Clifford algebras as twisted group algebras and the Arf invariant

Instituto Universitario de Matemáticas y Aplicaciones
Universidad de Zaragoza

Alberto Elduque

*(based on joint work with
Adrián Rodrigo-Escudero)*

Goal

To connect

- the work of Bahturin and Zaicev on one hand, and Rodrigo-Escudero on the other, on real graded-division algebras,
- Ovsienko's work on real Clifford algebras and quadratic forms over $\mathbb{F}_2$, and
- the results of **Albuquerque** and Majid on Clifford algebras as twisted group algebras.

Summary

1. Graded-division algebras and twisted group algebras
2. Clifford algebras as twisted group algebras
3. Quadratic forms over $\mathbb{F}_2$
4. From quadratic forms over $\mathbb{F}_2$ to real algebras

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Group-gradings

Definition

Let $\mathcal{A}$ be an algebra (over a field $\mathbb{F}$) and let G be an *abelian* group.

- A G-grading on $\mathcal{A}$ is a vector space decomposition $\Gamma : \mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g$ such that $\mathcal{A}_g \mathcal{A}_h \subseteq \mathcal{A}_{gh}$ for any $g, h \in G$.
- The nonzero elements in $\mathcal{A}_g$ are said to be homogeneous of degree g.
- The support of Γ is the set $\{g \in G \mid \mathcal{A}_g \neq 0\}$.

Graded-division algebras

Given and algebra with a G-grading Γ, $(\mathcal{D}, \Gamma)$ is a graded-division algebra if the left and right multiplications by any homogeneous element are bijections.

If $\mathcal{D}$ is associative, this is equivalent to $\mathcal{D}$ being unital and every nonzero homogeneous element being invertible.
In this case $1 \in \mathcal{D}_e$ and for $0 \neq X \in \mathcal{D}_g$, $X^{-1} \in \mathcal{D}_{g^{-1}}$.
The support of Γ is then a subgroup of G.

Twisted group algebras

Definition

Given a group G, a field $\mathbb{F}$ and a map $\sigma : G \times G \to \mathbb{F}^\times$, the twisted group algebra $\mathbb{F}^\sigma G$ is the algebra over $\mathbb{F}$ with a basis consisting of a copy of G: $\{\varepsilon_g : g \in G\}$, and with (bilinear) multiplication given by

$$\varepsilon_g \varepsilon_h := \sigma(g, h)\varepsilon_{gh}$$

for any $g, h \in G$.

$\mathbb{F}^\sigma G$ is a graded-division algebra.

Twisted group algebras

Remark

Any G-graded-division algebra (not necessarily associative), with homogeneous components of dimension 1, is a twisted group algebra, isomorphic to $\mathbb{F}^\sigma T$, for a suitable σ, where T is the support of the grading.

Twisted group algebras

Examples

With $\mathbb{F} = \mathbb{R}$, **Albuquerque** and Majid considered the classical algebras of complex numbers, quaternions and octonions as the twisted group algebras $\mathbb{R}^\sigma T$ with:

- $T = \mathbb{Z}_2$ and $\sigma(x, y) = (-1)^{xy}$ for the complex numbers.
- $T = \mathbb{Z}_2^2$ and $\sigma\big((x_1, x_2), (y_1, y_2)\big) = (-1)^{x_1 y_1 + (x_1 + x_2) y_2}$ for the real associative division algebra $\mathbb{H}$ of quaternions.
- $T = \mathbb{Z}_2^3$ and

$$\sigma\big((x_1, x_2, x_3), (y_1, y_2, y_3)\big) = (-1)^{y_1 x_2 x_3 + x_1 y_2 x_3 + x_1 x_2 y_3 + \sum_{1 \leq i \leq j \leq 3} x_i y_j}$$

for the real non-associative division algebra $\mathbb{O}$ of octonions.

Twisted group algebras

Associative case

The twisted group algebra $\mathbb{F}^\sigma G$ is associative if, and only if, σ is a 2-cocycle:

$$\sigma(g,h)\sigma(gh,k) = \sigma(h,k)\sigma(g,hk), \quad \forall g,h,k \in G,$$

with values in $\mathbb{F}^\times$.

