varphi \in Q\}.$$

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- The G-gradings on $\mathcal{A}$ up to isomorphism are in bijection with the conjugacy classes by $\operatorname{Aut}\mathcal{A}$ of the homomorphism of algebraic groups $\hat{G} \to \operatorname{Aut}\mathcal{A}$.
- Algebras with isomorphic groups of automorphisms present "the same" classification of fine gradings up to equivalence, and of gradings up to isomorphism.

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  An explicit (and irredundant) description of the corresponding fine gradings has been given for some classical simple Lie algebras of small rank (Patera, Pelantova, Svobodova).

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- The classification of the fine gradings, up to equivalence, on the classical simple Lie algebras (including D_4!) has been recently completed (E. 2010).

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  (The results are involved. For instance, there appear 17 non equivalent fine gradings for D_4.)

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Up to equivalence, there appear only the Cartan grading and the $\mathbb{Z}_2^3$-grading induced by the corresponding grading of the octonions.

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Apart from the Cartan grading, there appear three other non-equivalent fine gradings.

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E_7, E_8: ??

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$$\Gamma : \mathcal{A} = \oplus_{g\in G}\mathcal{A}_g \qquad \Rightarrow \qquad \begin{aligned} \rho_\Gamma : \mathcal{A} &\longrightarrow \mathcal{A}\otimes\mathbb{F}G \\ x_g &\mapsto x_g \otimes g \end{aligned}$$

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$$\Gamma_\rho : \mathcal{A} = \oplus_{g\in G}\mathcal{A}_g \qquad \Leftarrow \qquad \rho : \mathcal{A} \to \mathcal{A}\otimes \mathbb{F}G$$

$$(\mathcal{A}_g = \{x \in \mathcal{A} : \rho(x) = x \otimes g\})$$

Gradings and comodule algebras

A comodule algebra map

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induces a *generic automorphism* of $\mathbb{F}G$-algebras

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All the information on the grading Γ attached to ρ is contained in this single automorphism!

Affine group schemes

An *affine group scheme* over $\mathbb{F}$ is a representable functor

$$\mathbf{G}\colon \mathrm{Alg}_{\mathbb{F}} \to \mathrm{Grp},$$

where

- $\mathrm{Alg}_{\mathbb{F}}$ is the category of unital commutative associative algebras over the field $\mathbb{F}$,
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Recall that a functor $\mathbf{F}\colon \mathrm{Alg}_{\mathbb{F}} \to \mathrm{Grp}$ is said to be *representable* if there exists an object $\mathcal{A}$ in $\mathrm{Alg}_{\mathbb{F}}$ such that $\mathbf{F}$ is naturally isomorphic to $\mathrm{Hom}_{\mathrm{Alg}_{\mathbb{F}}}(\mathcal{A}, -)$.

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We will denote the representing object of $\mathbf{G}$ by $\mathbb{F}[\mathbf{G}]$. For $\mathcal{R}$ in $\mathrm{Alg}_{\mathbb{F}}$, the elements of $\mathbf{G}(\mathcal{R})$, i.e., homomorphisms $\mathbb{F}[\mathbf{G}] \to \mathcal{R}$, will be called the *$\mathcal{R}$-points* of $\mathbf{G}$.

Affine group schemes. Examples

- Let V be an n-dimensional vector space. The general linear group scheme is the functor

$$\mathbf{GL}(V)(\mathcal{R}) = \{\varphi \in \mathrm{End}_{\mathcal{R}}(V \otimes \mathcal{R}) \mid \varphi \text{ is invertible}\}.$$

  This defines an affine group scheme $\mathbf{GL}(V)$. If we fix a basis in V, we can identify V with $\mathbb{F}^n$. Then the representing object is $\mathbb{F}[X_{ij}, D^{-1}]$ where $D = \det(X_{ij})$, $i, j = 1, \ldots, n$.

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- Let $\mathcal{U}$ be a nonassociative algebra over $\mathbb{F}$, $\dim \mathcal{U} = n < \infty$. Then, for any $\mathcal{R}$ in $\mathrm{Alg}_{\mathbb{F}}$, the tensor product $\mathcal{U} \otimes \mathcal{R}$ is an algebra over $\mathcal{R}$. The *automorphism group scheme* is the functor

$$\mathbf{Aut}(\mathcal{U})(\mathcal{R}) = \mathrm{Aut}_{\mathcal{R}}(\mathcal{U} \otimes \mathcal{R}).$$

Affine group schemes and Hopf algebras

Since the sets $\mathbf{G}(\mathcal{R})$ are endowed with multiplication that makes them groups, the representing object $\mathcal{A} = \mathbb{F}[\mathbf{G}]$ should also carry some additional structure.

