Symmetric Composition Algebras and Freudenthal's Magic Square

1. **Tits construction.**
2. **Symmetric composition algebras.**
3. **Freudenthal's Magic Square.**
4. **New construction of the exceptional simple Lie algebras.**

1. Tits construction.

Throughout F will denote a field of characteristic $\neq 2, 3$.

- **C a unital composition algebra over F:**

$$a^2 - \mathrm{tr}(a)a + n(a)1 = 0,$$

$$n(ab) = n(a)n(b),$$

$$D_{a,b} = [L_a, L_b] + [L_a, R_b] + [R_a, R_b] \in \mathrm{Der}(C).$$

- **J a unital Jordan algebra over F with a *normalized trace*:**

$$xy = yx, \quad x^2(yx) = (x^2y)x$$

$$t(1) = 1, \qquad t\big((J, J, J)\big) = 0,$$

$$xy = t(xy)1 + x * y,$$

$$d_{x,y} = [l_x, l_y] \in \mathrm{Der}(J).$$

$$\boxed{\mathcal{T}(C, J) := D_{C,C} \oplus (C^0 \otimes J^0) \oplus d_{J,J}}$$

with the anticommutative product [,] specified by

- $D_{C,C}$ **and** $d_{J,J}$ **are Lie subalgebras,**
- $[D_{C,C}, d_{J,J}] = 0,$
- $[D, a \otimes x] = D(a) \otimes x, \qquad [d, a \otimes x] = a \otimes d(x),$
- $[a \otimes x, b \otimes y] = t(xy)D_{a,b} + [a, b] \otimes x * y$
  $$+ 2\,\mathrm{tr}(ab)d_{x,y}.$$

1. Tits construction.

$\mathcal{T}(C,J)$ **is a Lie algebra if and only if**

$$\text{(i)}\quad 0=\sum_{\text{cyclic}}\operatorname{tr}\big([a_1,a_2]a_3\big)d_{(x_1*x_2),x_3},$$

$$\text{(ii)}\quad 0=\sum_{\text{cyclic}}t\big((x_1*x_2)x_3\big)D_{[a_1,a_2],a_3}$$

$$\begin{aligned}\text{(iii)}\quad 0=\sum_{\text{cyclic}}\big(&D_{a_1,a_2}(a_3)\otimes t\big(x_1x_2\big)x_3\\ &+[[a_1,a_2],a_3]\otimes(x_1*x_2)*x_3\\ &+2\operatorname{tr}\big(a_1a_2\big)a_3\otimes d_{x_1,x_2}(x_3)\big)\end{aligned}$$

In particular, this happens if J satisfies the Cayley-Hamilton equation $ch_3(x)=0$, where

$$\begin{aligned}ch_3(x)=x^3-3t(x)x^2+\big(\tfrac{9}{2}t(x)^2-\tfrac{3}{2}t(x^2)\big)x\\ -\Big(t(x^3)-\tfrac{9}{2}t(x^2)t(x)+\tfrac{9}{2}t(x)^3\Big)1\end{aligned}$$

1. Tits construction.

Freudenthal's Magic Square

$\boldsymbol{C \setminus J}$	$\boldsymbol{F}$	$\boldsymbol{H_3(F)}$	$\boldsymbol{H_3(K)}$	$\boldsymbol{H_3(Q)}$	$\boldsymbol{H_3(C)}$
$\boldsymbol{F}$	$\mathbf{0}$	$\mathbf{A_1}$	$\mathbf{A_2}$	$\mathbf{C_3}$	$\mathbf{F_4}$
$\boldsymbol{K}$	$\mathbf{0}$	$\mathbf{A_2}$	$\mathbf{A_2 \oplus A_2}$	$\mathbf{A_5}$	$\mathbf{E_6}$
$\boldsymbol{Q}$	$\mathbf{A_1}$	$\mathbf{C_3}$	$\mathbf{A_5}$	$\mathbf{D_6}$	$\mathbf{E_7}$
$\boldsymbol{C}$	$\mathbf{G_2}$	$\mathbf{F_4}$	$\mathbf{E_6}$	$\mathbf{E_7}$	$\mathbf{E_8}$

1. Tits construction.

Remark:

Replace Jordan algebra by Jordan superalgebra above.

