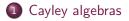
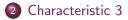
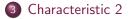
Octonions in low characteristics

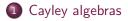
Alberto Elduque

Universidad de Zaragoza









2 Characteristic 3

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Definition

A composition algebra over a field \mathbb{F} is a triple (C, \cdot, n) where

- C is a vector space over \mathbb{F} ,
- $x \cdot y$ is a bilinear multiplication $C \times C \rightarrow C$,
- n: C → F is a multiplicative (n(x · y) = n(x)n(y) ∀x, y ∈ C) nonsingular quadratic form.

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The unital composition algebras are called Hurwitz algebras.

Hurwitz algebras form a class of degree two algebras:

$$x^{\cdot 2} - n(x, 1)x + n(x)1 = 0$$

for any x. $(n(x, y) := n(x + y) - n(x) - n(y).)$

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They are endowed with an involution, the standard conjugation:

$$\bar{x} = n(x,1)1 - x,$$

satisfying

$$\overline{\overline{x}} = x$$
, $x + \overline{x} = n(x, 1)1$, $x \cdot \overline{x} = \overline{x} \cdot x = n(x)1$.

Cayley-Dickson doubling process

Let (B, \cdot, n) be an associative Hurwitz algebra, and let λ be a nonzero scalar in the ground field \mathbb{F} . Consider the direct sum of two copies of B:

$$C = B \oplus Bu$$
,

with the following multiplication and nondegenerate quadratic form that extend those on B:

$$(a + bu) \cdot (c + du) = (a \cdot c + \lambda \overline{d} \cdot b) + (d \cdot a + b \cdot \overline{c})u,$$

$$n(a + bu) = n(a) - \lambda n(b).$$

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Then (C, \cdot, n) is again a Hurwitz algebra, which is denoted by $CD(B, \lambda)$

Notation: $CD(A, \mu, \lambda) := CD(CD(A, \mu), \lambda).$

Every Hurwitz algebra over a field \mathbb{F} is isomorphic to one of the following:

- (i) The ground field \mathbb{F} .
- (ii) A quadratic commutative and associative separable algebra K(μ) = F1 + Fv, with v² = v + μ and 4μ + 1 ≠ 0. The norm is given by its generic norm.
 If char F ≠ 2, these are the algebras CD(F, α).
- (iii) A quaternion algebra $Q(\mu, \beta) = CD(K(\mu), \beta)$. (These four dimensional algebras are associative but not commutative.)
- (iv) A Cayley algebra (or algebra of octonions) $C(\mu, \beta, \gamma) = CD(K(\mu), \beta, \gamma)$. (These eight dimensional algebras are alternative, but not associative.)

Corollary

Every Hurwitz algebra over a field \mathbb{F} of characteristic $\neq 2$ is obtained by applying the Cayley-Dickson doubling process to \mathbb{F} at most three times.

Proposition

Two Hurwitz algebras are isomorphic if and only if their norms are isometric.

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For each dimension 2, 4 or 8, there is a unique, up to isomorphism, Hurwitz algebra with isotropic norm.

- $\mathbb{F} \times \mathbb{F}$ with $n((\alpha, \beta)) = \alpha \beta$,
- $Mat_2(\mathbb{F})$ with n = det,
- $\mathbb{C}_s := CD(Mat_2(\mathbb{F}), 1)$ (the split Cayley algebra).

If char $\mathbb{F} \neq 2$ and \mathbb{C} is a Cayley algebra, then $\mathbb{C} = \mathbb{F}1 \oplus \mathbb{C}_0$, where \mathbb{C}_0 is the subspace orthogonal to $\mathbb{F}1$. For $x, y \in \mathbb{C}_0$, $[x, y] := xy - yx \in \mathbb{C}_0$ and

$$xy = -\frac{1}{2}n(x,y)1 + \frac{1}{2}[x,y].$$

Besides,

$$[[x, y], y] = 2n(x, y)y - 2n(y, y)x,$$

so the multiplication in ${\mathfrak C}$ and its norm are determined by the bracket in ${\mathfrak C}_0.$

Theorem (Sagle 1962, Kuzmin 1968, Filippov 1976)

If char $\mathbb{F} \neq 2,3$, the anticommutative algebra \mathcal{C}_0 is a central simple non-Lie Malcev algebra, and any such algebra is, up to isomorphism, of this form.

