

RECONSTRUCTION OF GRADED GROUPOIDS FROM GRADED STEINBERG ALGEBRAS

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ABSTRACT. We show how to reconstruct a graded ample Hausdorff groupoid with topologically principal neutrally-graded component from the ring structure of its graded Steinberg algebra over any commutative integral domain with 1, together with the embedding of the canonical abelian subring of functions supported on the unit space. We deduce that diagonal-preserving ring isomorphism of Leavitt path algebras implies C^* -isomorphism of C^* -algebras for graphs E and F in which every cycle has an exit.

1. INTRODUCTION

Since the introduction by Abrams–Aranda-Pino [1] and, independently, by Ara–Moreno–Pardo [5] of Leavitt path algebras as algebraic analogues of graph C^* -algebras, there has been a great deal of interest in the many parallels between the two theories. Both graph C^* -algebras and Leavitt path algebras are defined as universal objects for sets of generators and relations that encode the structure of the underlying graph; and many structural results for graph algebras have direct analogues for Leavitt path algebras, and vice versa. But it is not immediately clear why: the proofs tend to require very different techniques, and there is no obvious mechanism, using the generators-and-relations formalism, for deducing a given result about graph C^* -algebras from the corresponding result about Leavitt path algebras or vice-versa. In particular (see question 6 on Tomforde’s Graph Algebra Problem Page [29]) a key unresolved conjecture, due to Abrams and Tomforde, is that if E and F are graphs whose complex Leavitt path algebras are isomorphic as rings, then they have isomorphic C^* -algebras. In this paper, we make some progress on this question by studying diagonal-preserving isomorphisms of Steinberg algebras. Specifically, we confirm a slight weakening of Abrams and Tomforde’s conjecture: if E and F are graphs in which every cycle has an entrance and there is a ring-isomorphism $L_{\mathbb{C}}(E) \cong L_{\mathbb{C}}(F)$ that respects the canonical diagonal, then there is a diagonal-preserving isomorphism $C^*(E) \cong C^*(F)$.

The algebras that we now call Steinberg algebras were first introduced by Steinberg in [26] as a groupoid-based approach to the study of inverse-semigroup algebras. Working independently, Clark–Farthing–Sims–Tomforde [8], developed the same class of algebras to provide a groupoid-based approach to the study of Leavitt path algebras. Groupoids generalise groups by allowing for a partially-defined multiplication. The groupoid C^* -algebra of a groupoid \mathcal{G} is a norm completion of the algebra of continuous, compactly supported functions from \mathcal{G} to \mathbb{C} under a natural convolution product. When the groupoid \mathcal{G} is *ample*, meaning that it is totally disconnected as a topological space, the Steinberg

Date: January 11, 2016.

The first and second-named authors were partially supported by the grants DGI MICIIN MTM2011-28992-C02-01 and MINECO MTM2014-53644-P. The second author is supported by the Beatriu de Pinós fellowship (2014 BP-A 00123). This research was supported by the Australian Research Council grant DP150101598.

algebra $A_R(\mathcal{G})$ of \mathcal{G} over a ring R is the algebra of locally constant functions from \mathcal{G} to R under the multiplication product; in particular, $A_{\mathbb{C}}(\mathcal{G})$ is, more or less by definition, a dense subalgebra of $C^*(\mathcal{G})$. Each directed graph determines a groupoid, whose groupoid C^* -algebra, in the sense of Renault [24], coincides with the graph C^* -algebra; in fact this is how graph C^* -algebras were originally constructed [17]. The reasoning of Remark 4.4 of [8] shows that the Steinberg algebra of this same groupoid, over any ring R , coincides with the Leavitt path algebra of E over the same ring (see [9, Example 3.2]).

Because the unit space $\mathcal{G}^{(0)}$ of an ample groupoid is a clopen set, the algebra $C_0(\mathcal{G}^{(0)})$ of continuous complex-valued functions vanishing at infinity on $\mathcal{G}^{(0)}$ is a commutative C^* -subalgebra of $C^*(\mathcal{G})$, and the algebra D of locally constant functions from $\mathcal{G}^{(0)}$ to R is a commutative subalgebra of $A_R(\mathcal{G})$. In [25], building on previous work of Kumjian [14], Renault studied the pair $(C^*(\mathcal{G}), C_0(\mathcal{G}^{(0)}))$ under the hypothesis that \mathcal{G} is topologically principal (for graph groupoids, this corresponds to every cycle in the underlying graph having an exit [16, Lemma 3.4]). One consequence of his powerful theory of Cartan subalgebras of C^* -algebras is that the groupoid \mathcal{G} can be recovered from the pair $(C^*(\mathcal{G}), C_0(\mathcal{G}^{(0)}))$. It follows, in particular, that if \mathcal{G} and \mathcal{H} are ample groupoids, then there is an isomorphism $C^*(\mathcal{G}) \cong C^*(\mathcal{H})$ that carries $C_0(\mathcal{G}^{(0)})$ to $C_0(\mathcal{H}^{(0)})$ if and only if $\mathcal{G} \cong \mathcal{H}$. These results have been used recently to prove remarkable results about continuous orbit equivalence rigidity for symbolic dynamical systems [7, 20, 21].