Here, a *normalized trace* satisfies

$$t(1) = 1, \qquad t(J_{\bar{1}}) = 0, \qquad t\big((J, J, J)\big) = 0.$$

Only the following Jordan superalgebras do the job:

i) $J(V) = F1 \oplus V$ **of a supersymmetric bilinear form such that** $V = V_{\bar{1}}$ **and** $\dim V = 2$**, and**

i) $D_\mu = (Fe \oplus Ff) \oplus (Fx \oplus Fy)$ $(\mu \neq 0, -1)$**, with multiplication given by**

$$e^2 = e, \qquad f^2 = f, \qquad ef = 0$$

$$ex = \tfrac{1}{2}x = fx, \quad ey = \tfrac{1}{2}y = fy,$$

$$xy = e + \mu f = -yx.$$

1. Tits construction.

Extension of Freudenthal's Magic Square

$\boldsymbol{C \setminus J}$	$\boldsymbol{J(V)}$	$\boldsymbol{D_\mu}$ $(\mu\neq 0,-1)$
$\boldsymbol{F}$	$\mathbf{A_1}$	$\mathbf{B(0,1)}$
$\boldsymbol{K}$	$\mathbf{B(0,1)}$	$\mathbf{A(1,0)}$
$\boldsymbol{Q}$	$\mathbf{B(1,1)}$	$\mathbf{D(2,1;}\boldsymbol{\mu}\mathbf{)}$
$\boldsymbol{C}$	$\mathbf{G(3)}$	$\mathbf{F(4)}$ $(\mu=2,1/2)$

1. Tits construction.

More symmetric constructions:

Vinberg (1966):

$$\mathcal{V}(C, C') = \mathrm{Der}\, C \oplus \mathrm{Der}\, C' \oplus \mathrm{Skew}\big(\mathrm{Mat}_3(C \otimes C'), *\big)$$

(involved multiplication)

Allison-Faulkner (1993), Barton-Sudbery (2000), Landsberg-Manivel (2002):

$$\mathcal{BS}(C, C') = \mathrm{Tri}(C) \oplus \mathrm{Tri}(C') \oplus 3 \text{ copies of } C \otimes C'$$

$$\mathrm{Tri}(C) = \{(f, g, h) \in \mathfrak{so}(C)^3 : \\ f(xy) = h(x)y + xg(y)\ \forall x, y \in C\}$$

1. Tits construction.

2. Symmetric composition algebras.

$(S, *, q)$ **such that:** $$\begin{cases} q(x*y) = q(x)q(y) \\ q(x*y, z) = q(x, y*z) \end{cases}$$

Examples:

i) (Okubo-Myung 1980) (C, n) **Hurwitz algebra (unital composition algebra)**

$$\begin{cases} x*y = \bar{x}\bar{y} \\ q(x) = n(x) \end{cases}$$

para-Hurwitz algebra $\bar{C}$

ii) (Okubo 1978) $S = \mathfrak{sl}_3(\mathbb{C}), \quad \omega = e^{2\pi i/3},$

$$\begin{cases} x*y = \omega xy - \omega^2 yx - \dfrac{\omega - \omega^2}{3}\,\mathrm{tr}(xy)1 \\ q(x) = -\dfrac{1}{2}\,\mathrm{tr}(x^2) \end{cases}$$

Classification:

(Okubo, Osborn, Myung, Pérez-Izquierdo, E.)

Essentially, the symmetric composition algebras are either

para-Hurwitz algebras, or

Okubo algebras (related to ii)).

Okubo algebras are described in terms of separable associative algebras of degree 3 (possibly with involution of second kind) over fields of characteristic $\neq 3$, and in an 'ad hoc' way in characteristic 3.

