Tits construction, Freudenthal Magic Square, and modular simple Lie superalgebras

Alberto Elduque

Universidad de Zaragoza

Lie Algebras and their Applications

in honor of Professor Hyo Chul Myung on his 70th birthday

Exceptional Lie algebras

$$G_2, \quad F_4, \quad E_6, \quad E_7, \quad E_8$$

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$G_2 = \mathfrak{der}\, \mathbb{O}$ (Cartan 1914)

$F_4 = \mathfrak{der}\, H_3(\mathbb{O})$

$E_6 = \mathfrak{str}_0\, H_3(\mathbb{O})$

(Chevalley-Schafer 1950)

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then

$$\mathcal{T}(C, J) = \mathfrak{der}\, C \oplus (C_0 \otimes J_0) \oplus \mathfrak{der}\, J$$

is a Lie algebra (char $\neq 2, 3$) under a suitable Lie bracket:

$$[a \otimes x, b \otimes y] = \frac{1}{3} tr(xy) D_{a,b} + \Big([a, b] \otimes \big(xy - \frac{1}{3} tr(xy) 1\big)\Big) + 2t(ab) d_{x,y}.$$

Freudenthal Magic Square

$\mathcal{T}(C,J)$	$H_3(k)$	$H_3(k \times k)$	$H_3(\mathrm{Mat}_2(k))$	$H_3(C(k))$
k	A_1	A_2	C_3	F_4
$k \times k$	A_2	$A_2 \oplus A_2$	A_5	E_6
$\mathrm{Mat}_2(k)$	C_3	A_5	D_6	E_7
$C(k)$	F_4	E_6	E_7	E_8

Tits construction rearranged

$$J = H_3(C') \simeq k^3 \oplus \left(\oplus_{i=0}^{2} \iota_i(C')\right),$$

$$J_0 \simeq k^2 \oplus \left(\oplus_{i=0}^{2} \iota_i(C')\right),$$

$$\mathfrak{der}\, J \simeq \mathfrak{tri}(C') \oplus \left(\oplus_{i=0}^{2} \iota_i(C')\right),$$

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$$\begin{aligned}
\mathcal{T}(C,J) &= \mathfrak{der}\, C \oplus (C_0 \otimes J_0) \oplus \mathfrak{der}\, J \\
&\simeq \mathfrak{der}\, C \oplus (C_0 \otimes k^2) \oplus \left(\oplus_{i=0}^{2} C_0 \otimes \iota_i(C')\right) \oplus \left(\mathfrak{tri}(C') \oplus \left(\oplus_{i=0}^{2}\iota_i(C')\right)\right) \\
&\simeq \left(\mathfrak{tri}(C) \oplus \mathfrak{tri}(C')\right) \oplus \left(\oplus_{i=0}^{2}\iota_i(C \otimes C')\right)
\end{aligned}$$

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$$\begin{aligned}\mathcal{T}(C, J) &= \mathfrak{der}\, C \oplus (C_0 \otimes J_0) \oplus \mathfrak{der}\, J \\ &\simeq \mathfrak{der}\, C \oplus (C_0 \otimes k^2) \oplus \left(\oplus_{i=0}^{2} C_0 \otimes \iota_i(C')\right) \oplus \left(\mathfrak{tri}(C') \oplus \left(\oplus_{i=0}^{2} \iota_i(C')\right)\right) \\ &\simeq \left(\mathfrak{tri}(C) \oplus \mathfrak{tri}(C')\right) \oplus \left(\oplus_{i=0}^{2} \iota_i(C \otimes C')\right)\end{aligned}$$

(Vinberg, Allison-Faulkner, Barton-Sudbery, Landsberg-Manivel)

Symmetric composition algebras

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$$(S, *, q)$$

$$\begin{cases} q(x * y) = q(x)q(y), \\ q(x * y, z) = q(x, y * z). \end{cases}$$

Symmetric composition algebras: examples

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- Para-Hurwitz algebras: C Hurwitz algebra with norm q and standard involution $^-$, but with new multiplication

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$$x * y = \bar{x}\bar{y}.$$

- Okubo algebras: In characteristic $\neq 3$ these are the forms of $(\mathfrak{sl}_3, *, q)$ with

$$x * y = \omega xy - \omega^2 yx - \frac{\omega - \omega^2}{3}\,\mathrm{tr}(xy)1,$$

$$q(x) = \frac{1}{2}\,\mathrm{tr}(x^2), \quad q(x, y) = \mathrm{tr}(xy).$$

(ω a cubic root of 1.)