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- The group multiplication defines a natural map of functors $\mathbf{G} \times \mathbf{G} \to \mathbf{G}$, which, in view of Yoneda's Lemma, gives rise to a homomorphism
$$\Delta\colon \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}.$$
The associativity of group multiplication translates to the property
$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta.$$

- The existence of identity element in each $\mathbf{G}(\mathcal{R})$ can be expressed as a natural map from the trivial group scheme to $\mathbf{G}$, which gives rise to a homomorphism
$$\varepsilon\colon \mathcal{A} \to \mathbb{F}.$$
The definition of identity element translates to the property
$$(\varepsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id} = (\mathrm{id} \otimes \varepsilon) \circ \Delta,$$
where we identified $\mathbb{F} \otimes \mathcal{A}$ and $\mathcal{A} \otimes \mathbb{F}$ with $\mathcal{A}$.

- The existence of inverses can be expressed as a natural map $\mathbf{G} \to \mathbf{G}$, which gives rise to a homomorphism

$$S\colon \mathcal{A} \to \mathcal{A}.$$

The definition of inverse translates to the property

$$m \circ (S \otimes \mathrm{id}) \circ \Delta = \eta \circ \varepsilon = m \circ (\mathrm{id} \otimes S) \circ \Delta,$$

where $m\colon \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ is the multiplication map and $\eta\colon \mathbb{F} \to \mathcal{A}$ is the map $\lambda \to \lambda 1_{\mathcal{A}}$.

Affine group schemes and Hopf algebras

Theorem

There is a one-to-one correspondence between affine group schemes and commutative Hopf algebras.

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Example

Let G be a group. The group algebra $\mathbb{F}G$ becomes a Hopf algebra if we declare all elements of G group-like: $\Delta(g) = g \otimes g$ for any $g \in G$.
If G is abelian, then $\mathbb{F}G$ is commutative and hence gives rise to an affine group scheme, which we will denote by G^D. For any $\mathcal{R}$ in $\mathrm{Alg}_{\mathbb{F}}$, we have

$$G^D(\mathcal{R}) = \mathrm{Hom}_{\mathrm{Alg}_{\mathbb{F}}}(\mathbb{F}G, \mathcal{R}) \simeq \mathrm{Hom}(G, \mathcal{R}^{\times})$$

In particular, $G^D(\mathbb{F})$ is the group of characters of the group G.

$$\Gamma : \mathcal{A} = \oplus_{g\in G}\mathcal{A}_g \quad \Leftrightarrow \quad \rho_\Gamma : \mathcal{A} \to \mathcal{A}\otimes \mathbb{F}G$$

(comodule algebra structure)

Affine group schemes and gradings

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Now,

$$\underset{\text{(comodule algebra)}}{\rho_\Gamma : \mathcal{A} \to \mathcal{A}\otimes \mathbb{F}G} \quad \Longleftrightarrow \quad \underset{\text{(morphism of affine group schemes)}}{\eta_\Gamma : G^D \to \mathbf{Aut}\,\mathcal{A}}$$

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(comodule algebra) (morphism of affine group schemes)

For any $\varphi \in G^D(\mathcal{R})$, $\eta_\Gamma(\varphi) \in \mathrm{Aut}_{\mathcal{R}}(\mathcal{A}\otimes\mathcal{R})$ is given by:

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and ρ_Γ is recovered as

$$\rho_\Gamma(x) = \eta_\Gamma(id_{\mathbb{F}G})(x\otimes 1) \qquad \Big(\eta_\Gamma(id_{\mathbb{F}G}) \in \mathrm{Aut}_{\mathbb{F}G}(\mathcal{A}\otimes\mathbb{F}G)\Big)$$

Message:

It is not enough to deal with $\hat{G}$ and $\operatorname{Aut}\mathcal{A}$, but also with their extensions to unital commutative and associative $\mathbb{F}$-algebras.

Gradings and affine group schemes

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It is not enough to deal with $\hat{G}$ and $\operatorname{Aut}\mathcal{A}$, but also with their extensions to unital commutative and associative $\mathbb{F}$-algebras.

If $\mathbf{Aut}\,\mathcal{A} \cong \mathbf{Aut}\,\mathcal{B}$, then the problem of the classification of fine gradings up to equivalence, and of gradings up to isomorphism, on $\mathcal{A}$ and $\mathcal{B}$ are equivalent.

Actually, assume that $\Phi : \mathbf{Aut}\,\mathcal{A} \longrightarrow \mathbf{Aut}\,\mathcal{A}'$ is a homomorphism of affine group schemes (a natural transformation).

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Then any grading $\Gamma : \mathcal{A} = \oplus_{g\in G}\mathcal{A}_g$ induces a grading $\Gamma' : \mathcal{A}' = \oplus_{g\in G}\mathcal{A}'_g$ by means of:

$$\eta_{\Gamma'} = \Phi \circ \eta_\Gamma : G^D \to \mathbf{Aut}\,\mathcal{A}'.$$

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If $\Gamma_1 : \mathcal{A} = \oplus_{g\in G}\mathcal{A}_g$ and $\Gamma_2 : \mathcal{A} = \oplus_{h\in H}\mathcal{A}'_h$ are *weakly isomorphic* (i.e., there is an automorphism $\psi \in \operatorname{Aut}\mathcal{A}$ and a group isomorphism $\varphi : G \to H$ such that $\psi(\mathcal{A}_g) = \mathcal{A}'_{\varphi(g)}$ for any $g \in G$), then the induced gradings Γ'_1 and Γ'_2 on $\mathcal{A}'$ are weakly isomorphic too through the automorphism $\Phi_{\mathbb{F}}(\psi) \in \operatorname{Aut}\mathcal{A}'$ and $\varphi : G \to H$.

