

# **UNIT 7: MINIMIZING COSTS AND MAXIMIZING PROFITS**

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# UNIT 7: MINIMIZING COSTS AND MAXIMIZING PROFITS

1. Minimizing costs
2. Conditional factor demands
3. Cost functions
4. Long-run and short-run costs
5. Marginal income, marginal cost and maximizing profits



# 1. MINIMIZING COSTS

## Unit 6: How to produce...

- Technically efficient combination of factors

## Unit 7: ...at the minimum cost?

- Combination of production factors that, at given prices, lead to the minimum cost

### Example:

Productive processes that are technically efficient, aimed at producing 10 units of output. Prices of production factors:  $w=1$ ,  $r=3$

Process	Labour (L)	Capital (K)	Output	Cost= $w \cdot L + r \cdot K$
A	2	3	10	$11 = 1 \cdot 2 + 3 \cdot 3$
<b>B</b>	<b>3</b>	<b>2</b>	<b>10</b>	<b><math>9 = 1 \cdot 3 + 3 \cdot 2</math></b>
D	1	4	10	$13 = 1 \cdot 1 + 3 \cdot 4$

The **production process B** is **economically efficient**: it allows a given amount of output at the minimum cost.

# 1. MINIMIZING COSTS

The firm's goal is maximization of profits, determining the amount of output ( $q$ ) and the necessary quantities of inputs ( $L$  and  $K$ ). In doing so, the firm proceeds in **2 steps**:

## 1. Cost-minimization

The **optimum quantity of Labour ( $L^*$ ) and capital ( $K^*$ )**, subject to a given amount of output, are determined, as well as the **cost function**.

## 2. Profit-maximization

Following the previous step, **the optimum quantity of output ( $q^*$ )** is determined.



# 1. MINIMIZING COSTS

We analyze the cost minimization by the firm from a **long-run perspective**. That is, considering that **all the productive factors are variable (freely eligible)**

Knowing the **production function  $q=f(L,K)$ , which covers all the production processes that are technically efficient**, we assume that the firm combines labour and capital to produce a given amount of output  $q^0$ .

**Given the prices of labour ( $w=w^0$ ) and capital ( $r=r^0$ ), the production cost of each combination of productive factors ( $L,K$ ) is given by:**

$$C = w^0 \cdot L + r^0 \cdot K$$

The firm aims to produce the amount of output  $q^0$  by choosing **the combination of inputs that minimizes the production cost (economically efficient)**:

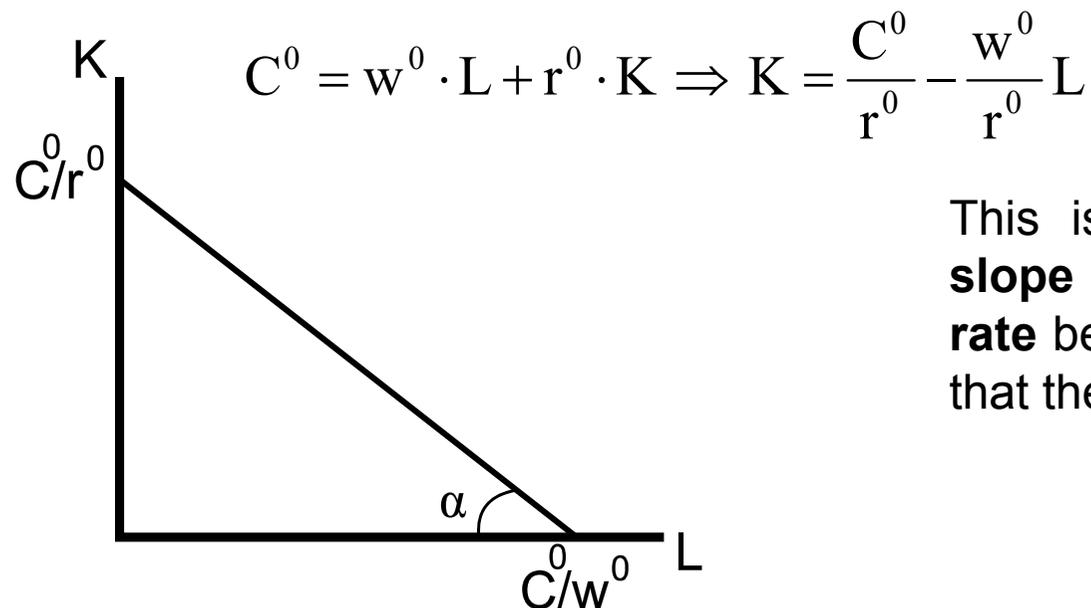
$$\begin{aligned} \min C &= w^0 \cdot L + r^0 \cdot K \\ \text{s.t. } q^0 &= f(L,K) \end{aligned}$$

# 1. MINIMIZING COSTS

We can represent graphically the solution of the cost-minimization problem:

The objective function of the problem of cost minimization can be represented with the map of isocost lines

(Isocost line: set of combinations of productive factors that, given  $w^0$  and  $r^0$  leads to the same production cost):



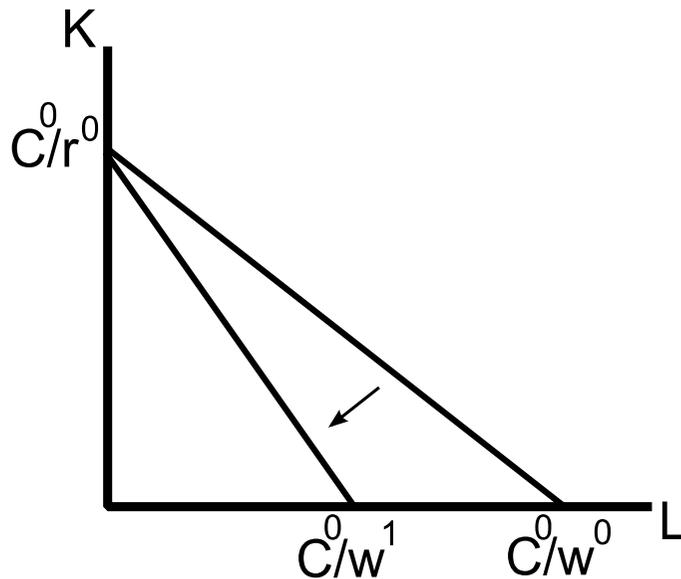
This is a linear equation, whose **slope** measures the **substitution rate** between labour and capital, so that the cost is constant:

$$-\left. \frac{dK}{dL} \right|_{C=C^0} = \frac{w^0}{r^0} = \text{tg}(\alpha)$$

# 1. MINIMIZING COSTS

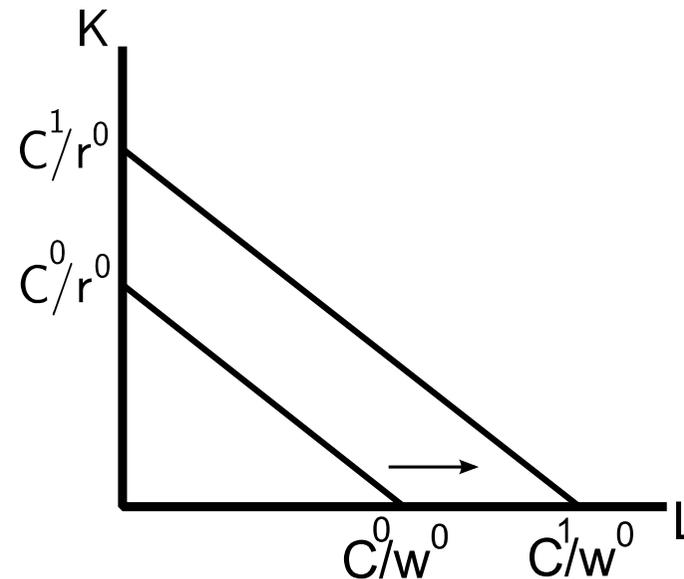
- Prices of productive factors

$$\uparrow w \quad (w^1 > w^0)$$
$$\text{slope} = \frac{w^1}{r^0}$$



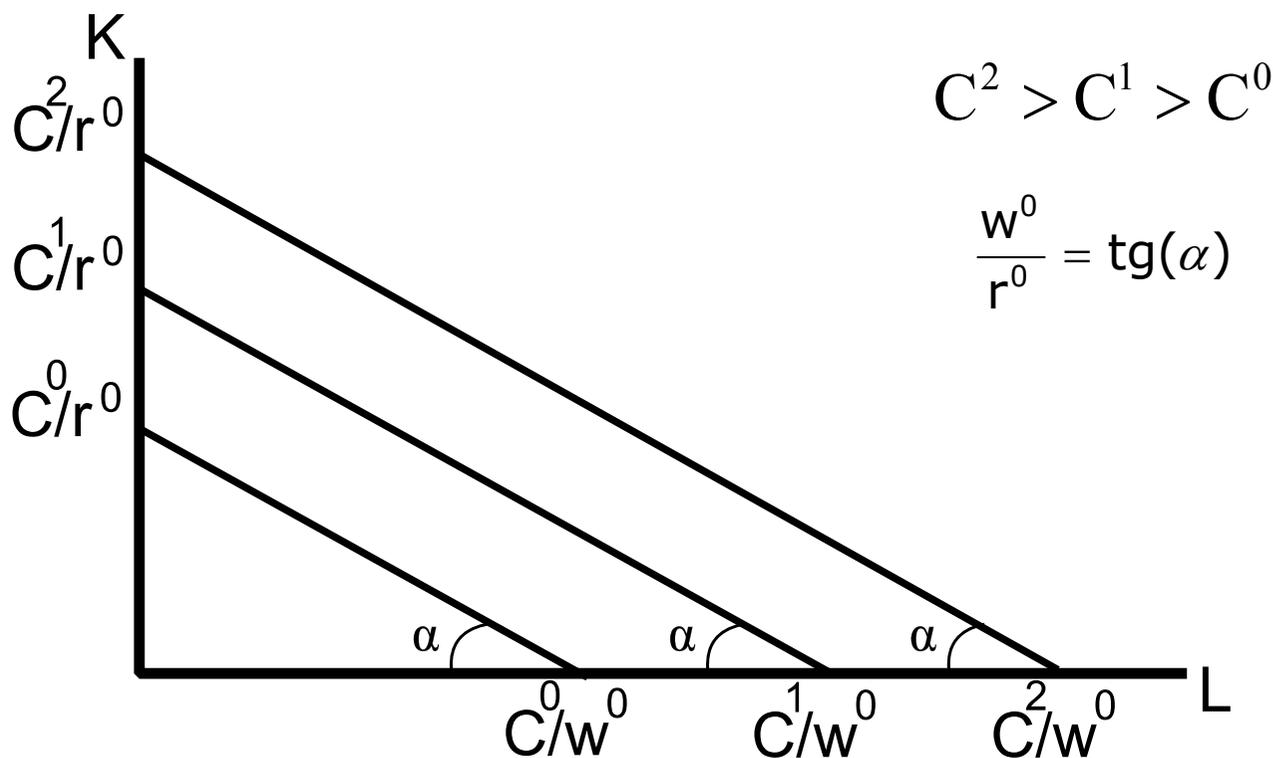
- Cost level

$$\uparrow C \quad (C^1 > C^0)$$
$$\text{slope} = \frac{w^0}{r^0}$$



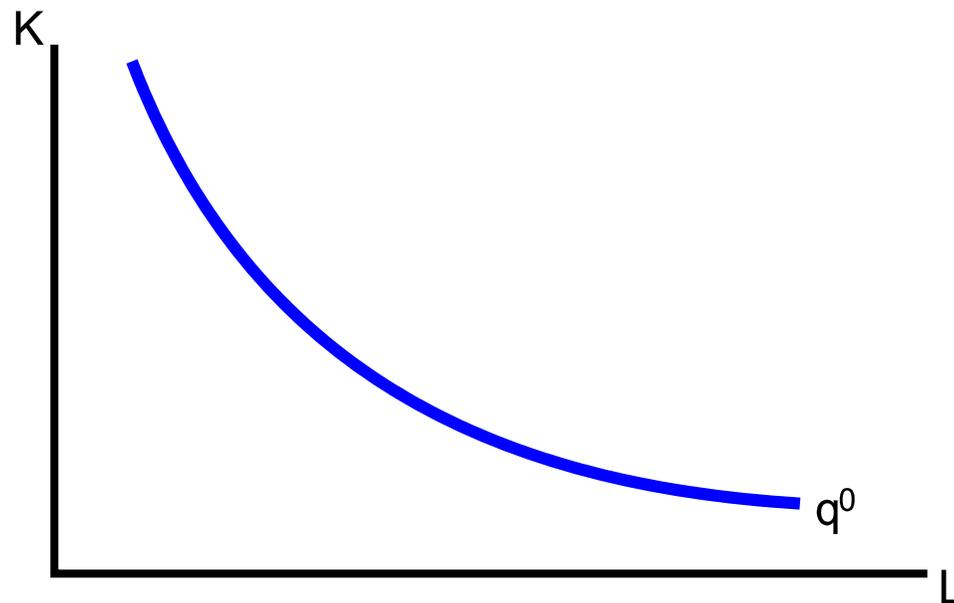
# 1. MINIMIZING COSTS

**Map of isocost lines:** given the prices of the productive factors ( $w^0$  and  $r^0$ ), we consider all the possible costs of production for the firm. There is a different isocost line for each combination of productive factors:



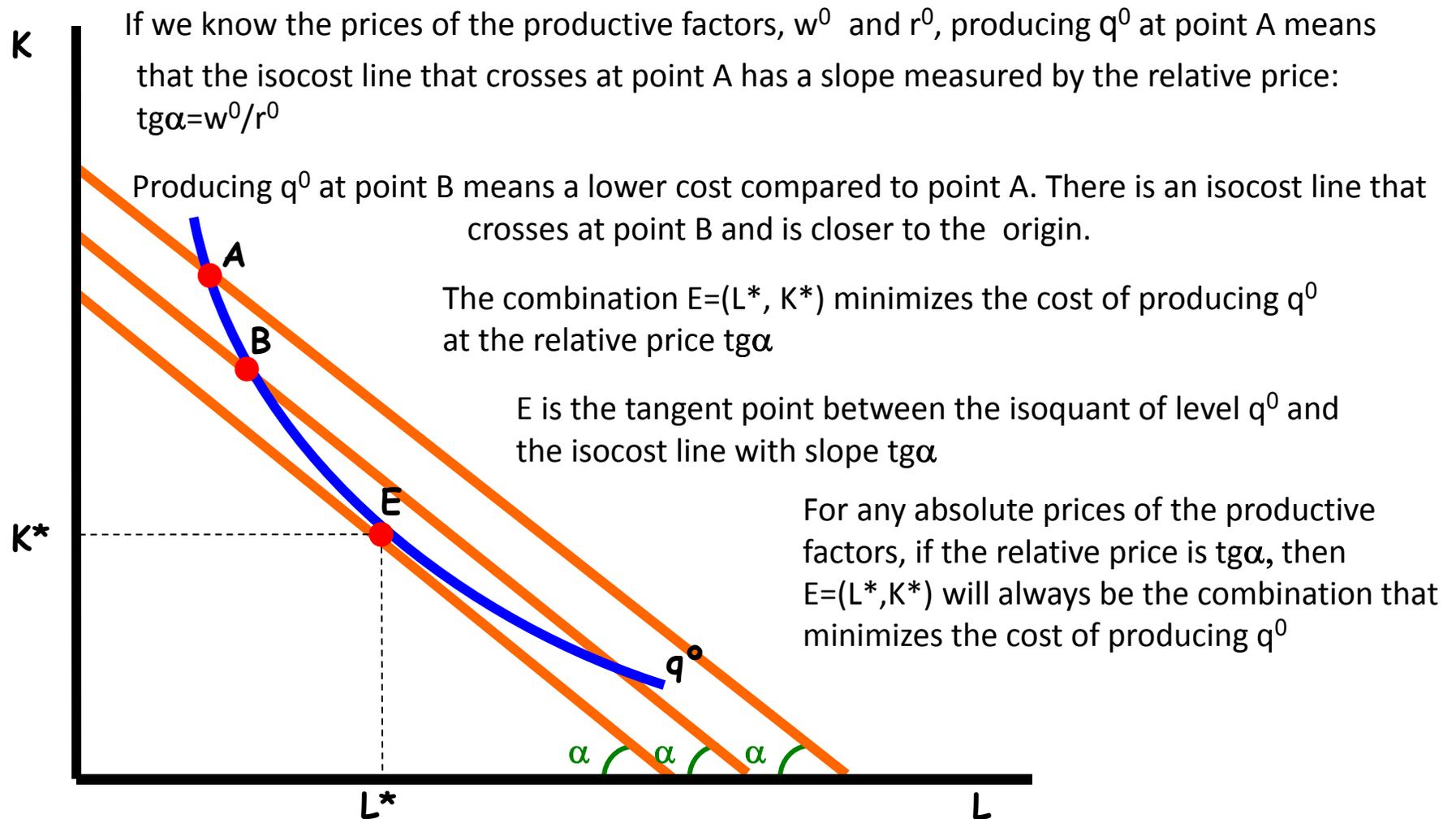
# 1. MINIMIZING COSTS

The constraint of the problem of cost-minimization by the firm is the isoquant curve corresponding to the production level  $q^0$ . That is to say, all the combinations of labour and capital that are technically efficient to obtain  $q^0$



# 1. MINIMIZING COSTS

Given the production function, the isoquant curve of level  $q^0$  represents all the combinations of inputs that are technically efficient.