A different definition is needed in characteristic 3.

Symmetric composition algebras: classification

Theorem (Okubo, Osborn, Myung, Pérez-Izquierdo, E.)

With some exceptions in dimension 2*, any symmetric composition algebra is either*

- *a para-Hurwitz algebra (dimension* 1, 2, 4 *or* 8*), or*
- *an Okubo algebra (dimension* 8*).*

Triality algebra

$(S, *, q)$ a symmetric composition algebra

$$\mathfrak{tri}(S) = \{(d_0, d_1, d_2) \in \mathfrak{so}(S, q)^3 : \\ d_0(x * y) = d_1(x) * y + x * d_2(y) \ \forall x, y \in S\}$$

is the triality Lie algebra of S.

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$$\mathfrak{tri}(S) = \begin{cases} 0 & \text{if } \dim S = 1, \\ \text{2-dim'l abelian} & \text{if } \dim S = 2, \\ \mathfrak{so}(S_0, q)^3 & \text{if } \dim S = 4, \\ \mathfrak{so}(S, q) & \text{if } \dim S = 8. \end{cases}$$

The Lie algebra $\mathfrak{g}(S, S')$

Let S and S' be two symmetric composition algebras. Consider

$$\mathfrak{g}(S, S') = \big(\mathfrak{tri}(S) \oplus \mathfrak{tri}(S')\big) \oplus \big(\oplus_{i=0}^{2} \iota_i(S \otimes S')\big),$$

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where $\iota_i(S \otimes S')$ is just a copy of $S \otimes S'$, with bracket given by:

- $\mathfrak{tri}(S) \oplus \mathfrak{tri}(S')$ is a Lie subalgebra of $\mathfrak{g}(S, S')$,
- $[(d_0, d_1, d_2), \iota_i(x \otimes x')] = \iota_i\big(d_i(x) \otimes x'\big)$,
- $[(d'_0, d'_1, d'_2), \iota_i(x \otimes x')] = \iota_i\big(x \otimes d'_i(x')\big)$,
- $[\iota_i(x \otimes x'), \iota_{i+1}(y \otimes y')] = \iota_{i+2}\big((x * y) \otimes (x' * y')\big)$ (indices modulo 3),
- $[\iota_i(x \otimes x'), \iota_i(y \otimes y')] = q'(x', y')\theta^i(t_{x,y}) + q(x, y)\theta'^i(t'_{x',y'})$,

  where $t_{x,y} = \big(q(x, .)y - q(y, .)x, \frac{1}{2}q(x, y)1 - r_x l_y, \frac{1}{2}q(x, y)1 - l_x r_y\big)$

Freudenthal Magic Square again (2004)

dim S'

$\mathfrak{g}(S,S')$	1	2	4	8
1	A_1	A_2	C_3	F_4
2	A_2	$A_2 \oplus A_2$	A_5	E_6
4	C_3	A_5	D_6	E_7
8	F_4	E_6	E_7	E_8

dim S

(characteristic $\neq 2, 3$)

Freudenthal Magic Square (char 3)

dim S'

$\mathfrak{g}(S,S')$	1	2	4	8
1	A_1	$\tilde{A}_2$	C_3	F_4
2	$\tilde{A}_2$	$\tilde{A}_2 \oplus \tilde{A}_2$	$\tilde{A}_5$	$\tilde{E}_6$
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4	C_3	$\tilde{A}_5$	D_6	E_7
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- $\tilde{A}_2$ denotes a form of $\mathfrak{pgl}_3$, so $[\tilde{A}_2, \tilde{A}_2]$ is a form of $\mathfrak{psl}_3$.
- $\tilde{A}_5$ denotes a form of $\mathfrak{pgl}_6$, so $[\tilde{A}_5, \tilde{A}_5]$ is a form of $\mathfrak{psl}_6$.
- $\tilde{E}_6$ is not simple, but $[\tilde{E}_6, \tilde{E}_6]$ is a codimension 1 simple ideal.

Composition superalgebras

Definition

A superalgebra $C = C_{\bar 0} \oplus C_{\bar 1}$, endowed with a regular quadratic superform $q = (q_{\bar 0}, b)$, called the *norm*, is said to be a *composition superalgebra* in case

$$q_{\bar 0}(x_{\bar 0} y_{\bar 0}) = q_{\bar 0}(x_{\bar 0}) q_{\bar 0}(y_{\bar 0}),$$

$$b(x_{\bar 0} y, x_{\bar 0} z) = q_{\bar 0}(x_{\bar 0}) b(y, z) = b(y x_{\bar 0}, z x_{\bar 0}),$$

$$b(xy, zt) + (-1)^{|x||y|+|x||z|+|y||z|} b(zy, xt) = (-1)^{|y||z|} b(x, z) b(y, t),$$

The unital composition superalgebras are termed *Hurwitz superalgebras*.