Example

G affine group scheme, its *Lie algebra* is defined by:

$$\mathrm{Lie}(\mathbf{G}) := \ker\bigl(\mathbf{G}(\mathbb{F}[\tau]) \to \mathbf{G}(\mathbb{F})\bigr) \simeq \mathfrak{Der}(\mathbb{F}[\mathbf{G}], \mathbb{F})$$

($\mathbb{F}[\tau] = \mathbb{F}1 + \mathbb{F}\tau$, $\tau^2 = 0$, is the algebra of dual numbers; $\mathbb{F}$ is a module for $\mathbb{F}[\mathbf{G}]$ by means of the augmentation ε.)

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The *adjoint map* $\mathrm{Ad}\colon \mathbf{G} \to \mathbf{Aut}(\mathrm{Lie}(\mathbf{G}))$ is defined as follows:

$$\mathrm{Ad}_{\mathcal{R}}(g) = (x \mapsto gxg^{-1}) \quad \text{for all} \quad g \in \mathbf{G}(\mathcal{R}),\ x \in \mathrm{Lie}(\mathbf{G}) \otimes \mathcal{R},$$

where $\mathrm{Lie}(\mathbf{G}) \otimes \mathcal{R}$ is identified with a subset in $\mathbf{G}(\mathcal{R}[\tau])$.

Example

For $\mathbf{G} = \mathbf{Aut}\,\mathcal{A}$, $\mathrm{Lie}(\mathbf{G}) = \mathfrak{Der}(\mathcal{A})$, so

$$\mathrm{Ad} : \mathbf{Aut}\,\mathcal{A} \to \mathbf{Aut}\big(\mathfrak{Der}(\mathcal{A})\big)$$

is a homomorphism, and any grading $\Gamma : \mathcal{A} = \oplus_{g\in G}\mathcal{A}_g$ induces a grading

$$\Gamma' : \mathfrak{Der}(\mathcal{A}) = \oplus_{g\in G}\,\mathfrak{Der}(\mathcal{A})_g,$$

$$\mathfrak{Der}(\mathcal{A})_g = \{d \in \mathfrak{Der}(\mathcal{A}) : d(\mathcal{A}_h) \subseteq \mathcal{A}_{gh}\ \forall h \in G\}.$$

Automorphism group schemes of classical simple Lie algebras

- **Orthogonal and symplectic Lie algebras**

$$\mathbf{Aut}\,\mathfrak{so}_n(\mathbb{F}) \cong \mathbf{Aut}(\mathrm{Mat}_n(\mathbb{F}), \varphi_o), \qquad n \geq 5,\ n \neq 8,$$

$$\mathbf{Aut}\,\mathfrak{sp}_n(\mathbb{F}) \cong \mathbf{Aut}(\mathrm{Mat}_n(\mathbb{F}), \varphi_s), \qquad n \text{ even},\ n \geq 4,$$

where φ_o is an orthogonal involution and φ_s a symplectic involution.

Automorphism group schemes of classical simple Lie algebras

- **Special linear Lie algebras**

$$\mathbf{Aut}\,\mathfrak{sl}_2(\mathbb{F}) \cong \mathbf{Aut}\,\mathrm{Mat}_2(\mathbb{F}),$$

$$\mathbf{Aut}\,\mathfrak{psl}_n(\mathbb{F}) \cong \overline{\mathbf{Aut}}\,\mathrm{Mat}_n(\mathbb{F}), \qquad n \geq 3, \text{ unless } n = 3 = \operatorname{char}\mathbb{F},$$

where $\overline{\mathbf{Aut}}\,\mathrm{Mat}_n(\mathbb{F})$ denotes the affine group scheme of automorphisms and antiautomorphisms of $\mathrm{Mat}_n(\mathbb{F})$.

Therefore, the problem of classification of gradings on the classical Lie algebras (other than D_4 and, for $\operatorname{char} \mathbb{F} = 3$, A_2), reduces to a problem about certain gradings on algebras of matrices.

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There is a graded division algebra $\mathcal{D}$ and a right graded module V for $\mathcal{D}$ such that

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($\mathcal{D}$ is a tensor product of matrix algebras graded by Pauli matrices.)

Gradings on matrix algebras

Sketch of proof:

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- $\mathcal{I}$ minimal $\Rightarrow e\mathcal{A}e$ is a graded division algebra (graded Schur Lemma).
- Take $V = \mathcal{I}$ and $\mathcal{D} = e\mathcal{A}e$.

