## Some simple modular Lie superalgebras

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Freudenthal-Tits Magic Square

A supermagic rectangle

A supermagic square

 $Bouarroudj\hbox{-} Grozman\hbox{-} Leites \ Classification$ 

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# Exceptional Lie algebras

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$$G_2$$
,  $F_4$ ,  $E_6$ ,  $E_7$ ,  $E_8$ 

$$G_2=\mathfrak{der}\,\mathbb{O}$$
 (Cartan 1914) 
$$F_4=\mathfrak{der}\,H_3(\mathbb{O})$$
 (Chevalley-Schafer 1950) 
$$E_6=\mathfrak{str}_0\,H_3(\mathbb{O})$$

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then

$$\mathcal{T}(C,J) = \mathfrak{der} C \oplus (C_0 \otimes J_0) \oplus \mathfrak{der} J$$

is a Lie algebra (char  $\neq 2,3$ ) under a suitable Lie bracket:

$$[a\otimes x,b\otimes y]=\frac{1}{3}tr(xy)D_{a,b}+\left([a,b]\otimes\left(xy-\frac{1}{3}tr(xy)1\right)\right)+2t(ab)d_{x,y}.$$

# Freudenthal-Tits Magic Square

_	$\mathcal{T}(C,J)$	$H_3(k)$	$H_3(k \times k)$	$H_3(Mat_2(k))$	$H_3(C(k))$
	k	$A_1$	$A_2$	<i>C</i> <sub>3</sub>	F <sub>4</sub>
	$k \times k$	$A_2$	$A_2$ $A_2 \oplus A_2$	$A_5$	<i>E</i> <sub>6</sub>
	$Mat_2(k)$ $C(k)$	<i>C</i> <sub>3</sub>	$A_5$	$D_6$	E <sub>7</sub>
	C(k)	F <sub>4</sub>	$E_6$	E <sub>7</sub>	<i>E</i> <sub>8</sub>

$$J = H_3(C') \simeq k^3 \oplus \left( \bigoplus_{i=0}^2 \iota_i(C') \right),$$
  
 $J_0 \simeq k^2 \oplus \left( \bigoplus_{i=0}^2 \iota_i(C') \right),$   
 $\operatorname{der} J \simeq \operatorname{tri}(C') \oplus \left( \bigoplus_{i=0}^2 \iota_i(C') \right),$ 

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$$\simeq \operatorname{der} C \oplus (C_0 \otimes k^2) \oplus \left( \bigoplus_{i=0}^2 C_0 \otimes \iota_i(C') \right) \oplus \left( \operatorname{tri}(C') \oplus \left( \bigoplus_{i=0}^2 \iota_i(C') \right) \right)$$

$$\simeq \left( \operatorname{tri}(C) \oplus \operatorname{tri}(C') \right) \oplus \left( \bigoplus_{i=0}^2 \iota_i(C \otimes C') \right)$$

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$$\simeq \left( \operatorname{tri}(C) \oplus \operatorname{tri}(C') \right) \oplus \left( \bigoplus_{i=0}^2 \iota_i(C \otimes C') \right)$$

$$\operatorname{tri}(C) = \{(d_0, d_1, d_2) \in \mathfrak{so}(C)^3 : d_0(x \bullet y) = d_1(x) \bullet y + x \bullet d_2(y) \ \forall x, y\}$$

$$(x \bullet y = \bar{x}\bar{y}) \text{ is the triality Lie algebra of } C.$$

The product in

$$\mathfrak{g}(\mathit{C},\mathit{C}') = \big(\mathfrak{tri}(\mathit{C}) \oplus \mathfrak{tri}(\mathit{C}')\big) \oplus \big(\oplus_{i=0}^2 \iota_i(\mathit{C} \otimes \mathit{C}')\big),$$

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is given by:

