Nonassociative systems and irreducible homogeneous spaces I

(joint work with P. Benito and F. Martín)

1. **Nomizu's Theorem.**

2. **Examples.**

3. **Lie-Yamaguti algebras.**

4. **Classification results.**

1. Nomizu's Theorem.

Homogeneous spaces:

- G Lie group
- M differentiable manifold
- $G \times M \to M$, $(a, m) \mapsto a.m = \tau_a m$, a smooth and transitive action.

M is a homogeneous space

Fix $p \in M$, then $K = \{a \in G : a.p = p\}$ is a closed subgroup of G and $M \simeq G/K$.

M is said to be *reductive* if $\exists \mathfrak{m}$ subspace of $\mathfrak{g} = \mathrm{Lie}(G)$ such that:

$$\begin{cases} \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m} \quad (\mathfrak{k} = \mathrm{Lie}(K)) \\ (\mathrm{Ad}\, K)(\mathfrak{m}) \subseteq \mathfrak{m} \qquad (\Longrightarrow [\mathfrak{k}, \mathfrak{m}] \subseteq \mathfrak{m} \\ \qquad\qquad\qquad\qquad \Longleftarrow \quad \text{if } K \text{ is connected}) \end{cases}$$

$$\begin{aligned} \mathfrak{m} &\longleftrightarrow \mathrm{T}_p M \\ X &\mapsto \left.\frac{d}{dt}\right|_{t=0} (\exp tX).p \end{aligned}$$

Affine connections:

$$M \simeq G/K,\ \chi(M) = \{\text{smooth vector fields on } M\}$$

$$\begin{aligned} \nabla : \chi(M) \times \chi(M) &\longrightarrow \chi(M) \\ (X, Y) &\mapsto \nabla_X(Y) \end{aligned}$$

is said to be an *affine connection* if:

(i) ∇ is $\mathcal{C}^\infty(M)$-linear in the first component and $\mathbb{R}$-linear in the second.

(ii) $\nabla_X(fY) = (Xf)Y + f\nabla_X Y$.

If

(iii) $d_{\tau_a}(\nabla_X Y) = \nabla_{d\tau_a X} d_{\tau_a} Y$

then ∇ is said to be *G-invariant*.

1. Nomizu's Theorem.

<u>Theorem</u> (Nomizu, 1954)

There is a bijection

$$\left\{\begin{array}{c}\text{Invariant}\\ \text{affine}\\ \text{connections}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\alpha : \mathfrak{m}\times\mathfrak{m}\to\mathfrak{m}\\ \quad \alpha \text{ is bilinear}\\ \quad \text{and } \mathrm{Ad}\,K\text{-invariant}\end{array}\right\}$$

Sketch of proof:

$$X \in \mathfrak{m} \quad \longleftrightarrow \quad X_m^+ = \left.\frac{d}{dt}\right|_{t=0} (\exp tX).m$$

Given an invariant ∇, consider the *Nomizu operator*

$$L_{X^+} = \nabla_{X^+} - \mathrm{ad}_{X^+}$$

and

$$\alpha : \mathfrak{m}\times\mathfrak{m} \to \mathfrak{m} \qquad \alpha(X,Y) = \left(L_{X^+}Y^+\right)_p \in \mathrm{T}_p M \simeq \mathfrak{m}.$$

1. Nomizu's Theorem.

Purely algebraic problem:

Given a reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$

($\mathfrak{k}$ is a subalgebra of $\mathfrak{g}$, and $\mathfrak{m}$ a subspace with $[\mathfrak{k}, \mathfrak{m}] \subseteq \mathfrak{m}$)

determine the nonassociative algebras defined on $\mathfrak{m}$ with a fixed Lie subalgebra of derivations: $\mathrm{ad}_{\mathfrak{k}}$.

$$\mathrm{Hom}_{\mathfrak{k}}(\mathfrak{m} \otimes \mathfrak{m}, \mathfrak{m}) \ ?$$

Geometric properties are expressed in algebraic terms:

Torsion: $T(X, Y) = \alpha(X, Y) - \alpha(Y, X) - [X, Y]_{\mathfrak{m}}$

Curvature: $R(X, Y)Z = \alpha(X, \alpha(Y, Z)) - \alpha(Y, \alpha(X, Z)) - \alpha([X, Y]_{\mathfrak{m}}, Z) - [[X, Y]_{\mathfrak{k}}, Z]$

There appear naturally two distinguished connections:

Canonical $\alpha(X, Y) = 0$

Natural $\alpha(X, Y) = \frac{1}{2}[X, Y]_{\mathfrak{m}}$ **(symmetric)**

1. Nomizu's Theorem.

2. Examples.

(i) Symmetric spaces:

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m} \quad \text{is a } \mathbb{Z}_2\text{-grading}$$

In the *irreducible* case, $\mathfrak{g}$ is simple and

Theorem (Laquer, Benito-Draper-E.)