Example: fine gradings on $\mathrm{Mat}_8(\mathbb{F})$

Up to equivalence, the fine gradings on $\mathrm{Mat}_8(\mathbb{F})$, with their universal grading groups, are:

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Up to equivalence, the fine gradings on $\mathrm{Mat}_8(\mathbb{F})$, with their universal grading groups, are:

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- $\mathbb{Z}^3 \times \mathbb{Z}_2^2$**-grading:** $\mathcal{D} = \mathrm{Mat}_2(\mathbb{F})$, $\dim_{\mathcal{D}} V = 4$.
- $\mathbb{Z} \times \mathbb{Z}_2^4$**-grading:** $\mathcal{D} = \mathrm{Mat}_2(\mathbb{F}) \otimes \mathrm{Mat}_2(\mathbb{F})$, $\dim_{\mathcal{D}} V = 2$.
- $\mathbb{Z}_2^6$**-grading:** $\mathcal{D} = \mathrm{Mat}_2(\mathbb{F}) \otimes \mathrm{Mat}_2(\mathbb{F}) \otimes \mathrm{Mat}_2(\mathbb{F})$, $\dim_{\mathcal{D}} V = 1$.
- $\mathbb{Z} \times \mathbb{Z}_4^2$**-grading:** $\mathcal{D} = \mathrm{Mat}_4(\mathbb{F})$, $\dim_{\mathcal{D}} V = 2$.
- $\mathbb{Z}_2^2 \times \mathbb{Z}_4^2$**-grading:** $\mathcal{D} = \mathrm{Mat}_2(\mathbb{F}) \otimes \mathrm{Mat}_4(\mathbb{F})$, $\dim_{\mathcal{D}} V = 1$.
- $\mathbb{Z}_8^2$**-grading:** $\mathcal{D} = \mathrm{Mat}_8(\mathbb{F})$, $\dim_{\mathcal{D}} V = 1$.

Gradings on matrix algebras. Antiautomorphisms.

$\mathcal{A} = \mathsf{Mat}_n(\mathbb{F}), \qquad \Gamma : \mathcal{A} = \oplus_{g\in G}\mathcal{A}_g$

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If φ is an antiautomorphism of $\mathcal{A}$ preserving the grading, and such that φ^2 acts as a scalar on each homogeneous space (in particular, if φ is an involution on $\mathcal{A}$ preserving the grading), then there is a graded division algebra $\mathcal{D}$ with a graded involution τ, and a right graded module V for D endowed with a *balanced homogeneous hermitian form*

$$B : V \times V \to \mathcal{D},$$

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$$\mathcal{A} \cong \mathrm{End}_{\mathcal{D}}(V)$$

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The involution τ imposes severe restriction on $\mathcal{D}$, which must be a tensor product of algebras of 2×2 matrices (quaternion algebras) each one graded over $\mathbb{Z}_2^2$.

5 Classical simple Lie algebras

6 Octonions and G_2

7 The Albert algebra and F_4

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Orthogonal and symplectic Lie algebras

Orthogonal and symplectic Lie algebras

$$\mathbf{Aut}\,\mathfrak{so}_n(\mathbb{F}) \cong \mathbf{Aut}(\mathrm{Mat}_n(\mathbb{F}), \varphi_o), \qquad n \geq 5,\ n \neq 8,$$

$$\mathbf{Aut}\,\mathfrak{sp}_n(\mathbb{F}) \cong \mathbf{Aut}(\mathrm{Mat}_n(\mathbb{F}), \varphi_s), \qquad n \text{ even},\ n \geq 4.$$

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With $\mathcal{A} = \mathrm{Mat}_n(\mathbb{F})$ and φ an involution of $\mathcal{A}$:

$$\Gamma : \mathcal{A} = \oplus_{g\in G}\mathcal{A}_g \quad \text{with } \varphi(\mathcal{A}_g) = \mathcal{A}_g\ \forall g \in G$$

$$\Leftrightarrow\ \rho_\Gamma : \mathcal{A} \to \mathcal{A}\otimes\mathbb{F}G \text{ ‘commuting with } \varphi\text{’}$$

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It is enough to study gradings on matrix algebras which are compatible with an involution.

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In all these cases, $\mathfrak{g} \cong \mathcal{K}(\mathcal{A}, \varphi) = \{x \in \mathcal{A} : \varphi(x) = -x\}$, for $\mathcal{A} = \mathrm{Mat}_n(\mathbb{F})$ and φ an involution, and

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There is a graded division algebra $\mathcal{D}$ with a graded involution τ, and a right graded module V, endowed with a hermitian form $B : V \times V \to \mathcal{D}$ such that $\mathcal{A} \cong \mathrm{End}_{\mathcal{D}}(V)$ and φ corresponds to the adjoint map relative to B.

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Some freedom exists in the election of τ and B, and it is important to diagonalize B in a way compatible with the grading on V.

Example: fine gradings on $\mathfrak{sp}_8(\mathbb{F})$

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- $\boxed{\mathcal{D} = \mathbb{F}:}$ B is a nondegenerate skew-symmetric form, unique up to isomorphism. Take a symplectic homogeneous basis $\{u_i, v_i : 1 \leq i \leq 4\}$ of V ($B(u_i, v_j) = \delta_{ij} = -B(v_j, u_i)$, all the other pairings equal to 0). This gives a fine $\mathbb{Z}^4$-grading: $\deg(u_1) = (1, 0, 0, 0) = -\deg(v_1)$, ...