# 1. MINIMIZING COSTS

The isocost line that is closest to the origin is tangent to the isoquant corresponding to  $q^0$  : their slopes are equal at the combination of inputs that minimize the cost ( $L^*, K^*$ )

Slope of the isoquant: 
$$-\left. \frac{dK}{dL} \right|_{q=q^0} = MRTS_L^K(L, K) = \frac{f_L(L, K)}{f_K(L, K)}$$

Slope of the isocost: 
$$-\left. \frac{dK}{dL} \right|_{C=C^0} = \frac{w^0}{r^0}$$

At minimum cost, that is to say, **at the equilibrium ( $L^*, K^*$ )**:

**Tangency condition:**

$$MRTS_L^K(L^*, K^*) = \frac{f_L(L^*, K^*)}{f_K(L^*, K^*)} = \frac{w^0}{r^0}$$

**Production restriction:**

$$q^0 = f(L^*, K^*)$$

# 1. MINIMIZING COSTS

Analytically, the problem of cost-minimization consists, of solving the following **conditioned optimization problem**:

$$\begin{aligned} \min_{L, K} C &= w^0 \cdot L + r^0 \cdot K \\ \text{s.t. } q^0 &= f(L, K) \end{aligned}$$

where:

- Endogenous variables: L y K
- Exogenous variables:  $w^0$ ,  $r^0$  y  $q^0$

To solve this problem, we need to formulate the following Lagrangian:

$$\Phi = \Phi(L, K, \mu) = w^0 \cdot L + r^0 \cdot K + \mu \cdot [q^0 - f(L, K)]$$

and, consequently, the problem becomes:

$$\min_{L, K, \mu} \Phi(L, K, \mu)$$



# 1. MINIMIZING COSTS

$$\text{Min}\Phi = \Phi(L, K, \mu) = w^0 \cdot L + r^0 \cdot K + \mu \cdot [q^0 - f(L, K)]$$

The **first-order conditions (FOCs)** form a system of 3 equations with 3 unknown quantities(L,K, $\mu$ ):

$$\Phi_L = \frac{\partial \Phi}{\partial L} = 0 \Rightarrow w^0 - \mu \frac{\partial f}{\partial L} = 0 \Rightarrow w^0 = \mu \cdot f_L$$

$$\Phi_K = \frac{\partial \Phi}{\partial K} = 0 \Rightarrow r^0 - \mu \frac{\partial f}{\partial K} = 0 \Rightarrow r^0 = \mu \cdot f_K$$

$$\Phi_\mu = \frac{\partial \Phi}{\partial \mu} = 0 \Rightarrow q^0 - f(L, K) = 0 \Rightarrow q^0 = f(L, K)$$

The second-order condition (SOC) is automatically fulfilled under the strict convexity of the isoquant curves



# 1. MINIMIZING COSTS

Using the first two FOCs we can formulate the **Law of Equality of the Weighted Marginal Producty (LEWMP)**, equivalent to the tangency condition between the isocost line and the isoquant curve:

$$\text{MRTS}_L^K(L^*, K^*) = \frac{f_L}{f_K} = \frac{w^0}{r^0} \Leftrightarrow \boxed{\text{LEWMP: } \frac{f_L}{w^0} = \frac{f_K}{r^0}}$$

The firm minimizes its cost when the additional output generated by the last monetary unit spent on each input is the same.

From the LEWMP and the expression of the isoquant curve of level  $q^0$  we obtain the solution to the problem of cost-minimization: given the prices, the quantities of labour ( $L^*$ ) and capital ( $K^*$ ) that must be used to produce  $q^0$  at the minimum cost:

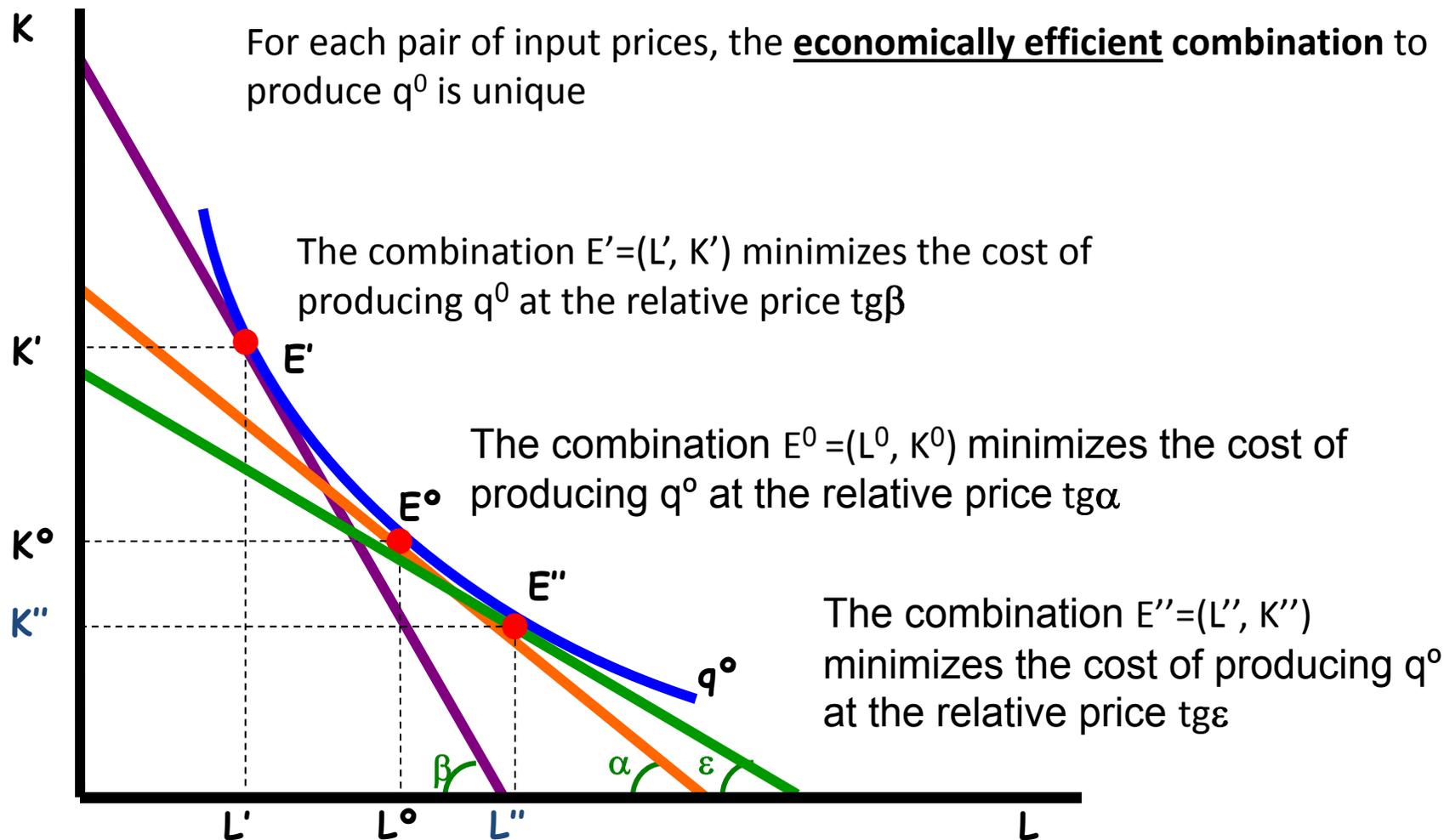
$$\left. \begin{array}{l} \frac{f_L}{f_K} = \frac{w^0}{r^0} \\ q^0 = f(L, K) \end{array} \right\} \Rightarrow (L^*, K^*)$$

We can then determine the lowest cost of producing  $q^0$  at the given prices:

$$C^* = w^0 \cdot L^* + r^0 \cdot K^*$$

# 1. MINIMIZING COSTS

Given the production function, the isoquant curve of level  $q^0$  contains all the input combinations that are **technically efficient** to produce  $q^0$



## 2. CONDITIONAL FACTOR DEMANDS

We can formulate the problem of cost-minimization without giving specific values to the exogenous variables (w,r,q):

$$\begin{aligned} \min C &= w \cdot L + r \cdot K \\ \text{s.t. } q &= f(L,K) \end{aligned}$$

Applying the usual solution method, we obtain the **Conditional Factor Demands(CFDs)**:

$$L = L(w, r, q)$$

$$K = K(w, r, q)$$

that determine the quantities of labour and capital needed to obtain, at any given prices, a specific level of production at the minimum cost.

### 3. COST FUNCTIONS

If we now replace the CFDs in the general form of the expression of the cost:

$$C = w \cdot L + r \cdot K = w \cdot L(w, r, q) + r \cdot K(w, r, q)$$

we obtain the **long-run cost function**:

$$C = C(w, r, q)$$

This function gives, for any prices of the inputs, **the minimum cost in the long-run** that the firm needs to produce any given amount of output:

- It combines technical and economic aspects
- It captures the situation where the firm minimizes its cost.
- It does not mean the profit is maximized, as we will see later....



### 3. COST FUNCTIONS

Some **properties** of the cost function in the long-run:

$$C = C(w, r, q)$$

1. It is **increasing with the price of the inputs** ( $w$  y  $r$ )

- If the price of any of the inputs increases, the firm will have a higher cost to produce the same amount of output

2. It is **increasing with the quantity of output** ( $q$ ), for given prices of the inputs

- Given  $w$  and  $r$ , a higher quantity of output can be produced only if the firm assumes a higher cost.

3. It is **homogeneous of degree 1** in the **input prices**

$$\forall r, w, q, \lambda > 0: C(\lambda \cdot w, \lambda \cdot r, q) = \lambda \cdot C(w, r, q)$$



### 3. COST FUNCTIONS

Taking the conditional factor demands (CFD)  $L=L(w,r,q)$ ,  $K=K(w,r,q)$ , we obtain the cost function in the long-run  $C=C(w,r,q)$  and, if we know the price of the inputs ( $w=w^0, r=r^0$ ), we will obtain the Conditional Factor Demands at the given prices:

$$L = L(q)$$

$$K = K(q)$$

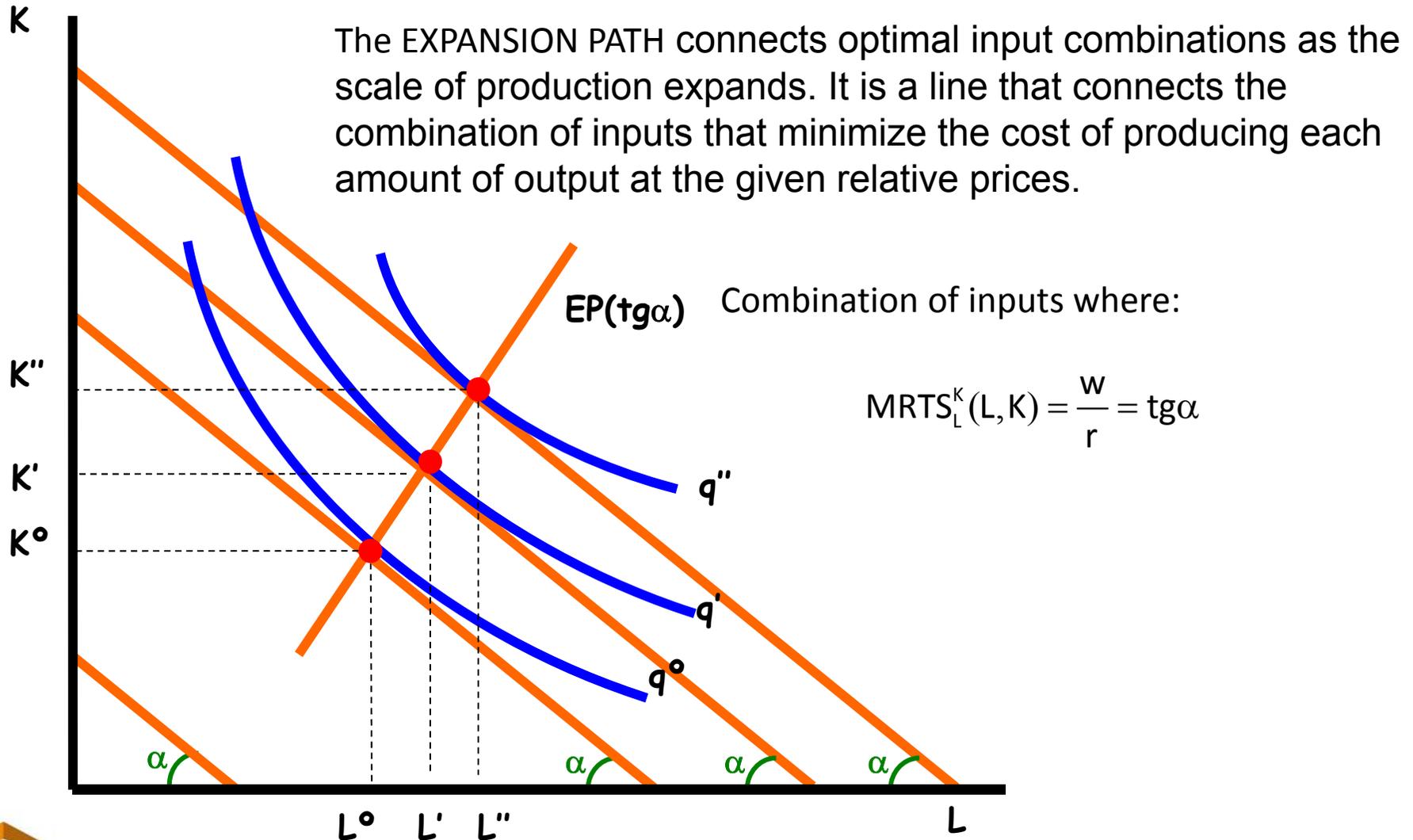
Then, we can obtain the **cost function in the long-run**:

$$C = C_L(q)$$

The cost function in the long-run measures, for any given prices of inputs, the minimum cost that is needed to produce any amount of output in the long-run (when L and K are considered to be variable).

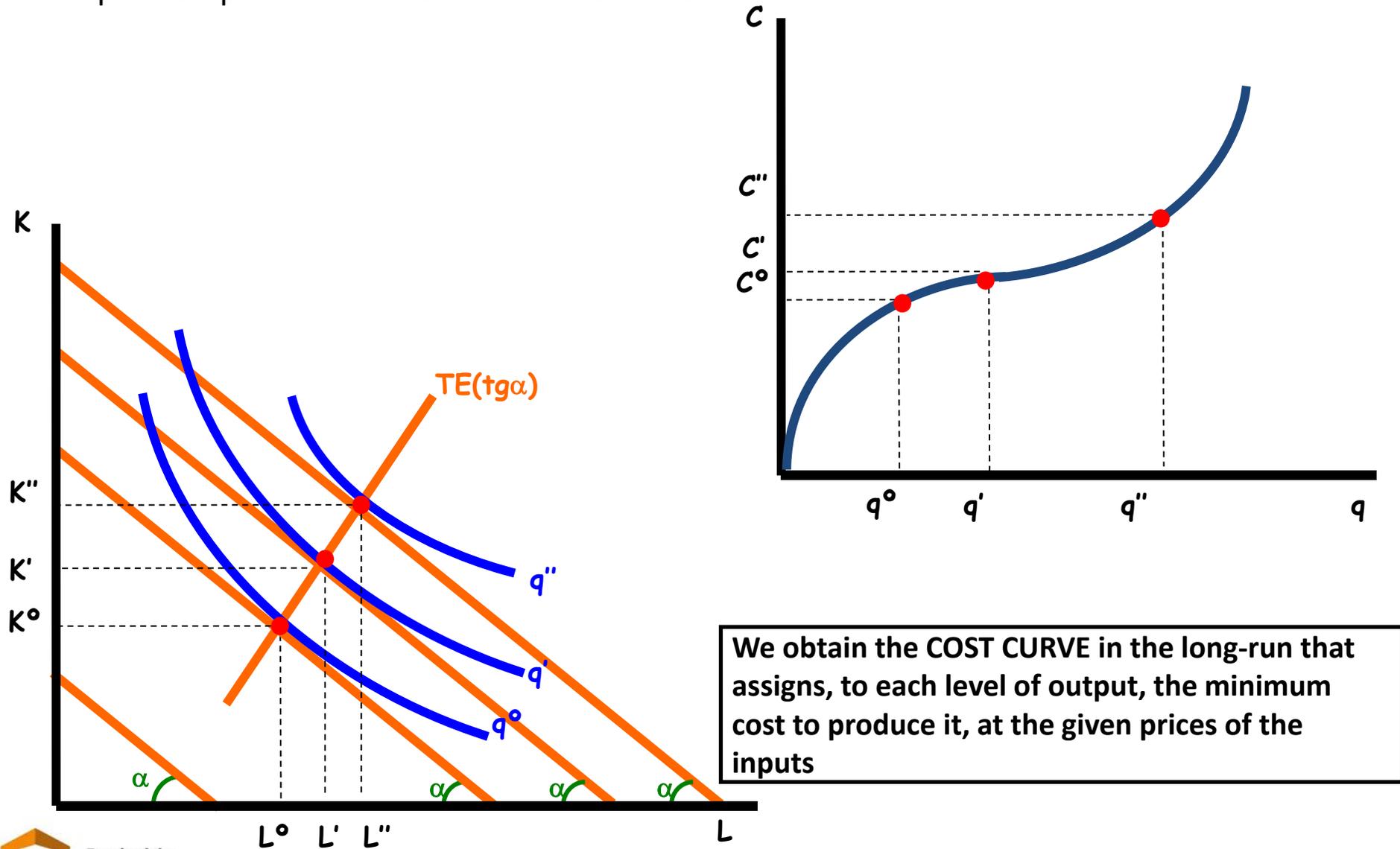
### 3. COST FUNCTIONS

Given the production function and the relative price  $tg\alpha$ , for each amount of output, the combination of inputs that minimizes the cost is determined



### 3. COST FUNCTIONS

Given the production function and the price of the inputs, the expansion path can be determined as follows:



We obtain the COST CURVE in the long-run that assigns, to each level of output, the minimum cost to produce it, at the given prices of the inputs

### 3. COST FUNCTIONS

Considering the cost function in the long-run, we can define:

**Average cost:** cost per unit of produced output

$$AC_L(q) = \frac{CL(q)}{q}$$

- It is the slope of the vector which links each point of the cost curve in the long-run to the origin
- It reaches its optimum (minimum) when the vector is tangent to the curve
- **Minimum Efficient Scale ( $q^{MES}$ ):** amount of output that minimizes the average cost