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ be a $\mathbb{Z}_2$-graded central simple Lie algebra over a field of characteristic 0. Then $\mathrm{Hom}_{\mathfrak{k}}(\mathfrak{m} \otimes \mathfrak{m}, \mathfrak{m}) = 0$ unless either:

(a) $\mathfrak{k}$ is simple of type $\neq A$ and $\mathfrak{m} \cong \mathfrak{k}$ as $\mathfrak{k}$-modules. In this case $\mathrm{Hom}_{\mathfrak{k}}(\mathfrak{m} \otimes \mathfrak{m}, \mathfrak{m}) \cong \mathrm{Hom}_{\mathfrak{k}}(\mathfrak{k} \otimes \mathfrak{k}, \mathfrak{k})$, which is spanned by the Lie bracket.

(b) There exists a central simple Jordan algebra of degree $n \geq 3$ such that, up to isomorphism, $\mathfrak{k} = \mathrm{Der}\, J$, $\mathfrak{m} = J_0 = \{x \in J : \mathrm{tr}(x) = 0\}$. In this case, depending on J, $\dim \mathrm{Hom}_{\mathfrak{k}}(\mathfrak{m} \otimes \mathfrak{m}, \mathfrak{m}) = 1$ or 2.

In the real compact case, case (a) corresponds to the the compact Lie groups other that SU_n and case (b) to the symmetric spaces SU_n/SO_n, SU_{2n}/SP_{2n}, E_6/F_4 and the compact groups SU_n.

(ii) Spheres and octonions

Borel (1949): the only spheres that appear as homogeneous spaces in nonclassical ways are:

$$S^6 = G_2/SU(3),\ S^7 = Spin_7/G_2,\ S^{15} = Spin_9/Spin_7.$$

$\mathbb{O} = \mathbb{C} \oplus \mathbb{C}^3$ **(algebra of octonions)**

$$(\alpha + u)(\beta + v) = (\alpha\beta - \sigma(u,v)) + (\alpha v + \bar{\beta} u + u * v)$$

$$\begin{cases} \sigma : \mathbb{C}^3 \times \mathbb{C}^3 \to \mathbb{C} \quad \text{usual hermitian form} \\ \sigma(u, v * w) = \det(u, v, w) \\ \overline{\alpha + u} = \bar{\alpha} - u \\ t(x) = x + \bar{x} \in \mathbb{R}, \quad n(x) = x\bar{x} = |\alpha|^2 + ||u||^2 \in \mathbb{R} \end{cases}$$

$$\boldsymbol{S^6 \simeq \{x \in \mathbb{O} : t(x) = 0, n(x) = 1\}}$$

$$\mathfrak{g} = \mathbf{Der}\,\mathbb{O},\ \ \mathfrak{k} = \{\boldsymbol{d} \in \mathbf{Der}\,\mathbb{O} : \boldsymbol{d}(\mathbb{C}) = \mathbf{0}\}$$

$$\boldsymbol{S^7 \simeq \{x \in \mathbb{O} : n(x) = 1\}}$$

$$\mathfrak{g} = \mathbf{Der}(\mathbb{O}, (\boldsymbol{x\bar{y}})\boldsymbol{z}),\ \ \mathfrak{k} = \mathbf{Der}\,\mathbb{O} = \{\boldsymbol{d} \in \mathfrak{g} : \boldsymbol{d}(\mathbf{1}) = \mathbf{0}\}$$

$$\boldsymbol{S^{15} \simeq \{(x, y) \in \mathbb{O} \times \mathbb{O} : n(x) + n(y) = 1\}}$$