In this paper, we establish an analogue of Renault's reconstruction theorem for Steinberg algebras over any commutative integral domain with 1. We also extend Renault's ideas to deal with graded algebras, and groupoids \mathcal{G} admitting a discrete-group-valued cocycle c for which only the kernel $c^{-1}(e)$ is assumed to be topologically principal. Specifically, we prove that if \mathcal{G} is ample and Hausdorff, $c : \mathcal{G} \rightarrow \Gamma$ is a 1-cocycle taking values in a discrete group, and R is a commutative integral domain with 1, then D is a maximal commutative subring of the ring $A_R(\mathcal{G})$, and we can recover \mathcal{G} from the pair $(A_R(\mathcal{G}), D)$ regarded as a Γ -graded ring with distinguished commutative subring. As a direct consequence, we deduce the following. Suppose that E and F are directed graphs in which every cycle has an exit, and suppose that there is a commutative integral domain R with 1 for which there is a ring isomorphism $\pi : L_R(E) \cong L_R(F)$ that such $\pi(s_\mu s_{\mu^*}) s_\eta s_{\eta^*} = s_\eta s_{\eta^*} \pi(s_\mu s_{\mu^*})$ for every path μ in E and every path η in F . Then there is a diagonal-preserving isomorphism $C^*(E) \cong C^*(F)$. We also make a little progress on a vexing question appearing as part of question 1 on Tomforde's problem page [29]: are the Leavitt path algebras $L_{2,K}$ and $L_{2-,K}$ isomorphic for a field K ? This question has been settled in the situation where the field K is replaced by the ring \mathbb{Z} and the isomorphism is required to be a $*$ -isomorphism in [13]. However, the original question remains open. Our results are strong enough to show that there is no diagonal-preserving ring-isomorphism between these two algebras.

A similar result about Leavitt path algebras, of which we became aware late in the preparation of this work, was obtained recently by Brown–Clark–an Huef [6]. Neither our result nor theirs is a direct generalisation of the other, though: their theorem requires a $*$ -ring isomorphism and that E be row-finite with no sources, whereas ours requires only a ring isomorphism and does not insist that E should be row-finite or have no sources; but our result requires that every cycle in E have an exit whereas theirs does not. But our result has many further applications; for example, to Kumjian–Pask algebras of higher-rank graphs, to algebras associated to Cantor minimal systems, to the algebras $L^{ab}(E, C)$

associated to separated graphs by Ara and Exel in [4], and to all groupoids arising from partial actions of countable discrete groups on totally disconnected metrisable spaces (many examples of this situation are described in [11]). Indeed, by Steinberg's result [26, Theorem 6.3], relating inverse-semigroup algebras to Steinberg algebras, our result applies to the algebras of all inverse semigroups whose associated universal groupoids are both Hausdorff and topologically principal; and [26, Theorem 5.17] implies that the universal groupoid is Hausdorff if and only if the inverse semigroup is a weak semilattice. We do not explore all of these potential applications in this paper.

The paper is organised as follows. We recall some basic facts about groupoids, inverse semigroups, and Steinberg algebras in Section 2. We then prove our main result in Section 3; our approach decomposes into five steps, and we devote a subsection to each one. The first step is to define and analyse the *normalisers* of the subring D in $A_R(\mathcal{G})$. The second step is to show that these normalisers form a natural inverse semigroup S . The third step is to show that we can recover the inverse semigroup of compact open bisections of the groupoid \mathcal{G} as a natural quotient of S . The fourth step is to show that the ring structure of $A_R(\mathcal{G})$ determines an action of this quotient on the Stone spectrum of D which is isomorphic to the canonical action of the inverse semigroup of compact open bisections on the unit space of \mathcal{G} . The final step is to prove that D is a maximal abelian subring so that any isomorphism $A_R(\mathcal{G}) \cong A_R(\mathcal{H})$ that carries $D_{\mathcal{G}}$ into the commutator of the corresponding subring $D_{\mathcal{H}}$ must in fact restrict to an isomorphism $D_{\mathcal{G}} \cong D_{\mathcal{H}}$. Combining all of these elements yields our main result. We go on to explore some consequences of our main result in Section 4. We first describe explicitly what our results say for topologically principal groupoids \mathcal{G} . We then detail our conclusions about diagonal-preserving ring isomorphisms of Leavitt path algebras, and apply them to obtain some new results on the well-known open problem of whether the algebras $L_{2,R}$ and $L_{2-,R}$ are isomorphic for a commutative ring R with 1. Finally, we detail what our results say about graded isomorphisms of Kumjian–Pask algebras.