**Marginal cost:** the change in cost when the output changes by one unit:

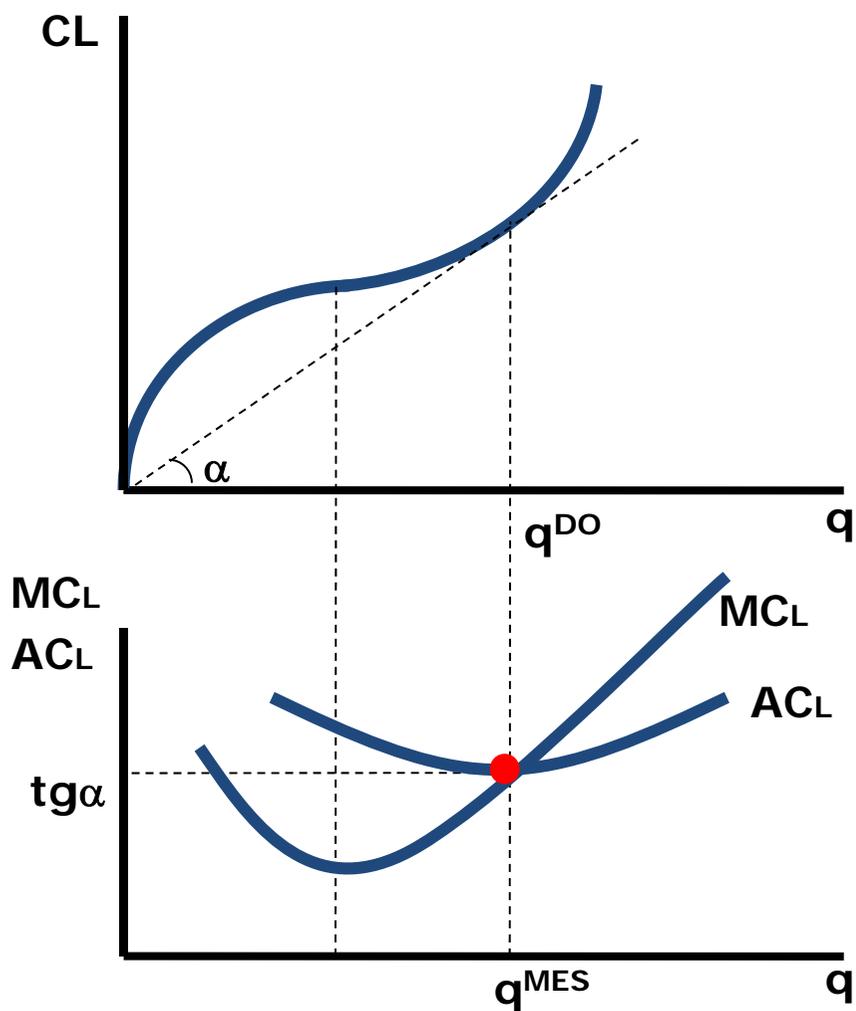
$$MC_L(q) = \frac{\Delta C_L(q)}{\Delta q} \stackrel{\Delta q \rightarrow 0}{=} \frac{dC_L(q)}{dq}$$

- It measures the slope of the cost curve in the long-run
- It reaches its minimum at the inflexion point



### 3. COST FUNCTIONS

Cost curve in the long-run (general case):

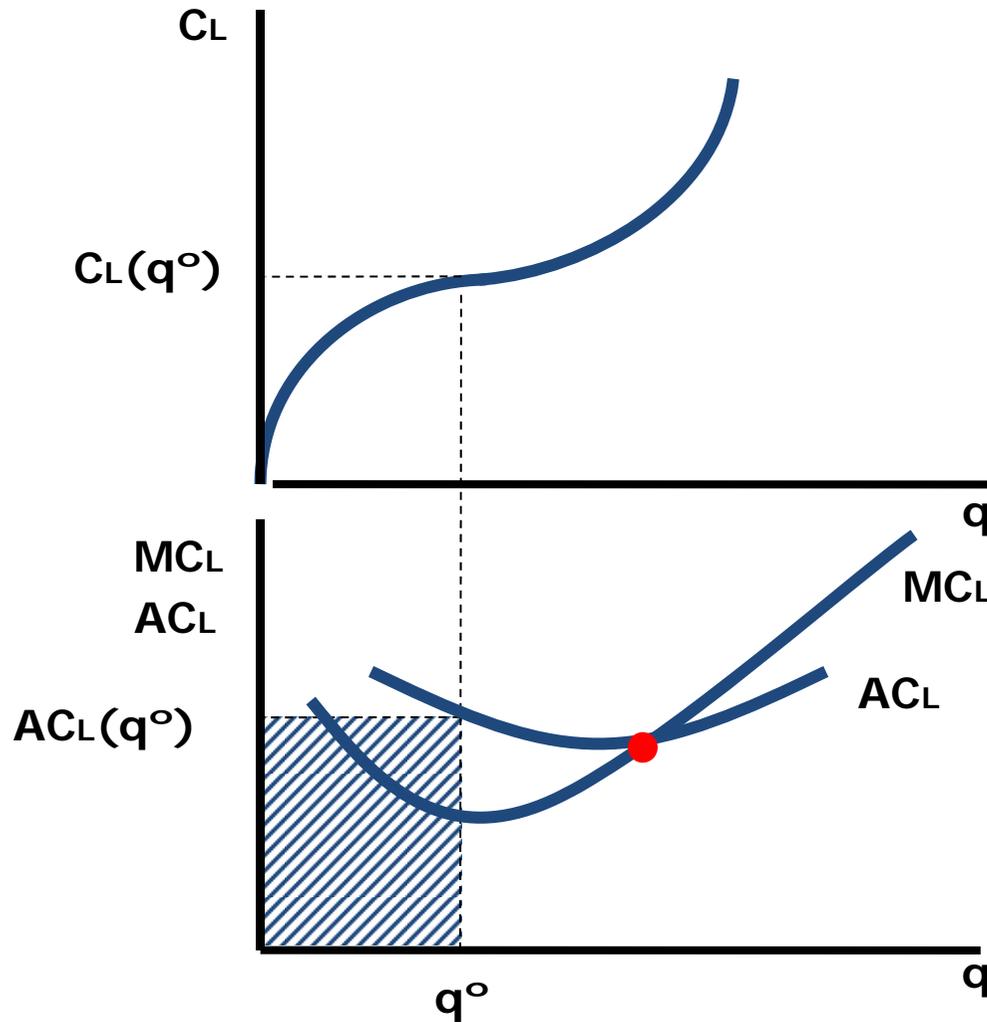


Minimum Efficient Scale ( $q^{\text{MES}}$ ): level of production where the average cost in the long-run reaches its minimum, and is also equal to the marginal cost:

$$AC_L(q^{\text{MES}}) = MC_L(q^{\text{MES}})$$

### 3. COST FUNCTIONS

The total cost of producing  $q^0$  is the area of base  $q^0$  and height  $AC_L(q^0)$ :

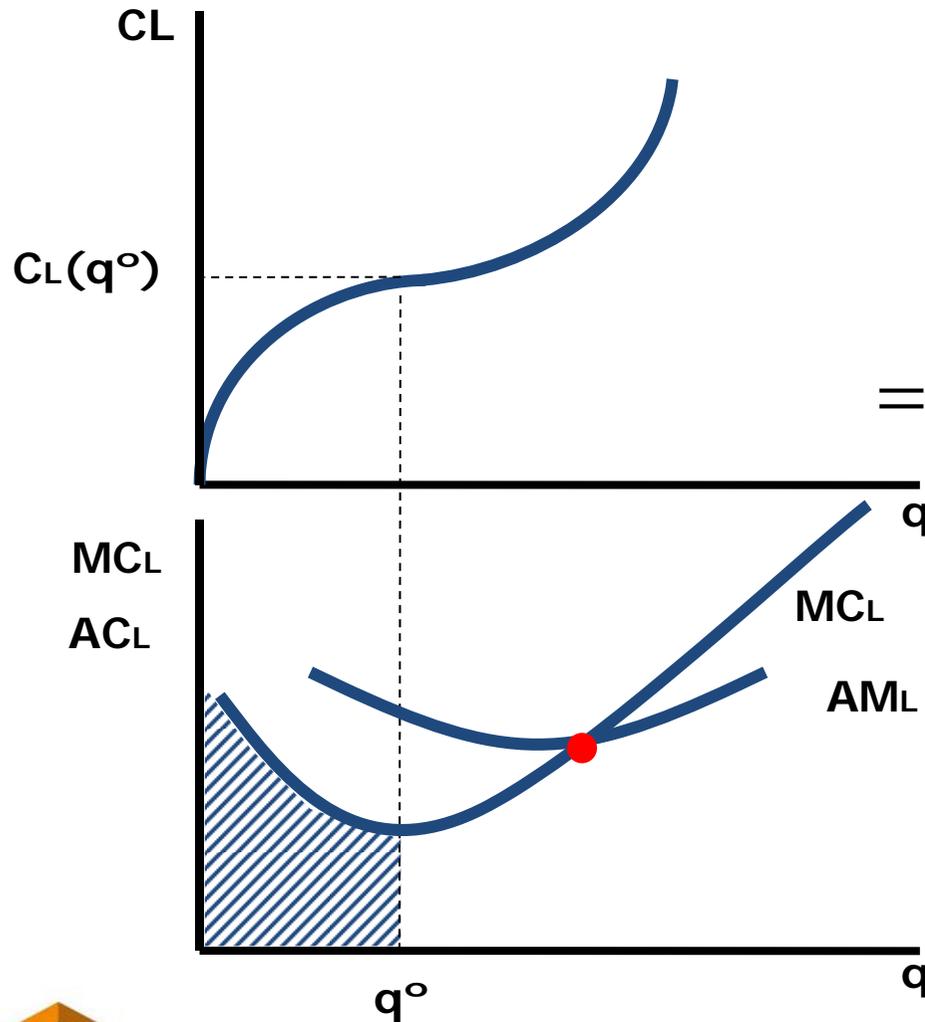


$$AC_L(q^0) = \frac{C_L(q^0)}{q^0} \Rightarrow$$

$$C_L(q^0) = AC_L(q^0) \cdot q^0$$

### 3. COST FUNCTIONS

The total cost of producing  $q^0$  can also be calculated as the area under the cost curve between  $q=0$  and  $q=q^0$

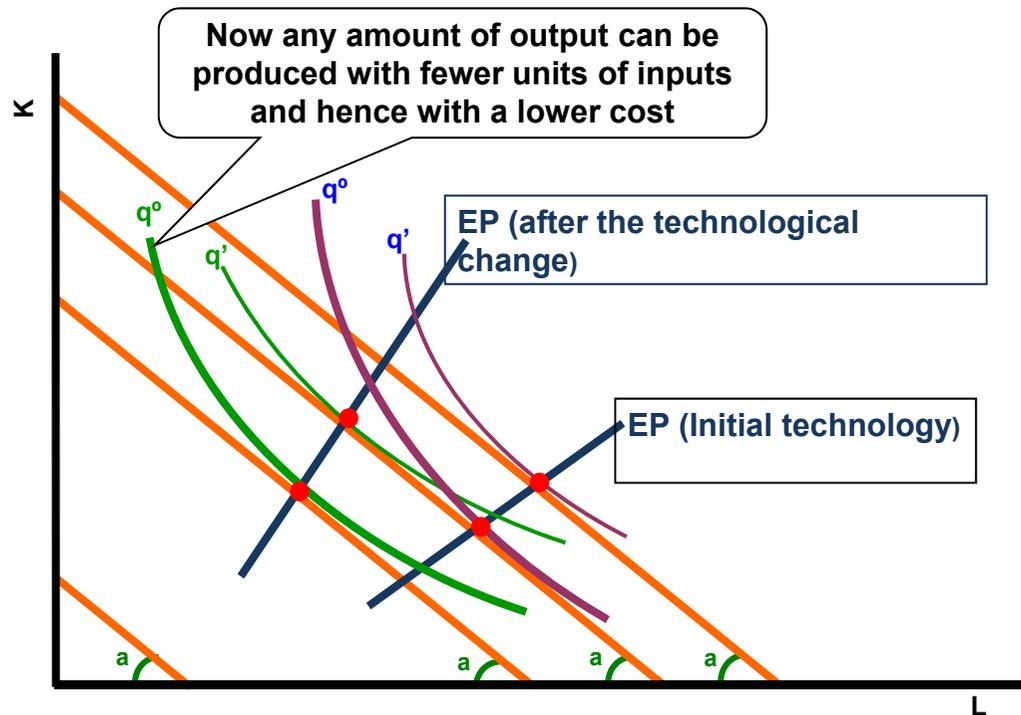


$$\int_0^{q^0} MC_L(q) dq = [C_L(q)]_0^{q^0} =$$

$$= C_L(q^0) - \underbrace{C_L(0)}_0 = C_L(q^0)$$

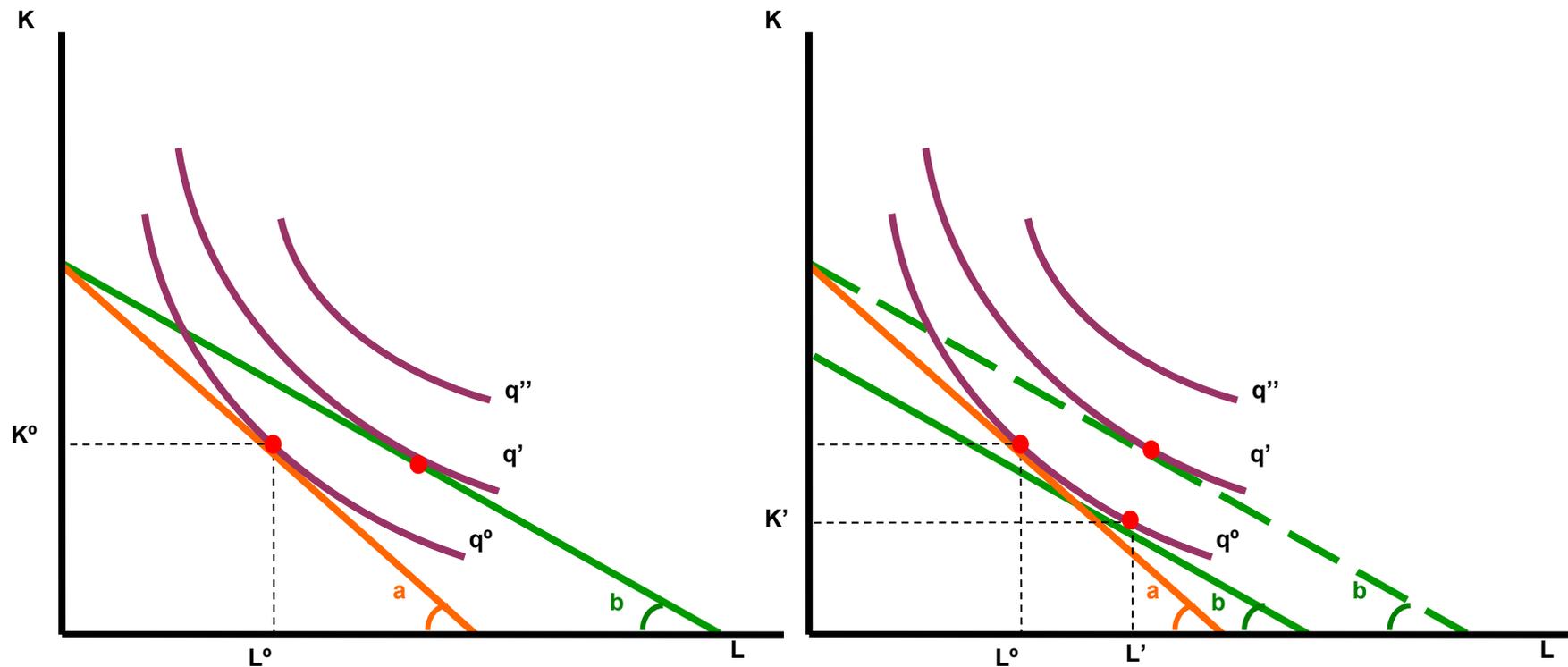
### 3. COST FUNCTIONS

If there are technological improvements, with the same quantities of inputs, the firm will obtain a higher amount of output (to produce any amount of output the firm will need fewer units of inputs)  $\Rightarrow$  the production function changes  $\Rightarrow$  the isoquant map changes  $\Rightarrow$  the MRTS changes  $\Rightarrow$  the Expansion Path changes  $\Rightarrow$  the cost function changes:



### 3. COST FUNCTIONS

If there is a change in the relative price of the inputs  $\Rightarrow$  the Expansion Path changes  $\Rightarrow$  the cost function changes:

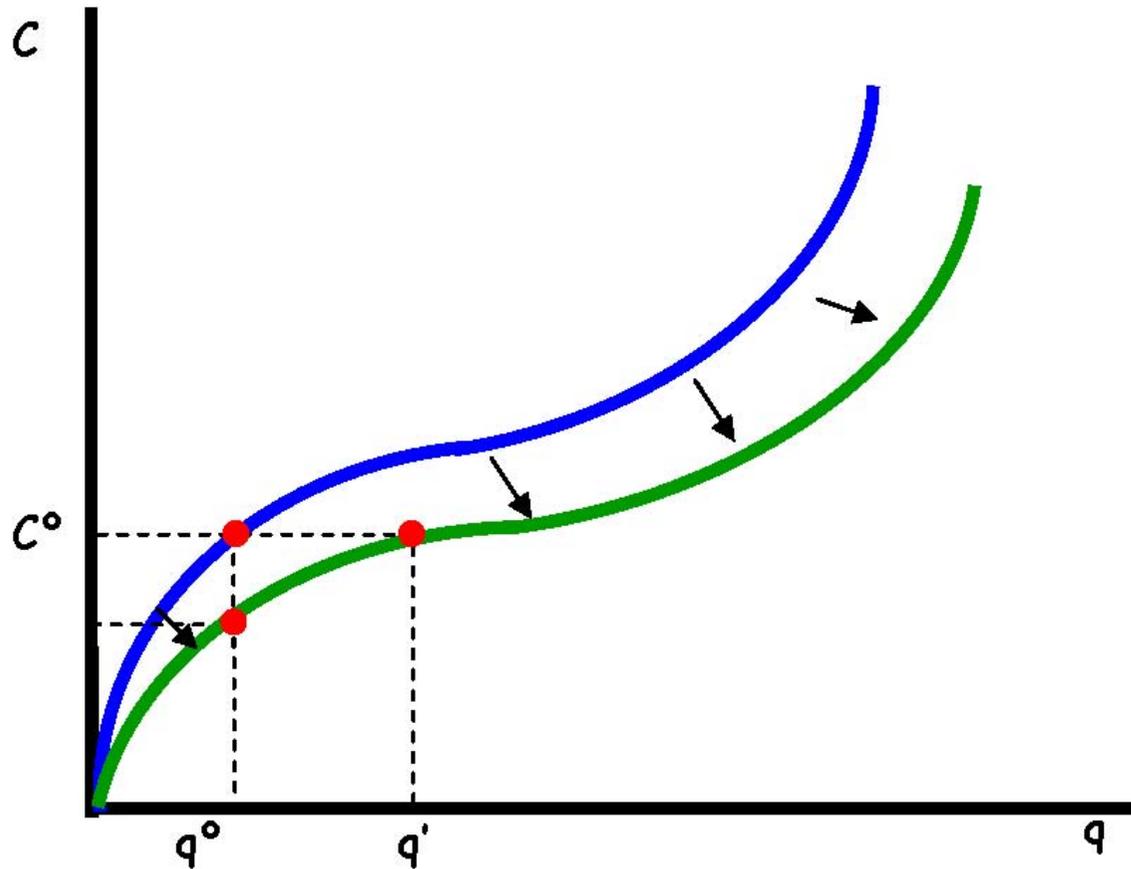


With the same cost needed to produce  $q^0$ , now  $q'$  can be produced

To produce  $q^0$  with minimum cost,  $(L', K')$  must be used now, leading to a lower cost

