VECTOR CROSS PRODUCTS

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Abstract. The classical result that r-fold vector cross products exist only for d-dimensional vector spaces with $r = 1$ and d even; $r = 2$ and $d = 3$ or 7; $r = 3$ and $d = 8$; and $r = d - 1$ for arbitrary d will be explained. Vector cross products will then be used to construct exceptional Lie superalgebras.

1. Bilinear vector cross products

We all are familiar with the usual vector cross product $\times$ in $\mathbb{R}^3$, which satisfies:

$$\begin{cases} u \times v \text{ is bilinear,} \\ u \times v \perp u, v, \quad (\text{so } (u \times v) \cdot w \text{ is skew symmetric, and so is } u \times v) \\ (u \times v) \cdot (u \times v) = \left| \begin{smallmatrix} u\cdot u & u\cdot v \\ v\cdot u & v\cdot v \end{smallmatrix} \right| \end{cases}$$

Definition 1. Let V be a d-dimensional vector space over a field F of characteristic $\neq 2$, endowed with a nondegenerate symmetric bilinear form $(.\mid .)$. A bilinear map $\times : V \times V \to V$ is called a *vector cross product* if it satisfies the following conditions:

$$(u \times v \mid u) = (u \times v) \mid v) = 0, \tag{1}$$

$$(u \times v \mid u \times v) = \begin{vmatrix} (u \mid u) & (u \mid v) \\ (v \mid u) & (v \mid v) \end{vmatrix}, \tag{2}$$

for any $u, v \in V$.

Our main purpose in this talk is to provide a proof of the following result:

Theorem 1. *Let $\times$ be a vector cross product on the vector space V. Then* $\dim V = 1, 3$ *or* 7.

It is interesting to note that in 1943, Beno Eckmann gave a proof of this result for real euclidean spaces, but where the map $\times$ is not supposed to be bilinear, but continuous (which is manifestly a weaker condition). His proof used algebraic topology. It was in 1967 when R.B. Brown and A. Gray gave the first proof of the result above (actually, of an extension of it we will consider later on). A completely new and surprising proof was given in 1996 by M. Rost (later simplified by K. Meyberg in 2002). This proof is completely elementary, but it is valid only over fields of characteristic 0:

Surprising proof: (a variation of Rost's proof)

By (2), for any $a, b \in V$, $a \times (a \times b) = (a \mid b)a - (a \mid a)b$, or

$$l_a^2 = a \otimes a - (a \mid a)id, \tag{3}$$

for any a, where l_a (respectively $a \otimes b$) denotes the linear map $V \to V$ such that $l_a(b) = a \times b$ (respectively $(a \otimes b)(c) = (b \mid c)a$). Hence, for any $a, b, c \in V$,

$$(a \otimes b) \circ l_c = a \otimes (b \times c), \quad l_a \circ (b \otimes c) = (a \times b) \otimes c. \tag{4}$$

(The first assertion holds because of the skew symmetry of the trilinear map $(u \times v \mid w)$, implied by (1).) By linearization of (3) we get

$$a \times (c \times b) + c \times (a \times b) = (a \mid b)c + (b \mid c)a - 2(a \mid c)b, \tag{5}$$

or

$$l_{a \times b} + l_a l_b = 2b \otimes a - a \otimes b - (a \mid b)id. \tag{6}$$

Now, using (6) a couple of times, we obtain:

$$\begin{aligned} l_a l_b l_a &= -l_{a\times b} l_a - a \otimes (b \times a) - (a \mid b) l_a & (7)\\ &= l_{(a\times b)\times a} - a \otimes (a \times b) + (a \times b) \otimes a - (a \mid b) l_a & (8)\\ &= (a \mid a) l_b - 2(a \mid b) l_a - a \otimes (a \times b) + (a \times b) \otimes a & (9)\end{aligned}$$

Extend scalars if necessary, so that you can choose a basis $\{e_i\}_{i=1}^d$ of V with $(e_i \mid e_i) = 1$ and $(e_i \mid e_j) = 0$ for any $i \neq j$. Hence, for any $x \in V$, $x = \sum_{i=1}^d (x \mid e_i)e_i$. Consider the linear map:

$$\begin{aligned} S : \mathrm{End}(V) &\longrightarrow \mathrm{End}(V)\\ f \quad &\mapsto \sum_{i=1}^d l_{e_i} \circ f \circ l_{e_i}. \end{aligned}$$