2. PRELIMINARIES

2.1. Groupoids and Inverse Semigroups. We give a very brief introduction to Hausdorff ample groupoids. For more detail, see [10, 22].

A groupoid \mathcal{G} is a small category with inverses. We denote the set of identity morphisms of \mathcal{G} by $\mathcal{G}^{(0)}$, and call it the *unit space* of \mathcal{G} . So $\mathcal{G}^{(0)} = \{\gamma\gamma^{-1} : \gamma \in \mathcal{G}\}$. For $\gamma \in \mathcal{G}$, we write $r(\gamma) := \gamma\gamma^{-1}$ and $s(\gamma) := \gamma^{-1}\gamma$. So $r, s : \mathcal{G} \rightarrow \mathcal{G}^{(0)}$ satisfy $r(\gamma)\gamma = \gamma = \gamma s(\gamma)$ for all $\gamma \in \mathcal{G}$. A pair $(\alpha, \beta) \in \mathcal{G} \times \mathcal{G}$ is then *composable* if and only if $s(\alpha) = r(\beta)$. We write $\mathcal{G}^{(2)}$ for the set of all composable pairs.

For $U, V \subseteq \mathcal{G}$, we write

$$(2.1) \quad UV = \{\alpha\beta \mid \alpha \in U, \beta \in V \text{ and } r(\beta) = s(\alpha)\}.$$

Given units $u, v \in \mathcal{G}^{(0)}$, we write, as usual (e.g. [24]), \mathcal{G}_u for $s^{-1}(u)$ and \mathcal{G}^v for $r^{-1}(v)$. We then write \mathcal{G}_u^v for $\mathcal{G}^v \cap \mathcal{G}_u$. The *isotropy group* at the unit $u \in \mathcal{G}^{(0)}$ is then the group \mathcal{G}_u^u . We say u has trivial isotropy if $\mathcal{G}_u^u = \{u\}$. The *isotropy subgroupoid* of \mathcal{G} is $\text{Iso}(\mathcal{G}) := \bigcup_{u \in \mathcal{G}^{(0)}} \mathcal{G}_u^u$.

We say that \mathcal{G} is a topological groupoid if it is endowed with a topology under which the unit space is Hausdorff in the relative topology, the range, source and inverse maps are continuous, and the composition map is continuous with respect to the subspace topology on $\mathcal{G}^{(2)} \subseteq \mathcal{G} \times \mathcal{G}$. This implies, in particular, that both $\mathcal{G}^{(0)}$ and $\text{Iso}(\mathcal{G})$ are closed in \mathcal{G} . A

bisection of \mathcal{G} is a subset $U \subseteq \mathcal{G}$ such that $r|_U$ and $s|_U$ are homeomorphisms. We say that \mathcal{G} is *ample* if r and s are local homeomorphisms and $\mathcal{G}^{(0)}$ has a basis of compact open sets. If \mathcal{G} is ample, then $\mathcal{G}^{(0)}$ is clopen in \mathcal{G} and \mathcal{G} admits a basis of compact open bisections. The composition map in an ample Hausdorff groupoid is an open map. In this note, we will uniquely deal with Hausdorff groupoids, so that compact open sets are clopen.

We will work frequently with *topologically principal* groupoids in which the set of units with trivial isotropy is dense in $\mathcal{G}^{(0)}$.

Recall (see e.g. [10, 18, 22, 26] for more details) that an *inverse semigroup* is a semigroup S such that for each $s \in S$ there exists a unique element $s^* \in S$ satisfying $ss^*s = s$ and $s^*ss^* = s^*$. We denote by $E(S)$ the set of idempotents of S , which is automatically a commutative semigroup. There is a natural order on $E(S)$ given by $e \leq f$ if and only if $ef = e$, and this order extends to a partial order on S given by $s \leq t$ if $s = et$ for some idempotent e (in which case we may always take $e = ss^*$). Given an ample Hausdorff groupoid \mathcal{G} , the collection $S_{\mathcal{G}}$ of compact open bisections of \mathcal{G} forms an inverse semigroup under the multiplication given by (2.1), with $U^* = U^{-1} = \{\gamma^{-1} : \gamma \in U\}$. We then have $E(S_{\mathcal{G}}) := \{U \in S_{\mathcal{G}} : U \subseteq \mathcal{G}^{(0)}\}$, and the product in $E(S_{\mathcal{G}})$ agrees with the intersection operation on subsets of $\mathcal{G}^{(0)}$.