$$\mathbf{dim\,Hom}_{\mathfrak{k}}(\mathfrak{m} \otimes \mathfrak{m}, \mathfrak{m}) = \begin{cases} \mathbf{2} & \textbf{for } \boldsymbol{S^6} \\ \mathbf{1} & \textbf{for } \boldsymbol{S^7} \\ \mathbf{3} & \textbf{for } \boldsymbol{S^{15}} \end{cases}$$

$$(\mathfrak{m}, [\ ,\ ]_{\mathfrak{m}})\ \textbf{is a}\ \begin{cases} \textbf{\textit{vector color algebra}} \textbf{ for } \boldsymbol{S^6} \\ \textbf{simple non-Lie Malcev algebra for } \boldsymbol{S^7} \end{cases}$$

2. Examples.

3. Lie-Yamaguti algebras.

The key information is located in two multiplications on $\mathfrak{m}$:

$$\begin{cases} \text{binary:} \quad x \cdot y = [x,y]_{\mathfrak{m}} \\ \\ \text{ternary:} \quad [x,y,z] = [[x,y]_{\mathfrak{k}}, z] \end{cases}$$

<u>Definition</u> (Kinyon-Weinstein 2001)

A *Lie-Yamaguti algebra* $(\mathfrak{m}, \cdot, [\ ,\ ,\ ])$ is a vector space $\mathfrak{m}$ equipped with a bilinear operation $\cdot : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ and a trilinear operation $[\ ,\ ,\ ] : \mathfrak{m} \times \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ such that, for all $x, y, z, u, v, w \in \mathfrak{m}$:

(LY1) $x \cdot x = 0$,

(LY2) $[x, x, y] = 0$,

(LY3) $\sum_{(x,y,z)} \Big([x,y,z] + (x \cdot y) \cdot z \Big) = 0$,

(LY4) $\sum_{(x,y,z)} [x \cdot y, z, t] = 0$,

(LY5) $[x, y, u \cdot v] = [x, y, u] \cdot v + u \cdot [x, y, v]$,

(LY6) $[x, y, [u, v, w]] = [[x, y, u], v, w] + [u, [x, y, v], w] + [u, v, [x, y, w]]$.

Previously named *General Lie Triple Systems* (Yamaguti 1958) or *Lie triple algebras* (Kikkawa 1975).

$(\mathfrak{m}, \cdot, [\ ,\ ,\ ])$ Lie-Yamaguti algebra:

$\forall x, y \in \mathfrak{m}$, the operator

$$\begin{aligned} D(x,y) : \mathfrak{m} &\to \mathfrak{m} \\ z &\mapsto [x,y,z] \end{aligned}$$

is a derivation of both the binary and the ternary products, and

$$[D(x,y), D(z,t)] = D([x,y,z],t) + D(z,[x,y,t])$$

so that $D(\mathfrak{m},\mathfrak{m})$ is a Lie subalgebra of $\mathfrak{gl}(\mathfrak{m})$.

Then

$$\mathcal{L}(\mathfrak{m}) = D(\mathfrak{m},\mathfrak{m}) \oplus \mathfrak{m}$$

is a Lie algebra with

- **$D(\mathfrak{m},\mathfrak{m})$ is a Lie subalgebra,**
- **$[D(x,y), z] = [x,y,z]$,**
- **$[x,y] = D(x,y) + x \cdot y$,**

called the *standard enveloping Lie algebra* of $\mathfrak{m}$.

Therefore

Lie-Yamaguti algebra $\equiv$ component $\mathfrak{m}$ of
a reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$

4. Classification results.

Simple Lie-Yamaguti algebras?

Out of reach, by now, because of the generality of the concept.

A Lie-Yamaguti algebra is said to be *irreducible* if $\mathfrak{m}$ is an irreducible module for $D(\mathfrak{m}, \mathfrak{m})$.

$$\boxed{\text{Irreducible} \Longrightarrow \text{simple.}}$$

Real irreducible LY algebras correspond to the *isotropy irreducible homogeneous spaces* studied by Wolf (1968).

Irreducible Lie-Yamaguti algebras?

<u>Purpose</u>: To classify the irreducible Lie-Yamaguti algebras, while showing their connections to other nonassociative algebraic systems.

