An extended Freudenthal Magic Square in characteristic 3

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From Lie Algebras to Quantum Groups
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Exceptional Lie algebras

$G_2,\ F_4,\ E_6,\ E_7,\ E_8$

$G_2 = \mathfrak{der}\,\mathbb{O}$ (Cartan 1914)

$F_4 = \mathfrak{der}\,H_3(\mathbb{O})$

$E_6 = \mathfrak{str}_0\,H_3(\mathbb{O})$ (Chevalley-Schafer 1950)

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Tits construction (1966)

- C a Hurwitz algebra (unital composition algebra),
- J a central simple Jordan algebra of degree 3,

Then

$$\mathcal{T}(C,J) = \mathfrak{der}\, C \oplus (C_0 \otimes J_0) \oplus \mathfrak{der}\, J$$

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Freudenthal Magic Square

$\mathcal{T}(C,J)$	$H_3(k)$	$H_3(k\times k)$	$H_3(\mathrm{Mat}_2(k))$	$H_3(C(k))$
k	A_1	A_2	C_3	F_4
$k\times k$	A_2	$A_2\oplus A_2$	A_5	E_6
$\mathrm{Mat}_2(k)$	C_3	A_5	D_6	E_7
$C(k)$	F_4	E_6	E_7	E_8

Tits construction rearranged

$$J = H_3(C') \simeq k^3 \oplus \left(\oplus_{i=0}^{2} \iota_i(C')\right),$$

$$J_0 \simeq k^2 \oplus \left(\oplus_{i=0}^{2} \iota_i(C')\right),$$

$$\mathfrak{der}\, J \simeq \mathfrak{tri}(C') \oplus \left(\oplus_{i=0}^{2} \iota_i(C')\right),$$

$$\begin{aligned}\mathcal{T}(C, J) &= \mathfrak{der}\, C \oplus (C_0 \otimes J_0) \oplus \mathfrak{der}\, J \\ &\simeq \mathfrak{der}\, C \oplus (C_0 \otimes k^2) \oplus \left(\oplus_{i=0}^{2} C_0 \otimes \iota_i(C')\right) \oplus \left(\mathfrak{tri}(C') \oplus \left(\oplus_{i=0}^{2} \iota_i(C')\right)\right) \\ &\simeq \left(\mathfrak{tri}(C) \oplus \mathfrak{tri}(C')\right) \oplus \left(\oplus_{i=0}^{2} \iota_i(C \otimes C')\right)\end{aligned}$$

(Barton-Sudbery, Landsberg-Manivel, Allison-Faulkner)

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Symmetric composition algebras

Nicer formulas are obtained if symmetric composition algebras are used, instead of the more classical Hurwitz algebras.

$$(S, *, q)$$

$$\begin{cases} q(x * y) = q(x)q(y), \\ q(x * y, z) = q(x, y * z). \end{cases}$$

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Symmetric composition algebras: examples

- Para-Hurwitz algebras: C Hurwitz algebra with norm q and standard involution $^-$, but with new multiplication

$$x * y = \bar{x}\bar{y}.$$

- Okubo algebras: In characteristic $\neq 3$ these are the forms of $(\mathfrak{sl}_3, *, q)$ with

$$x * y = \omega xy - \omega^2 yx - \frac{\omega - \omega^2}{3}\operatorname{tr}(xy)1,$$

$$q(x) = \frac{1}{2}\operatorname{tr}(x^2), \quad q(x, y) = \operatorname{tr}(xy).$$

(ω a cubic root of 1.)

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Symmetric composition algebras: classification

Theorem (Okubo, Osborn, Myung, Pérez-Izquierdo, E.)

With some exceptions in dimension 2*, any symmetric composition algebras is either*

- *a para-Hurwitz algebra (dimension* 1, 2, 4 *or* 8*), or*
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Triality algebra

$(S, *, q)$ a symmetric composition algebra

$$\mathfrak{tri}(S) = \{(d_0, d_1, d_2) \in \mathfrak{so}(S, q)^3 : \\ d_0(x * y) = d_1(x) * y + x * d_2(y) \ \forall x, y \in S\}$$

is the triality Lie algebra of S.

$$\mathfrak{tri}(S) = \begin{cases} 0 & \text{if } \dim S = 1, \\ \text{2-dim'l abelian} & \text{if } \dim S = 2, \\ \mathfrak{so}(S_0, q)^3 & \text{if } \dim S = 4, \\ \mathfrak{so}(S, q) & \text{if } \dim S = 8. \end{cases}$$