### 3. COST FUNCTIONS

Technical improvement or decrease in the price of any of the inputs  $\Rightarrow$  the cost function moves downwards:

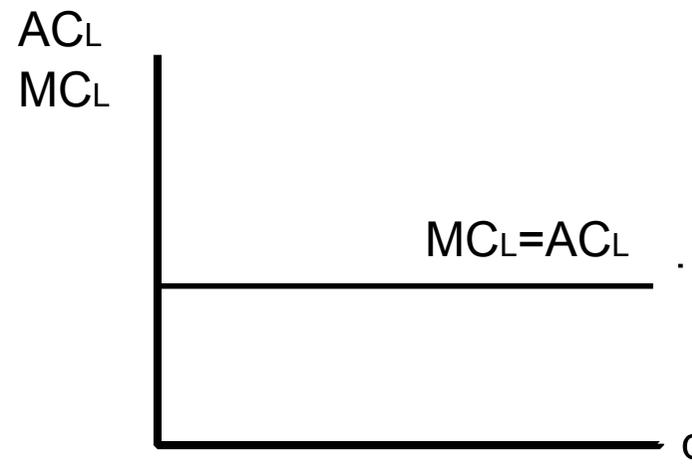
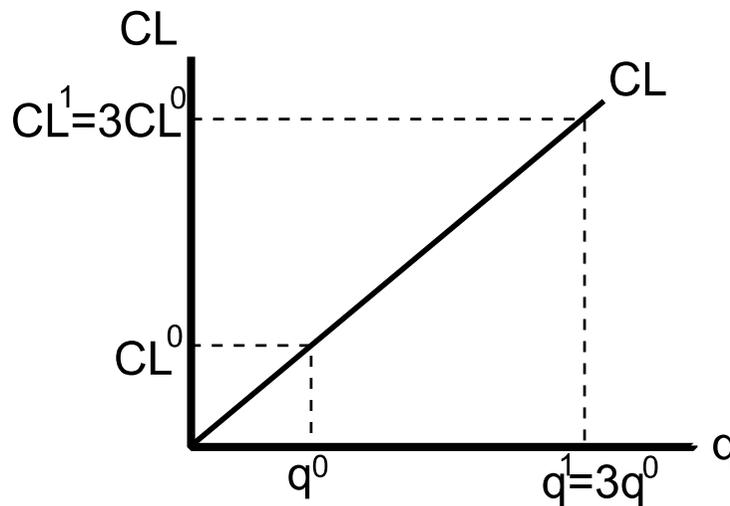


### 3. COST FUNCTIONS

The form of the cost curves is related to the type of returns to scale of the production function:

1. If the technology has **constant returns to scale**:

$$\begin{aligned} q^0 &= f(L, K) \rightarrow q^1 = f(\lambda \cdot L, \lambda \cdot K) \\ C_L^0 &= w \cdot L + r \cdot K \rightarrow C_L^1 = w \cdot \lambda \cdot L + r \cdot \lambda \cdot K \\ \text{so that } q^1 &= \lambda \cdot q^0, \text{ and thus } C_L^1 = \lambda \cdot C_L^0 \end{aligned}$$

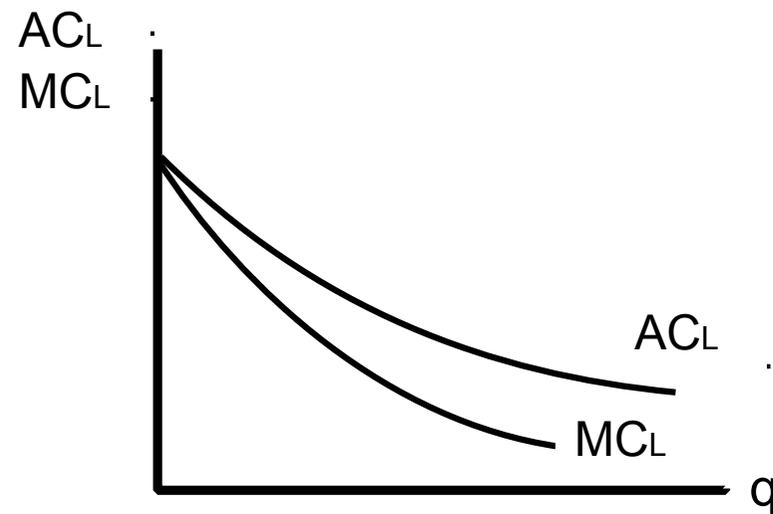
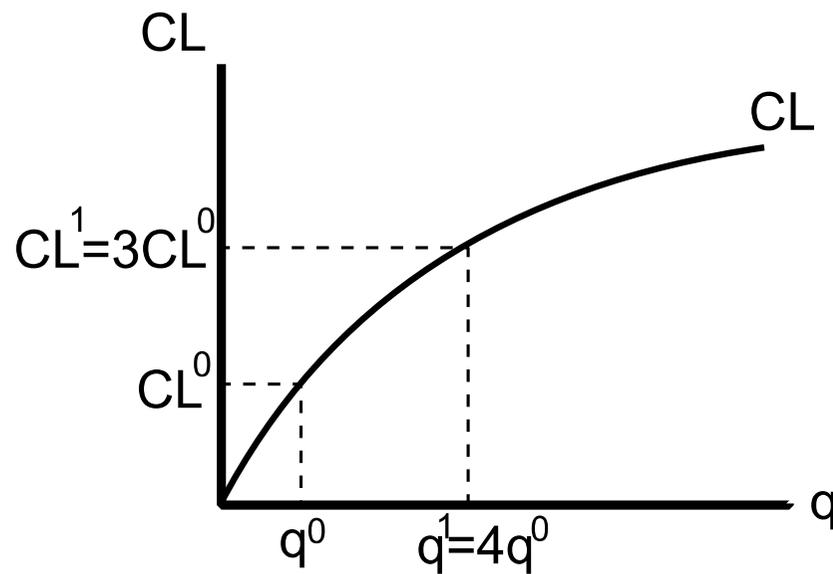


### 3. COST FUNCTIONS

2. If the technology presents **increasing returns to scale**:

$$q^0 = f(L, K) \rightarrow q^1 = f(\lambda \cdot L, \lambda \cdot K)$$
$$C_L^0 = w \cdot L + r \cdot K \rightarrow C_L^1 = w \cdot \lambda \cdot L + r \cdot \lambda \cdot K$$

so that  $q^1 > \lambda \cdot q^0$ , and thus  $C_L^1 = \lambda \cdot C_L^0$



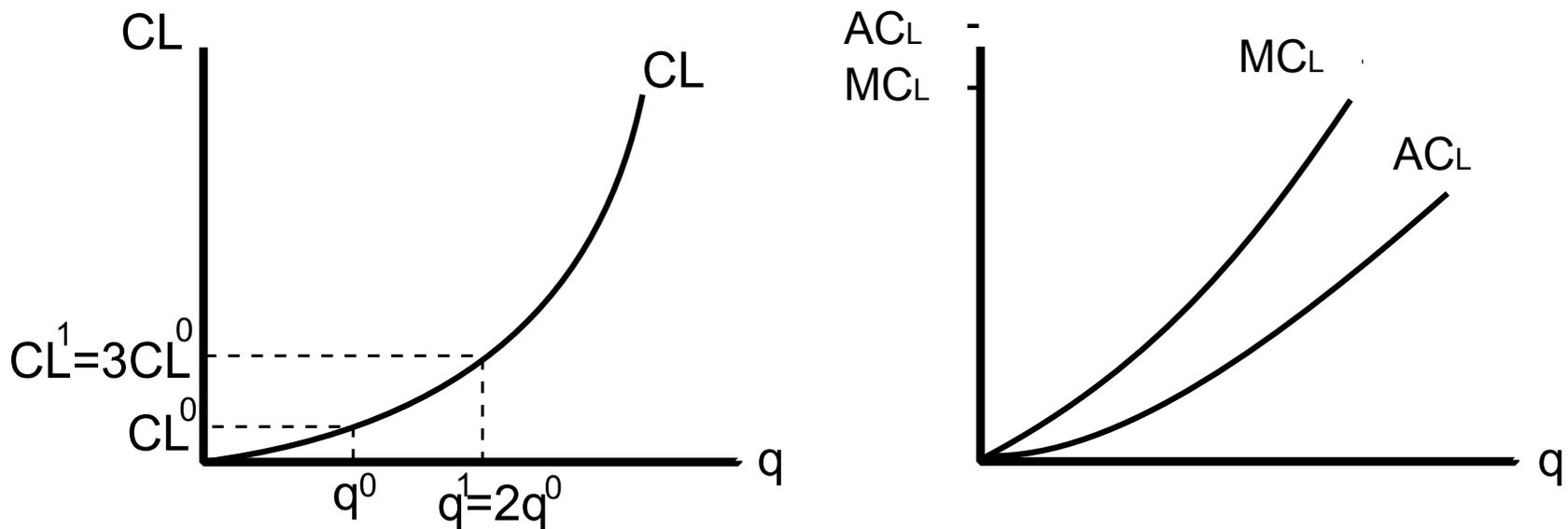
### 3. COST FUNCTIONS

3. In the case that the technology presents **decreasing returns to scale**:

$$q^0 = f(L, K) \rightarrow q^1 = f(\lambda \cdot L, \lambda \cdot K)$$

$$C_L^0 = w \cdot L + r \cdot K \rightarrow C_L^1 = w \cdot \lambda \cdot L + r \cdot \lambda \cdot K$$

so that  $q^1 < \lambda \cdot q^0$ , and thus  $C_L^1 = \lambda \cdot C_L^0$



## 4. LONG-RUN AND SHORT-RUN COSTS

We have analyzed the problem of cost-minimization, considering that all the productive factors are variable (long-run). We have obtained the Conditioned Factor Demands (CFD) and the cost function in the long-run.

We now suppose we are in the short-run. That is, we also want to minimize the production cost, but now we assume that the firm has a certain quantity of **capital (fixed factor)** available, and that the firm cannot modify it, and thus the firm is able to decide only the amount of **labour (variable factor)**. We formulate the problem as follows:

$$\begin{aligned} \min C &= w \cdot L + r \cdot \bar{K} \\ \text{s.t. } q &= f(L, \bar{K}) \end{aligned}$$

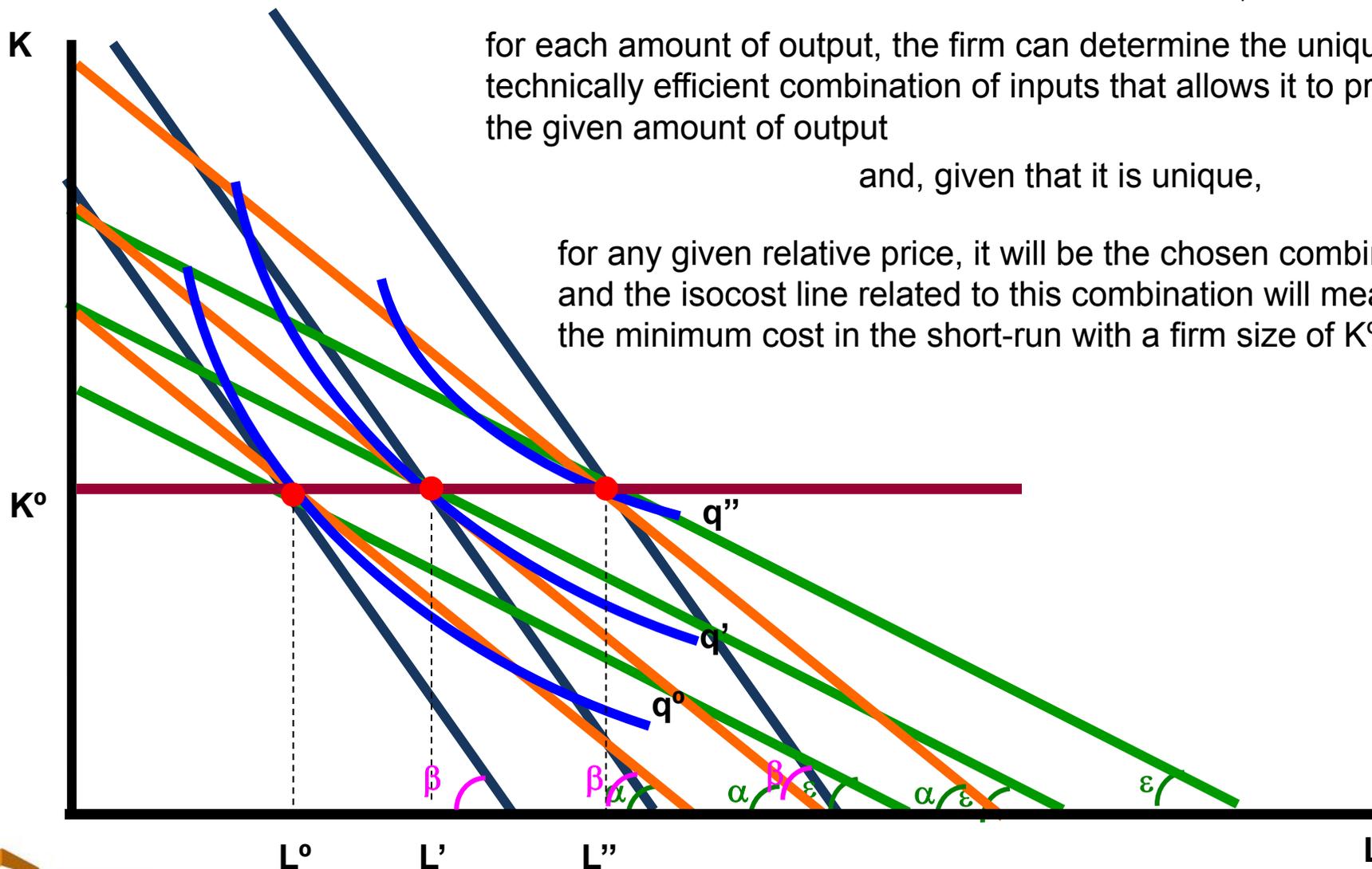
# 4. LONG-RUN AND SHORT-RUN COSTS

Given the production function  
and the size of the firm:  $K^0$ ,

for each amount of output, the firm can determine the unique  
technically efficient combination of inputs that allows it to produce  
the given amount of output

and, given that it is unique,

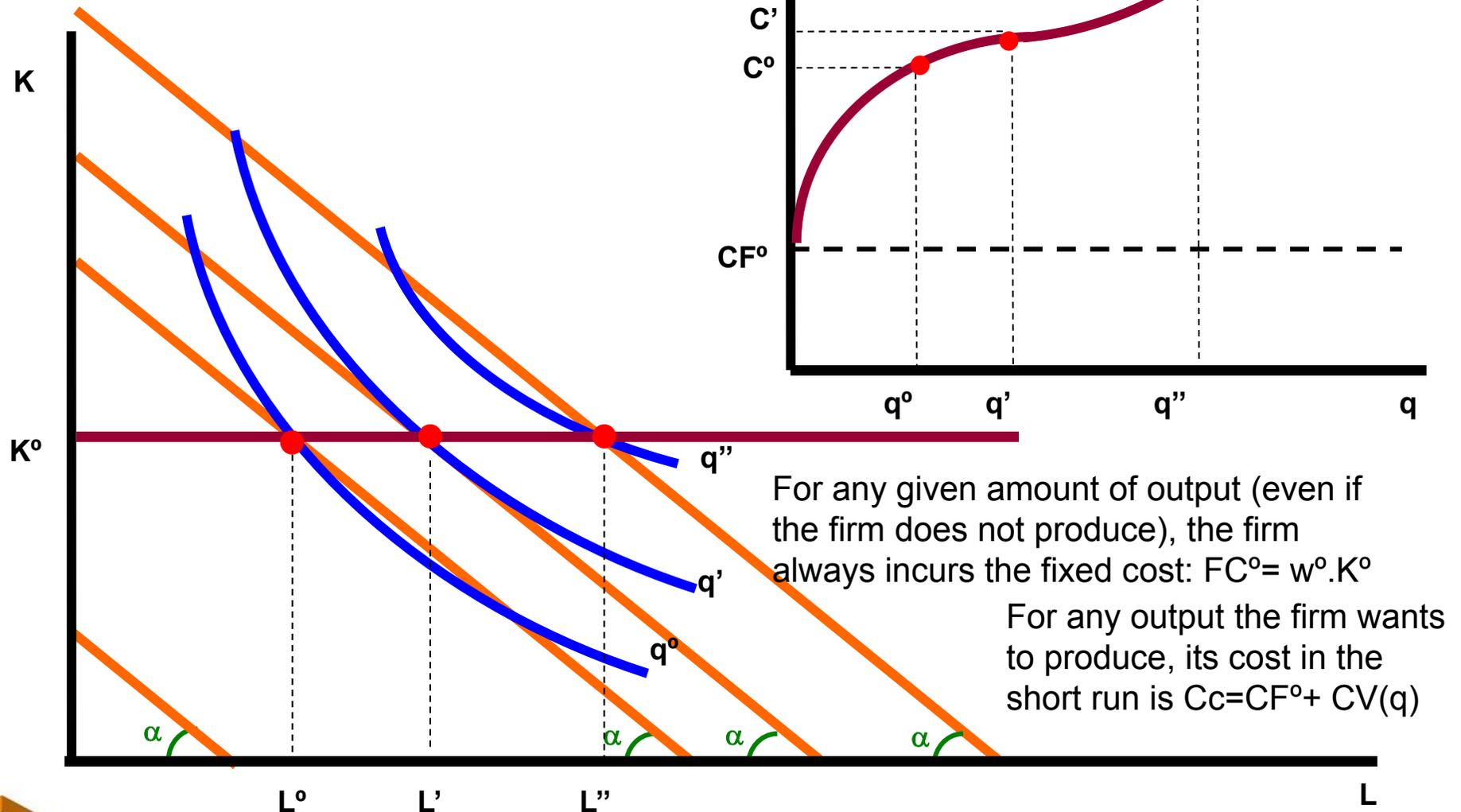
for any given relative price, it will be the chosen combination,  
and the isocost line related to this combination will measure  
the minimum cost in the short-run with a firm size of  $K^0$ :