Triality:

$(S, *, q)$ symmetric composition algebra

$$\mathfrak{tri}(S,*,q) = \{(d_0, d_1, d_2) \in \mathfrak{so}(S,q)^3 : \\ d_0(x*y) = d_1(x)*y + x*d_2(y)\ \forall x,y \in S\}$$

(triality algebra)

$$\begin{aligned} \theta : \mathfrak{tri}(S,*,q) &\longrightarrow \mathfrak{tri}(S,*,q) \\ (d_0, d_1, d_2) &\mapsto (d_2, d_0, d_1) \end{aligned}$$

is an automorphism of order 3.

$$\mathrm{Fix}(\theta) = \mathrm{Der}\, S = \begin{cases} A_2 & \text{Okubo algebras (char} \neq 3) \\ 0, 0, A_1 \text{ or } G_2 & \text{para-Hurwitz} \end{cases}$$

Principle of Local Triality:

- **For any $d_0 \in \mathfrak{so}(S,q)$, there are $d_1, d_2 \in \mathfrak{so}(S,q)$ such that $(d_0, d_1, d_2) \in \mathfrak{tri}(S,q)$:**

$$\begin{aligned} S \times S &\longrightarrow \mathfrak{tri}(S,*,q) \\ (x,y) &\mapsto \quad t_{x,y} \end{aligned}$$

where, if $\sigma_{x,y} = q(x,-)y - q(y,-)x$,

$$t_{x,y} = \left(\sigma_{x,y}, \frac{1}{2}q(x,y)1 - r_x l_y, \frac{1}{2}q(x,y)1 - l_x r_y\right)$$

- **If $\dim S = 8$, then there is uniqueness above.**

That is, the projections

$$\begin{aligned} \mathfrak{tri}(S,*,q) &\longrightarrow \mathfrak{so}(S,q) \\ (d_0, d_1, d_2) &\mapsto \quad d_i \end{aligned}$$

are isomorphisms.

3. Freudenthal's Magic Square.

$(S, *, q)$, $(S', *, q')$ **symmetric composition algebras**

$$\mathfrak{g}(S, S') = \mathfrak{tri}(S, *, q) \oplus \mathfrak{tri}(S', *, q') \oplus \Big(\oplus_{i=0}^{2} \iota_i (S \otimes S') \Big)$$

with anticommutative multiplication given by:

- $\mathfrak{tri}(S, *, q)$ **and** $\mathfrak{tri}(S', *, q')$ **are Lie subalgebras of** $\mathfrak{g}(S, S')$, **with** $[\mathfrak{tri}(S, *, q), \mathfrak{tri}(S', *, q')] = 0$.
- $[(d_0, d_1, d_2), \iota_i(a \otimes x)] = \iota_i\big(d_i(a) \otimes x\big)$, **same on second slot.**
- $[\iota_i(a \otimes x), \iota_{i+1}(b \otimes y)] = \iota_{i+2}\big((a*b) \otimes (x*y)\big)$ **(indices modulo 3).**
- $[\iota_i(a \otimes x), \iota_i(b \otimes y)] = q'(x, y)\theta^i(t_{a,b}) + q(a, b)\theta'^i(t'_{x,y})$.

Then

$$\mathfrak{g}(S, S') \text{ is a Lie algebra}$$

and

dim S

dim S'		1	2	4	8
	1	$\mathbf{A_1}$	$\mathbf{A_2}$	$\mathbf{C_3}$	$\mathbf{F_4}$
	2	$\mathbf{A_2}$	$\mathbf{A_2 \oplus A_2}$	$\mathbf{A_5}$	$\mathbf{E_6}$
	4	$\mathbf{C_3}$	$\mathbf{A_5}$	$\mathbf{D_6}$	$\mathbf{E_7}$
	8	$\mathbf{F_4}$	$\mathbf{E_6}$	$\mathbf{E_7}$	$\mathbf{E_8}$

3. Freudenthal's Magic Square.

Remarks:

- **The symmetry of the construction makes very easy to check that $\mathfrak{g}(S, S')$ is indeed a Lie algebra.**

- $\mathcal{T}(C, J_{C'})$, $\mathcal{V}(C, C')$, $\mathcal{BS}(C, C') \quad \equiv \quad \mathfrak{g}(\bar{C}, \bar{C}')$.