Composition superalgebras: examples (Shestakov)

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$$B(1,2) = k1 \oplus V,$$

char $k = 3$, V a two dim'l vector space with a nonzero alternating bilinear form $\langle .|.\rangle$, with

$$1x = x1 = x, \quad uv = \langle u|v\rangle 1, \qquad q_{\bar{0}}(1) = 1, \quad b(u,v) = \langle u|v\rangle,$$

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Fix a symplectic basis $\{u, v\}$ of V and $\lambda \in k$.
$\varphi : 1 \mapsto 1,\ u \mapsto u + \lambda v,\ v \mapsto v,$ is an automorphism of $B(1,2)$, $\varphi^3 = 1$ and

$$S_{1,2}^{\lambda} = B(1,2) \quad \text{with same norm but} \quad x * y = \varphi(\bar{x})\varphi^2(\bar{y})$$

is a symmetric composition superalgebra.

Composition superalgebras: examples (Shestakov)

$$B(4,2) = \mathrm{End}_k(V) \oplus V,$$

k and V as before, $\mathrm{End}_k(V)$ is equipped with the symplectic involution $f \mapsto \bar{f}$, $(\langle f(u)|v\rangle = \langle u|\bar{f}(v)\rangle)$, the multiplication is given by:

- the usual multiplication (composition of maps) in $\mathrm{End}_k(V)$,
- $v \cdot f = f(v) = \bar{f} \cdot v$ for any $f \in \mathrm{End}_k(V)$ and $v \in V$,
- $u \cdot v = \langle .|u\rangle v \ \big(w \mapsto \langle w|u\rangle v\big) \in \mathrm{End}_k(V)$ for any $u, v \in V$,

and with quadratic superform

$$q_{\bar{0}}(f) = \det f, \quad b(u,v) = \langle u|v\rangle,$$

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$S_{4,2}$ will denote the associated para-Hurwitz superalgebra.

Theorem (E.-Okubo 02)

Composition superalgebras: classification

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- *Any unital composition superalgebra is either:*
  - *a Hurwitz algebra,*
  - *a $\mathbb{Z}_2$-graded Hurwitz algebra in characteristic* 2*,*
  - *isomorphic to either* $B(1,2)$ *or* $B(4,2)$ *in characteristic* 3*.*

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- *Any symmetric composition superalgebra is either:*
  - *a symmetric composition algebra,*
  - *a $\mathbb{Z}_2$-graded symmetric composition algebra in characteristic* 2*,*
  - *isomorphic to either $S_{1,2}^{\lambda}$ or $S_{4,2}$ in characteristic* 3*.*

Freudenthal Supermagic Square (char 3, Cunha-E.)

$\mathfrak{g}(S, S')$	S_1	S_2	S_4	S_8	$S_{1,2}$	$S_{4,2}$
S_1	A_1	$\tilde{A}_2$	C_3	F_4	(6,8)	(21,14)
S_2		$\tilde{A}_2 \oplus \tilde{A}_2$	$\tilde{A}_5$	$\tilde{E}_6$	(11,14)	(35,20)
S_4			D_6	E_7	(24,26)	(66,32)
S_8				E_8	(55,50)	(133,56)
$S_{1,2}$					(21,16)	(36,40)
$S_{4,2}$						(78,64)

Lie superalgebras in Freudenthal Supermagic Square

	$S_{1,2}$	$S_{4,2}$
S_1	$\mathfrak{psl}_{2,2}$	$\mathfrak{sp}_6 \oplus (14)$
S_2	$\big(\mathfrak{sl}_2 \oplus \mathfrak{pgl}_3\big) \oplus \big((2) \otimes \mathfrak{psl}_3\big)$	$\mathfrak{pgl}_6 \oplus (20)$
S_4	$\big(\mathfrak{sl}_2 \oplus \mathfrak{sp}_6\big) \oplus \big((2) \otimes (13)\big)$	$\mathfrak{so}_{12} \oplus spin_{12}$
S_8	$\big(\mathfrak{sl}_2 \oplus \mathfrak{f}_4\big) \oplus \big((2) \otimes (25)\big)$	$\mathfrak{e}_7 \oplus (56)$
$S_{1,2}$	$\mathfrak{so}_7 \oplus 2spin_7$	$\mathfrak{sp}_8 \oplus (40)$
$S_{4,2}$	$\mathfrak{sp}_8 \oplus (40)$	$\mathfrak{so}_{13} \oplus spin_{13}$

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All these Lie superalgebras are simple, with the exception of $\mathfrak{g}(S_2, S_{1,2})$ and $\mathfrak{g}(S_2, S_{4,2})$, both of which contain a codimension one simple ideal.