In what follows:

> k **algebraically closed field of characteristic 0,**
>
> $(\mathfrak{m}, \cdot, [\ ,\ ,\ ])$ **irreducible LY algebra over** k**,**
>
> $\mathfrak{d} = D(\mathfrak{m}, \mathfrak{m})$, $\mathfrak{g} = \mathcal{L}(\mathfrak{m})$ **(the standard enveloping Lie algebra),**

<u>Theorem</u>

(i) $\mathfrak{d}$ **is a maximal subalgebra of** $\mathfrak{g}$**,**

(ii) $\mathfrak{d}$ **is semisimple,**

(iii) Either $\mathfrak{g}$ **is simple and** $\mathfrak{m} = \mathfrak{d}^{\perp}$ **(orthogonal relative to the Killing form), or** $\mathfrak{d}$ **is simple and** $\mathfrak{m}$ **is, up to isomorphism, the adjoint module for** $\mathfrak{d}$**.**

Therefore, the classification splits into:

- ***adjoint case*****:** $\mathfrak{m}$ **is the adjoint module for** $\mathfrak{d}$**,**
- ***nonsimple case*****:** $\mathfrak{d}$ **is not simple,**
- ***generic case*****: both** $\mathfrak{g}$ **and** $\mathfrak{d}$ **are simple.**

Adjoint case:

$\mathfrak{m} \cong \mathfrak{d}$, $\mathfrak{d}$ is a simple Lie algebra,

$\mathrm{Hom}_{\mathfrak{d}}(\mathfrak{d} \wedge \mathfrak{d}, \mathfrak{d})$ is spanned by the Lie bracket,

so the bilinear maps $\cdot : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ and $D : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{d}$ are given by

$$x \cdot y = \alpha[x, y],$$

$$D(x, y) = \beta\, \mathrm{ad}_{[x,y]}\,.$$

Theorem

In the adjoint case, $\mathfrak{m}$ has the structure of a simple Lie algebra and either:

- the LY algebra $\mathfrak{m}$ is the Lie triple system associated to this simple Lie algebra ($\alpha = 0$, $\beta = 1$), or
- $\alpha = 1$ and $\beta \neq 0$.

Also, either $\mathcal{L}(\mathfrak{m}) \cong \mathfrak{m} \oplus \mathfrak{m}$ (as Lie algebra) or $\mathcal{L}(\mathfrak{m}) \cong k[t]/(t^2) \otimes_k \mathfrak{m}$ (this is the only case in which $\mathcal{L}(\mathfrak{m})$ is not semisimple).

Nonsimple case:

Some examples:

(1) $\mathfrak{m} = \mathfrak{sl}(V_1) \otimes \mathfrak{sl}(V_2)$ ($\dim V_i \geq 2$, i=1,2),

$\mathfrak{g} = \mathfrak{sl}(V_1 \otimes V_2)$ and $\mathfrak{d} = \mathfrak{sl}(V_1) \oplus \mathfrak{sl}(V_2)$.

$$(x_1 \otimes x_2) \cdot (y_1 \otimes y_2) = \frac{1}{2}\big([x_1, y_1] \otimes (x_2 \star y_2) + (x_1 \star y_1) \otimes [x_2, y_2]\big)$$

$$\left(x_i \star y_i = x_i y_i + y_i x_i - \frac{2\,\mathrm{tr}(x_i y_i)}{\dim V_i} Id\right)$$

$$[x_1 \otimes x_2, y_1 \otimes y_2, z_1 \otimes z_2] = (y_1, z_1, x_1) \otimes \left(\frac{\mathrm{tr}(x_2 y_2)}{\dim V_2} z_2\right) + \left(\frac{\mathrm{tr}(x_1 y_1)}{\dim V_1} z_1\right) \otimes (y_2, z_2, x_2)$$

$$\big((x, y, z) = (x \circ y) \circ z - x \circ (y \circ z)\big)$$

$$\big(x \circ y = xy + yx\big)$$

(2) $\mathfrak{m} = \mathrm{Skew}(U, \varphi) \otimes \mathrm{Skew}(V, \gamma)$

$\dim U = 2$, φ nondegenerate skew-symmetric form,

$\dim V \geq 3$, γ nondegenerate (skew-)symmetric form,

$\mathfrak{g} = \mathrm{Skew}(U \otimes V, \varphi \otimes \gamma)$,

$\mathfrak{d} = \mathrm{Skew}(U, \varphi) \oplus \mathrm{Skew}(V, \gamma)$.

$$(x_1 \otimes x_2) \cdot (y_1 \otimes y_2) = \frac{1}{2}[x_1, y_1] \otimes x_2 \star y_2$$

$[x_1 \otimes x_2, y_1 \otimes y_2, z_1 \otimes z_2]$ as in (1).

