# Gradings on simple Lie algebras

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Gradings on Lie algebras have been extensively used since the beginning of Lie theory:

- the Cartan grading on a complex semisimple Lie algebra is the $\mathbb{Z}^r$-grading ($r$ being the rank) whose homogeneous components are the root spaces relative to a Cartan subalgebra (which is the zero component),
- symmetric spaces are related to $\mathbb{Z}_2$-gradings,
- Kac–Moody Lie algebras to gradings by a finite cyclic group,
- the theory of Jordan algebras and pairs to 3-gradings on Lie algebras, etc.

In 1989, a systematic study of gradings on Lie algebras was started by Patera and Zassenhaus.

Fine gradings on the classical simple complex Lie algebras, other than $D_4$, by arbitrary abelian groups were considered by Havlícek, Patera, and Pelantova in 1998.

The arguments there are computational and the problem of classification of fine gradings is not completely settled. The complete classification, up to equivalence, of fine gradings on all classical simple Lie algebras (including $D_4$) over algebraically closed fields of characteristic zero has been obtained quite recently.

For any abelian group $G$, the classification of all $G$-gradings, up to isomorphism, on the classical simple Lie algebras other than $D_4$ over algebraically closed fields of characteristic different from two has been achieved in 2010 by Bahturin and Kochetov, using methods developed in the last years by a number of authors.

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Gradings on the octonions, on the Albert algebra, and on some other algebraic structures, are instrumental in obtaining a classification of the gradings on the exceptional simple Lie algebras.

1. Definitions and examples

2. Characteristic 0

3. Gradings and affine group schemes

4. Classification results

1 Definitions and examples

2 Characteristic 0

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## Gradings

$G$ abelian group, $\mathcal{A}$ algebra over a field $\mathbb{F}$.

$G$-grading on $\mathcal{A}$:

$$\Gamma : \mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g,$$

$$\mathcal{A}_g \mathcal{A}_h \subseteq \mathcal{A}_{gh} \qquad \forall g, h \in G.$$

# Examples

**Cartan grading:**

$$\mathfrak{g} = \mathfrak{h} \oplus \left( \oplus_{\alpha \in \boldsymbol{\Phi}} \mathfrak{g}_\alpha \right)$$

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This is a grading by $\mathbb{Z}^n$, $n = \operatorname{rank} \mathfrak{g}$.

# Examples

**Pauli matrices:** $\mathcal{A} = \mathrm{Mat}_n(\mathbb{F})$

$$X = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & \epsilon & 0 & \dots & 0 \\ 0 & 0 & \epsilon^2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \epsilon^{n-1} \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}$$

($\epsilon$ a primitive $n$th root of 1)

$$X^n = 1 = Y^n, \qquad YX = \epsilon XY$$

$$\mathcal{A} = \bigoplus_{(\bar{\imath},\bar{\jmath}) \in \mathbb{Z}_n \times \mathbb{Z}_n} \mathcal{A}_{(\bar{\imath},\bar{\jmath})}, \qquad \mathcal{A}_{(\bar{\imath},\bar{\jmath})} = \mathbb{F} X^i Y^j.$$

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This grading induces a grading on $\mathfrak{sl}_n(\mathbb{F})$.

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- The universal grading group of $\Gamma$ is the group $U(\Gamma)$ generated by $\mathrm{Supp}\,\Gamma$ subject to the relations $g_1 g_2 = g_3$ if $0 \neq \mathcal{A}_{g_1}\mathcal{A}_{g_2} \subseteq \mathcal{A}_{g_3}$.

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The grading $\Gamma$ is then a grading too by $U(\Gamma)$.

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- The quotient $W(\Gamma) = \mathrm{Aut}(\Gamma)/\,\mathrm{Stab}(\Gamma)$ is the Weyl group of $\Gamma$.

# $W(\Gamma)$ acts by automorphisms on $U(\Gamma)$

Each $\varphi \in \mathsf{Aut}(\Gamma)$ determines a self-bijection $\alpha$ of $\mathrm{Supp}\,\Gamma$ that induces an automorphism of the universal grading group $U(\Gamma)$. Then, there appears a natural group homomorphism:

$$\mathsf{Aut}(\Gamma) \to \mathsf{Aut}(U(\Gamma))$$

with kernel $\mathrm{Stab}(\Gamma)$.

Thus, the Weyl group embeds naturally in $\mathsf{Aut}(U(\Gamma))$, i.e., there is a natural action of the Weyl group on $U(\Gamma)$ by automorphisms.

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**Remark**

$\mathrm{Diag}(\Gamma)$ is isomorphic to the group of characters of $U(\Gamma)$.

# Fine gradings

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- $\Gamma$ is a refinement of $\Gamma'$ if for any $g \in G$ there is a $g' \in G'$ such that $\mathcal{A}_g \subseteq \mathcal{A}_{g'}$.
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**Remark**

Any grading is a coarsening of a fine grading.

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The homogeneous components are the eigenspaces for the action of $\hat{G}$!!

# MAD subgroups

**Theorem**

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The complete classification of the fine gradings up to equivalence on the classical Lie algebras (including $D_4$) was obtained in 2010.

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- $E_6$: Draper-Viruel (2016).
- $E_7$, $E_8$: Recent work by Jun Yu classifying conjugacy classes of certain subgroups of the compact Lie groups classifies, in particular, the fine gradings on $E_7$ and $E_8$ over $\mathbb{C}$.
  This is enough to classify these gradings over arbitrary algebraically closed fields of characteristic 0 (E. 2014).