Then,

$$S(id) = \sum_i l_{e_i}^2 = \sum_i e_i \otimes e_i - \sum_i (e_i \mid e_i) id = (1-d)id,$$
(because of (6))

$$S(a \otimes b) = \sum_i l_{e_i} \circ (a \otimes b) \circ l_{e_i} = \sum_i (e_i \times a) \otimes (b \times e_i) = l_a l_b,$$

$$\begin{aligned}\Bigl(\text{because } \sum (e_i \times a) \otimes (b \times e_i)\,(c) &= \sum (b \times e_i \mid c) e_i \times a\\ &= a \times \sum_i \bigl(e_i \mid b \times c\bigr) e_i = a \times (b \times c)\,\Bigr)\end{aligned}$$

$$S(l_b) = (d - 2 - 1 - 1) l_b = (d-4) l_b, \quad \text{(because of (9))}$$

$$\begin{aligned} S(l_a l_b) &= -S(l_{a\times b}) + 2S(a \otimes b) - S(a \otimes b) - (a \mid b) S(id) \quad \text{(using (6))}\\ &= -(d-4) l_{a\times b} + 2 l_b l_a - l_a l_b - (1-d)(a \mid b) id. \end{aligned}$$

With all this, for any $x \in V$ let us compute $g = \sum_{i,j=1}^{d} l_{e_i} l_x l_{e_j} l_{e_i} l_{e_j}$ in two ways:

$$\textbf{(i)} \quad g = \sum_i l_{e_i} l_x S(l_{e_i}) = (d-4)S(l_x) = (d-4)^2 l_x,$$

$$\begin{aligned}\textbf{(ii)} \quad g &= \sum_j S(l_x l_{e_j}) l_{e_j} \\ &= \sum_j \Big(-(d-4) l_{x\times e_j} l_{e_j} + 2 l_{e_j} l_x l_{e_j} - l_x l_{e_j}^2 - (1-d)(x \mid e_j) l_{e_j} \Big) \\ &= -(d-4)S(l_x) + 2S(l_x) - l_x S(id) - (1-d) l_x \\ &= \Big(-(d-4)^2 + 2(d-4) + 2(d-1) \Big) l_x\end{aligned}$$

where we have used that, by (7), $-\sum_j l_{x\times e_j} l_{e_j} = \sum_j l_{e_j \times x} l_{e_j} = -\sum_j l_{e_j} l_x l_{e_j} - \sum_j e_j \otimes (x \times e_j) - \sum_j (e_j \mid x) l_{e_j} = -S(l_x)$, because $-\sum_j e_j \otimes (x \times e_j) = \Big(\sum_j (e_j \otimes e_j)\Big) \circ l_x = l_x = \sum_j (e_j \mid x) l_{e_j}$, by (4).

Therefore, we conclude that $\Big((d-4)^2 - (d-4) - (d-1)\Big) l_x = 0$, or

$$\boxed{(d-3)(d-7)l_x = 0}$$

for any $x \in V$. Hence, if the characteristic of the ground field is 0, either $l_x = 0$ for any $x \in V$, which is possible only if $d = \dim V = 1$, because of (2), or $d = 3$, or $d = 7$, as required.

Brown-Gray's method:

Consider the vector space $A = F1 \oplus V$ and define a multiplication and a nondegenerate quadratic form on A by means of:

$$\begin{aligned}(\alpha 1 + u)(\beta 1 + v) &= \Big(\alpha\beta - (u \mid v)\Big) 1 + \Big(\alpha v + \beta u + u \times v\Big), \\ q(\alpha 1 + u) &= \alpha^2 + (u \mid u),\end{aligned}$$

for any $\alpha, \beta \in F$ and $u, v \in V$. Then

$$\begin{aligned}q\big((\alpha 1 + u)(\beta 1 + v)\big) \\ &= \Big(\alpha\beta - (u \mid v)\Big)^2 + (\alpha v + \beta u + u \times v \mid \alpha v + \beta u + u \times v) \\ &= \Big(\alpha^2 + (u \mid u)\Big)\Big(\beta^2 + (v \mid v)\Big) \quad \text{(by (2))} \\ &= q(\alpha 1 + u) q(\beta 1 + v)\end{aligned}$$

That is, A is a unital algebra, with a nondegenerate quadratic form satisfying $q(xy) = q(x)q(y)$ for any $x, y \in A$. In other words, A is a *composition algebra.*

A classical theorem, due to Hurwitz* (1898) and Jacobson (1958), asserts that in this case the dimension of A is restricted to $1, 2, 4$ or 8, and that A is a 'variation' of the classical algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ (Hamilton quaternions) and $\mathbb{O}$ (the octonions). In particular, the dimension of V is restricted to $1, 3$ or 7.

*Hurwitz considered the real and complex cases.