$\mathfrak{g}(S_{1,2}, S)$

$$\mathfrak{tri}(S_{1,2}) = \{(d, d, d) : d \in \mathfrak{osp}(S_{1,2})\} \simeq \mathfrak{osp}(S_{1,2}) \simeq \mathfrak{sp}(V) \oplus V.$$

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$$\begin{aligned}\mathfrak{g}(S_{1,2}, S) &= \big(\mathfrak{tri}(S_{1,2}) \oplus \mathfrak{tri}(S)\big) \oplus \big(\oplus_{i=0}^{2} \iota_i(S_{1,2} \otimes S)\big) \\ &= \big((\mathfrak{sp}(V) \oplus V) \oplus \mathfrak{tri}(S)\big) \oplus \big(\oplus_{i=0}^{2} \iota_i(1 \otimes S)\big) \oplus \big(\oplus_{i=0}^{2} \iota_i(V \otimes S)\big)\end{aligned}$$

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$$\begin{aligned}\mathfrak{g}(S_{1,2}, S)_{\bar 0} &= \mathfrak{sp}(V) \oplus \mathfrak{tri}(S) \oplus \bigl(\oplus_{i=0}^{2} \iota_i(S)\bigr) \\ &\simeq \mathfrak{sp}(V) \oplus \mathfrak{der}\, J \qquad (J = H_3(\bar{S}), \text{ degree 3 Jordan algebra}) \\ \mathfrak{g}(S_{1,2}, S)_{\bar 1} &= V \oplus \bigl(\oplus_{i=0}^{2} \iota_i(V \otimes S)\bigr) \\ &\simeq V \otimes \bigl(k \oplus (\oplus_{i=0}^{2} \iota_i(S))\bigr)\end{aligned}$$

$\mathfrak{g}(S_{1,2}, S)$ (cont.)

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Thus,

$$\begin{cases} \mathfrak{g}(S_{1,2}, S)_{\bar{0}} \simeq \mathfrak{sp}(V) \oplus \mathfrak{Der}\, J, & \text{(direct sum of ideals)} \\ \mathfrak{g}(S_{1,2}, S)_{\bar{1}} \simeq V \otimes \hat{J}. \end{cases}$$

$$k \oplus \left(\oplus_{i=0}^{2} \iota_i(S)\right) \simeq J_0/k1 =: \hat{J}$$

Thus,

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$\hat{J}$ is then an orthogonal triple system with

$$[\hat{x}\hat{y}\hat{z}] = \big(x \circ (y \circ z) - y \circ (x \circ z)\big)^{\wedge}$$

$(\hat{x} = x + k1)$

$\mathfrak{g}(S_{1,2}, S)$ (cont.)

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Theorem (Cunha-E.)

The Lie superalgebra $\mathfrak{g}(S_{1,2}, S)$ is the Lie superalgebra associated to the orthogonal triple system $\hat{J} = J_0/k1$, for $J = H_3(\bar{S})$.

$\mathfrak{g}(S_{4,2}, S)$

Given any $\mathbb{Z}_2$-graded Lie algebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ with

$$\begin{cases} \mathfrak{g}_0 = \mathfrak{sp}(V) \oplus \mathfrak{s} & \text{(direct sum of ideals)}, \\ \mathfrak{g}_1 = V \otimes T & \text{(as a module for } \mathfrak{g}_0\text{)}, \end{cases}$$

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then

$$[u \otimes x, v \otimes y] = (x|y)\big(\langle u|.\rangle v + \langle v|.\rangle u\big) + \langle u|v\rangle d_{x,y}$$

for some alternating bilinear form $(.|.) : T \times T \to k$ and skewsymmetric bilinear map $d_{.,.} : T \times T \to \mathfrak{s}$.

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Theorem (E. 06)

In characteristic 3, $\mathfrak{s} \oplus T$ *is a Lie superalgebra with the natural bracket.*

$\mathfrak{g}(S_{4,2}, S)$ (cont.)