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The Lie algebra $\mathfrak{g}(S, S')$

Let S and S' be two symmetric composition algebras. Consider

$$\mathfrak{g}(S, S') = \bigl(\mathfrak{tri}(S) \oplus \mathfrak{tri}(S')\bigr) \oplus \bigl(\oplus_{i=0}^{2} \iota_i(S \otimes S')\bigr),$$

where $\iota_i(S \otimes S')$ is just a copy of $S \otimes S'$, with bracket given by:

- $\mathfrak{tri}(S) \oplus \mathfrak{tri}(S')$ is a Lie subalgebra of $\mathfrak{g}(S, S')$,
- $[(d_0, d_1, d_2), \iota_i(x \otimes x')] = \iota_i\bigl(d_i(x) \otimes x'\bigr)$,
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- $[\iota_i(x \otimes x'), \iota_i(y \otimes y')] = q'(x', y')\theta^i(t_{x,y}) + q(x, y)\theta'^i(t'_{x',y'})$,

  where $t_{x,y} = \bigl(q(x, .)y - q(y, .)x, \frac{1}{2}q(x, y)1 - r_x l_y, \frac{1}{2}q(x, y)1 - l_x r_y\bigr)$

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Freudenthal Magic Square again (2004)

dim S'

$\mathfrak{g}(S,S')$	1	2	4	8
1	A_1	A_2	C_3	F_4
2	A_2	$A_2 \oplus A_2$	A_5	E_6
4	C_3	A_5	D_6	E_7
8	F_4	E_6	E_7	E_8

dim S

(characteristic $\neq 3$)

Freudenthal Magic Square (char 3)

dim S'

$\mathfrak{g}(S,S')$	1	2	4	8
1	A_1	$\tilde{A}_2$	C_3	F_4
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dim S

- $\tilde{A}_2$ denotes a form of $\mathfrak{pgl}_3$, so $[\tilde{A}_2, \tilde{A}_2]$ is a form of $\mathfrak{psl}_3$.
- $\tilde{A}_5$ denotes a form of $\mathfrak{pgl}_6$, so $[\tilde{A}_5, \tilde{A}_5]$ is a form of $\mathfrak{psl}_6$.
- $\tilde{E}_6$ is not simple, but $[\tilde{E}_6, \tilde{E}_6]$ is a codimension 1 simple ideal.

Freudenthal Magic Square (char 3)

$\mathfrak{g}(S,S')$		dim S': 1	2	4	8
dim S	1	A_1	$\tilde{A}_2$	C_3	F_4
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Composition superalgebras

Definition

A superalgebra $C = C_{\bar{0}} \oplus C_{\bar{1}}$, endowed with a regular quadratic superform $q = (q_{\bar{0}}, b)$, called the *norm*, is said to be a *composition superalgebra* in case

$$q_{\bar{0}}(x_{\bar{0}}y_{\bar{0}}) = q_{\bar{0}}(x_{\bar{0}})q_{\bar{0}}(y_{\bar{0}}),$$

$$b(x_{\bar{0}}y, x_{\bar{0}}z) = q_{\bar{0}}(x_{\bar{0}})b(y, z) = b(yx_{\bar{0}}, zx_{\bar{0}}),$$

$$b(xy, zt) + (-1)^{|x||y|+|x||z|+|y||z|}b(zy, xt) = (-1)^{|y||z|}b(x, z)b(y, t),$$

The unital composition superalgebras are termed *Hurwitz superalgebras*.

Composition superalgebras: examples

$$B(1,2) = k1 \oplus V,$$

char $k = 3$, V a two dim'l vector space with a nonzero alternating bilinear form $\langle .|. \rangle$, with

$$1x = x1 = x, \quad uv = \langle u|v \rangle 1, \qquad q_{\bar{0}}(1) = 1, \quad b(u,v) = \langle u|v \rangle,$$

is a Hurwitz superalgebra.

Fix a symplectic basis $\{u, v\}$ of V and $\lambda \in k$.
$\varphi : 1 \mapsto 1,\ u \mapsto u + \lambda v,\ v \mapsto v,$ is an automorphism of $B(1,2)$, $\varphi^3 = 1$ and

$$S^{\lambda}_{1,2} = B(1,2) \quad \text{with same norm but} \quad x * y = \varphi(\bar{x})\varphi^2(\bar{y})$$

is a symmetric composition superalgebra.