4. Classification results.

(3) Related to the Tits construction of the exceptional simple Lie algebras

$A \setminus J$	$H_3(k)$	$H_3(K)$	$H_3(Q)$	$H_3(C)$
Q				$\mathbf{E}_7$
C	$\mathbf{F}_4$	$\mathbf{E}_6$	$\mathbf{E}_7$	$\mathbf{E}_8$

$\mathfrak{m} = Q_0 \otimes H_3(C)_0$ **or** $C_0 \otimes H_3(B)_0$ $(B = k, K, Q$ **or** $C)$

$\mathfrak{d} = A_1 \times F_4$ **or** $G_2 \times L$ **with** $L = A_1, A_2, C_3$ **or** F_4**.**

$\mathfrak{g}$ **is given in the table.**

4. Classification results.

(4) $\mathfrak{m} = V_1 \otimes V_2$, where V_1 and V_2 are vector spaces endowed with nondegenerate bilinear forms γ_i ($i = 1, 2$), both symmetric or skew-symmetric, $\dim V_i \geq 3$ (resp. $\dim V_i \geq 2$) if γ_i is symmetric (resp. skew-symmetric),

$$\mathfrak{g} = \mathrm{Skew}(V_1 \oplus V_2, \gamma_1 \perp \gamma_2),$$

$$\mathfrak{d} = \mathrm{Skew}(V_1, \gamma_1) \oplus \mathrm{Skew}(V_2, \gamma_2)$$

$$(x_1 \otimes x_2) \cdot (y_1 \otimes y_2) = 0$$

$$[x_1 \otimes x_2, y_1 \otimes y_2, z_1 \otimes z_2] =$$
$$\gamma_{x_1,y_1}(z_1) \otimes \gamma_2(x_2, y_2)z_2 + \gamma_1(x_1, y_1)z_1 \otimes \gamma_{x_2,y_2}(z_2)$$
$$\Big(\gamma_{x_i,y_i}(z_i) = \gamma_i(x_i, z_i)y_i - \gamma_i(z_i, y_i)x_i\Big)$$

4. Classification results.

(5) The Lie triple systems corresponding to the symmetric pairs

$$(G_2, A_1 \times A_1), \quad (F_4, A_1 \times C_3), \quad (E_6, A_1 \times A_5),$$

$$(E_7, A_1 \times D_6), \quad (E_8, A_1 \times E_7)$$

Theorem

The list above exhausts all the possibilities in the 'nonsimple case'.

Remarks:

- **Examples (4) and (5) are Lie triple systems. The ones in (5) are related to *Freudenthal triple systems*.**

- **In example (1) consider $J = \mathrm{Mat}_{p\times q}(k)$, with $p = \dim V_1$ and $q = \dim V_2$. J is a *Jordan triple system* ($J \cong V_1 \otimes V_2$)**

$$instr(J) \cong \mathfrak{sl}_p \oplus \mathfrak{sl}_q \oplus k,$$
$$instr_0(J) = [instr(J), instr(J)] \cong \mathfrak{sl}_p \oplus \mathfrak{sl}_q,$$

The pair $(\mathfrak{g}, \mathfrak{d})$ is $(\mathfrak{sl}(J), instr_0(J))$.

- **In example (2):**

If γ is skew-symmetric, $T = U \otimes V$ is a simple Lie triple system and $(\mathfrak{g}, \mathfrak{d}) = (\mathfrak{sp}(T), \mathrm{Der}(T))$. Moreover, V is a $(1, 1)$ *balanced Freudenthal Kantor triple system of symplectic type*.

If γ is symmetric, $T = U \otimes V$ is a simple Lie triple system and $(\mathfrak{g}, \mathfrak{d}) = (\mathfrak{so}(T), \mathrm{Der}(T))$. Moreover, V is a $(-1, -1)$ *balanced Freudenthal Kantor triple system of orthogonal type*.