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- $\boxed{\mathcal{D} = \mathrm{Mat}_2(\mathbb{F}):}$ Here τ may be taken to be the standard symplectic involution on $\mathcal{D}$, and B an hermitian form. There are three different diagonalizations with diagonal $(1,1,1,1)$, $(1,1,\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right))$, and $(\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right))$. They give fine gradings over $\mathbb{Z}_2^5$, $\mathbb{Z}_2^3 \times \mathbb{Z}$ and $\mathbb{Z}_2^2 \times \mathbb{Z}^2$.

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- $\boxed{\mathcal{D} = \mathrm{Mat}_2(\mathbb{F}) \otimes \mathrm{Mat}_2(\mathbb{F}) \otimes \mathrm{Mat}_2(\mathbb{F}):}$ A final fine grading over $\mathbb{Z}_2^6$.

Special linear Lie algebras

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$$\mathbf{Aut}\,\mathfrak{sl}_2(\mathbb{F}) \cong \mathbf{Aut}\,\mathrm{Mat}_2(\mathbb{F}),$$

$$\mathbf{Aut}\,\mathfrak{psl}_n(\mathbb{F}) \cong \overline{\mathbf{Aut}}\,\mathrm{Mat}_n(\mathbb{F}), \qquad n \geq 3, \text{ unless } n = 3 = \operatorname{char}\mathbb{F}$$

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Here $\overline{\mathbf{Aut}}$ denotes the affine group scheme of automorphisms and antiautomorphisms.

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Assume $n \neq 3$ if $\operatorname{char} \mathbb{F} = 3$. Let $\mathfrak{g} = \mathfrak{psl}_n(\mathbb{F}) = [\mathcal{A}, \mathcal{A}]/Z(\mathcal{A}) \cap [\mathcal{A}, \mathcal{A}]$ with $\mathcal{A} = \mathrm{Mat}_n(\mathbb{F})$.

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- For $n = 2$, any grading on $\mathfrak{g}$ is induced by a grading on $\mathcal{A}$: $\mathfrak{g}_g = \mathfrak{g} \cap \mathcal{A}_g \ \forall g \in G$.

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- For $n = 2$, any grading on $\mathfrak{g}$ is induced by a grading on $\mathcal{A}$: $\mathfrak{g}_g = \mathfrak{g} \cap \mathcal{A}_g \ \forall g \in G$.
- For $n > 2$, any grading on $\mathfrak{g}$ is either
  - induced by a grading on $\mathcal{A}$, or
  - there are $h \in G$, $h^2 = 1$, a $\bar{G} = G/\langle h \rangle$-grading on $\mathcal{A}$, an antiautomorphism φ such that φ preserves the $\bar{G}$-grading and its square acts as a scalar on each homogeneous component, and a character $\chi \in \hat{G}$ with $\chi(h) = -1$, such that

$$\mathfrak{g}_g = \{x \in \mathcal{A}_{\bar{g}} : -\varphi(x) = \chi(g)x\} \cap [\mathcal{A}, \mathcal{A}] \pmod{Z(\mathcal{A})}.$$

Type A_2, char $\mathbb{F} = 3$.

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The exceptional case $\mathfrak{psl}_3(\mathbb{F})$ in characteristic 3 is dealt with in an unexpected way, as $\mathfrak{psl}_3(\mathbb{F}) \cong [\mathbb{O}, \mathbb{O}]$ here, and its gradings are in bijection with the gradings on the octonions. (E. 1998)

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In characteristic 3 there are no simple Lie algebras of type G_2.
(Note that $\mathfrak{Der}(\mathbb{O})$ is no longer simple in characteristic 3, as it contains the simple ideal $\mathrm{ad}_{[\mathbb{O},\mathbb{O}]} \cong \mathfrak{psl}_3(\mathbb{F})$.)

Octonions

Theorem (E. 1998)

Up to equivalence, the fine gradings on $\mathbb{O}$ are

- *the Cartan grading (weight space decomposition relative to a Cartan subalgebra of $\mathfrak{g}_2 = \mathfrak{Der}(\mathbb{O})$), and*
- *the $\mathbb{Z}_2^3$-grading given by the Cayley-Dickson doubling process.*

Sketch of proof:

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- The Cayley-Hamilton equation: $x^2 - n(x,1)x + n(x)1 = 0$, implies that the norm has a well behavior relative to the grading:

$$n(\mathbb{O}_g) = 0 \text{ unless } g^2 = e, \quad n(\mathbb{O}_g, \mathbb{O}_h) = 0 \text{ unless } gh = e.$$

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$$n(\mathbb{O}_g) = 0 \text{ unless } g^2 = e, \quad n(\mathbb{O}_g, \mathbb{O}_h) = 0 \text{ unless } gh = e.$$

- If there is a $g \in \mathrm{Supp}\,\Gamma$ with either order > 2 or $\dim \mathbb{O}_g \geq 2$, there are elements $x \in \mathbb{O}_g$, $y \in \mathbb{O}_{g^{-1}}$ with $n(x) = 0 = n(y)$, $n(x,y) = 1$. Then $e_1 = x\bar{y}$ and $e_2 = y\bar{x}$ are orthogonal primitive idempotents in $\mathbb{O}_e$, and one uses the corresponding Peirce decomposition to check that, up to equivalence, our grading is the Cartan grading.