Two twisted group algebras $\mathbb{F}^\sigma G$ and $\mathbb{F}^{\sigma'} G$ are isomorphic as G-graded algebras if, and only if, σ and σ' are cohomologous:

$$\exists \mu : G \to \mathbb{F}^\times \text{ s.t. } \mu(gh)\sigma(g,h) = \mu(g)\mu(h)\sigma'(g,h), \ \forall g,h \in G.$$

Twisted group algebras

Associative case

Given a 2-cocycle $\sigma \in Z^2(G, \mathbb{F}^\times)$, consider the map

$$\begin{aligned}\beta : G \times G &\to \mathbb{F}^\times \\ (g, h) &\mapsto \sigma(g,h)\sigma(h,g)^{-1}.\end{aligned}$$

In other words, β is determined by the *commutativity condition*:

$$\varepsilon_g \varepsilon_h = \beta(g,h)\varepsilon_h \varepsilon_g$$

for any $g, h \in G$, in the twisted group algebra $\mathbb{F}^\sigma G$.

β is an alternating bicharacter:

- $\beta(g,g) = 1$ for any $g \in G$.
- β is multiplicative on each slot.

Twisted group algebras

Another way of looking at them

Theorem

Let G be a finitely generated abelian group, and let $\beta : G \times G \to \mathbb{F}^\times$ be an alternating bicharacter on G, then there exists a 2-cocycle $\sigma \in Z^2(G, \mathbb{F}^\times)$ such that $\beta(g,h) = \sigma(g,h)\sigma(h,g)^{-1}$.

Moreover, if $G = \langle g_1 \rangle \times \cdots \times \langle g_N \rangle$, for a finite number of elements $g_1, \ldots, g_N$, with g_i of order $m_i \in \mathbb{N}_{\geq 2}$ for $i = 1, \ldots, r$, and g_i of infinite order for $i = r+1, \ldots, N$; and if $\mu_1, \ldots, \mu_r \in \mathbb{F}^\times$ are chosen arbitrarily, then the 2-cocycle σ above can be taken so that $\mathbb{F}^\sigma G$ is isomorphic, as a G-graded algebra, to the algebra:

$$\begin{aligned}\mathcal{A}_{\mathbb{F}}(N, \underline{m}, \underline{\mu}, \beta) := \operatorname{alg}\Big\langle x_1, \ldots, x_N \mid x_i^{m_i} = \mu_i,\ i = 1, \ldots, r;\\ x_i x_j = \beta(g_i, g_j) x_j x_i,\ i, j = 1, \ldots, N\Big\rangle,\end{aligned}$$

which is a G-graded-division algebra ($\deg(x_i) = g_i\ \forall i = 1$).

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Clifford algebras

Definition

Let V be an $\mathbb{F}$-vector space of finite dimension N, and let $Q : V \to \mathbb{F}$ be a quadratic form on V.

The Clifford algebra $\mathrm{Cl}(V(, Q)$ is the quotient of the tensor algebra $T(V)$ by the ideal $I(Q)$ generated by the elements $v \otimes v - Q(v)1$, $v \in V$.

Clifford algebras

If the characteristic of the ground field $\mathbb{F}$ is not 2, Q is nondegenerate, and $\{v_1, \ldots, v_N\}$ is an orthogonal basis of V, then

$$\begin{aligned}\mathrm{Cl}(V, Q) \cong \mathrm{alg}\,\big\langle x_1, \ldots, x_N \mid x_i^2 = \mu_i,\ i = 1, \ldots, N;\\ x_i x_j = -x_j x_i,\ i \neq j\big\rangle = \mathcal{A}_{\mathbb{F}}(N, \underline{2}, \underline{\mu}, \beta)\end{aligned}$$

where

- $\underline{2} = (2, 2, \ldots, 2)$ (N components),
- $\underline{\mu} = \big(Q(v_1), Q(v_2), \ldots, Q(v_N)\big)$,
- β is the alternating bicharacter on the cartesian product $G = C_2^N$ ($G = \langle g_1 \rangle \times \cdots \times \langle g_N \rangle$ with g_i or order 2 for any $i = 1, \ldots, N$), with $\beta(g_i, g_j) = -1$ for any $i \neq j$.

Clifford algebras

- $\mathcal{A}_{\mathbb{F}}(N, \underline{2}, \underline{\mu}, \beta)$ is graded-isomorphic to a twisted group algebra $\mathbb{F}^\sigma G$ and, in particular, its dimension is 2^N.
- Therefore $\mathrm{Cl}(V, Q)$ is a G-graded-division algebra, with $\deg(v_i) = g_i$ for $i = 1, \ldots, N$ (here we identify $v_i \in V$ with the class of v_i modulo $I(Q)$ in $\mathrm{Cl}(V, Q)$).
- This is the grading by the group $C_2^N \cong \mathbb{Z}_2^N$ that **Albuquerque** and Majid considered in 2002.