- **For E_8, three different constructions are obtained:**

$$\mathfrak{g}(pH, pH), \quad \mathfrak{g}(pH, Ok), \quad \mathfrak{g}(Ok, Ok),$$

each one displays different aspects of E_8.

4. New construction of the exceptional simple Lie algebras.

$\dim_F V = 2$,

$(|)$ nondegenerate and skew symmetric on V,

$\mathfrak{sp}(V) = \mathfrak{sl}(V)$

Then

$$V \otimes V \simeq \mathrm{End}_F(V)\,(= Q), \quad a \otimes b \mapsto (a|-)b$$

$$C = Q \oplus Q \simeq V \otimes V \ \oplus \ V \otimes V$$

The split para-Cayley algebra is $S_8 = V \otimes V \ \oplus \ V \otimes V$ with

$$(a \otimes b, c \otimes d) * (a' \otimes b', c' \otimes d') = \Big((b|a')b' \otimes a - (d|d')c \otimes c', -(a|c')b \otimes d' - (c|b')a' \otimes d\Big)$$

$$q_8\big((a \otimes b, c \otimes d), (a' \otimes b', c' \otimes d')\big) = (a|a')(b|b') + (c|c')(d|d')$$

$\mathfrak{tri}(S_8, *, q_8)$ ($\simeq \mathfrak{so}(S_8, q_8)$)?

$\mathfrak{sp}(V)^4$ **acts naturally on** $S_8 = V \otimes V \ \oplus \ V \otimes V$:

$$\rho : \mathfrak{sp}(V)^4 \hookrightarrow \mathfrak{so}(S_8, q_8)$$

Also consider

$$\rho : V^{\otimes 4} \hookrightarrow \mathfrak{so}(S_8, q_8).$$

given by

$$\begin{aligned} &\rho(v_1 \otimes v_2 \otimes v_3 \otimes v_4)\Big((a \otimes b, c \otimes d)\Big) \\ &\quad = \Big((v_3|c)(v_4|d)v_1 \otimes v_2, -(v_1|a)(v_2|b)v_3 \otimes v_4\Big). \end{aligned}$$

Then

$$\mathfrak{so}(S_8, q_8) \simeq \mathfrak{sp}(V)^4 \oplus V^{\otimes 4}$$

with a 'natural' bracket.

Consider the order 3 automorphism θ**:**

$$\theta\big((s_1, s_2, s_3, s_4)\big) = (s_3, s_1, s_2, s_4)$$

$$\theta(v_1 \otimes v_2 \otimes v_3 \otimes v_4) = v_3 \otimes v_1 \otimes v_2 \otimes v_4$$

$\forall f \in \mathfrak{d} = \mathfrak{sp}(V)^4 \oplus V^{\otimes 4}$, $\forall x, y \in S_8$:

$$\boxed{\rho(f)(x * y) = \rho\big(\theta^{-1}(f)\big)(x) * y + x * \rho\big(\theta^{-2}(f)\big)(y)}$$

so (with $\rho_i = \rho \circ \theta^{-i}$)

$$\mathfrak{tri}(S_8, *, q_8) = \big\{\big(\rho_0(f), \rho_1(f), \rho_2(f)\big) : f \in \mathfrak{d}\big\} \ (\simeq \mathfrak{d}).$$

As $\mathfrak{sp}(V)^4$-module, S_8 is isomorphic to:

$$\begin{aligned} &V_1 \otimes V_2 \ \oplus \ V_3 \otimes V_4 \quad \text{through } \rho_0, \\ &V_2 \otimes V_3 \ \oplus \ V_1 \otimes V_4 \quad \text{through } \rho_1, \\ &V_1 \otimes V_3 \ \oplus \ V_2 \otimes V_4 \quad \text{through } \rho_2, \end{aligned}$$

where V_i denotes the natural module for the i^{th} copy of $\mathfrak{sp}(V)$ in $\mathfrak{sp}(V)^4$, annihilated by the other copies.