## 4. LONG-RUN AND SHORT-RUN COSTS

Given the production function, the size of the firm:  $K^0$  and the input prices  $w^0, r^0$  ( $w^0/r^0 = \gamma/\alpha$ ):

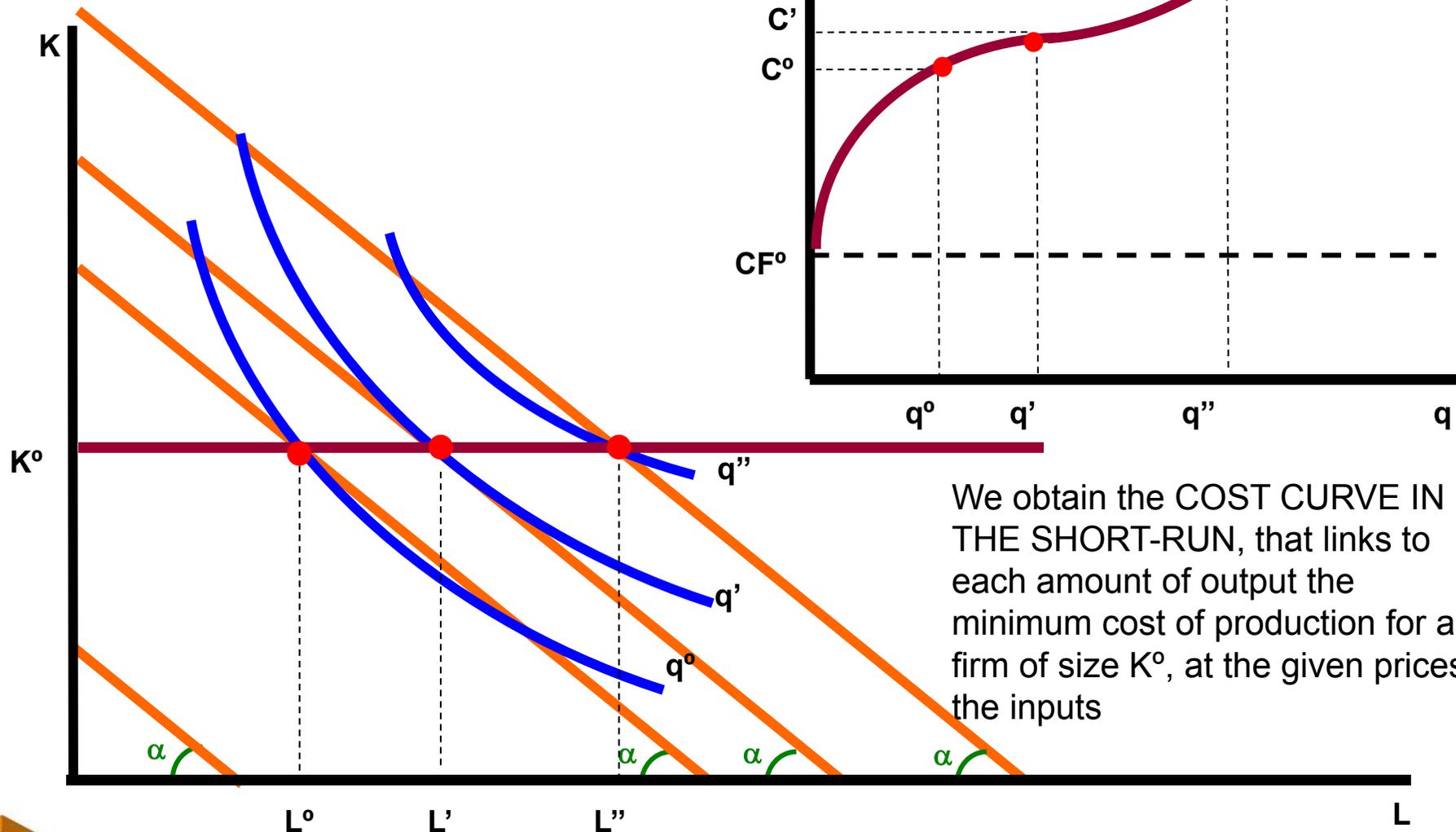
To each output, we associate the cost in the short-run of producing it for a firm of size  $K^0$  :



## 4. LONG-RUN AND SHORT-RUN COSTS

Given the production function, the size of the firm:  $K^0$  and the input prices  $w^0, r^0$  ( $w^0/r^0 = \text{tg}\alpha$ ):

To each output we associate the cost in the short-run of producing it for a firm of size  $K^0$ :



We obtain the COST CURVE IN THE SHORT-RUN, that links to each amount of output the minimum cost of production for a firm of size  $K^0$ , at the given prices of the inputs



## 4. LONG-RUN AND SHORT-RUN COSTS

Analytically, under this framework of short-run, from the production function the conditioned demand of the variable input can be determined:

$$q = f(L, K) \xrightarrow{K=\bar{K}} q = f(L, \bar{K}) \Rightarrow L = L(q, \bar{K})$$

If we substitute the conditioned demand in the expression of the cost, we obtain the cost function in the short-run:

$$CT = w \cdot L(q, \bar{K}) + r \cdot \bar{K} = CT(w, r, q, \bar{K})$$

- It is increasing in the price of the inputs and the amount of output and, given the input prices, the **cost curve in the short-run** can be obtained:

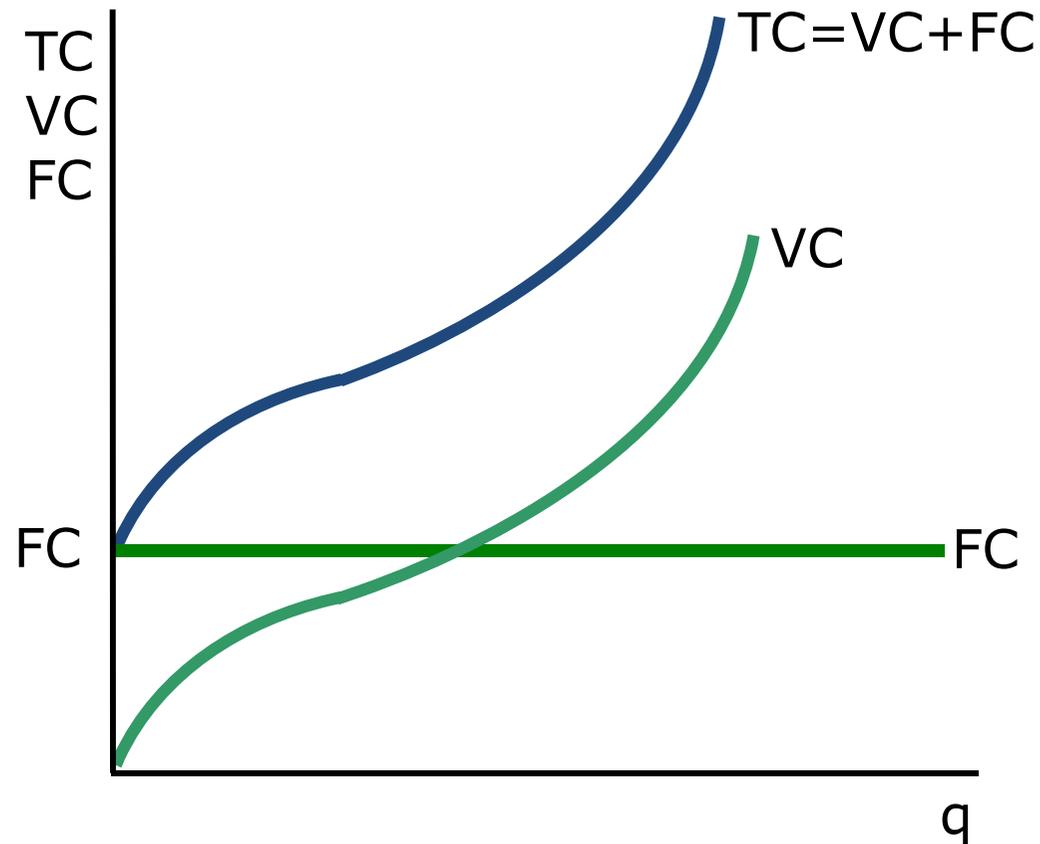
$$TC(q) = VC(q) + FC$$

where:

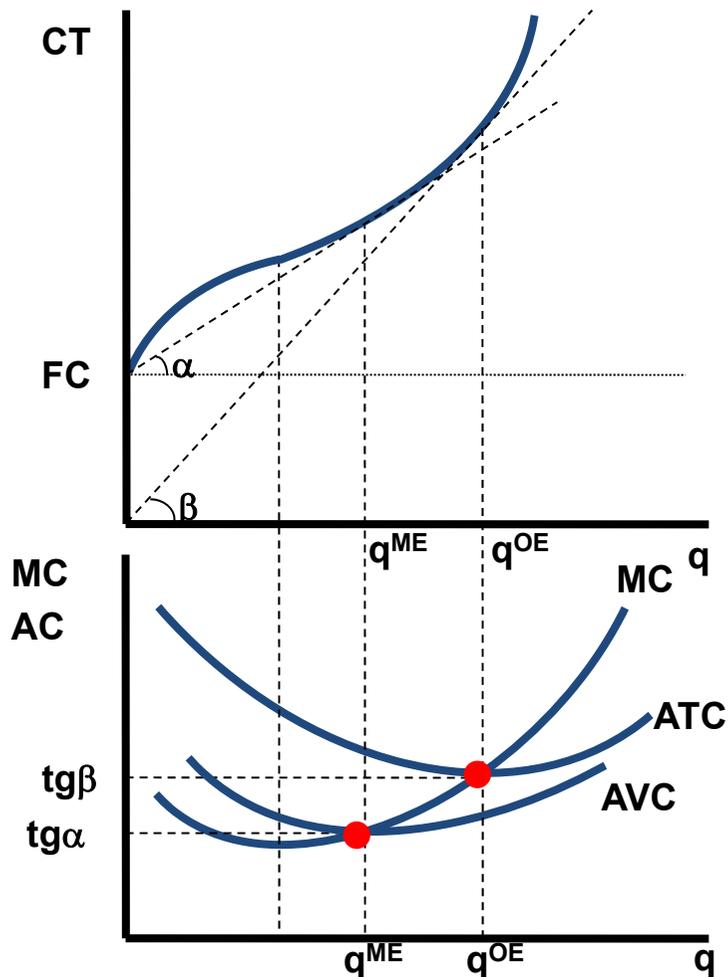
- TC is the total cost (depends on the level of production)
- VC is the variable cost (depends on the level of production)
- Fc is the fixed cost (does not depend on the level of production)

## 4. LONG-RUN AND SHORT-RUN COSTS

It can be shown that, given the **law of diminishing marginal returns**, the cost curve in the short run goes from concave to convex:



## 4. LONG-RUN AND SHORT-RUN COSTS



$$AVC(q) = \frac{VC(q)}{q}$$

$$ATC(q) = \frac{TC(q)}{q} = \frac{VC(q)}{q} + \frac{FC}{q}$$

$$MC(q) = \frac{dTC(q)}{dq} = \frac{d\left[\frac{q}{q}VC(q) + \frac{q}{q}FC\right]}{dq} = \frac{dVC(q)}{dq}$$

**Minimum average variable cost ( $q^{MAVC}$ ):** amount of output that minimizes the average variable cost:

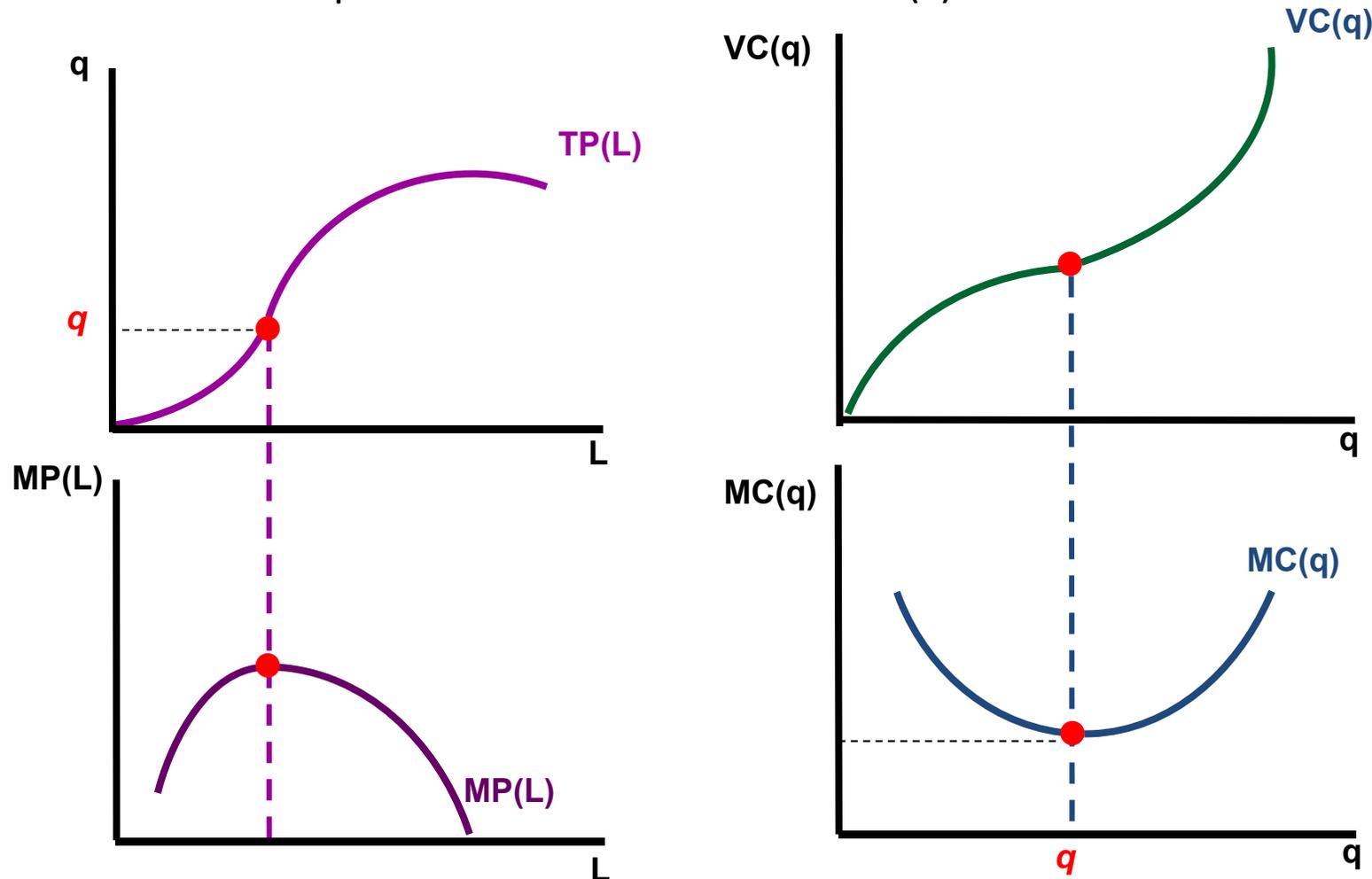
$$MC(q^{MAVC}) = AVC(q^{MAVC})$$

**Minimum average total cost ( $q^{MATC}$ ):** amount of output that minimizes the average total cost:

$$MC(q^{MATC}) = ATC(q^{MATC})$$

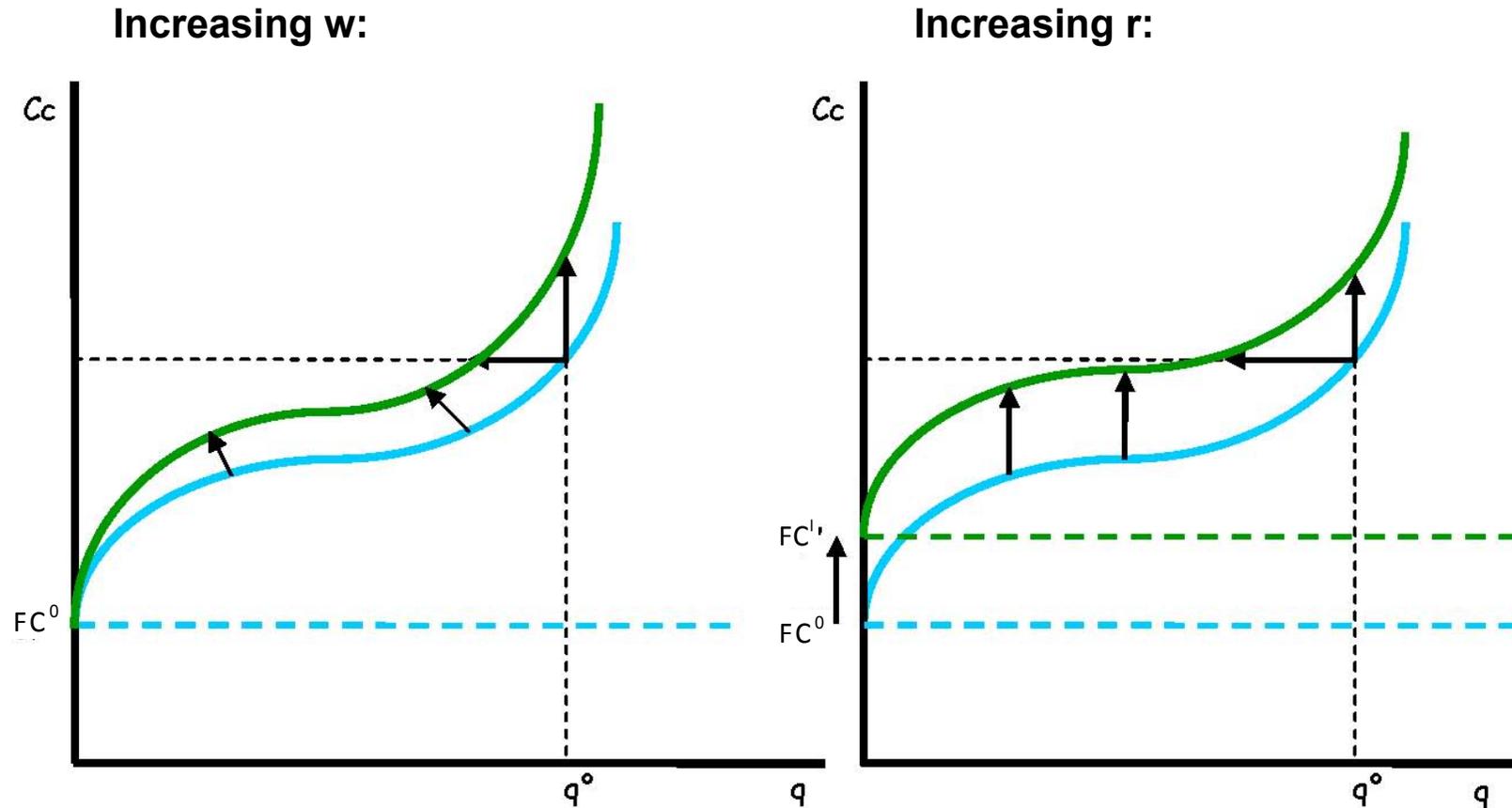
## 4. LONG-RUN AND SHORT-RUN COSTS

There is a relationship between the shape of the cost curve in the short-run and the total product of the variable factor (L):