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- Otherwise $\dim \mathbb{O}_g = 1$ and $g^2 = e$ for any $g \in \mathrm{Supp}\,\Gamma$. We get the $\mathbb{Z}_2^3$-grading.

$\mathbb{O}$ contains canonical bases:

$$\mathcal{B} = \{e_1, e_2, u_1, u_2, u_3, v_1, v_2, v_3\}$$

with

$$n(e_1, e_2) = n(u_i, v_i) = 1, \quad \text{otherwise } 0.$$

$$e_1^2 = e_1, \quad e_2^2 = e_2,$$
$$e_1 u_i = u_i e_2 = u_i, \quad e_2 v_i = v_i e_1 = v_i, \quad (i = 1, 2, 3)$$
$$u_i v_i = -e_1, \quad v_i u_i = -e_2, \quad (i = 1, 2, 3)$$
$$u_i u_{i+1} = -u_{i+1} u_i = v_{i+2},\ v_i v_{i+1} = -v_{i+1} v_i = u_{i+2},\ \text{(indices modulo 3)}$$

otherwise 0.

Cartan grading on the Octonions

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otherwise 0.

The Cartan grading is determined by:

$$\deg u_1 = -\deg v_1 = (1, 0), \quad \deg u_2 = -\deg v_2 = (0, 1),$$

$\mathbb{Z}_2^3$-grading: Octonions as a twisted group algebra

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Theorem (Albuquerque-Majid 1999)

The octonion algebra is the twisted group algebra

$$\mathbb{O} = \mathbb{F}_\sigma[\mathbb{Z}_2^3],$$

where

$$e^\alpha e^\beta = \sigma(\alpha, \beta) e^{\alpha+\beta}$$

for $\alpha, \beta \in \mathbb{Z}_2^3$, *with*

$$\sigma(\alpha, \beta) = (-1)^{\psi(\alpha,\beta)},$$

$$\psi(\alpha, \beta) = \beta_1\alpha_2\alpha_3 + \alpha_1\beta_2\alpha_3 + \alpha_1\alpha_2\beta_3 + \sum_{i \le j} \alpha_i\beta_j.$$

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This allows to consider the algebra of octonions as an "associative algebra in a suitable category".

G_2

Up to equivalence, the fine gradings on $\mathfrak{g}_2$ are

- the Cartan grading, and
- a $\mathbb{Z}_2^3$-grading with $(\mathfrak{g}_2)_0 = 0$ and where $(\mathfrak{g}_2)_g$ is a Cartan subalgebra of $\mathfrak{g}_2$ for any $0 \neq g \in \mathbb{Z}_2^3$.

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$\mathbf{Aut}\,\mathfrak{f}_4 \cong \mathbf{Aut}\,\mathbb{A}$ $\qquad$ ($\mathbb{A} = H_3(\mathbb{O})$ is the Albert algebra)

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The gradings on $\mathfrak{f}_4 = \mathfrak{Der}(\mathbb{A})$ are induced by gradings on $\mathbb{A}$.

Albert algebra

$$\mathbb{A} = H_3(\mathbb{O}, *) = \left\{ \begin{pmatrix} \alpha_1 & \bar{a}_3 & a_2 \\ a_3 & \alpha_2 & \bar{a}_1 \\ \bar{a}_2 & a_1 & \alpha_3 \end{pmatrix} \mid \alpha_1, \alpha_2, \alpha_3 \in \mathbb{F},\ a_1, a_2, a_3 \in \mathbb{O} \right\}$$

$$= \mathbb{F}E_1 \oplus \mathbb{F}E_2 \oplus \mathbb{F}E_3 \oplus \iota_1(\mathbb{O}) \oplus \iota_2(\mathbb{O}) \oplus \iota_3(\mathbb{O}),$$

where

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad E_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\iota_1(a) = 2\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \bar{a} \\ 0 & a & 0 \end{pmatrix}, \quad \iota_2(a) = 2\begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ \bar{a} & 0 & 0 \end{pmatrix}, \quad \iota_3(a) = 2\begin{pmatrix} 0 & \bar{a} & 0 \\ a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

Albert algebra

The multiplication in $\mathbb{A}$ is given by $X \circ Y = \frac{1}{2}(XY + YX)$.

Then E_i are orthogonal idempotents with $E_1 + E_2 + E_3 = 1$. The rest of the products are as follows:

$$E_i \circ \iota_i(a) = 0, \quad E_{i+1} \circ \iota_i(a) = \frac{1}{2}\iota_i(a) = E_{i+2} \circ \iota_i(a),$$

$$\iota_i(a) \circ \iota_{i+1}(b) = \iota_{i+2}(\bar{a}\bar{b}), \quad \iota_i(a) \circ \iota_i(b) = 2n(a,b)(E_{i+1} + E_{i+2}),$$

for any $a, b \in \mathbb{O}$, with $i = 1, 2, 3$ taken modulo 3.