Remark: If S_4 denotes the split para-quaternion algebra, $S_4 = V \otimes V$, then

$$\mathfrak{tri}(S_4, *, q_4) = \Big\{\big(\tilde\rho_0(f), \tilde\rho_1(f), \tilde\rho_2(f)\big) : f \in \mathfrak{sp}(V)^3\Big\}$$

($\tilde\rho_i$ is the restriction of ρ_i: $\tilde\rho_i : \mathfrak{sp}(V)^3 \hookrightarrow \mathfrak{so}(V \otimes V)$).

4. Exceptional simple Lie algebras.

Some notation:

$$\mathfrak{sp}(V)^n = \mathfrak{sp}(V_1) \oplus \cdots \oplus \mathfrak{sp}(V_n).$$

$\forall \sigma \in \mathbb{Z}_2^n$ let

$$V(\sigma) = V_{i_1} \otimes \cdots \otimes V_{i_r}$$

where $i_1 < \cdots < i_r$ are the indices of the slots with nonzero component of σ. By convention $V(0) = \mathfrak{sp}(V)^n$.

For any $\sigma, \tau \in \mathbb{Z}_2^n$ define:

$$\boxed{\varphi_{\sigma,\tau} : V(\sigma) \times V(\tau) \to V(\sigma + \tau)}$$

by means of

$$\varphi_{(1,1,1,0),(1,0,1,1)}(v_1 \otimes v_2 \otimes v_3, w_1 \otimes w_3 \otimes w_4)$$
$$= (v_1|w_1)(v_3|w_3)v_2 \otimes w_4$$

$$\varphi_{(1,0,1,0),(1,0,1,0)}(v_1 \otimes v_3, w_1 \otimes w_3)$$
$$= \Big((v_3|w_3)\gamma_{v_1,w_1}, 0, (v_1|w_1)\gamma_{v_3,w_3}, 0\Big) \in \mathfrak{sp}(V)^4$$

(where $\gamma_{v,w} = \frac{1}{2}\Big((v|.)w + (w|.)v\Big) \in \mathfrak{sp}(V)$)

$$\varphi_{0,(1,0,1,0)}\big((s_1, s_2, s_3, s_4), v_1 \otimes v_3\big)$$
$$= -\varphi_{(1,0,1,0),0}\big(v_1 \otimes v_3, (s_1, s_2, s_3, s_4)\big)$$
$$= s_1(v_1) \otimes v_3 + v_1 \otimes s_3(v_3).$$

$$\begin{aligned}
\mathbf{E_7} &= \mathfrak{g}(S_8, S_4) \\
&= \mathfrak{tri}(S_8, *, q_8) \oplus \mathfrak{tri}(S_4, *, q_4) \oplus \Big(\oplus_{i=0}^{2} \iota_i(S_8 \otimes S_4)\Big) \\
&= \Big(\mathfrak{sp}(V)^4 \oplus V_1 \otimes V_2 \otimes V_3 \otimes V_4\Big) \oplus \mathfrak{sp}(V)^3 \\
&\qquad \oplus (V_1 \otimes V_2 \oplus V_3 \otimes V_4) \otimes (V_5 \otimes V_6) \\
&\qquad \oplus (V_2 \otimes V_3 \oplus V_1 \otimes V_4) \otimes (V_6 \otimes V_7) \\
&\qquad \oplus (V_1 \otimes V_3 \oplus V_2 \otimes V_4) \otimes (V_5 \otimes V_7) \\
&= \bigoplus_{\sigma \in \mathcal{S}_{E_7}} V(\sigma),
\end{aligned}$$

with

$$\begin{aligned}
\mathcal{S}_{E_7} = \Big\{ &\emptyset, \{1,2,3,4\}, \\
&\{1,2,5,6\}, \{3,4,5,6\}, \\
&\{2,3,6,7\}, \{1,4,6,7\}, \\
&\{1,3,5,7\}, \{2,4,5,7\} \Big\}
\end{aligned}$$