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Quadratic forms over $\mathbb{F}_2$

Here we will consider quadratic forms $\mathbf{q} : W \to \mathbb{F}_2$ defined on a finite dimensional vector space W over the field $\mathbb{F}_2$ of two elements.

Let $b_{\mathbf{q}} : W \times W \to \mathbb{F}_2$ be the associated bilinear form.
As $2 = 0$ in $\mathbb{F}_2$, $b_{\mathbf{q}}$ is alternating: $b_{\mathbf{q}}(w, w) = 0$ for any $w \in W$.

Two quadratic forms $\mathbf{q} : W \to \mathbb{F}_2$ and $\mathbf{q}' : W' \to \mathbb{F}_2$ are equivalent if there is a linear isomorphism $\varphi : W \to W'$ such that $\mathbf{q}'\big(\varphi(w)\big) = \mathbf{q}(w)$ for any $w \in W$.

The orthogonal sum $\mathbf{q} \perp \mathbf{q}'$ is the quadratic form on $W \times W'$ given by

$$(\mathbf{q} \perp \mathbf{q}')(w, w') = \mathbf{q}(w) + \mathbf{q}'(w')$$

for any $w \in W$ and $w' \in W'$.

Quadratic forms over $\mathbb{F}_2$

Classification (Dickson 1901)

Theorem

Any regular *quadratic form* $\mathbf{q} : \mathbb{F}_2^N \to \mathbb{F}_2$ *is equivalent to one of the following:*

N even $(N = 2n)$

- $\mathbf{q}_0^N(x_1, \ldots, x_{2n}) = x_1x_2 + x_3x_4 + \cdots + x_{2n-1}x_{2n}$,
- $\mathbf{q}_1^N(x_1, \ldots, x_{2n}) =$
  $x_1x_2 + x_3x_4 + \cdots + x_{2n-1}x_{2n} + x_1 + x_2$.

N odd $(N = 2n + 1)$

- $\mathbf{q}_0^N(x_1, \ldots, x_{2n+1}) = x_1x_2 + x_3x_4 + \cdots + x_{2n-1}x_{2n}$,
- $\mathbf{q}_1^N(x_1, \ldots, x_{2n+1}) =$
  $x_1x_2 + x_3x_4 + \cdots + x_{2n-1}x_{2n} + x_1 + x_2$,
- $\mathbf{q}_2^N(x_1, \ldots, x_{2n+1}) =$
  $x_1x_2 + x_3x_4 + \cdots + x_{2n-1}x_{2n} + x_{2n+1}$.

Arf invariant

The Arf invariant of a quadratic form $\mathbf{q} : W \to \mathbb{F}_2$ on a finite dimensional vector space W over $\mathbb{F}_2$ is the *democratic invariant* defined by:

$$\mathrm{Arf}(\mathbf{q}) := \begin{cases} 1 & \text{if } |\{w \in W \mid \mathbf{q}(w) = 0\}| > |\{w \in W \mid \mathbf{q}(w) = 1\}|, \\ 0 & \text{if } |\{w \in W \mid \mathbf{q}(w) = 0\}| = |\{w \in W \mid \mathbf{q}(w) = 1\}|, \\ -1 & \text{if } |\{w \in W \mid \mathbf{q}(w) = 0\}| < |\{w \in W \mid \mathbf{q}(w) = 1\}|. \end{cases}$$

$$\mathrm{Arf}(\mathbf{q}_0^N) = 1, \qquad \mathrm{Arf}(\mathbf{q}_1^N) = -1, \qquad \mathrm{Arf}(\mathbf{q}_2^N) = 0 \ (N \text{ odd}),$$

so regular quadratic forms are determined, up to equivalence, by their Arf invariant.