- $tri(C) \oplus tri(C')$  is a Lie subalgebra of g(C, C'),
- $[(d_0,d_1,d_2),\iota_i(x\otimes x')]=\iota_i(d_i(x)\otimes x'),$
- $[(d'_0, d'_1, d'_2), \iota_i(x \otimes x')] = \iota_i(x \otimes d'_i(x')),$
- $[\iota_i(x \otimes x'), \iota_{i+1}(y \otimes y')] = \iota_{i+2}((\bar{x}\bar{y}) \otimes (\bar{x}'\bar{y}')) \text{ (indices modulo 3)}.$
- $[\iota_i(x \otimes x'), \iota_i(y \otimes y')] = q'(x', y')\theta^i(t_{x,y}) + q(x,y)\theta'^i(t'_{x',y'}),$

where

$$t_{x,y} = (q(x,.)y - q(y,.)x, \frac{1}{2}q(x,y)1 - R_{\bar{x}}R_y, \frac{1}{2}q(x,y)1 - L_{\bar{x}}L_y)$$

(Vinberg, Allison-Faulkner, Barton-Sudbery, Landsberg-Manivel)



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		$\operatorname{dim} C'$			
	$\mathfrak{g}(C,C')$	1	2	4	8
	1	$A_1$	$ ilde{\mathcal{A}}_2$	$C_3$	$F_4$
	2	$\tilde{A}_2$	$\tilde{\textit{A}}_2 \oplus \tilde{\textit{A}}_2$	$\tilde{A}_5$	$\tilde{E}_6$
dim C	4	<i>C</i> <sub>3</sub>	$ \begin{array}{ccc} 2 & 4 \\ \tilde{A}_2 & C_3 \\ \tilde{A}_2 \oplus \tilde{A}_2 & \tilde{A}_5 \\ \tilde{A}_5 & D_6 \\ \tilde{E}_6 & E_7 \end{array} $	E <sub>7</sub>	
	8	F <sub>4</sub>	$ ilde{E}_6$	E <sub>7</sub>	$E_8$

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dim C	4	<i>C</i> <sub>3</sub>	$ ilde{\mathcal{A}}_5$	$D_6$	E <sub>7</sub>	
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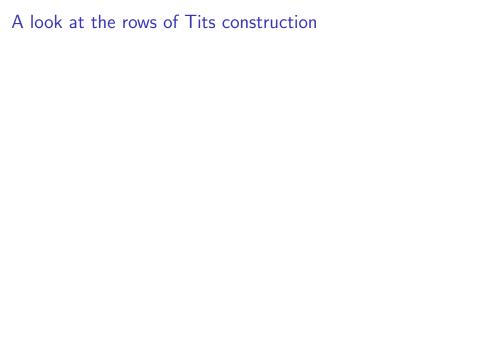
- $ightharpoonup \tilde{A}_2$  denotes a form of  $\mathfrak{pgl}_3$ , so  $[\tilde{A}_2, \tilde{A}_2]$  is a form of  $\mathfrak{pgl}_3$ .
- ightharpoonup  $ilde{A}_5$  denotes a form of  $\mathfrak{pgl}_6$ , so  $[ ilde{A}_5, ilde{A}_5]$  is a form of  $\mathfrak{pgl}_6$ .
- ightharpoonup  $ilde{E}_6$  is not simple, but  $[ ilde{E}_6, ilde{E}_6]$  is a codimension 1 simple ideal.

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Bouarroudj-Grozman-Leites Classification



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Third row dim C = 4, so C = Q is a quaternion algebra and

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Up to now, everything works for arbitrary Jordan algebras in characteristic  $\neq 2$ , and even for Jordan superalgebras.

Fourth row dim C = 8. If the characteristic is  $\neq 2, 3$ , then  $\mathfrak{der} C = \mathfrak{g}_2$  is simple of type  $G_2$ ,  $C_0$  is its smallest nontrivial irreducible module, and

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- $ightharpoonup \mathcal{T}(C, D_2) \simeq F(4).$
- $ightharpoonup \mathcal{T}(C, K_{10})$  in characteristic 5!! This is a new simple modular Lie superalgebra, whose even part is  $\mathfrak{so}_{11}$  and odd part its spin module.