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$$\Gamma : \mathcal{A} = \bigoplus_{g\in G} \mathcal{A}_g \qquad \Rightarrow \qquad \begin{aligned} \eta : \mathcal{A} &\longrightarrow \mathcal{A} \otimes \mathbb{F}G \\ x_g &\mapsto x_g \otimes g \end{aligned}$$

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**Theorem**

*$G$-grading* $\longleftrightarrow$ *comodule algebra over the group algebra* $\mathbb{F}G$.

# Gradings and comodule algebras

A comodule algebra map

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induces a *generic automorphism* of $\mathbb{F}G$-algebras

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All the information on the grading $\Gamma$ attached to $\eta$ is contained in this single automorphism!

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where

$$\begin{aligned} G^D : \mathrm{Alg}_{\mathbb{F}} &\longrightarrow \mathrm{Grp} \\ R &\mapsto G^D(R) = \mathrm{Hom}_{\mathrm{Alg}_{\mathbb{F}}}(\mathbb{F}G, R) \simeq \mathrm{Hom}_{\mathrm{Grp}}(G, R^\times), \end{aligned}$$

$$\begin{aligned} \mathbf{Aut}(\mathcal{A}) : \mathrm{Alg}_{\mathbb{F}} &\longrightarrow \mathrm{Grp} \\ R &\mapsto \mathrm{Aut}_{R\text{-alg}}(\mathcal{A} \otimes_{\mathbb{F}} R). \end{aligned}$$

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$$\rho_R(f)\big(x_g \otimes r\big) = x_g \otimes f(g)r$$

for $f \in G^D(R) = \mathrm{Hom}_{\mathrm{Alg}_{\mathbb{F}}}(\mathbb{F}G, R)$, $x_g \in \mathcal{A}_g$ and $r \in R$.

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Conversely,

$$\rho : G^D \to \mathbf{Aut}(\mathcal{A}) \quad \Longrightarrow \quad \eta : \mathcal{A} \hookrightarrow \mathcal{A} \otimes \mathbb{F}G \xrightarrow{\rho_{\mathbb{F}G}(\mathrm{id})} \mathcal{A} \otimes_{\mathbb{F}} \mathbb{F}G.$$

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**Message:**

It is not enough to deal with $\hat{G}$ and $\operatorname{Aut}\mathcal{A}$, but also with their extensions to unital commutative and associative $\mathbb{F}$-algebras.

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Given a morphism $\mathbf{Aut}(\mathcal{A}) \to \mathbf{Aut}(\mathcal{B})$, any grading on $\mathcal{A}$ induces a grading on $\mathcal{B}$.

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**Example**

$$\mathrm{Ad} : \mathbf{Aut}(\mathcal{A}) \to \mathbf{Aut}\big(\mathfrak{Der}(\mathcal{A})\big).$$

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**Example**

$$\mathrm{Ad} : \mathbf{Aut}(\mathcal{A}) \to \mathbf{Aut}\big(\mathfrak{Der}(\mathcal{A})\big).$$

If $\mathbf{Aut}(\mathcal{A}) \cong \mathbf{Aut}(\mathcal{B})$, the problems of classifying fine gradings on $\mathcal{A}$ and on $\mathcal{B}$ up to equivalence (or the problem of classifying gradings up to isomorphism) are equivalent.

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$$\mathbf{Aut}(\mathcal{L}) \cong \mathbf{Aut}(M_r(\mathbb{F}), \text{involution}).$$

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$$\mathbf{Aut}(\mathcal{L}) \cong \mathbf{Aut}(M_r(\mathbb{F}), \text{involution}).$$

- $A_n$:

$$\mathbf{Aut}(\mathcal{L}) \cong \mathbf{Aut}\big(M_r(\mathbb{F})^{(+)}\big),$$

  ("Affine group scheme of automorphisms and antiautomorphisms of the matrix algebra")

# Classical Lie algebras

Gradings on matrix algebras (with involution) have been dealt with by Bahturin et al.

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$D_4$ requires a different treatment in the modular case (E.-Kochetov 2015)

# Octonions and $G_2$

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There are, up to equivalence, two fine gradings on the octonions (E. 1998):

- The Cartan grading, obtained as the eigenspace decomposition for a maximal torus in $\mathbf{Aut}(\mathbb{O})$.
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# Octonions and $G_2$

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The induced $\mathbb{Z}_2^3$-grading on the simple Lie algebra of type $G_2$ satisfies that $\mathcal{L}_0 = 0$ and $\mathcal{L}_\alpha$ is a Cartan subalgebra of $\mathcal{L}$ for any $0 \neq \alpha \in \mathbb{Z}_2^3$.

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- The Cartan grading, obtained as the eigenspace decomposition for a maximal torus in $\mathbf{Aut}(\mathbb{A})$.
- A $\mathbb{Z} \times \mathbb{Z}_2^3$-grading related to the fine $\mathbb{Z}_2^3$-grading on the octonions
- A $\mathbb{Z}_2^5$-grading obtained by combining a natural $\mathbb{Z}_2^2$-grading on $3 \times 3$ hermitian matrices with the fine grading over $\mathbb{Z}_2^3$ of $\mathbb{O}$.
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The induced $\mathbb{Z}_3^3$-grading on the simple Lie algebra of type $F_4$ satisfies that $\mathcal{L}_0 = 0$ and $\mathcal{L}_\alpha \oplus \mathcal{L}_{-\alpha}$ is a Cartan subalgebra of $\mathcal{L}$ for any $0 \neq \alpha \in \mathbb{Z}_3^3$.

**Open problem:**

Fine gradings on $E_6$, $E_7$, $E_8$ in the modular case?

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A. Elduque and M. Kochetov.
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Thanks