## 4. LONG-RUN AND SHORT-RUN COSTS

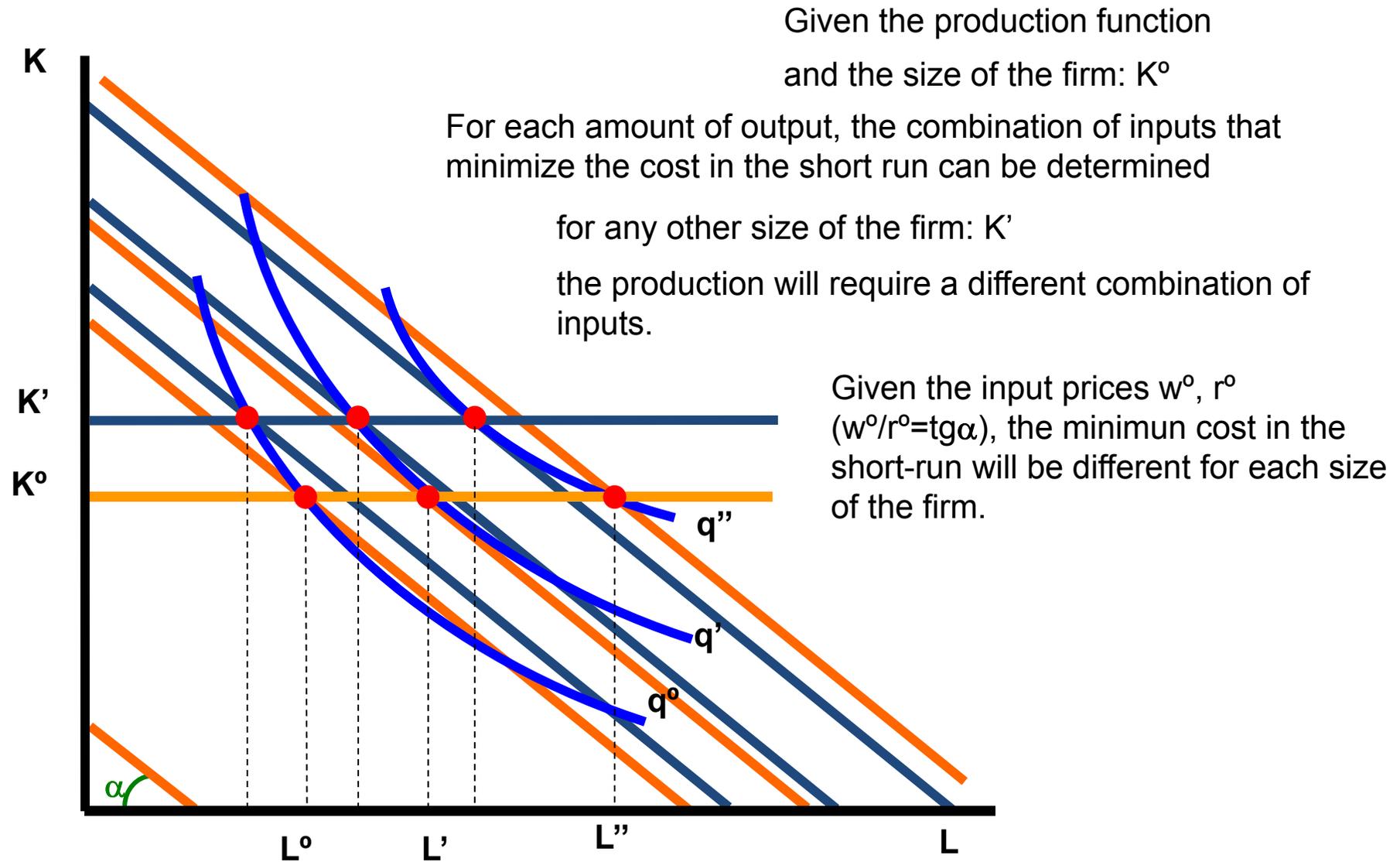
Changes in the prices of the inputs move the cost curves in the short-run:



The fixed cost held constant, the variable cost increases

The variable cost held constant, the fixed cost increases.

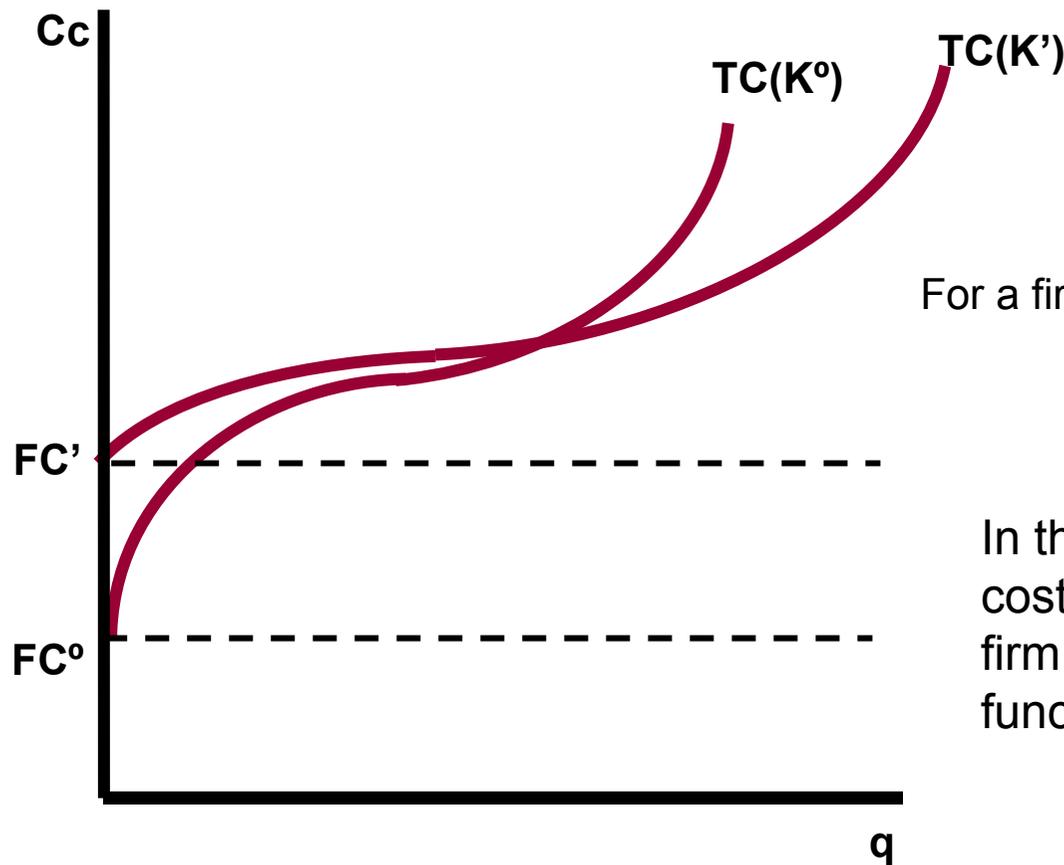
# 4. LONG-RUN AND SHORT-RUN COSTS



## 4. LONG-RUN AND SHORT-RUN COSTS

Given the production function (the same for all firms) and the price of the inputs  $w^0$ ,  $r^0$  (the same for all firms), we obtain, for each size of the firm, the corresponding cost function in the short-run, which minimizes the short-run cost of producing the output:

For a firm of size  $K^0$ :



For a firm of size  $K'$ :

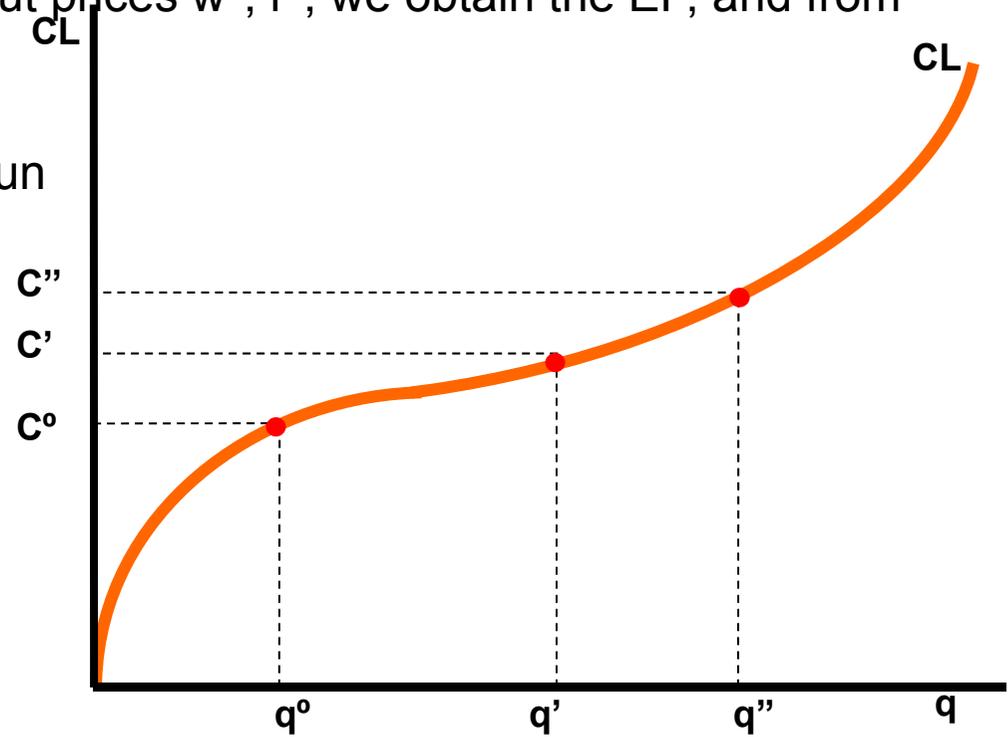
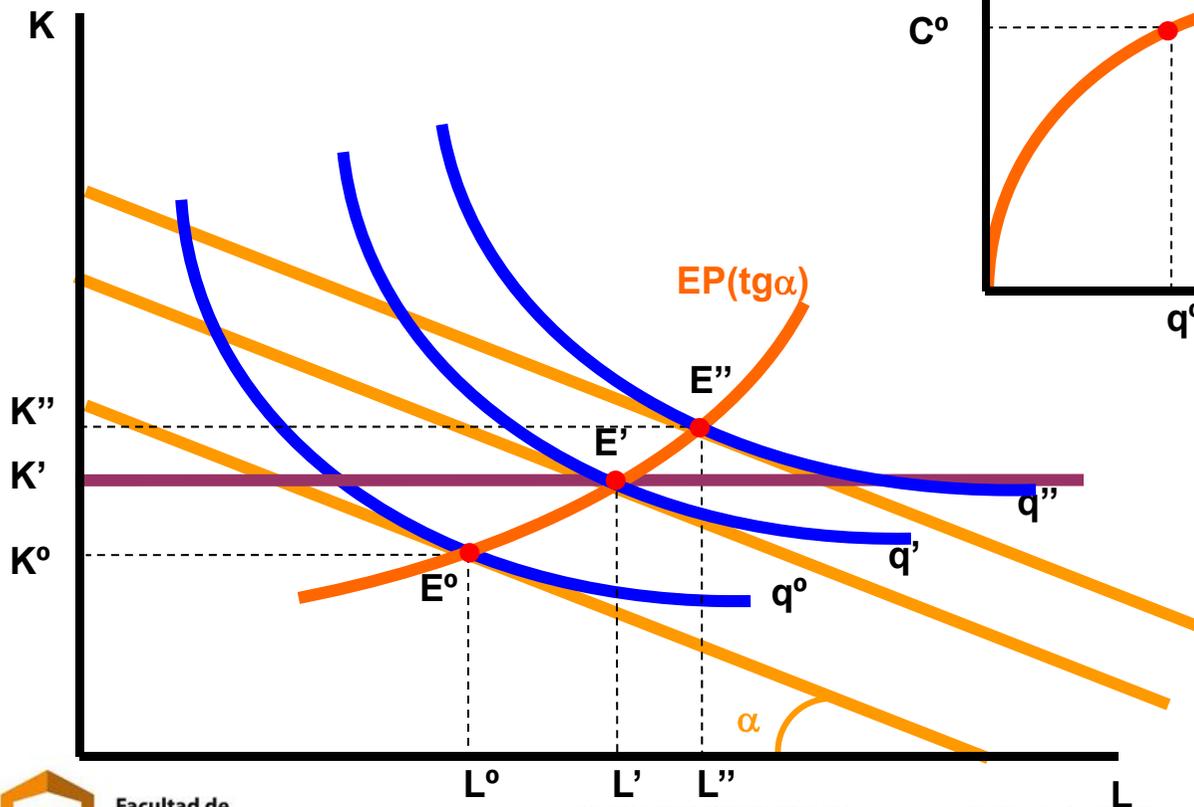
In the short-run, there is only one cost function for a given size of the firm, and there will be as many cost functions as possible firm sizes

## 4. LONG-RUN AND SHORT-RUN COSTS

Given the production function and the input prices  $w^0$ ,  $r^0$ , we obtain the EP, and from this...

we obtain the cost function in the long run

If we consider a firm of size  $K'$ :



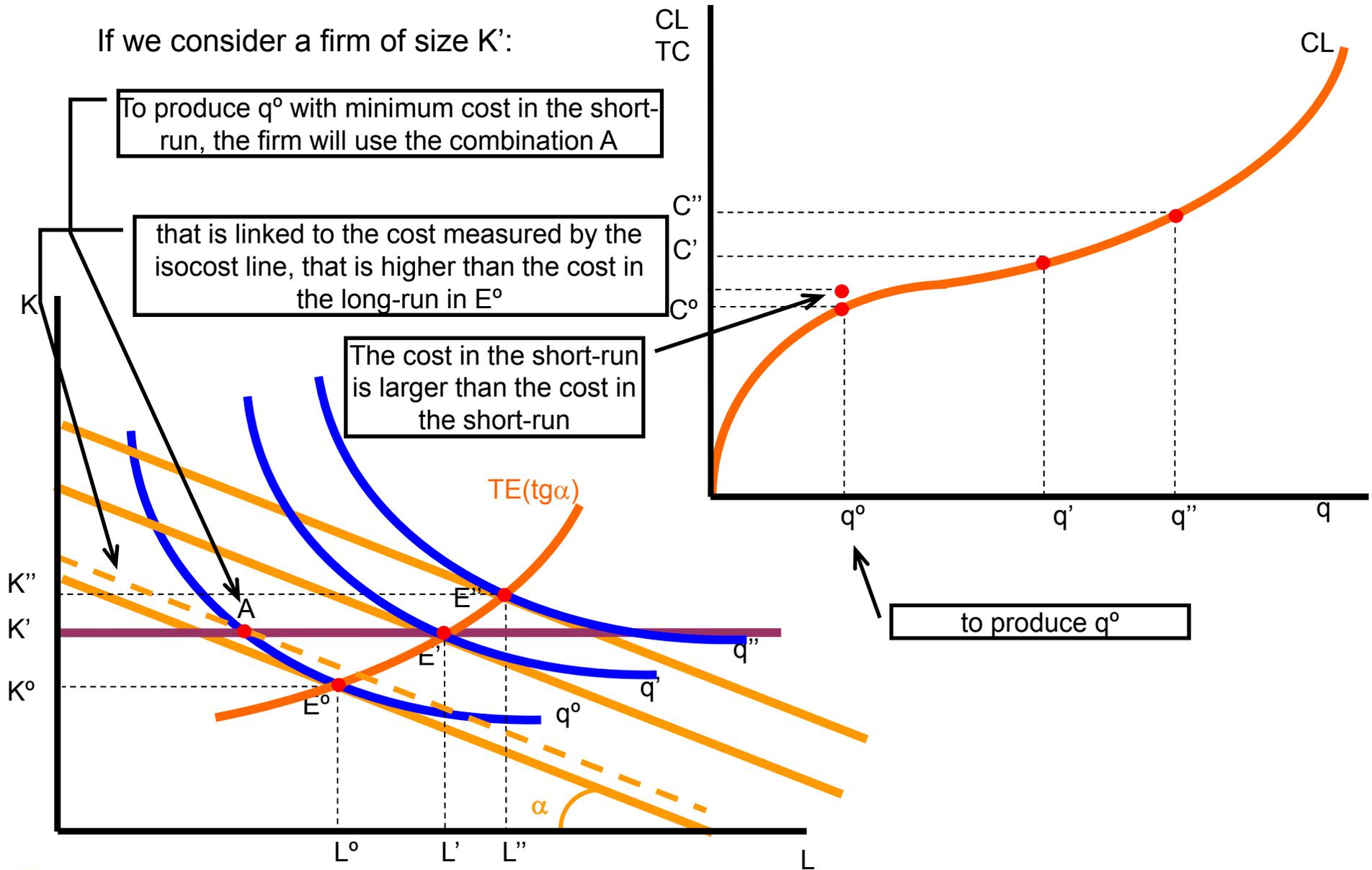
# 4. LONG-RUN AND SHORT-RUN COSTS

If we consider a firm of size  $K'$ :