2. r-fold vector cross products

Throughout the years, the concept of vector cross product was extended as follows:

Definition 2. Let V be a d-dimensional vector space over a field F of characteristic $\neq 2$, endowed with a nondegenerate symmetric bilinear form $(.\mid .)$. A multilinear map $X : V^r \to V$ $(1 \leq r \leq d)$ is called a (r-fold) *vector cross product* if it satisfies the following conditions:

$$\Bigl(X(u_1,\dots,u_r)\mid u_i\Bigr) = 0 \text{ for any } i=1,\dots,r, \tag{10}$$

$$\Bigl(X(u_1,\dots,u_r)\mid X(u_1,\dots,u_r)\Bigr) = \det\Bigl((u_i\mid u_j)\Bigr), \tag{11}$$

for any $u_1,\dots,u_r \in V$.

What are the possibilities for this extended definition?

Eckmann (1943) and Whitehead (1963) solved this problem in the 'continuous' case over real euclidean spaces, while Brown and Gray (1967) solved the multilinear case. Before stating their result, let us look at some particular cases:

$\mathbf{r=1}$: In this case, $X : V \to V$ is an isometry (11) such that $X(u)$ is orthogonal to u for any $u \in V$, that is, $X^* = -X$, where X^* denotes the adjoint relative to the bilinear form $((X(u)\mid v) = (u \mid X^*(v))$ for any $u, v \in V)$. Hence $X^2 = -XX^* = -id$ and X is just a 'complex structure' on V. In particular, the dimension of V has to be even (for any $u \in V$ with $(u \mid u) \neq 0$, $V = (Fu \oplus F(Xu)) \oplus (Fu \oplus F(Xu))^\perp$ and $(Fu \oplus F(Xu))^\perp$ is closed under X, so we may repeat the process).

Note that in $\mathbb{R}^d$, with the standard inner product, such an X gives a tangent unit vector field on the sphere S^{d-1}.

$\mathbf{r=d-1}$: Let us extend scalars and fix an orthonormal basis $\{e_1,\dots,e_{r+1}\}$ of V. This provides a multilinear map

$$V^r \longrightarrow V^*$$
$$(v_1,\dots,v_r) \mapsto \bigl(v \mapsto \det(v_1|\dots|v_r|v)\bigr)$$

(the determinant of the matrix whose columns are the coordinates of the vectors $v_1,\dots,v_r,v$ in the chosen basis). Besides, the nondegenerate bilinear form $(.\mid .)$ on V provides a linear isomorphism $V \to V^*$, $v \mapsto (v\mid .)$.

Composing these two maps, one gets a multilinear map $X : V^r \to V$, which satisfies trivially (10) (the determinant of a matrix with two equal columns is 0!) and which satisfies (11) too.

Observe that for $d=3$ and $r=2$, the construction above gives exactly the usual vector cross product on $\mathbb{R}^3$.

Are there any other possibilities for $r \geq 3$?

By extending scalars, we may assume that our ground field is algebraically closed. First note that if there is an r-fold vector cross product X on a vector space of dimension d, v is a fixed vector with $(v,v)=1$, and $W = (Fv)^\perp$, then $\tilde{X} : W^{r-1} \to W$, defined by $\tilde{X}(w_1, \dots, w_{r-1}) = X(v, w_1, \dots, w_{r-1})$, provides an $(r-1)$-fold vector cross product on W.

Therefore, for $r = 3$ we have to consider only the case of $d = 8$.

$\mathbf{r = 3,\ d = 8:}$ Assume that $X : V \times V \times V \to V$ is a 3-fold vector cross product on a vector space V of dimension 8, and take a vector $e \in V$ with $(e \mid e) \neq 0$. Consider the bilinear multiplication and the quadratic form on V given by

$$\begin{aligned} xy &= (e \mid e)^{-1}\Big(X(x, e, y) + (x \mid e)y + (y \mid e)x - (x \mid y)e\Big), \\ q(x) &= \frac{(x \mid x)}{(e \mid e)}, \end{aligned}$$

for any $x, y \in V$. Then it follows that $ex = xe = x$ for any x, so this is a unital algebra, and also $q(xy) = q(x)q(y)$ for any $x, y \in V$ (this follows from (11)). Moreover, it can be checked that one of the following formulas hold:

$$\begin{aligned} X(a, b, c) &= (e \mid e)(a\bar{b})c - (a \mid b)c - (b \mid c)a + (a \mid c)b, \text{ or} \\ X(a, b, c) &= (e \mid e)a(\bar{b}c) - (a \mid b)c - (b \mid c)a + (a \mid c)b, \end{aligned} \tag{12}$$

where $x \mapsto \bar{x}$ denotes the standard conjugation in the composition algebra defined on V. This gives two different types of 3-fold vector cross products.