## A supermagic rectangle

$\mathcal{T}(C,J)$	H <sub>3</sub> (k)	$H_3(k \times k)$	$H_3(Mat_2(k))$	$H_3(C(k))$	J(V)	$D_t$	$\kappa_{10}$
k	$A_1$	$A_2$	C <sub>3</sub>	F <sub>4</sub>	$A_1$	B(0, 1)	$B(0,1)\oplus B(0,1)$
$k \times k$	A <sub>2</sub>	$A_2 \oplus A_2$	$A_5$	E <sub>6</sub>	B(0, 1)	A(1,0)	C(3)
$Mat_2(k)$	C <sub>3</sub>	$A_5$	$D_6$	E <sub>7</sub>	B(1, 1)	D(2, 1; t)	F(4)
<i>C</i> ( <i>k</i> )	F <sub>4</sub>	<i>E</i> <sub>6</sub>	E <sub>7</sub>	E <sub>8</sub>	G(3)	(t=2)	$\frac{\mathcal{T}(C(k), K_{10})}{(char 5)}$

## A supermagic rectangle: the new columns

$\mathcal{T}(C,J)$	J(V)	$D_t$	K <sub>10</sub>
k	$A_1$	B(0,1)	$B(0,1)\oplus B(0,1)$
$k \times k$	B(0,1)	A(1,0)	C(3)
$Mat_2(k)$	B(1,1)	D(2,1;t)	F(4)
<i>C</i> ( <i>k</i> )	G(3)	F(4) $(t=2)$	$\mathcal{T}(C(k), K_{10})$ (char 5)

If the characteristic is 3 and dim C=8, then  $\mathfrak{der} C$  is no longer simple, but contains the simple ideal ad  $C_0$  (a form of  $\mathfrak{psl}_3$ ). It makes sense to consider:

$$\tilde{\mathcal{T}}(C,J) = \operatorname{ad} C_0 \oplus (C_0 \otimes J_0) \oplus \operatorname{der} J 
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The simple commutative alternative algebras are just the fields, so nothing interesting appears here.

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- (i) fields,
- (ii) J(V), the Jordan superalgebra of a superform on a two dimensional odd space V,
- (iii)  $B = B(\Gamma, D, 0) = \Gamma \oplus \Gamma u$ , where
  - $ightharpoonup \Gamma$  is a commutative associative algebra,
  - ▶  $D \in \mathfrak{der} \Gamma$  such that  $\Gamma$  is D-simple,
  - ▶ a(bu) = (ab)u = (au)b, (au)(bu) = aD(b) D(a)b,  $\forall a, b \in \Gamma$ .

#### Example (Divided powers)

$$\Gamma = \mathcal{O}(1; n) = \operatorname{span}\left\{t^{(r)}: 0 \le r \le 3^n - 1\right\},$$

$$t^{(r)}t^{(s)} = \binom{r+s}{r}t^{(r+s)},$$

$$D(t^{(r)})=t^{(r-1)}.$$

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- $ightharpoonup ilde{\mathcal{T}}ig(C(k),J(V)ig)$  is a simple Lie superalgebra specific of characteristic 3 of (super)dimension 10|14,
- ▶  $\tilde{T}(C(k), \mathcal{O}(1, n) \oplus \mathcal{O}(1, n)u) = \text{Bj}(1; n|7)$  is a simple Lie superalgebra of (super)dimension  $2^3 \times 3^n | 2^3 \times 3^n$ .

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Both simple Lie superalgebras have been considered in a completely different way by Bouarroudj and Leites (2006).