To produce  $q^0$  with minimum cost in the short-run, the firm will use the combination A

that is linked to the cost measured by the isocost line, that is higher than the cost in the long-run in  $E^0$

The cost in the short-run is larger than the cost in the short-run



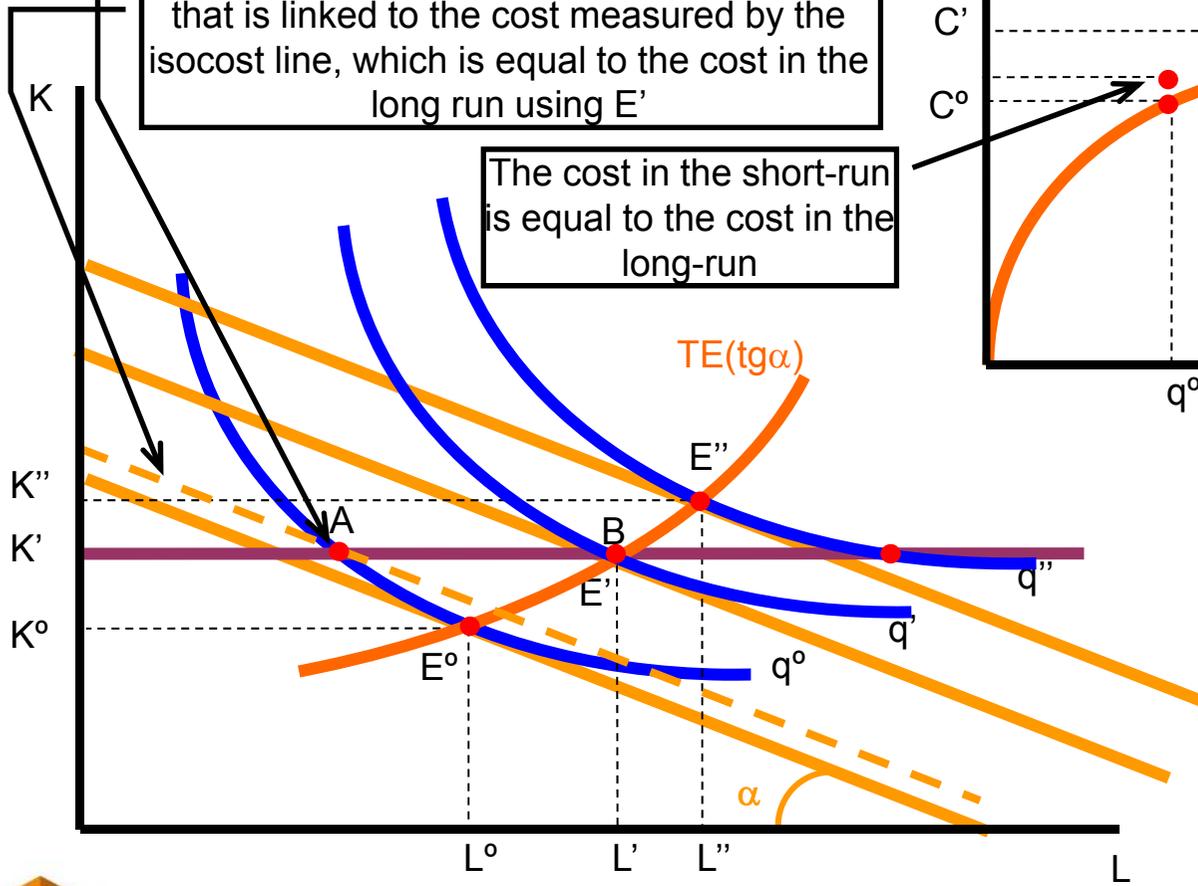
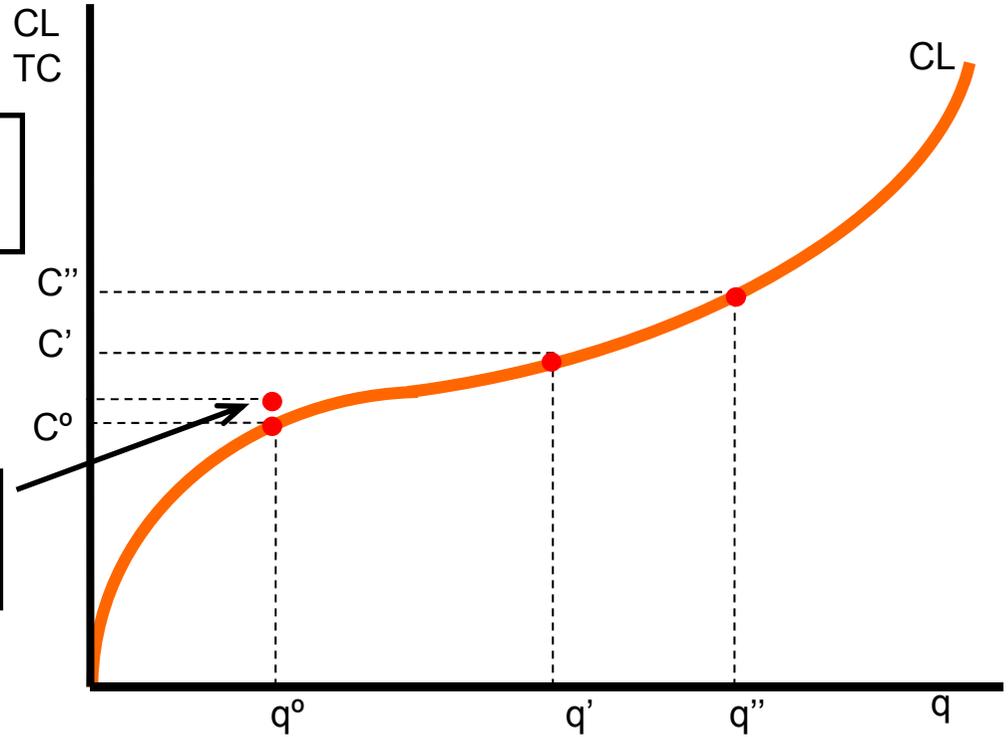
# 4. LONG-RUN AND SHORT-RUN COSTS

If we consider a firm of size  $K'$ :

To produce  $q'$  with the minimum cost in the short run, the firm will use the combination  $B=E'$

that is linked to the cost measured by the isocost line, which is equal to the cost in the long run using  $E'$

The cost in the short-run is equal to the cost in the long-run



To produce  $q'$

# 4. LONG-RUN AND SHORT-RUN COSTS

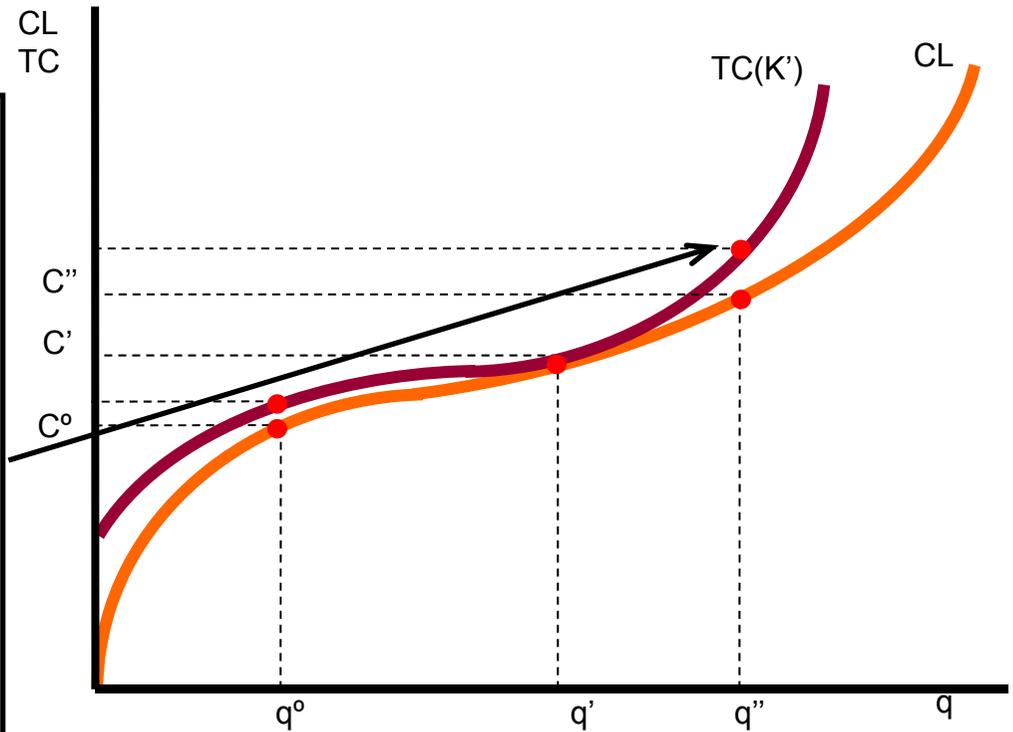
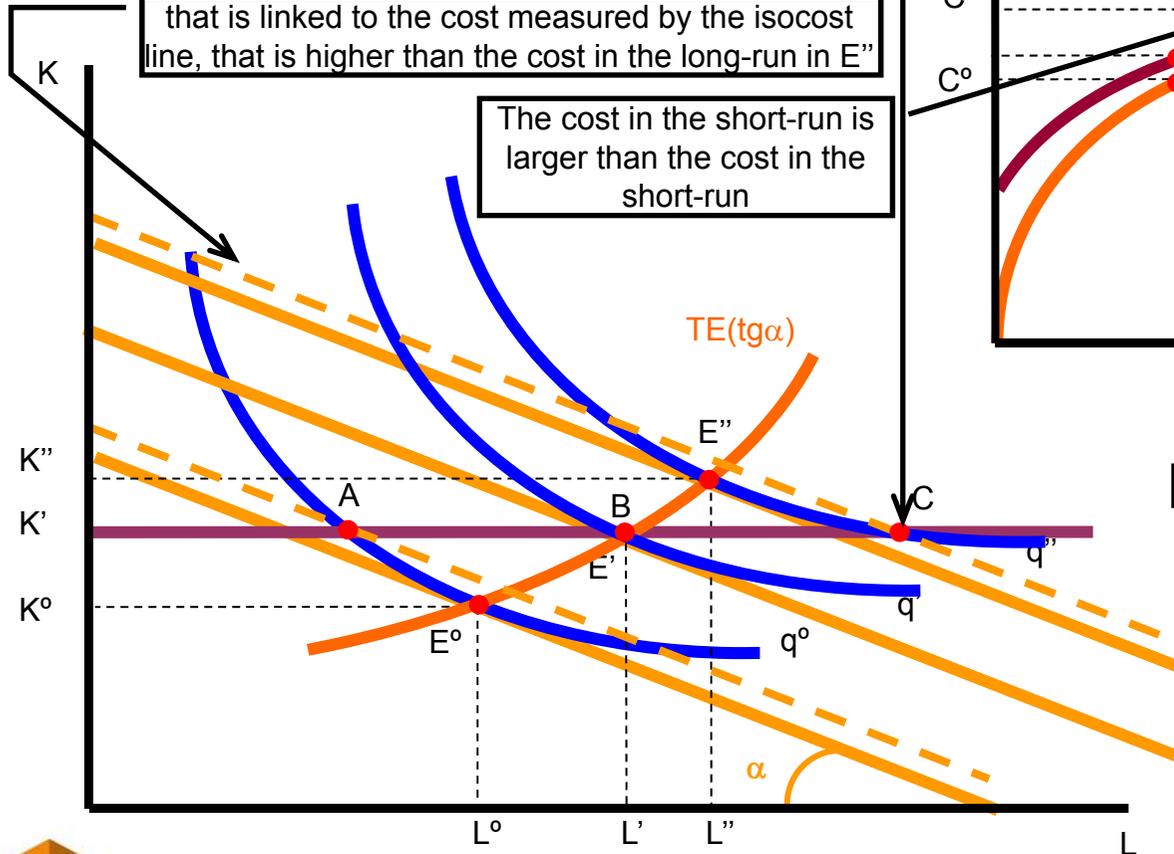
If we consider a firm of size  $K'$ :

To produce  $q''$  with the minimum cost in the short run, the firm will use the combination  $C$

The cost function in the short-run for a firm of size  $K'$  has been obtained, in relation to the cost function in the long-run

that is linked to the cost measured by the isocost line, that is higher than the cost in the long-run in  $E''$

The cost in the short-run is larger than the cost in the short-run

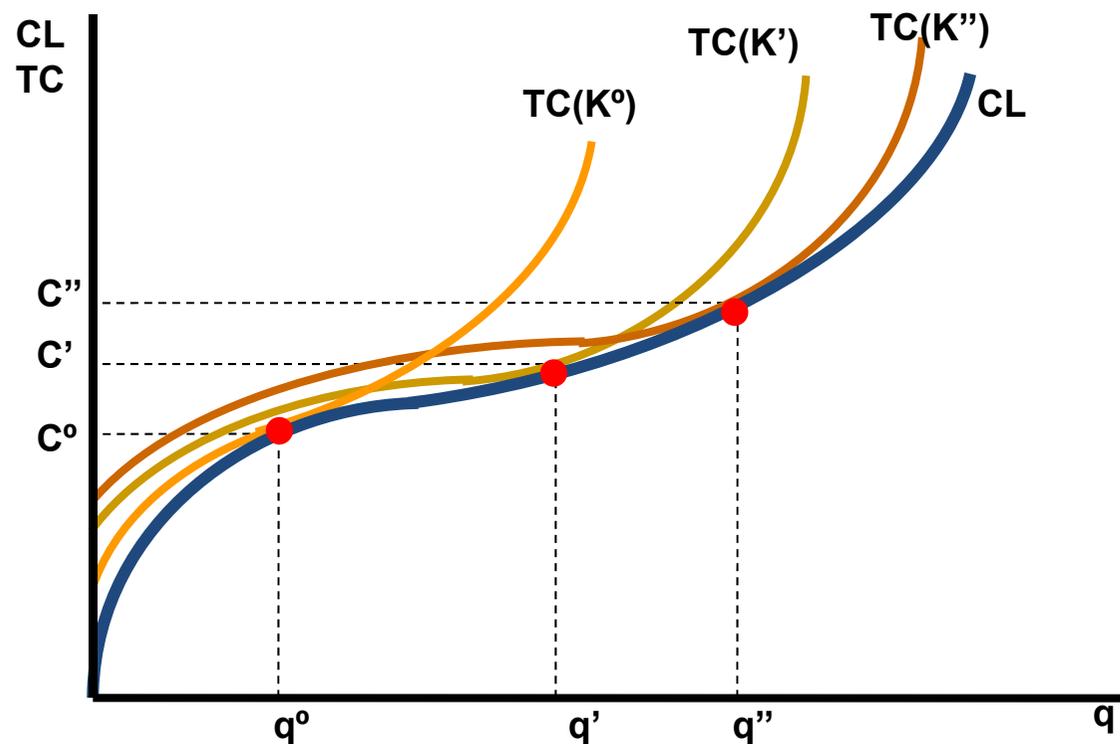


to produce  $q''$

Only for  $q'$  costs in the short and long-run are equal, for any other amount of input the cost in the short-run is larger than the cost in the short-run, as there is a size of the firm that better suits, compared to  $K'$

## 4. LONG-RUN AND SHORT-RUN COSTS

Every cost curve in the long-run  $TC(q)$  (with fixed  $K$ ) is always above the cost curve in the short-run  $CL(q)$ , and they equalize for only one level of output, which is the level of output where the firm would have chosen that  $K$  in the long-run:



The long-run cost curve is the envelope of all the short-run cost curves

## 5. MARGINAL REVENUE, MARGINAL COST AND PROFIT MAXIMIZATION

We have analyzed the problem of cost-minimization. If we solve it, for a given amount of output ( $q$ ), we can determine, given the technology and the price of the inputs:

- The combination of inputs that minimize the cost
- The minimum cost

The next step consists of, assuming **that the goal of the firm is to maximize its profit**, in order to answer the following questions:

- How much to produce?
- At what prices?

That is, what is the amount of production  $q^*$  that maximizes the profit?