It turns out that, with suitable identifications:

$$\mathfrak{g}(S_8, S) = (\mathfrak{sp}(V) \oplus \mathfrak{g}(S_{4,2}, S)_{\bar{0}}) \oplus (V \otimes \mathfrak{g}(S_{4,2}, S)_{\bar{1}}).$$

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Corollary (Cunha-E.)

Let S be a para-Hurwitz algebra, then $\mathfrak{g}(S_{4,2}, S)$ is the Lie superalgebra attached to a symplectic triple system.

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Let S be a para-Hurwitz algebra, then $\mathfrak{g}(S_{4,2}, S)$ is the Lie superalgebra attached to a symplectic triple system.

Remark

$$\mathfrak{g}(S_{4,2}, S)_{\bar{1}} \simeq \begin{pmatrix} k & J \\ J & k \end{pmatrix}, \quad J = H_3(\bar{S}) \simeq k^3 \oplus \left(\oplus_{i=0}^{2} \iota_i(S)\right).$$

Conclusion on $\mathfrak{g}(S_{1,2}, S)$ and $\mathfrak{g}(S_{4,2}, S)$

	S_1	S_2	S_4	S_8
$S_{1,2}$	Lie superalgebras attached to orthogonal triple systems $\hat{J} = J_0/k1$			
$S_{4,2}$	Lie superalgebras attached to symplectic triple systems $\begin{pmatrix} k & J \\ J & k \end{pmatrix}$			

(J a degree 3 central simple Jordan algebra)

Some comments

- Only $\mathfrak{g}(S_{1,2}, S_1) \simeq \mathfrak{psl}_{2,2}$ has a counterpart in Kac's classification in characteristic 0. The other Lie superalgebras in Freudenthal Supermagic Square, or their derived algebras, are new simple Lie superalgebras, specific of characteristic 3.

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- The simple Lie superalgebra $[\mathfrak{g}(S_{1,2}, S_2), \mathfrak{g}(S_{1,2}, S_2)]$ has recently appeared, in a completely different way, in work of Bouarroudj and Leites.

A look at the rows of Tits construction

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Third row dim $C = 4$, so $C = Q$ is a quaternion algebra and

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Up to now, everything works for arbitrary Jordan algebras in characteristic $\neq 2$, and even for Jordan superalgebras.

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- $\mathcal{T}(C, K_{10})$ in characteristic 5!!
  This is a new simple modular Lie superalgebra, whose even part is $\mathfrak{so}_{11}$ and odd part its spin module.

Fourth row, characteristic 3

If the characteristic is 3 and $\dim C = 8$, then $\mathfrak{Der}\, C$ is no longer simple, but contains the simple ideal $\mathrm{ad}\, C_0$ (a form of $\mathfrak{psl}_3$). It makes sense to consider:

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The simple commutative alternative algebras are just the fields, so nothing interesting appears here.

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(iii) $B = B(\Gamma, D, 0) = \Gamma \oplus \Gamma u$, where

- Γ is a commutative associative algebra,
- $D \in \mathfrak{Der}\,\Gamma$ such that Γ is D-simple,
- $a(bu) = (ab)u = (au)b, \quad (au)(bu) = aD(b) - D(a)b, \ \forall a, b \in \Gamma.$

Example (Divided powers)

$$\Gamma = \mathcal{O}(1; n) = \operatorname{span}\left\{ t^{(r)} : 0 \leq r \leq 3^n - 1 \right\},$$

$$t^{(r)} t^{(s)} = \binom{r+s}{r} t^{(r+s)},$$

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Bouarroudj-Leites superalgebras

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Over an algebraically closed field of characteristic 3:

- $\tilde{\mathcal{T}}(C, B(1,2)) = [\mathfrak{g}(S_2, S_{1,2}), \mathfrak{g}(S_2, S_{1,2})]$,
- $\tilde{\mathcal{T}}(C, \mathcal{O}(1,n) \oplus \mathcal{O}(1,n)u) = \mathrm{Bj}(1; n|7)$ is a simple Lie superalgebra of dimension $2^4 \times 3^n$ introduced in a completely different way by Bouarroudj and Leites (2006).

New simple modular Lie superalgebras

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- $\mathfrak{g}(S_r, S_{4,2})$ $(r = 1, 4, 8)$ and $\mathfrak{g}(S_2, S_{4,2})'$, related to some symplectic triple systems.
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That's all. Thanks