Given a finite-dimensional simple Lie algebra \mathfrak{g} of type X_r over the complex numbers, and a Chevalley basis \mathfrak{B} , let $\mathfrak{g}_{\mathbb{Z}}$ be the \mathbb{Z} -span of \mathfrak{B} (a Lie algebra over \mathbb{Z}). The Lie algebra $\mathfrak{g}_{\mathbb{F}} := \mathfrak{g}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{F}$ is the Chevalley algebra of type X_r .

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Theorem

- The Chevalley algebra of type G_2 is isomorphic to $\mathfrak{Der}(\mathcal{C}_s)$.
- For any Cayley algebra C, the Lie algebra Det(C) is a twisted form of the Chevalley algebra Det(C_s).
- If char $\mathbb{F} \neq 2, 3$, then $\mathfrak{Der}(\mathbb{C})$ is simple.

Cayley algebras and simple Lie algebras of type G_2

Theorem (Jacobson 1931, Barnes 1961)

If char $\mathbb{F} \neq 2, 3$,

- Any twisted form of the Chevalley algebra of type G₂ is isomorphic to $\mathfrak{Der}(\mathbb{C})$ for a Cayley algebra \mathbb{C} .
- Two Cayley algebras C_1 and C_2 are isomorphic if and only if their Lie algebras of derivations are isomorphic.

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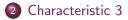
Sketch of a 'modern' proof

For any Cayley algebra $\ensuremath{\mathfrak{C}},$ the adjoint map

$$\begin{split} \operatorname{Ad}: \operatorname{\mathsf{Aut}}(\operatorname{\mathfrak{C}}) &\longrightarrow \operatorname{\mathsf{Aut}}\bigl(\operatorname{\mathfrak{Der}}(\operatorname{\mathfrak{C}})\bigr) \\ f &\mapsto \operatorname{Ad}(f): d \mapsto \mathit{fd} f^{-1}, \end{split}$$

is an isomorphism of affine group schemes.







In any algebra A, the Jacobian of the elements x_1, x_2, x_3 is

$$J(x_1, x_2, x_3) := [[x_1, x_2], x_3] + [[x_2, x_3], x_1] + [[x_3, x_1], x_2].$$

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Expand to get:

$$\begin{aligned} J(x_1, x_2, x_3) &= \left((x_1 x_2 - x_2 x_1) x_3 - x_3 (x_1 x_2 - x_2 x_1) \right) + \text{cyclically} \\ &= \left((x_1 x_2) x_3 - x_1 (x_2 x_3) \right) + \cdots \\ &= \sum_{\sigma \in \Sigma_3} (-1)^{\sigma} (x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}), \end{aligned}$$

where (x, y, z) := (xy)z - x(yz) is the associator.

But in any Cayley algebra \mathcal{C} ,

$$x^2y = x(xy)$$
 and $yx^2 = (yx)x$

for any x, y. That is, (x, x, y) = 0 = (y, x, x), and hence:

$$J(x_1, x_2, x_3) = \sum_{\sigma \in \Sigma_3} (-1)^{\sigma} (x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}) = 6(x_1, x_2, x_3).$$

Let \mathbb{C} be a Cayley algebra over a field \mathbb{F} of characteristic 3.

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- $\mathfrak{Det}(\mathbb{C})$ contains a unique proper ideal: $\operatorname{ad}(\mathbb{C}_0)$, isomorphic to \mathbb{C}_0 , and the quotient $\mathfrak{Det}(\mathbb{C})/\operatorname{ad}(\mathbb{C}_0)$ is isomorphic again to \mathbb{C}_0 .

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- \mathcal{C}_0 is a twisted form of the projective special linear algebra $\mathfrak{psl}_3(\mathbb{F})$.
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- $\mathfrak{Der}(\mathbb{C})$ contains a unique proper ideal: $ad(\mathbb{C}_0)$, isomorphic to \mathbb{C}_0 , and the quotient $\mathfrak{Der}(\mathbb{C})/ad(\mathbb{C}_0)$ is isomorphic again to \mathbb{C}_0 .

Remark

Actually, there are no 'prime' non-Lie Malcev algebras over fields of characteristic 3.

In spite of this strange behavior, still we get:

Theorem

Let ${\mathfrak C}$ be a Cayley algebra over a field ${\mathbb F}$ of characteristic 3, the adjoint map

$$\begin{aligned} \operatorname{Ad}: \operatorname{Aut}(\mathcal{C}) &\longrightarrow \operatorname{Aut}(\mathfrak{Der}(\mathcal{C})) \\ f &\mapsto \operatorname{Ad}(f): d \mapsto \mathit{fd} f^{-1} \end{aligned}$$

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is an isomorphism of affine group schemes.