$\mathbb{Z}_2^5$-grading:

$\mathbb{A}$ is naturally $\mathbb{Z}_2^2$-graded with

$$\mathbb{A}_{(\bar{0},\bar{0})} = \mathbb{F}E_1 + \mathbb{F}E_2 + \mathbb{F}E_3,$$

$$\mathbb{A}_{(\bar{1},\bar{0})} = \iota_1(\mathbb{O}), \qquad \mathbb{A}_{(\bar{0},\bar{1})} = \iota_2(\mathbb{O}), \qquad \mathbb{A}_{(\bar{1},\bar{1})} = \iota_3(\mathbb{O}).$$

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This $\mathbb{Z}_2^2$-grading may be combined with the fine $\mathbb{Z}_2^3$-grading on $\mathbb{O}$ to obtain a fine $\mathbb{Z}_2^5$-grading:

$$\deg E_i = (\bar{0},\bar{0},\bar{0},\bar{0},\bar{0}),\ i = 1,2,3,$$
$$\deg \iota_1(x) = (\bar{1},\bar{0},\deg x),\ \deg \iota_2(x) = (\bar{0},\bar{1},\deg x),\ \deg \iota_3(x) = (\bar{1},\bar{1},\deg x).$$

$\mathbb{Z}\times\mathbb{Z}_2^3$-grading:

Take an element $\mathbf{i}\in\mathbb{F}$ with $\mathbf{i}^2=-1$ and consider the following elements in $\mathbb{A}$:

$$\begin{aligned}
&E=E_1,\ \widetilde{E}=1-E=E_2+E_3,\\
&\nu(a)=\mathbf{i}\iota_1(a)\quad\text{for all}\quad a\in\mathbb{O}_0,\\
&\nu_\pm(x)=\iota_2(x)\pm\mathbf{i}\iota_3(\bar{x})\quad\text{for all}\quad x\in\mathbb{O},\\
&S^\pm=E_3-E_2\pm\frac{\mathbf{i}}{2}\iota_1(1).
\end{aligned}$$

$\mathbb{A}$ is then 5-graded:

$$\mathbb{A}=\mathbb{A}_{-2}\oplus\mathbb{A}_{-1}\oplus\mathbb{A}_0\oplus\mathbb{A}_1\oplus\mathbb{A}_2,$$

with $\mathbb{A}_{\pm2}=\mathbb{F}S^\pm$, $\mathbb{A}_{\pm1}=\nu_\pm(\mathbb{O})$, and $\mathbb{A}_0=\mathbb{F}E\oplus\Big(\mathbb{F}\widetilde{E}\oplus\nu(\mathbb{O}_0)\Big)$.

$\mathbb{Z}\times\mathbb{Z}_2^3$-grading:

The $\mathbb{Z}_2^3$-grading on $\mathbb{O}$ combines with this $\mathbb{Z}$-grading

$$\mathbb{A} = \mathbb{F}S^- \oplus \nu^-(\mathbb{O}) \oplus \mathbb{A}_0 \oplus \nu^+(\mathbb{O}) \oplus \mathbb{F}S^+$$

to give a fine $\mathbb{Z}\times\mathbb{Z}_2^3$-grading as follows:

$$\begin{aligned}
\deg S^\pm &= (\pm 2, \bar{0}, \bar{0}, \bar{0}),\\
\deg \nu_\pm(x) &= (\pm 1, \deg x),\\
\deg E &= 0 = \deg \widetilde{E},\\
\deg \nu(a) &= (0, \deg a),
\end{aligned}$$

for homogeneous elements $x\in\mathbb{O}$ and $a\in\mathbb{O}_0$.

$\mathbb{Z}_3^3$-grading (char $\mathbb{F} \neq 3$):

Consider the order three automorphism τ of $\mathbb{O}$:

$$\tau(e_i) = e_i,\ i = 1,2, \quad \tau(u_j) = u_{j+1},\ \tau(v_j) = v_{j+1},\ j = 1,2,3,$$

and define a new multiplication on $\mathbb{O}$:

$$x * y = \tau(\bar{x})\tau^2(\bar{y}).$$

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This is the *Okubo algebra*, which is $\mathbb{Z}_3^2$-graded by setting

$$\deg e_1 = (\bar{1}, \bar{0}) \quad \text{and} \quad \deg u_1 = (\bar{0}, \bar{1}).$$

Gradings on $\mathbb{A}$

$\mathbb{Z}_3^3$-grading (char $\mathbb{F} \neq 3$):

Define $\tilde{\iota}_i(x) = \iota_i(\tau^i(x))$ for all $i = 1, 2, 3$ and $x \in \mathbb{O}$. Then the multiplication in the Albert algebra

$$\mathcal{A} = \oplus_{i=1}^3 \bigl(\mathbb{F}E_i \oplus \tilde{\iota}_i(\mathbb{O})\bigr)$$

becomes:

$$E_i^{\circ 2} = E_i, \quad E_i \circ E_{i+1} = 0,$$
$$E_i \circ \tilde{\iota}_i(x) = 0, \quad E_{i+1} \circ \tilde{\iota}_i(x) = \frac{1}{2}\tilde{\iota}_i(x) = E_{i+2} \circ \tilde{\iota}_i(x),$$
$$\tilde{\iota}_i(x) \circ \tilde{\iota}_{i+1}(y) = \tilde{\iota}_{i+2}(x * y), \quad \tilde{\iota}_i(x) \circ \tilde{\iota}_i(y) = 2n(x, y)(E_{i+1} + E_{i+2}),$$

for $i = 1, 2, 3$ and $x, y \in \mathbb{O}$.