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$$B(4,2) = \mathrm{End}_k(V) \oplus V,$$

k and V as before, and where $\mathrm{End}_k(V)$ is equipped with the symplectic involution $f \mapsto \bar{f}$, $(\langle f(u)|v\rangle = \langle u|\bar{f}(v)\rangle)$, with multiplication given by:

- the usual multiplication (composition of maps) in $\mathrm{End}_k(V)$,
- $v \cdot f = f(v) = \bar{f} \cdot v$ for any $f \in \mathrm{End}_k(V)$ and $v \in V$,
- $u \cdot v = \langle .|u\rangle v \; \big(w \mapsto \langle w|u\rangle v\big) \in \mathrm{End}_k(V)$ for any $u, v \in V$,

and with quadratic superform

$$q_{\bar{0}}(f) = \det f, \quad b(u,v) = \langle u|v\rangle,$$

is a Hurwitz superalgebra.

$S_{4,2}$ will denote the associated para-Hurwitz superalgebra.

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Composition superalgebras: classification

Theorem (E.-Okubo 02)

- *Any unital composition superalgebra is either:*
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Freudenthal Magic Supersquare (Cunha-E.)

$\mathfrak{g}(S,S')$	S_1	S_2	S_4	S_8	$S_{1,2}$	$S_{4,2}$
S_1	A_1	$\tilde{A}_2$	C_3	F_4	(6,8)	(21,14)
S_2		$\tilde{A}_2 \oplus \tilde{A}_2$	$\tilde{A}_5$	$\tilde{E}_6$	(11,14)	(35,20)
S_4			D_6	E_7	(24,26)	(66,32)
S_8				E_8	(55,50)	(133,56)
$S_{1,2}$					(21,16)	(36,40)
$S_{4,2}$						(78,64)

Lie superalgebras in Freudenthal Magic Supersquare

	$S_{1,2}$	$S_{4,2}$
S_1	$\mathfrak{psl}_{2,2}$	$\mathfrak{sp}_6 \oplus (14)$
S_2	$\big(\mathfrak{sl}_2 \oplus \mathfrak{pgl}_3\big) \oplus \big((2) \otimes \mathfrak{psl}_3\big)$	$\mathfrak{pgl}_6 \oplus (20)$
S_4	$\big(\mathfrak{sl}_2 \oplus \mathfrak{sp}_6\big) \oplus \big((2) \otimes (13)\big)$	$\mathfrak{so}_{12} \oplus spin_{12}$
S_8	$\big(\mathfrak{sl}_2 \oplus \mathfrak{f}_4\big) \oplus \big((2) \otimes (25)\big)$	$\mathfrak{e}_7 \oplus (56)$
$S_{1,2}$	$\mathfrak{so}_7 \oplus 2spin_7$	$\mathfrak{sp}_8 \oplus (40)$
$S_{4,2}$	$\mathfrak{sp}_8 \oplus (40)$	$\mathfrak{so}_{13} \oplus spin_{13}$

All these Lie superalgebras are simple, with the exception of $\mathfrak{g}(S_2, S_{1,2})$ and $\mathfrak{g}(S_2, S_{4,2})$, both of which contain a codimension one simple ideal.

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$\mathfrak{g}(S_{1,2}, S)$

$$\mathfrak{tri}(S_{1,2}) = \{(d,d,d) : d \in \mathfrak{osp}(S_{1,2})\} \simeq \mathfrak{osp}(S_{1,2}) \simeq \mathfrak{sp}(V) \oplus V.$$

$$\begin{aligned}\mathfrak{g}(S_{1,2}, S) &= \big(\mathfrak{tri}(S_{1,2}) \oplus \mathfrak{tri}(S)\big) \oplus \big(\oplus_{i=0}^{2} \iota_i(S_{1,2} \otimes S)\big)\\ &= \big((\mathfrak{sp}(V) \oplus V) \oplus \mathfrak{tri}(S)\big) \oplus \big(\oplus_{i=0}^{2} \iota_i(1 \otimes S)\big) \oplus \big(\oplus_{i=0}^{2} \iota_i(V \otimes S)\big)\end{aligned}$$

$$\begin{aligned}\mathfrak{g}(S_{1,2}, S)_{\bar{0}} &= \mathfrak{sp}(V) \oplus \mathfrak{tri}(S) \oplus \big(\oplus_{i=0}^{2} \iota_i(S)\big)\\ &\simeq \mathfrak{sp}(V) \oplus \mathfrak{der}\, J \qquad (J = H_3(\bar{S}))\end{aligned}$$

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$$k \oplus \left(\oplus_{i=0}^{2} \iota_i(S)\right) \simeq J_0/k1 =: \hat{J}$$