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Bouarroudj-Grozman-Leites Classification

## Composition superalgebras

#### Composition superalgebras

#### Definition

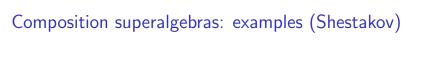
A superalgebra  $C=C_{\bar 0}\oplus C_{\bar 1}$ , endowed with a regular quadratic superform  $q=(q_{\bar 0},b)$ , called the *norm*, is said to be a *composition superalgebra* in case

$$q_{\bar{0}}(x_{\bar{0}}y_{\bar{0}})=q_{\bar{0}}(x_{\bar{0}})q_{\bar{0}}(y_{\bar{0}}),$$

$$b(x_{\bar{0}}y,x_{\bar{0}}z)=q_{\bar{0}}(x_{\bar{0}})b(y,z)=b(yx_{\bar{0}},zx_{\bar{0}}),$$

$$b(xy,zt) + (-1)^{|x||y|+|x||z|+|y||z|}b(zy,xt) = (-1)^{|y||z|}b(x,z)b(y,t),$$

The unital composition superalgebras are termed *Hurwitz* superalgebras.



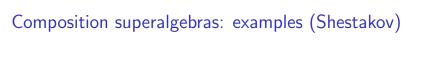
## Composition superalgebras: examples (Shestakov)

$$B(1,2) = k1 \oplus V$$
,

char  $k=3,\ V$  a two dim'l vector space with a nonzero alternating bilinear form  $\langle .|. \rangle$ , with

$$1x = x1 = x$$
,  $uv = \langle u|v\rangle 1$ ,  $q_{\bar{0}}(1) = 1$ ,  $b(u, v) = \langle u|v\rangle$ ,

is a Hurwitz superalgebra. (As a superalgebra, this is just our previous J(V).)



## Composition superalgebras: examples (Shestakov)

$$B(4,2)=\operatorname{End}_k(V)\oplus V,$$

k and V as before,  $\operatorname{End}_k(V)$  is equipped with the symplectic involution  $f\mapsto \bar f$ ,  $(\langle f(u)|v\rangle=\langle u|\bar f(v)\rangle)$ , the multiplication is given by:

- ▶ the usual multiplication (composition of maps) in  $End_k(V)$ ,
- $ightharpoonup v \cdot f = f(v) = \overline{f} \cdot v$  for any  $f \in \operatorname{End}_k(V)$  and  $v \in V$ ,
- ▶  $u \cdot v = \langle .|u\rangle v \ (w \mapsto \langle w|u\rangle v) \in \operatorname{End}_k(V)$  for any  $u, v \in V$ ,

and with quadratic superform

$$q_{\bar{0}}(f) = \det f, \quad b(u, v) = \langle u|v\rangle,$$

is a Hurwitz superalgebra.

## Composition superalgebras: classification

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#### Theorem (E.-Okubo 2002)

Any unital composition superalgebra is either:

- ▶ a Hurwitz algebra,
- ▶ a  $\mathbb{Z}_2$ -graded Hurwitz algebra in characteristic 2,
- ▶ isomorphic to either B(1,2) or B(4,2) in characteristic 3.

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Hurwitz superalgebras can be plugged into the symmetric construction of Freudenthal-Tits Magic Square  $\mathfrak{g}(C, C')$ .

## Supermagic Square (char 3, Cunha-E. 2007)

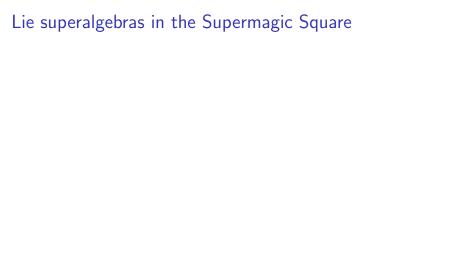
$\mathfrak{g}(C,C')$	k	$k \times k$	$Mat_2(k)$	C(k)	B(1, 2)	B(4,2)
k	$A_1$	$ ilde{\mathcal{A}}_2$	<i>C</i> <sub>3</sub>	$F_4$	6 8	21 14
$k \times k$		$\tilde{\textit{A}}_2 \oplus \tilde{\textit{A}}_2$	$ ilde{A}_5$	$ ilde{E}_6$	11 14	35 20
$Mat_2(k)$			$D_6$	E <sub>7</sub>	24 26	66 32
<i>C</i> ( <i>k</i> )				E <sub>8</sub>	55 50	133 56
B(1, 2)					21 16	36 40
B(4,2)						78 64