2.2. Graded Steinberg algebras. Let \mathcal{G} be a Hausdorff ample groupoid and let R be a commutative ring with 1. We write $A_R(\mathcal{G})$ for the space of all locally constant functions $f : \mathcal{G} \rightarrow R$ with compact support. This becomes an R -algebra under the convolution product

$$f * g(\gamma) = \sum_{r(\alpha)=r(\gamma)} f(\alpha)g(\alpha^{-1}\gamma) = \sum_{\alpha\beta=\gamma} f(\alpha)g(\beta),$$

and pointwise addition and R -action. For any involution on R (possibly the trivial one) the algebra $A_R(\mathcal{G})$ becomes a $*$ -algebra with $f^*(\gamma) = f(\gamma^{-1})^*$.

If \mathcal{U} denotes the collection of all compact open bisections of \mathcal{G} , then $A_R(\mathcal{G}) = \text{span}_R\{1_U \mid U \in \mathcal{U}\}$. Specifically, given $f \in A_R(\mathcal{G})$, the sets $f^{-1}(r)$, indexed by $r \in R$, are compact open sets, so each admit a finite cover by elements of \mathcal{U} . Since whether $U, V \in \mathcal{U}$ implies that $U \setminus V \in \mathcal{U}$, we can find, for each r such that $f^{-1}(r) \neq \emptyset$, a finite set $F_r \subseteq \mathcal{U}$ of mutually disjoint compact open bisections with $f^{-1}(r) = \bigsqcup_{U \in F_r} U$. We then have $f = \sum_{f^{-1}(r) \neq \emptyset} \sum_{U \in F_r} r \cdot 1_U$. That is, every element of $A_R(\mathcal{G})$ can be written as an R -linear combination of finitely many mutually disjoint compact open bisections. We will use this fact frequently, and, in this context, it will be useful to recall from [26, Proposition 4.5] that for $U, V \in \mathcal{U}$,

$$1_U * 1_V = 1_{UV} \quad \text{and} \quad 1_U^* = 1_{U^{-1}}.$$

Let Γ be a discrete group, and c a continuous homomorphism from \mathcal{G} to Γ (that is, $c : \mathcal{G} \rightarrow \Gamma$ is a continuous groupoid cocycle). By [9, Lemma 3.1] there is a Γ -grading of $A_R(\mathcal{G})$ such that

$$A_R(\mathcal{G})_g = \{f \in A_R(\mathcal{G}) : \text{supp}(f) \subseteq c^{-1}(g)\} \quad \text{for all } g \in \Gamma.$$

We say that a bisection U is *homogeneous* if $c(U)$ is a singleton. Each homogeneous piece $A_R(\mathcal{G})_g$ of the Steinberg algebra is then precisely the R -linear span of indicator functions of homogeneous bisections contained in $c^{-1}(g)$. As above, each element of $A_R(\mathcal{G})_g$ can be written as an R -linear combination of indicator functions of finitely many mutually disjoint such homogeneous bisections.

Since we can regard $\mathcal{G}^{(0)}$ as a subgroupoid of \mathcal{G} it makes sense to talk about the Steinberg algebra $A_R(\mathcal{G}^{(0)})$, which is just the commutative algebra of locally constant compactly supported functions from $\mathcal{G}^{(0)}$ to R under pointwise operations. Since $\mathcal{G}^{(0)}$ is clopen, there is an embedding $\iota : A_R(\mathcal{G}^{(0)}) \rightarrow A_R(\mathcal{G})$ such that $\iota(f)|_{\mathcal{G}^{(0)}} = f$ and $\iota(f)|_{\mathcal{G} \setminus \mathcal{G}^{(0)}} = 0$. We use this embedding to regard $A_R(\mathcal{G}^{(0)})$ as a commutative subalgebra of $A_R(\mathcal{G})$. This $A_R(\mathcal{G}^{(0)})$ contains local units for $A_R(\mathcal{G})$. Indeed, given $f \in A_R(\mathcal{G})$, the set $K := s(\text{supp}(f)) \cup r(\text{supp}(f)) \subseteq \mathcal{G}^{(0)}$ is compact and open, and $1_K * f = f = f * 1_K$.

To keep our notation uncluttered, we shall write $D_{\mathcal{G}}$, or just D when the groupoid is clear, for $A_R(\mathcal{G}^{(0)}) \subseteq A_R(\mathcal{G})$ throughout this note.