4. Exceptional simple Lie algebras.

The multiplication in $\mathfrak{g}(S_8, S_4)$ translates into:

$$[x_\sigma, y_\tau] = \epsilon_7(\sigma,\tau)\varphi_{\sigma,\tau}(x_\sigma, y_\tau)$$

for any $x_\sigma \in V(\sigma)$, $y_\tau \in V(\tau)$, where

$$\epsilon_7 : \mathcal{S}_{E_7} \times \mathcal{S}_{E_7} \to \{\pm 1\}$$

is given by the signs that appear in the multiplication table of the classical octonions $\mathbb{O}$ in the usual basis $\{1, i, j, k, l, il, jl, kl\}$, under the assignment:

$$\begin{array}{llll} \emptyset & \mapsto 1 & \{1,2,3,4\} & \mapsto l \\ \{1,2,5,6\} & \mapsto i & \{3,4,5,6\} & \mapsto il \\ \{2,3,6,7\} & \mapsto j & \{1,4,6,7\} & \mapsto jl \\ \{1,3,5,7\} & \mapsto k & \{2,4,5,7\} & \mapsto kl \end{array}$$

A precise and simple formula can be given explictly for ϵ_7 by following ideas of Albuquerque and Majid: $\mathbb{O} = \mathbb{R}_f[\mathbb{Z}_2^3]$ (twisted group algebra).

$$\mathbf{E_8} = \mathfrak{g}(\boldsymbol{S_8}, \boldsymbol{S_8})$$

$$= \bigoplus_{\sigma \in \mathcal{S}_{E_8}} V(\sigma),$$

with

$$\mathcal{S}_{E_8} = \Big\{\emptyset, \{1,2,3,4\}, \{5,6,7,8\}$$
$$\{1,2,5,6\}, \{3,4,5,6\}, \{1,2,7,8\}, \{3,4,7,8\},$$
$$\{2,3,6,7\}, \{1,4,6,7\}, \{2,3,5,8\}, \{1,4,5,8\},$$
$$\{1,3,5,7\}, \{2,4,5,7\}, \{1,3,6,8\}, \{2,4,6,8\}\Big\}$$

Here

$$\epsilon_8 : \mathcal{S}_{E_8} \times \mathcal{S}_{E_8} \to \{\pm 1\}$$

is given by the signs that appear in the multiplication table of $\mathbb{O} \otimes_{\mathbb{R}} \mathbb{R}[\varepsilon]$, with $\varepsilon^2 = 1$, in a suitable basis.

The construction thus obtained for $\mathbf{E}_7$ and $\mathbf{E}_8$ is related to a previous one by A. Grishkov (2001), obtained by giving the multiplication table of these Lie algebras in very specific bases.

Similar constructions appear for all the Lie algebras in Freudenthal's Magic Square:

$$\mathbf{F}_4 = \mathfrak{g}(S_8, F) = \bigoplus_{\sigma \in \mathcal{S}_{F_4}} V(\sigma),$$

with

$$\mathcal{S}_{F_4} = \Big\{\emptyset, \{1,2,3,4\}, \{1,2\}, \{2,3\}, \{1,3\}, \{3,4\}, \{1,4\}, \{2,4\}\Big\}$$

$$\mathbf{E_6} = \mathfrak{g}(S_8, S_2) = \bigoplus_{\sigma \in \mathcal{S}_{F_4}} \tilde{V}(\sigma),$$

where

$$\tilde{V}(\emptyset) = \mathfrak{sp}(V)^4 \oplus \mathfrak{t}_2,$$

($\mathfrak{t}_2$ a two dimensional abelian Lie algebra,)

$$\tilde{V}(\{1,2,3,4\}) = V(\{1,2,3,4\}),$$

$$\tilde{V}(\sigma) = V(\sigma) \otimes E \quad \text{otherwise.}$$

(E a suitable two dimensional $\mathfrak{t}_2$-module.)