Thus,

$$\begin{cases} \mathfrak{g}(S_{1,2}, S)_{\bar{0}} \simeq \mathfrak{sp}(V) \oplus \mathfrak{der}\, J, & \text{(direct sum of ideals)} \\ \mathfrak{g}(S_{1,2}, S)_{\bar{1}} \simeq V \otimes \hat{J}. \end{cases}$$

$\hat{J}$ is then an orthogonal triple system with

$$[\hat{x}\hat{y}\hat{z}] = \left(x \circ (y \circ z) - y \circ (x \circ z)\right)\hat{}$$

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$\mathfrak{g}(S_{4,2}, S)$

- $\dim V = 2$, $\langle .|. \rangle$ a nonzero alternating bilinear form.
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$$S_4 = \overline{\mathrm{Mat}_2(k)} \simeq \mathrm{End}_k(V) \simeq V_1 \otimes V_2 \ (\langle x|.\rangle y \leftrightarrow x \otimes y).$$

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$$
\begin{aligned}
\mathfrak{tri}(S_8) &\simeq \mathfrak{so}(8) \\
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$$\begin{aligned}\mathfrak{g}(S_8, S) \simeq \big(&(\oplus_{j=1}^4 \mathfrak{sp}(V_i)) \oplus (\otimes_{j=1}^4 V_i) \oplus \mathfrak{tri}(S)\big)\\ &\oplus \big((V_1 \otimes V_2) \,\oplus\, (V_3 \otimes V_4)\big) \otimes \iota_0(S)\\ &\oplus \big((V_2 \otimes V_3) \,\oplus\, (V_1 \otimes V_4)\big) \otimes \iota_1(S)\\ &\oplus \big((V_1 \otimes V_3) \,\oplus\, (V_2 \otimes V_4)\big) \otimes \iota_2(S)\end{aligned}$$

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then

$$[u \otimes x, v \otimes y] = (x|y)\big(\langle u|.\rangle v + \langle v|.\rangle u\big) + \langle u|v\rangle d_{x,y}$$

for some alternating bilinear form $(.|.) : T \times T \to k$ and skewsymmetric bilinear map $d_{.,.} : T \times T \to \mathfrak{s}$.
T becomes a symplectic triple system under $[xyz] = d_{x,y}(z)$.

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In characteristic 3, $\mathfrak{s} \oplus T$ is a Lie superalgebra with the natural bracket.

$\mathfrak{g}(S_{4,2}, S)$ (cont.)

Corollary (Cunha-E.)

Let S be a para-Hurwitz algebra, then:

- $T = (V_1 \otimes V_2 \otimes V_3) \oplus (V_3 \otimes \iota_0(S)) \oplus (V_1 \otimes \iota_1(S)) \oplus (V_2 \otimes \iota_2(S))$ *is a symplectic triple system.*
- $\mathfrak{g}(S_{4,2}, S)$ *is the Lie superalgebra attached to this triple system.*

Remark

$$T \simeq \begin{pmatrix} k & J \\ J & k \end{pmatrix}, \quad J = H_3(\bar{S}) \simeq k^3 \oplus \left(\oplus_{i=0}^{2} \iota_i(S)\right).$$

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Conclusion on $\mathfrak{g}(S_{1,2}, S)$ and $\mathfrak{g}(S_{4,2}, S)$

	S_1	S_2	S_4	S_8
$S_{1,2}$	Lie superalgebras attached to orthogonal triple systems $\hat{J} = J_0/k1$			
$S_{4,2}$	Lie superalgebras attached to symplectic triple systems $\begin{pmatrix} k & J \\ J & k \end{pmatrix}$			

(J a degree 3 central simple Jordan algebra)

Some final comments

- Only $\mathfrak{g}(S_{1,2}, S_1) \simeq \mathfrak{psl}_{2,2}$ has a counterpart in Kac's classification in characteristic 0. The other Lie superalgebras in Freudenthal Magic Supersquare, or their derived algebras, are new simple Lie superalgebras, specific of characteristic 3.
- $\mathfrak{g}(S_{1,2}, S_{1,2})$ and $\mathfrak{g}(S_{1,2}, S_{4,2})$ are related to some orthosymplectic triple systems (Cunha-E.).
- $\mathfrak{g}(S_{1,2}, S_{1,2})$ is related to a "null orthogonal triple system".
- The simple Lie superalgebra $[\mathfrak{g}(S_{1,2}, S_2), \mathfrak{g}(S_{1,2}, S_2)]$ has recently appeared, in a completely different way, in work of Bouarroudj and Leites.

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That's all. Thanks