3. RECONSTRUCTING THE GROUPOID

In this section we consider commutative integral domain R with 1, and graded groupoids \mathcal{G} endowed with a continuous cocycle c whose kernel is topologically principal. We show how to reconstruct (\mathcal{G}, c) from the pair $(A_R(\mathcal{G}), D)$, regarded as a graded ring with distinguished abelian subring. Our goal is the following result, which we prove at the end of the section. Throughout Γ is a group and e is its neutral element.

Theorem 3.1. *Let \mathcal{G} and \mathcal{H} be ample Hausdorff groupoids, R be a commutative integral domain with 1, $c : \mathcal{G} \rightarrow \Gamma$ and $d : \mathcal{H} \rightarrow \Gamma$ be gradings by a discrete group, and suppose that $c^{-1}(e)$ and $d^{-1}(e)$ are topologically principal. Let $D_{\mathcal{G}} \subseteq A_R(\mathcal{G})$ and $D_{\mathcal{H}} \subseteq A_R(\mathcal{H})$ be the abelian subalgebras consisting of functions supported on $\mathcal{G}^{(0)}$ and $\mathcal{H}^{(0)}$. Then, there is a graded isomorphism $\rho : A_R(\mathcal{G}) \rightarrow A_R(\mathcal{H})$ satisfying $\rho(D_{\mathcal{G}}) \subseteq D_{\mathcal{H}}$ if and only if there is an isomorphism $\bar{\rho} : \mathcal{G} \rightarrow \mathcal{H}$ such that $d \circ \bar{\rho} = c$.*

Remark 3.2. In theorem 3.1, we are regarding $A_R(\mathcal{G})$ as a ring. In particular, even if $R = \mathbb{C}$, so that $A_R(\mathcal{G})$ and $A_R(\mathcal{H})$ have natural $*$ -algebra structures, the existence of a diagonal-preserving ring isomorphism $A_R(\mathcal{G}) \rightarrow A_R(\mathcal{H})$ implies isomorphism of the groupoids \mathcal{G} and \mathcal{H} . *En passant* we observe that it implies that $A_R(\mathcal{G})$ and $A_R(\mathcal{H})$ are in fact isomorphic as $*$ -algebras.

3.1. The normaliser of $D_{\mathcal{G}}$. The first step in proving Theorem 3.1 is to define and study what we call the normalisers of $D_{\mathcal{G}}$. As said, we write D instead of $D_{\mathcal{G}}$ if there is no confusion for the subalgebra $A_R(\mathcal{G}^{(0)})$ of $A_R(\mathcal{G})$. This is based on Kumjian's work [14] on C^* -diagonals, and Renault's later work on Cartan subalgebras of C^* -algebras [25]. It is also related to Brown, Clark and an Huef's treatments of Leavitt path algebras [6] which, in turn, is based on Brownlowe, Carlsen and Whittaker's work on graph C^* -algebras [7]; but we must make adjustments for the lack of a $*$ -algebra structure here and to exploit the presence of a grading.

Let \mathcal{G} be an ample Hausdorff groupoid, and let $c : \mathcal{G} \rightarrow \Gamma$ be a grading of \mathcal{G} by a discrete group. Note that we have $D \subseteq A_R(\mathcal{G})_e$, the trivially-graded homogeneous subalgebra of $A_R(\mathcal{G})$ since $c(u) = c(u^2) = c(u)^2$ for each $u \in \mathcal{G}^{(0)}$.

We shall define what we call the graded normalisers of D , and construct from them an inverse semigroup. Later we will establish that an appropriate quotient of this inverse semigroup acts on the Stone spectrum of D by partial homeomorphisms, and prove that \mathcal{G} is isomorphic to the groupoid of germs for this action.

Definition 3.3. Let \mathcal{G} be an ample Hausdorff groupoid, and let $c : \mathcal{G} \rightarrow \Gamma$ be a grading of \mathcal{G} by a discrete group. A *normaliser* of D is a pair $(m, n) \in A_R(\mathcal{G}) \times A_R(\mathcal{G})$ satisfying the following two conditions:

- (N1) $mDn \cup nDm \subseteq D$; and
(N2) $mn = m$ and $nm = n$.

A normaliser (m, n) of D is *homogeneous* if m and n are both homogeneous elements of $A_R(\mathcal{G})$. We then have $m \in A_R(\mathcal{G})_g$ for some $g \in \Gamma$, and we say that (m, n) is *homogeneous of degree $g \in \Gamma$* . From (N2) it follows that $n \in A_R(\mathcal{G})_{g^{-1}}$.