# Supermagic Square (char 3, Cunha-E. 2007)

$\mathfrak{g}(C,C')$	k	$k \times k$	$Mat_2(k)$	C(k)	B(1, 2)	B(4, 2)
k	$A_1$	$ ilde{\mathcal{A}}_2$	<i>C</i> <sub>3</sub>	$F_4$	6 8	21 14
$k \times k$		$ ilde{\mathcal{A}}_2 \oplus  ilde{\mathcal{A}}_2$	$ ilde{\mathcal{A}}_5$	$ ilde{E}_6$	11 14	35 20
$Mat_2(k)$			$D_6$	E <sub>7</sub>	24 26	66 32
<i>C</i> ( <i>k</i> )				$E_8$	55 50	133 56
B(1, 2)					21 16	36 40
B(4,2)						78 64

Notation:  $\mathfrak{g}(n,m)$  will denote the superalgebra  $\mathfrak{g}(C,C')$ , with dim C=n, dim C'=m.

	B(1,2)	B(4,2)
k	$\mathfrak{psl}_{2,2}$	$\mathfrak{sp}_6 \oplus (14)$
$k \times k$	$\left(\mathfrak{sl}_2\oplus\mathfrak{pgl}_3 ight)\oplus\left(\left(2 ight)\otimes\mathfrak{psl}_3 ight)$	$\mathfrak{pgl}_6 \oplus (20)$
$Mat_2(k)$	$\left(\mathfrak{sl}_2 \oplus \mathfrak{sp}_6\right) \oplus \left((2) \otimes (13)\right)$	$\mathfrak{so}_{12} \oplus \mathit{spin}_{12}$
<i>C</i> ( <i>k</i> )	$\left(\mathfrak{sl}_2\oplus\mathfrak{f}_4\right)\oplus\left((2)\otimes(25)\right)$	$\mathfrak{e}_7 \oplus (56)$
B(1, 2)	so <sub>7</sub> ⊕2spin <sub>7</sub>	$\mathfrak{sp}_{8} \oplus (40)$
B(4,2)	$\mathfrak{sp}_8 \oplus (40)$	$\mathfrak{so}_{13} \oplus \mathit{spin}_{13}$



All these Lie superalgebras are simple, with the exception of  $\mathfrak{g}(2,3)$  and  $\mathfrak{g}(2,6)$ , both of which contain a codimension one simple ideal.

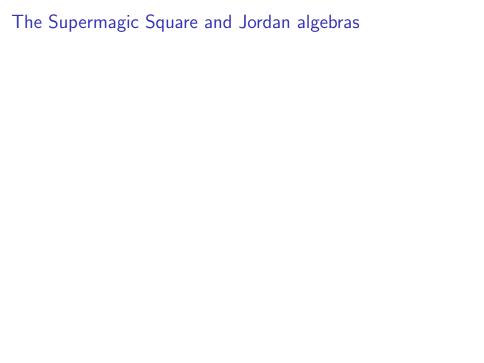
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The simple Lie superalgebra  $\mathfrak{g}(2,3)'=[\mathfrak{g}(2,3),\mathfrak{g}(2,3)]$  is isomorphic to our previous  $\tilde{\mathcal{T}}(C(k),J(V))$ .