And also,

$$\mathbf{C_3} = \mathfrak{g}(S_4, F) = \bigoplus_{\substack{\sigma \in \mathcal{S}_{F_4} \\ 4 \notin \sigma}} V(\sigma) \qquad \Big(\leq \mathbf{F_4}\Big),$$

$$\mathbf{A_5} = \mathfrak{g}(S_4, S_2) = \bigoplus_{\substack{\sigma \in \mathcal{S}_{F_4} \\ 4 \notin \sigma}} \tilde{V}(\sigma) \qquad \Big(\leq \mathbf{E_6}\Big),$$

$$\mathbf{D_6} = \mathfrak{g}(S_4, S_4) = \bigoplus_{\substack{\sigma \in \mathcal{S}_{E_7} \\ 4 \notin \sigma}} V(\sigma) \qquad \Big(\leq \mathbf{E_7}\Big).$$

Freudenthal triple systems:

Given a $\mathbb{Z}_2$-graded Lie algebra

$$\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$$

with

$$\begin{cases} \mathfrak{g}_{\bar{0}} = \mathfrak{s} \oplus \mathfrak{sp}(V) \\ \mathfrak{g}_{\bar{1}} = T \otimes V \end{cases}$$

Then

$$[x \otimes u, y \otimes v] = (u|v)d_{x,y} + \{x|y\}\gamma_{u,v}$$

where $d : T \times T \to \mathfrak{s}$ and $\{.|.\} : T \times T \to F$ are bilinear $\mathfrak{s}$-invariant maps, d being symmetric and $\{.|.\}$ alternating.

With

$$xyz = d_{x,y}(z) - \{x|z\}y - \{y|z\}x,$$

T becomes a Freudenthal triple system.

$$\mathbf{E_8} = (\mathbf{E_7} \oplus \mathfrak{sp}(V_8)) \bigoplus \left(\Big(\bigoplus_{\substack{\sigma \in \mathcal{S}_{E_8} \\ 8 \in \sigma}} V(\sigma \setminus \{8\}) \Big) \otimes V_8 \right)$$

$$\mathbf{E_7} = (\mathbf{D_6} \oplus \mathfrak{sp}(V_4)) \bigoplus \left(\Big(\bigoplus_{\substack{\sigma \in \mathcal{S}_{E_7} \\ 4 \in \sigma}} V(\sigma \setminus \{4\}) \Big) \otimes V_4 \right)$$

$$\mathbf{E_6} = (\mathbf{A_5} \oplus \mathfrak{sp}(V_4)) \bigoplus \left(\Big(\bigoplus_{\substack{\sigma \in \mathcal{S}_{F_4} \\ 4 \in \sigma}} \tilde{V}(\sigma \setminus \{4\}) \Big) \otimes V_4 \right)$$

$$\mathbf{F_4} = (\mathbf{C_3} \oplus \mathfrak{sp}(V_4)) \bigoplus \left(\Big(\bigoplus_{\substack{\sigma \in \mathcal{S}_{F_4} \\ 4 \in \sigma}} V(\sigma \setminus \{4\}) \Big) \otimes V_4 \right)$$

The simple exceptional Freudenthal triple systems, which usually appear as matrices $\begin{pmatrix} F & J \\ J & F \end{pmatrix}$, with J a central simple Jordan algebra of degree 3, are obtained in this way, and hence they are described too in terms of copies of $\mathfrak{sp}(V)$ and of V.

4. Exceptional simple Lie algebras.