If (m, n) is a normaliser, then (N2) implies that mn and nm are idempotents. We write $N(D)$ for the set of normalisers of D , and for $g \in \Gamma$ we write $N_g(D)$ for the set of normalisers that are homogeneous of degree g . We write $N_\star(D) := \bigcup_{g \in \Gamma} N_g(D)$ for the collection of all homogeneous normalisers. It is clear that if $(m, n) \in N_g(D)$ then $(n, m) \in N_{g^{-1}}(D)$.

Our first, and key, proposition characterises the homogeneous normalisers of D when $c^{-1}(e)$ is topologically principal. Throughout this paper, we write R^\times for the group of units of a ring R .

Proposition 3.4. *Let \mathcal{G} be an ample Hausdorff groupoid, and let $c : \mathcal{G} \rightarrow \Gamma$ be a grading of \mathcal{G} by a discrete group. Let R be a commutative ring with 1.*

- (1) *Let $U \subseteq c^{-1}(g)$ be a compact open bisection, and consider a decomposition $U = \bigsqcup_{V \in F} V$ of U into finitely many mutually disjoint compact open subsets indexed by F . Take units $\{a_V, V \in F\} \subseteq R^\times$, and let*

$$(m, n) := \left(\sum_{V \in F} a_V 1_V, \sum_{V \in F} a_V^{-1} 1_{V^{-1}} \right).$$

Then $(m, n) \in N(D)$.

- (2) *Suppose that R is an integral domain and $c^{-1}(e)$ is topologically principal. If $(m, n) \in N_g(D)$, then there exist a compact open bisection $U \subseteq c^{-1}(g)$, a decomposition $U = \bigsqcup_{V \in F} V$ of U into finitely many mutually disjoint compact open subsets indexed by F , and units $a_V \in R^\times, V \in F$ such that*

$$m = \sum_{V \in F} a_V 1_V \text{ and } n = \sum_{V \in F} a_V^{-1} 1_{V^{-1}}; \quad \text{and then } mn = 1_{r(U)}, \text{ and } nm = 1_{s(U)}.$$

Proof. (1) To show that (m, n) satisfies (N1), first recall that we have $D = \text{span}_R\{1_K : K \subseteq \mathcal{G}^{(0)} \text{ is compact open}\}$; hence, it suffices to show that each $m1_K n$ and each $n1_K m$ is contained in D . We check that $m1_K n \in D$; symmetry gives $n1_K m \in D$ too. Fixing a compact open $K \subseteq \mathcal{G}^{(0)}$, one has that

$$m1_K n = \sum_{V, W \in F} a_V a_W^{-1} 1_V 1_K 1_{W^{-1}} = \sum_{V, W \in F} a_V a_W^{-1} 1_{VK} 1_{(WK)^{-1}}.$$

Then, since $s(V) \cap s(W) = \emptyset$ for distinct $V, W \in F$, it follows that $1_{VK} 1_{(WK)^{-1}} = 0$ unless $V = W$. So we obtain

$$m1_K n = \sum_{V \in F} a_V a_V^{-1} 1_{VK(VK)^{-1}} = \sum_{V \in F} 1_{r(VK)} = 1_{r(UK)}$$

since the sets V are mutually disjoint and cover U . Similarly, $n1_K m = 1_{s(KU)}$. This establishes (N1). Moreover, applying the two identities just derived with $K = s(U) \cup r(U)$, this gives $mn = 1_{r(U)}$ and $nm = 1_{s(U)}$, which implies that each $1_V nm = mn 1_V = 1_V$. So we get that

$$mn = \sum_{V \in F} a_V 1_V 1_{s(U)} = \sum_{V \in F} a_V 1_{V s(U)} = \sum_{V \in F} a_V 1_V = m,$$

and similarly $nmn = n$, giving (N2).

(2) Let $U_m := \text{supp}(m) \subseteq c^{-1}(g)$, and let $U_n := \text{supp}(n) \subseteq c^{-1}(g^{-1})$. The sets U_m and U_n are compact open sets because $m, n \in A_R(\mathcal{G})$. We will show that U_m is a bisection and that $U_n = U_m^{-1}$. First observe that nm is an idempotent in $D = A_R(\mathcal{G}^{(0)})$. Since R is an integral domain, it has no nontrivial idempotents, and since multiplication in D is pointwise, we conclude that $nm = 1_K$ for some compact open set K . We claim that $K = s(U_m)$. For this, first suppose that $u \in K$. Then

$$0 \neq nm(u) = \sum_{\alpha\beta=u} n(\alpha)m(\beta) = \sum_{s(\alpha)=u} n(\alpha^{-1})m(\alpha).$$