#### The Supermagic Square and Jordan algebras

	k	$k \times k$	$Mat_2(k)$	<i>C</i> ( <i>k</i> )	
B(1, 2)	$\mathfrak{psl}_{2,2}$	$\big(\mathfrak{sl}_2\oplus\mathfrak{pgl}_3\big)\oplus\big((2)\otimes\mathfrak{psl}_3\big)$	$(\mathfrak{sl}_2 \oplus \mathfrak{sp}_6) \oplus ((2) \otimes (13))$	$\big(\mathfrak{sl}_2\oplus\mathfrak{f}_4\big)\oplus\big((2)\otimes(25)\big)$	
B(4, 2)	sp <sub>6</sub> ⊕(14)	$\mathfrak{pgl}_6 \oplus (20)$	$\mathfrak{so}_{12}\oplus \mathit{spin}_{12}$	e <sub>7</sub> ⊕ (56)	

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B(4, 2)	sp <sub>6</sub> ⊕(14)	$\mathfrak{pgl}_{6}\oplus (20)$	$\mathfrak{so}_{12}\oplus \mathit{spin}_{12}$	e <sub>7</sub> ⊕ (56)

$$\mathfrak{g}(3,r) = (\mathfrak{sl}_2 \oplus \mathfrak{der} J) \oplus ((2) \otimes \hat{J}),$$
 $r = 1, 2, 4, 8, \qquad \hat{J} = J_0/k1, \quad J = H_3(C).$ 

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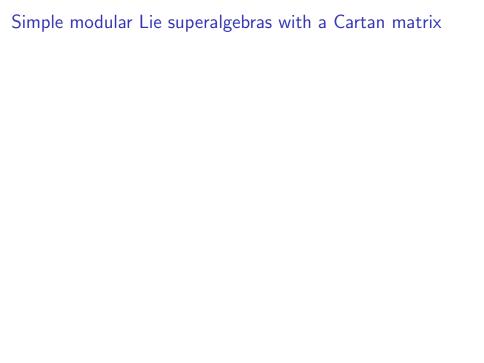
$$g(6,r) = (\mathfrak{der} T) \oplus T,$$
 $r = 1, 2, 4, 8, \qquad T = \begin{pmatrix} k & J \\ J & k \end{pmatrix}, \quad J = H_3(C).$ 

Freudenthal-Tits Magic Square

A supermagic rectangle

A supermagic square

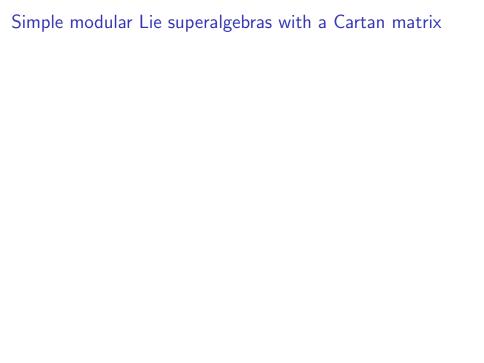
Bouarroudj-Grozman-Leites Classification



The finite dimensional modular Lie superalgebras with indecomposable symmetrizable Cartan matrices (or *contragredient Lie superalgebras*) over algebraically closed fields have been classified by Bouarroudj, Grozman and Leites (2009), under some extra technical hypotheses.

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For characteristic  $p \ge 3$ , apart from the Lie superalgebras obtained as the analogues of the Lie superalgebras in the classification in characteristic 0, by reducing the Cartan matrices modulo p, there are only the following exceptions:



1. Two exceptions in characteristic 5:  $\mathfrak{br}(2;5)$  and  $\mathfrak{el}(5;5)$ . (Dimensions 10|12 and 55|32.)

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- 2. The family of exceptions given by the Lie superalgebras in the Supermagic Square in characteristic 3.
- Another two exceptions in characteristic 3, similar to the ones in characteristic 5: br(2; 3) and εl(5; 3). (Dimensions 10|8 and 39|32.)

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That's all. Thanks