Arf invariant

Example

Take $p, q \in \mathbb{Z}_{\geq 0}$ such that $p + q = N$, and consider the regular quadratic form $\mathbf{q}_{p,q} : \mathbb{F}_2^N \to \mathbb{F}_2$ defined by:

$$\mathbf{q}_{p,q}(x_1, \ldots, x_N) = x_{p+1} + \cdots + x_N + \sum_{1 \leq i < j \leq N} x_i x_j.$$

Theorem

$$\begin{aligned}
\mathrm{Arf}(\mathbf{q}_{p,q}) &= \mathrm{sign}\left(\cos\frac{(p-q)\pi}{4} + \sin\frac{(p-q)\pi}{4}\right) \\
&= \begin{cases} 1 & \textit{if } p - q + 1 \equiv 1, 2, 3 \pmod 8, \\ 0 & \textit{if } p - q + 1 \equiv 0, 4 \pmod 8, \\ -1 & \textit{if } p - q + 1 \equiv 5, 6, 7 \pmod 8. \end{cases}
\end{aligned}$$

Multiplicative notation

Given a quadratic form $\mathbf{q} : \mathbb{F}_2^N \to \mathbb{F}_2$, its multiplicative version is the map $\boldsymbol{\mu} : C_2^N \to C_2$ given by

$$\boldsymbol{\mu}\left((-1)^{x_1}, \ldots, (-1)^{x_N}\right) = (-1)^{\mathbf{q}(x_1, \ldots, x_N)}$$

for $x_1, \ldots, x_N \in \mathbb{F}_2 = \{0, 1\}$.

Its *polar form* is given by

$$\beta_{\boldsymbol{\mu}}(r, s) = \boldsymbol{\mu}(rs)\boldsymbol{\mu}(r)\boldsymbol{\mu}(s)$$

which is an alternating bicharacter. Identifying C_2 with $\{1, -1\} \leq \mathbb{R}^\times$, we may thing of $\beta_{\boldsymbol{\mu}}$ as an alternating bicharacter with values in $\mathbb{R}^\times$.

The Arf invariant of $\boldsymbol{\mu}$ is, by definition, the Arf invariant of $\mathbf{q}$.

The multiplicative versions of the quadratic forms $\mathbf{q}_i^N$ will be denoted by $\boldsymbol{\mu}_i^N$, and the multiplicative version of $\mathbf{q}_{p,q}$ by $\boldsymbol{\mu}_{p,q}$.

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From quadratic forms over $\mathbb{F}_2$ to real algebras

Proposition

Given a multiplicative quadratic form $\boldsymbol{\mu} : C_2^N \to C_2$, the C_2^N-graded algebra $\mathcal{A}_{\mathbb{R}}(N, \underline{2}, \underline{\mu}, \beta_{\boldsymbol{\mu}})$ ($\underline{\mu} := \big(\boldsymbol{\mu}(\mathbf{e}_1), \ldots, \boldsymbol{\mu}(\mathbf{e}_N)\big)$, where $\mathbf{e}_i = (1, \ldots, 1, \overbrace{-1}^{i}, 1, \ldots, 1)$) is graded isomorphic to the twisted group algebra $\mathbb{R}^\sigma C_2^N$, where the 2-cocycle σ is given by

$$\sigma(q, q) = \boldsymbol{\mu}(q) \ \forall q \in C_2^N;$$

and

$$\sigma(\mathbf{e}_i, \mathbf{e}_j) = \begin{cases} 1 & \text{if } i < j, \\ \beta_{\boldsymbol{\mu}}(\mathbf{e}_i, \mathbf{e}_j) & \text{if } i > j. \end{cases}$$

$\mathcal{A}_{\mathbb{R}}(N, \underline{2}, \underline{\mu}, \beta_{\boldsymbol{\mu}})$ depends only on the quadratic form $\boldsymbol{\mu}$: $\mathcal{A}_{\mathbb{R}}(\boldsymbol{\mu})$.

From quadratic forms over $\mathbb{F}_2$ to real algebras

Theorem (Ovsienko 2016)

- $M_{2^n}(\mathbb{R}) \cong \mathcal{A}_{\mathbb{R}}(\boldsymbol{\mu}_0^{2n})$,
- $M_{2^{n-1}}(\mathbb{H}) \cong \mathcal{A}_{\mathbb{R}}(\boldsymbol{\mu}_1^{2n})$,
- $M_{2^n}(\mathbb{R}) \times M_{2^n}(\mathbb{R}) \cong \mathcal{A}_{\mathbb{R}}(\boldsymbol{\mu}_0^{2n+1})$,
- $M_{2^{n-1}}(\mathbb{H}) \times M_{2^{n-1}}(\mathbb{H}) \cong \mathcal{A}_{\mathbb{R}}(\boldsymbol{\mu}_1^{2n+1})$,
- $M_{2^n}(\mathbb{C}) \cong \mathcal{A}_{\mathbb{R}}(\boldsymbol{\mu}_2^{2n+1})$.