So $m(\alpha) \neq 0$ for some $\alpha \in \mathcal{G}_u$, giving $u \in s(\text{supp}(m)) = s(U_m)$. Hence $K \subseteq s(U_m)$. For the reverse inclusion, suppose that $u \in s(U_m)$; that is, there exists $\alpha \in U_m$ satisfying $u = s(\alpha)$. Then

$$0 \neq m(\alpha) = mnm(\alpha) = \sum_{\beta\gamma=\alpha} m(\beta)nm(\gamma) = \sum_{\beta\gamma=\alpha} m(\beta)1_K(\gamma).$$

Since $K \subseteq \mathcal{G}^{(0)}$, the rightmost sum in the preceding equation collapses to $m(\alpha)1_K(u)$. Moreover, as it is nonzero, we conclude that $u \in K$, and then, $nm = 1_{s(U_m)}$. The same argument applied to the homogeneous normaliser (n, m) shows that $mn = 1_{s(U_n)}$. Similarly, $mn = 1_{r(U_m)}$ and $nm = 1_{r(U_n)}$. Therefore, $s(U_m) = r(U_n)$ and $s(U_n) = r(U_m)$.

Now, fixing $u \in s(U_m)$, we show that the set $U_m u U_n$ consists of a single unit. Since

$$0 \neq 1_{s(U_m)}(u) = nm(u) = \sum_{\alpha\beta=u} n(\alpha)m(\beta) = \sum_{s(\alpha)=u} n(\alpha^{-1})m(\alpha),$$

there exists $\alpha_0 \in U_m u$ such that $\alpha_0^{-1} \in U_n$; we must show that $U_m u U_n = \{r(\alpha_0)\}$. Since inversion in \mathcal{G} is a homeomorphism, and since m, n are locally constant, we can choose an open bisection $V_{\alpha_0}^0$ such that $\alpha_0 \in V_{\alpha_0}^0 \subseteq U_m$, such that $(V_{\alpha_0}^0)^{-1} \subseteq U_n$, and such that m is constant on $V_{\alpha_0}^0$ and n is constant on $(V_{\alpha_0}^0)^{-1}$. Let $W_{\alpha_0^{-1}}^0 := (V_{\alpha_0}^0)^{-1}$.

The sets $U_m u \setminus \{\alpha_0\}$ and $u U_n \setminus \{\alpha_0^{-1}\}$ are finite and discrete because r and s are local homeomorphisms and U_m and U_n are compact. For each $\alpha \in U_m u \setminus \{\alpha_0\}$, choose an open bisection V_α^0 with $\alpha \in V_\alpha^0 \subseteq U_m$ such that m is constant on V_α^0 ; and for each $\beta \in u U_n \setminus \{\alpha_0^{-1}\}$ choose an open bisection $\beta \in W_\beta^0 \subseteq U_n$ with n constant on W_β^0 . The set $Y := \left(\bigcap_{\alpha \in U_m} s(V_\alpha^0) \right) \cap \left(\bigcap_{\beta \in u U_n} r(W_\beta^0) \right)$ is an open set. For each $\alpha \in U_m u$, put $V_\alpha := V_\alpha^0 Y$, and for each $\beta \in u U_n$, let $W_\beta := Y W_\beta^0$. Then:

- $W_{\alpha_0^{-1}} = V_{\alpha_0}^{-1}$;
- each V_α is an open bisection containing α , and each W_β is an open bisection containing β ;
- m is constant on each V_α and n is constant on each W_β ; and
- $s(V_\alpha) = Y = r(W_\beta)$ for all α, β .

Since $U_m \subseteq c^{-1}(g)$ and $U_n \subseteq c^{-1}(g^{-1})$, we have $V_\alpha W_\beta \subseteq c^{-1}(e)$ for all α, β . Since Y is open, and since $c^{-1}(e)$ is topologically principal, we can find $y \in Y$ such that $\mathcal{G}_y^y \cap c^{-1}(e) = \{y\}$. It follows that if $\mu, \nu \in U_m y$ are distinct then $r(\mu) \neq r(\nu)$ (because otherwise $\mu^{-1}\nu$ would belong to $\mathcal{G}_y^y \cap c^{-1}(e) \setminus \{y\}$), and similarly, if $\mu, \nu \in y U_n$ are distinct then $s(\mu) \neq s(\nu)$. So we can choose a compact open neighbourhood V'_η of each $\eta \in U_m y$ and a compact open neighbourhood W'_ζ of each $\zeta \in y U_n$ such that the $r(V'_\eta)$ are