Finally, working with composition algebras of dimension 8 (or Cayley algebras), it can be shown that there are no 4-fold vector cross products on vector spaces of dimension 9. Thus we arrive at the following result that summarizes the previous work:

Theorem 2. ***(Eckmann, Whitehead, Brown-Gray)*** *A vector cross product exists in precisely the following cases:*

$$\begin{aligned} &d \textit{ is even}, \quad r = 1, \\ &d \textit{ is arbitrary}, \quad r = d - 1, \\ &d = 3, 7, \quad r = 2, \\ &d = 8, \quad r = 3. \end{aligned}$$

3. Vector cross products and exceptional simple Lie superalgebras

The results in this section are based on joint work with N. Kamiya and S. Okubo.

First, we will attach some Lie algebras to several instances of vector cross products:

$\mathbf{d = 4,\ r = 3:}$ Consider a nonzero, skew-symmetric multilinear map $\Phi : V \times V \times V \times V \longrightarrow F$ (a 'determinant'); and define a skew-symmetric trilinear map $X : V \times V \times V \to V$ by means of

$$\Big(X(v_1, v_2, v_3) \mid v_4\Big) = \Phi(v_1, v_2, v_3, v_4)$$

for any $v_1, v_2, v_3, v_4 \in v$. Then it can be proven that

$$\Big(X(v_1, v_2, v_3) \mid X(w_1, w_2, w_3)\Big) = \mu \det\big((v_i \mid w_j)\big)$$

for some $0 \neq \mu \in F$. (It turns out that $\mu(F^\times)^2$ is the discriminant of $(.\mid .)$.)

Note that in this case, if $\mu \in F^2$, $\mu^{-\frac{1}{2}}X$ is a vector cross product. Now consider the operators

$$d_{u,v} = X(u,v,-) + \sigma_{u,v}$$

where $\sigma_{u,v}(w) = (u \mid w)v - (v \mid w)u$. Then $d_{V,V}$ is a semisimple Lie algebra (a direct sum of two copies of a simple Lie algebra of type A_1).

$\mathbf{d = 8,\ r = 3:}$ Let X be a 3-fold vector cross product on a vector space V and consider the operators

$$d_{u,v} = \frac{\epsilon}{3}X(u,v,-) + \sigma_{u,v},$$

where $\epsilon = \pm 1$ according to the type of X in (12). Again, $d_{V,V}$ is a Lie algebra, which is simple of type B_3.

$\mathbf{d = 7,\ r = 2:}$ Let $u \times v$ denote a (bilinear) vector cross product and consider the operators

$$d_{u,v} = \frac{1}{2}\Bigl(-l_{u\times v} + 3\sigma_{u,v}\Bigr).$$

As before, $d_{V,V}$ is a Lie algebra, which is simple of type G_2.

Finally, let (U,φ) be a two dimensional vector space U, endowed with a nonzero skew-symmetric bilinear form φ. For any $a,b \in U$, let $\varphi_{a,b} = \varphi(a,-)b + \varphi(b,-)a$. The symplectic Lie algebra $\mathfrak{sp}(U,\varphi)$ is spanned by these operators.

For any of the three classes of vector cross products above consider the superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar 0} \oplus \mathfrak{g}_{\bar 1}$, where

$$\mathfrak{g}_{\bar 0} = \mathfrak{sp}(U,\varphi) \oplus d_{V,V}, \quad \mathfrak{g}_{\bar 1} = U \otimes V,$$

and multiplication given by the usual Lie bracket on $\mathfrak{g}_{\bar 0}$, the natural action of $\mathfrak{g}_{\bar 0}$ on $\mathfrak{g}_{\bar 1}$, and

$$[a\otimes x, b\otimes y] = \langle u \mid v\rangle\, \varphi_{a,b} + \varphi(a,b)d_{u,v},$$

for any $a,b \in U$ and $x,y \in V$.

With this bracket, $\mathfrak{g}$ is then a Lie superalgebra.

In the classification of the simple finite dimensional Lie superalgebras by V. Kac (1977), there appear some infinite families and three exceptional cases: $D(2,1;\mu)$, $F(4)$ and $G(3)$. Then, with V_d denoting the vector space of dimension $d = 4, 8$ or 7, and with $\mathfrak{g}(V_d, \Phi, X$ or $\times)$ each of the Lie superalgebras defined above:

Theorem 3.
- $\mathfrak{g}(V_4, \Phi)$ *is a form of* $D(2,1;\mu)$,
- $\mathfrak{g}(V_8, X)$ *is a form of* $F(4)$,
- $\mathfrak{g}(V_7, \times)$ *is a form of* $G(3)$.

Also, in the three cases considered above, define a triple product on V as follows:

$$xyz = d_{x,y}z + (x \mid y)z$$

for any $x, y, z \in V$. Then V, with this triple product, becomes what is known by a $(-1,-1)$ *balanced Freudenthal-Kantor triple system* (BFKTS for short). In this way, three different classes of simple $(-1,-1)$-BFKTS's are obtained.

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