The profit function (revenue minus costs):

$$P(q) = R(q) - C(q)$$

## 5. MARGINAL REVENUE, MARGINAL COST AND PROFIT MAXIMIZATION

The elements to solve the profit-maximization problem are:

- 1.The technology
- 2.The price of the inputs
- 3.The market demand
- 4.The type of competition [Micro II]

Independently of the type of competition, the firm always has a Revenue function, which links, to each amount of output  $q$ , the income generated by the sale of the output (sell price  $\times$  produced units):

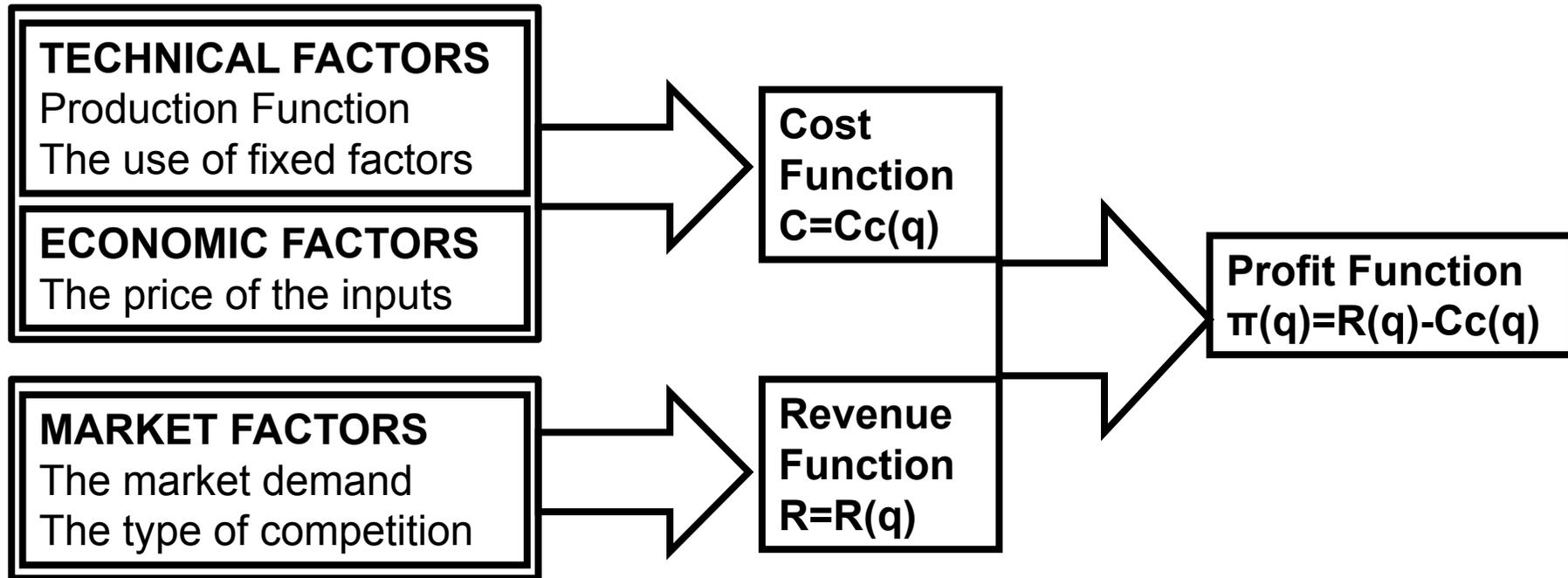
$$R = p \cdot q = R(q)$$

Goal: maximize the profit:

- Optimization problem without restrictions
- There is only one decision variable ( $q$ )

$$\max_q \Pi(q) = R(q) - C(q)$$

## 5. MARGINAL REVENUE, MARGINAL COST AND PROFIT MAXIMIZATION



## 5. MARGINAL REVENUE, MARGINAL COST AND PROFIT MAXIMIZATION

Analytically:  $\max_q \Pi(q) = R(q) - C(q)$

1. First Order Condition (FOC):

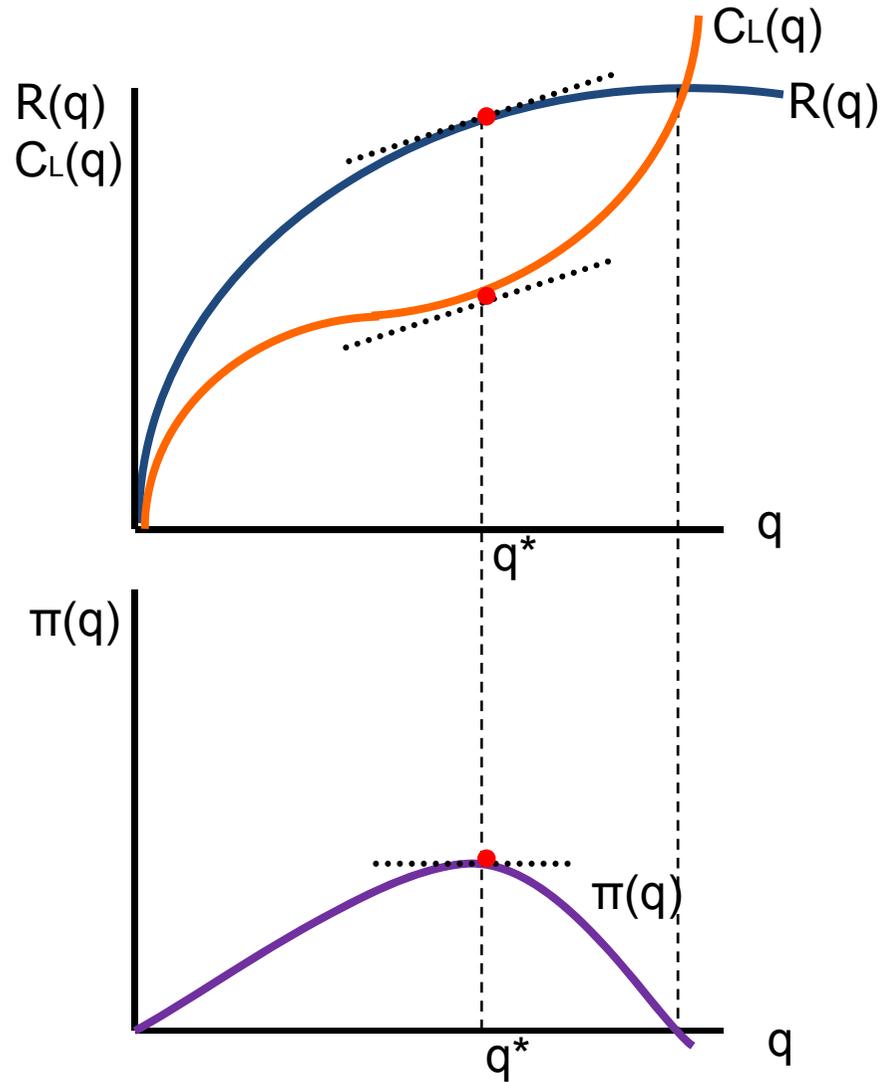
$$\frac{d\Pi(q)}{dq} = \frac{dR(q)}{dq} - \frac{dC(q)}{dq} = 0 \Leftrightarrow MR(q^*) = MC(q^*)$$

where MR and MC are the marginal revenue and marginal cost, respectively. Thus, in the optimum level of production ( $q^*$ ), where the profit is maximized, we observe that the last produced unit must add to the firm an additional cost that equals the marginal revenue obtained from that last unit

- The slopes of the marginal revenue and marginal cost are equal.
- If we consider a level of output where:

$$\begin{array}{l} MR(q) > MC(q) \Rightarrow \uparrow q \\ MR(q) < MC(q) \Rightarrow \downarrow q \end{array} \quad \begin{array}{l} \text{the firm must increase its level of production} \\ \text{the firm must decrease its level of production} \end{array}$$

# 5. MARGINAL REVENUE, MARGINAL COST AND PROFIT MAXIMIZATION



## 5. MARGINAL REVENUE, MARGINAL COST AND PROFIT MAXIMIZATION

### 2. Second Order Condition (SOC):

$$\frac{d^2\Pi(q)}{dq^2} < 0 \Rightarrow \frac{d^2R(q)}{dq^2} - \frac{d^2C(q)}{dq^2} < 0 \Leftrightarrow \frac{dMR(q^*)}{dq} - \frac{dMC(q^*)}{dq} < 0$$

- The profit function must be concave

### 3. Economic condition: the firm will produce only if the profit from producing is higher than or equal to the profit from no production (shutting down)

If  $\Pi(q^*) \geq \Pi(q = 0) \Rightarrow$  the firm must produce  $q^*$

If  $\Pi(q^*) < \Pi(q = 0) \Rightarrow$  the firm should shut down

## Exercises

1.- Consider the following production function:  $q = K^{1/2}L^{1/4}$  where K is capital and L is labor. The prices of the inputs are  $r=2$  and  $w=1$ , respectively:

- Define the concept of economic efficiency.
- Obtain the Conditional Factor Demands and the cost curve. What is the minimum cost in the long-run to produce 10 units of output? How many units of K and L must the firm use in order to be economically efficient?
- Represent graphically the cost curve in the long-run, and the Average Cost and Marginal Cost curves. Is there any relationship between the returns to scale of the technology and the form of these curves?
- Determine the elasticity function of the total cost.

2.- Consider the following production function:  $q = K^{2/3}L^{1/3}$  where K is capital and L is labor. The prices of the inputs are  $r=1$  and  $w=2$ , respectively:

- Represent graphically the cost curve in the long-run, and the Average Cost and Marginal Cost curves.
- Explain the relationship between the returns to scale and the cost curve in the long-run of the firm.



3.- The production function of a firm is given by  $q = K^{1/3}L^{1/3}$ . Currently the firm is producing a level of  $q=100$ , but due to a large increase in demand the firm wants to increase production by 300 units ( $q=400$ ). The firm has predicted that such an increase will lead to an increase of the long-run cost of 14,000 m.u. Is this prediction correct, knowing that the price of the inputs are  $w=r=1$ ? What is the combination of inputs that will allow the firm to obtain that increase in the output?

4.- Consider a firm that employs capital  $K$  and labor  $L$ . The price of the inputs is 1. Define the concept of Conditional Factor Demands. If the Conditional Factor Demands in the long-run of the firm are  $K = L = q^{3/2}$ , explain the type of returns to scale of the production function.

5.- The competitive firm CONSULTING&YOU is in the marketing business. The production cost in the long-run is given by the function  $C_L = q^2 + 10q$  (where  $q$  indicates the number of monthly reports). Due to a large increase in demand, the firm wants to double its production and has decided to double all its inputs. Is this correct?

6.- Let  $q = L^{1/2}K$  be the production function of a firm, where K is capital and L is labor. The prices of the inputs are  $r=2$  and  $w=2$ :

- a) Assume that in the short-run the input K is fixed at  $K=2$ . Obtain the cost curve in the short-run. Does the Law of Diminishing Returns applied?
- b) Obtain the Marginal Cost, Average Variable Cost, and Average Total Cost curves.
- c) What is the maximum level of output that in the short-run could be produced with 204 m.u.?
- d) Assume that the firm wants to produce 50 units of output. What is the minimum cost that allows the firm to produce that level in the short-run? How many units of K and L should the firm employ?

7.- Let  $q = L^{1/2}K$  be the production function of a firm. The price of K is  $r=4$  and of L is  $w=1$ .

- a) Assume that in the short-run the input K is fixed at  $K=2$ . Obtain the cost curve in the short-run, and the Marginal Cost, Average Variable Cost, and Average Total Cost curves.
- b) Obtain the cost curve in the long-run, and the Average Cost and Marginal Cost curves.
- c) Will the firm have a higher cost in the long-run or in the short-run if it wants to produce  $q=100$ ?
- d) What is the maximum level of output that in the long-run can be produced with 3,000 m.u.? What combination of factors should the firm employ?



**8.-** Obtain the Fixed Cost (FC) of a firm that in the Optimum Efficient Scale ( $q^{OES}$ ) produces  $q=5$ , knowing that the Marginal Cost is:

$$MC = 9q^2 - 30q + 50$$

**9.-** Given that the Marginal Cost function for a firm is  $MC = 3 + 8q + 15q^2$ , obtain the corresponding total cost curve if it is known that in  $q=4$ , the total cost of production is 896 m.u.

**10.-** A firm has a production technology represented by the function:

$$q = 10LK - 5L^2 - K^2$$

The prices of the inputs are:  $w=10$  and  $r=2$ . Determine the minimum cost the firm will have to produce 160,000 units of output.

**11.-** A firm has a production technology represented by the function  $q = 4L^{1/2}K^{1/2}$ . The prices of the inputs are  $w=5$  and  $r=20$ .

- a) Determine the minimum cost that allows the firm to produce 3,400 units of output. Represent it graphically.
- b) Obtain the Expansion Path. Represent it graphically.
- c) Obtain the Conditional Factor Demands functions.
- d) Obtain the cost functions. Represent them graphically.
- e) Answer the previous questions if now  $w=48$  and  $r=3$ .

**12.-** A firm has a production technology represented by the function  $q = L^{1/2}K^{1/2}$ . The prices of the inputs are  $w=32$  and  $r=2$ .

- a) Determine the minimum cost that allows the firm to produce 20 units of output.
- b) Obtain the maximum level of output that in the long-run can be produced with 6,400 m.u.
- c) Answer the previous questions if now  $w=8$  and  $r=2$ .



**13.-** Explain whether the following statements are true or false:

- a) A firm can be economically efficient and technically inefficient.
- b) A firm can be economically inefficient and technically efficient.
- c) If the prices of the inputs a firm employs are equal, cost-minimization will lead the firm to use equal amounts of inputs.
- d) The cost-minimization condition is characterized by the fact that the last monetary unit spent in each input increases the output by the same proportion.
- e) The Conditional Factor Demands functions indicate, given the price of the inputs, the units of inputs K and L that minimize the production cost of the firm, for each level of output.
- f) Any change in the price of one of the inputs will affect the production technology and the cost curve.
- g) For any level of output lower than the output associated with the Minimum Average Total Cost ( $q^{MATC}$ ), the Marginal Cost is higher than the Average Variable Cost.
- h) Since the Fixed Cost (FC) does not depend on the level of output, the average fixed cost is constant.
- i) With constant returns to scale, the Average Cost function is perfectly elastic.
- j) According to the “Law of Equality of the Weighted Marginal Productivity (LEWMP)” a firm minimizes its cost when the expenditure on each of the inputs is equal (the firm spends the same monetary units on each factor).
- k) The level of output associated with the Minimum Average Variable Cost ( $q^{MAVC}$ ) represents the output from where the Marginal Cost is higher than the Average Variable Cost.
- l) The area under the Marginal Cost function in the short-run measures the variable cost.
- m) The area under the Marginal Cost function in the long-run measures the total cost.



**14.-** Explain whether the following statements are true or false. With increasing returns to scale:

- a) If all the inputs, except one, increase by a given proportion, the output increases by a higher proportion.
- b) If the employment level of all the inputs is reduced by 50%, the output also decreases, but by a proportion less than 50%.
- c) The Law of Diminishing Returns can never be satisfied.

**15.-** A competitive firm produces a good “Q” employing two variable inputs,  $X_1$  and  $X_2$  whose prices are  $r_1=2$ ,  $r_2=6$ . As a consultant, the producer tells you: “currently, I am producing 100 units of output, employing 10 units of  $X_1$  and 20 of  $X_2$ . With the last employed unit of  $X_1$  the production increased by 4 units, while with the last employed unit of  $X_2$  the production increased by 8 units.” With this information, what would you recommend the producer do regarding the combination of employed inputs?

**16.-** Given a firm whose Marginal Product for inputs  $X_1$  and  $X_2$  are  $f_1=8$ ,  $f_2=4$  , and the prices of the inputs are  $r_1=4$ ,  $r_2=2$ . Explain the statement(s) you think is(are) true:

- a) It cannot produce more output at the same cost.
- b) It must increase  $X_2$  and decrease  $X_1$  in order to minimize costs.
- c) It must decrease  $X_2$  and increase  $X_1$  in order to minimize costs.
- d) It cannot produce the same level of output at a lower cost.

**17.-** A firm is able to produce 20 units of a good Q at minimum cost, employing the inputs K and L, whose prices are  $r=3$  and  $w=2$ , respectively. The firm argues that the increase in the production associated with the last monetary unit spent in input L is equal to 6:

- a) Calculate the Marginal Product of the last monetary unit spent in input L.
- b) Calculate the increase in the output associated with the last employed unit of L.

**18.-** A firm is producing 18,000 units of output per month, employing a combination of inputs that has a Marginal Rate of Technical Substitution (MRTS) lower than the ratio of the input prices. How is the firm able to reduce its costs?

**19.-** The combination of inputs that allows the firm to minimize the production cost is the one where:

- a) the isoquant curve and the isocost line are tangent.
- b) the value of the MRTS is equal to the ratio of prices.
- c) the last monetary unit spent in each input increases the output equally.
- d) the last unit employed of each input increases the output equally.

**20.-** Explain the “Law of Equality of the Weighted Marginal Productivity (LEWMP)”

**21.-** If the prices of the inputs employed by a firm are equal, cost-minimization will lead the firm to:

- a) employ the same units of all inputs.
- b) increase the level of input L so that the last monetary unit spent on it produces a greater increase in the level of output.
- c) employ a combination of inputs that makes equal the Marginal Product.
- d) increase the level of the input with the lowest Marginal Product.
- e) increase the level of the input with the highest Marginal Product.

**22.-** Explain which of the previous statements are true when the prices of the inputs are not equal.

**23.-** Consider a firm that has a production function  $q=f(L,K)$  presenting constant returns to scale. It is known that to produce 10 units of output the firm employs 6 units of K and 4 of L.

- a) If the prices of the inputs are  $r=2$  and  $w=2$ , respectively, how many units of K and L will the firm employ when its total cost is 100 m.u., and how much will it produce?
- b) Assume that the technology presents decreasing returns to scale. Will the firm be able to produce 15 units of output with a cost of 30 m.u.? And with a cost of 40 m.u.?

**24.-** a) Explain the conditions that determine the level of production that maximizes the revenue of a firm in the short-run and the long-run, for any type of market structure.

b) Consider a firm that in the short-run is producing a level of output so that  $MR > MC$ , and obtains a profit of 30,000 monetary units. Is the firm maximizing its profit? Why? Would you recommend the firm increase or decrease its level of production?

**25.-** Assume that you must advise a firm about the volume of output that maximizes the firm's profit in the short-run. The firm gives you the following information:

Level of Output..... 100 units  
Marginal Revenue..... 10 Euros  
Marginal Cost..... 10 Euros  
Total Revenue..... 1000 Euros  
Total Cost..... 1200 Euros  
Fixed Cost..... 300 Euros

What should you advise the firm?

- 1) Increase the level of output.
- 2) Decrease the level of output (to a level  $q > 0$ ).
- 3) Stop producing.
- 4) Do not change the level of output.

**26.-** Assume the production function  $q=f(L,K)$  is known, and also the prices of the inputs:  $w^0$  and  $r^0$ . Show graphically how the Expansion Path and the cost function in the long-run are affected if:

- a)  $W$  increases.
- b)  $R$  increases.

**27.-** Assume the production function  $q=f(L,K)$  is known, and also the prices of the inputs:  $w^0$  and  $r^0$ . Determine how the Conditional Factor Demand of the variable input in the short-run is affected, and show graphically how the cost functions in the short-run are affected for a firm of size  $K=K^0$ :

- a)  $W$  increases.
- b)  $R$  increases.

**28.-** Assume the production function  $q=f(L,K)$  is known, and also the prices of the inputs:  $w^0$  and  $r^0$ . Determine and show graphically how the Expansion Path, the Conditional Factor Demand, and the cost functions in the long-run are affected by an increase in the price of the inputs in the same proportion.