$\mathbb{Z}_3^3$-grading (char $\mathbb{F} \neq 3$):

Assume now char $\mathbb{F} \neq 3$. Then the $\mathbb{Z}_3^2$-grading on the Okubo algebra is determined by two commuting order 3 automorphisms $\varphi_1, \varphi_2 \in \mathrm{Aut}(\mathbb{O}, *)$:

$$\begin{aligned} \varphi_1(e_1) &= \omega e_1, \qquad & \varphi_1(u_1) &= u_1, \\ \varphi_2(e_1) &= e_1, \qquad & \varphi_2(u_1) &= \omega u_1, \end{aligned}$$

where ω is a primitive third root of unity in $\mathbb{F}$.

Gradings on $\mathbb{A}$

$\mathbb{Z}_3^3$-grading (char $\mathbb{F} \neq 3$):

The commuting order 3 automorphisms φ_1, φ_2 of $(\mathbb{O}, *)$ extend to commuting order 3 automorphisms of $\mathbb{A}$:

$$\varphi_j(E_i) = E_i, \quad \varphi_j\big(\tilde{\iota}_i(x)\big) = \tilde{\iota}_i(\varphi_j(x)).$$

On the other hand, the linear map $\varphi_3 \in \mathrm{End}(\mathcal{A})$ defined by

$$\varphi_3(E_i) = E_{i+1}, \quad \varphi_3\big(\tilde{\iota}_i(x)\big) = \tilde{\iota}_{i+1}(x),$$

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All the homogeneous components have dimension 1.

Theorem (Draper–Martín-González 2009 (char $= 0$), E.–Kochetov 2010)

Up to equivalence, the fine gradings of the Albert algebra are:

1. *The Cartan grading (weight space decomposition relative to a Cartan subalgebra of $\mathfrak{f}_4 = \mathfrak{Der}(\mathbb{A})$).*
2. *The $\mathbb{Z}_2^5$-grading obtained by combining the natural $\mathbb{Z}_2^2$-grading on 3×3 hermitian matrices with the fine grading over $\mathbb{Z}_2^3$ of $\mathbb{O}$.*
3. *The $\mathbb{Z} \times \mathbb{Z}_2^3$-grading obtained by combining a 5-grading and the $\mathbb{Z}_2^3$-grading on $\mathbb{O}$.*
4. *The $\mathbb{Z}_3^3$-grading with $\dim \mathbb{A}_g = 1\ \forall g$ (char $\mathbb{F} \neq 3$).*

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All the gradings up to isomorphism on $\mathbb{A}$ and $\mathfrak{f}_4$ have been classified too (E.–Kochetov).

The Albert algebra as a twisted group algebra

The Albert algebra as a twisted group algebra

Theorem

The Albert algebra is, up to isomorphism, the twisted group algebra

$$\mathbb{A} = \mathbb{F}_\sigma[\mathbb{Z}_3^3],$$

with

$$\sigma(\alpha,\beta) = \begin{cases} \omega^{\psi(\alpha,\beta)} & \text{if } \dim_{\mathbb{Z}_3}(\mathbb{Z}_3\alpha + \mathbb{Z}_3\beta) \leq 1, \\ -\frac{1}{2}\omega^{\psi(\alpha,\beta)} & \text{otherwise,} \end{cases}$$

where

$$\psi(\alpha,\beta) = (\alpha_2\beta_1 - \alpha_1\beta_2)(\alpha_3 - \beta_3).$$

5 Classical simple Lie algebras

6 Octonions and G_2

7 The Albert algebra and F_4

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- Bahturin, Kochetov and their students have classified gradings on some Cartan type simple modular Lie algebras. This requires different methods.
- A classification of gradings up to isomorphism on D_4 is missing even in characteristic 0, as it is the classification of fine gradings up to equivalence in prime characteristic.
- Many interesting gradings on E_6, E_7, E_8 are known, some of them are related either to gradings on octonions, the Okubo algebras, or the Albert algebra, but a classification is missing.
  The classification of fine gradings on E_6 in characteristic 0 has been announced by Draper and Viruel.

A commercial break

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Santos González

VI International Conference on

NON ASSOCIATIVE ALGEBRA AND ITS APPLICATIONS

A Conference in Honor of the 60th birthday of Santos González

November 1-5, 2011

The **VI International Conference on Non Associative Algebra and its Applications** will be held in Zaragoza (Spain), November 1-5, 2011.

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That's all. Thanks