The proof uses the fact that, even in characteristic 3, any derivation of $\mathfrak{Der}(\mathbb{C})$ is inner.

Corollary

Denote by Isom(Cayley), Isom(G_2), and Isom(\overline{A}_2), the sets of isomorphism classes of Cayley algebras, twisted forms of the Chevalley algebra of type G_2 , and twisted forms of $\mathfrak{psl}_3(\mathbb{F})$, respectively.

Then we have bijections:

$$\begin{array}{rcl} \mathsf{Isom}(\bar{A}_2) & \longleftrightarrow & \mathsf{Isom}(\mathsf{Cayley}) & \longleftrightarrow & \mathsf{Isom}(G_2) \\ [\mathfrak{C}_0] & \leftarrow & [\mathfrak{C}] & \to & [\mathfrak{Der}(\mathfrak{C})] \end{array}$$







Let \mathcal{C} be a Cayley algebra over a field \mathbb{F} of characteristic 2, then the Lie algebra $\mathfrak{Det}(\mathcal{C})$ is isomorphic to the projective special linear Lie algebra $\mathfrak{psl}_4(\mathbb{F})$.

Let \mathcal{C} be a Cayley algebra over a field \mathbb{F} of characteristic 2, then the Lie algebra $\mathfrak{Dec}(\mathcal{C})$ is isomorphic to the projective special linear Lie algebra $\mathfrak{psl}_4(\mathbb{F})$.

The isomorphism class of $\mathfrak{Der}(\mathcal{C})$ does not depend on $\mathcal{C}!!$

• Any $d \in \mathfrak{Der}(\mathbb{C})$ preserves \mathbb{C}_0 and $\mathbb{F}1$, and $1 \in \mathbb{C}_0!!!$

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- Hence d induces a linear map on the six-dimensional quotient C₀/F1, which is endowed with a nondegenerate alternating bilinear form induced by the norm n. (n(x, x) = 2n(x) = 0!!)
- This embeds $\mathfrak{Der}(\mathbb{C})$ into the symplectic Lie algebra $\mathfrak{sp}_6(\mathbb{F})$, and hence into $\mathfrak{sp}_6(\mathbb{F})^{(2)}$, which is isomorphic to $\mathfrak{psl}_4(\mathbb{F})$. But dim $\mathfrak{psl}_4(\mathbb{F}) = 14 = \dim \mathfrak{Der}(\mathbb{C})$.



Corollary

In characteristic 2, the Chevalley algebra of type G_2 is isomorphic to $\mathfrak{psl}_4(\mathbb{F})$ (the classical simple Lie algebra of type A_3).

Let \mathbb{F} be a field of characteristic 2. Then the affine group scheme of automorphisms of $\mathfrak{psl}_4(\mathbb{F})$ is isomorphic to the affine group scheme of automorphisms of the algebra with involution (Mat₆(\mathbb{F}), t_s), where t_s is the canonical symplectic involution.

Let \mathbb{F} be a field of characteristic 2. Then the affine group scheme of automorphisms of $\mathfrak{psl}_4(\mathbb{F})$ is isomorphic to the affine group scheme of automorphisms of the algebra with involution $(Mat_6(\mathbb{F}), t_s)$, where t_s is the canonical symplectic involution.

Sketch of proof

Any automorphism of $Mat_6(\mathbb{F})$ commuting with the involution t_s restricts to an automorphism of the Lie algebra $\mathfrak{sp}_6(\mathbb{F})^{(2)}$. This induces a closed embedding of group schemes. But the two group schemes involved are connected, smooth and of the same dimension.

Corollary

Let \mathbb{F} be a field of characteristic 2. The map

 $(\mathcal{B},\tau) \mapsto \operatorname{Skew}(\mathcal{B},\tau)^{(2)}$

that sends any central simple associative algebra of degree 6 over \mathbb{F} endowed with a symplectic involution (\mathfrak{B}, τ) to the second derived power of the Lie algebra of its skew-symmetric elements $\mathrm{Skew}(\mathfrak{B}, \tau)$, gives a bijection between the set of isomorphism classes of such pairs (\mathfrak{B}, τ) to the set of isomorphism classes of twisted forms over \mathbb{F} of the Lie algebra $\mathfrak{psl}_4(\mathbb{F})$.

A. Castillo-Ramírez and A. Elduque. Some special features of Cayley algebras, and G₂, in low characteristics.

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