From quadratic forms over $\mathbb{F}_2$ to real algebras

This result can be rephrased as follows:

Corollary

Let $\boldsymbol{\mu} : C_2^N \to C_2$ be a regular multiplicative quadratic form.

- *if N is even, $N = 2n$, and $\mathrm{Arf}(\boldsymbol{\mu}) = 1$, then $\mathcal{A}_{\mathbb{R}}(\boldsymbol{\mu}) \cong M_{2^n}(\mathbb{R})$,*
- *if N is even, $N = 2n$, and $\mathrm{Arf}(\boldsymbol{\mu}) = -1$, then $\mathcal{A}_{\mathbb{R}}(\boldsymbol{\mu}) \cong M_{2^{n-1}}(\mathbb{H})$,*
- *if N is odd, $N = 2n+1$, and $\mathrm{Arf}(\boldsymbol{\mu}) = 1$, then $\mathcal{A}_{\mathbb{R}}(\boldsymbol{\mu}) \cong M_{2^n}(\mathbb{R}) \times M_{2^n}(\mathbb{R})$,*
- *if N is odd, $N = 2n+1$, and $\mathrm{Arf}(\boldsymbol{\mu}) = -1$, then $\mathcal{A}_{\mathbb{R}}(\boldsymbol{\mu}) \cong M_{2^{n-1}}(\mathbb{H}) \times M_{2^{n-1}}(\mathbb{H})$,*
- *if N is odd, $N = 2n+1$, and $\mathrm{Arf}(\boldsymbol{\mu}) = 0$, then $\mathcal{A}_{\mathbb{R}}(\boldsymbol{\mu}) \cong M_{2^n}(\mathbb{C})$.*

$\mathrm{Cl}_{p,q}(\mathbb{R}) = \mathcal{A}_{\mathbb{R}}(\boldsymbol{\mu}_{p,q})$ and we have computed $\mathrm{Arf}(\boldsymbol{\mu}_{p,q})$, so:

Corollary

Let $p, q \in \mathbb{Z}_{\geq 0}$ and $N = p + q$. Then,

- $\mathrm{Cl}_{p,q}(\mathbb{R}) \cong M_{2^{N/2}}(\mathbb{R})$ *if* $p - q + 1 \equiv 1, 3 \pmod 8$.
- $\mathrm{Cl}_{p,q}(\mathbb{R}) \cong M_{2^{(N-2)/2}}(\mathbb{H})$ *if* $p - q + 1 \equiv 5, 7 \pmod 8$.
- $\mathrm{Cl}_{p,q}(\mathbb{R}) \cong M_{2^{(N-1)/2}}(\mathbb{R}) \times M_{2^{(N-1)/2}}(\mathbb{R})$ *if* $p - q + 1 \equiv 2 \pmod 8$.
- $\mathrm{Cl}_{p,q}(\mathbb{R}) \cong M_{2^{(N-1)/2}}(\mathbb{C})$ *if* $p - q + 1 \equiv 0, 4 \pmod 8$.
- $\mathrm{Cl}_{p,q}(\mathbb{R}) \cong M_{2^{(N-3)/2}}(\mathbb{H}) \times M_{2^{(N-3)/2}}(\mathbb{H})$ *if* $p - q + 1 \equiv 6 \pmod 8$.

Corollary

Let $p, q \in \mathbb{Z}_{\geq 0}$*, then the following* periodicity results *hold:*

1. $\mathrm{Cl}_{p+1,q+1}(\mathbb{R}) \cong \mathrm{Cl}_{p,q}(\mathbb{R}) \otimes_{\mathbb{R}} M_2(\mathbb{R})$.
2. $\mathrm{Cl}_{p+2,q}(\mathbb{R}) \cong \mathrm{Cl}_{q,p}(\mathbb{R}) \otimes_{\mathbb{R}} M_2(\mathbb{R})$.
3. $\mathrm{Cl}_{p,q+2}(\mathbb{R}) \cong \mathrm{Cl}_{q,p}(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{H}$.
4. $\mathrm{Cl}_{p+4,q}(\mathbb{R}) \cong \mathrm{Cl}_{p,q+4}(\mathbb{R})$.

Happy birthday!