mutually disjoint and the $s(W'_\zeta)$ are mutually disjoint; and we can assume that $V'_\eta \subseteq V_\alpha$ if $\eta \in V_\alpha$, and similarly for the W'_ζ . Let $X := (\bigcap_{\eta \in U_m y} s(V'_\eta)) \cap (\bigcap_{\zeta \in y U_n} r(W'_\zeta))$. Since $1_{r(V'_\eta)} 1_{r(V'_{\eta'})} = 0$ for distinct $\eta, \eta' \in U_m y$, and $1_{s(W'_\zeta)} 1_{s(W'_{\zeta'})} = 0$ for distinct $\zeta, \zeta' \in y U_n$, we have

$$(3.1) \quad (1_{r(V'_\eta)} m 1_X n 1_{s(W'_\zeta)}) (\eta \zeta) = m(\eta) n(\zeta)$$

for each $\eta \in U_m y$ and $\zeta \in y U_n$. Now we suppose, that $U_m u U_n$ is not a singleton, and derive a contraction. Either there exists $\alpha_1 \in U_m u \setminus \{\alpha_0\}$ or there is $\beta_1 \in u U_n \setminus \{\alpha_0^{-1}\}$. We consider the former case; the latter is similar. Let η_0 and η_1 be the unique elements of $V_{\alpha_0} y$ and $V_{\alpha_1} y$. Since $W_{\alpha_0^{-1}} = V_{\alpha_0}^{-1}$, the unique element of $y W_{\alpha_0^{-1}}$ is $\zeta_0 := \eta_0^{-1}$. Since (m, n) is a normaliser, we have $m 1_X n \in D$ and therefore $1_{r(V'_{\eta_1})} m 1_X n 1_{s(W'_{\zeta_0})} \in D$. But (3.1) and that m is constant on V_{α_1} and n is constant on $W_{\alpha_0^{-1}}$ gives

$$(1_{r(V'_{\eta_1})} m 1_X n 1_{s(W'_{\zeta_0})}) (\eta_1 \zeta_0) = m(\eta_1) n(\zeta_0) = m(\alpha_1) n(\alpha_0^{-1}).$$

We have $m(\alpha_1) \neq 0$ and $n(\alpha_0^{-1}) \neq 0$ because $\alpha_1 \in U_m = \text{supp}(m)$ and $\alpha_0^{-1} \in U_n = \text{supp}(n)$; since R is an integral domain, we deduce that $(1_{r(V'_{\eta_1})} m 1_X n 1_{s(W'_{\zeta_0})}) (\eta_1 \zeta_0) \neq 0$. Since $\zeta_0 = \eta_0^{-1} \neq \eta_1^{-1}$, we have $\eta_1 \zeta_0 \notin \mathcal{G}^{(0)}$, which contradicts $1_{r(V'_{\eta_1})} m 1_X n 1_{s(W'_{\zeta_0})} \in D$.

We have now established that U_m is a bisection, and symmetry shows that U_n is a bisection. We also showed at the beginning of the preceding paragraph that if $u \in s(U_m)$ then there exists $\alpha \in U_m u$ such that $\alpha^{-1} \in U_n$. Since we now also know that $U_m u$ is a singleton, we deduce that $U_m^{-1} \subseteq U_n$; and symmetry implies that in fact $U_n = U_m^{-1}$. Since m and n are locally constant, we can express $U_m = \bigsqcup_{V \in F} V$ where the V are mutually disjoint compact open sets such that m is constant (and nonzero) on V and n is constant on V^{-1} for each $V \in F$; say $m \equiv r_V$ on V and $n \equiv r'_V$ on V^{-1} . So $m = \sum_{V \in F} r_V 1_V$ and $n = \sum_{V \in F} r'_V 1_{V^{-1}}$. We just have to show that each $r_V r'_V = 1$. Since U is a bisection, the sets $r(V)$ are mutually disjoint, and so for $w \in r(U)$ there is a unique V such that $w \in r(V)$. We then have

$$mn(w) = \sum_{r(\alpha)=w} m(\alpha) n(\alpha^{-1}) = r_V r'_V.$$

We have $w \in r(U_m)$ and we proved that $mn = 1_{r(U_m)}$. Hence $r_V r'_V = 1$ as required. \square

Remark 3.5. (1) If $U \subseteq c^{-1}(g)$ is a compact open bisection, then $(1_U, 1_{U^{-1}}) \in N_g(D)$.

Hence $A_R(\mathcal{G}) = \text{span}_R \{m : (m, n) \in N_*(D)\}$.

(2) Since D has local units for $A_R(\mathcal{G})$, every $(m, n) \in N(D)$ satisfies $mn, nm \in D$.

3.2. The inverse semigroup S . We now use $N_*(D)$, the collection of all homogeneous normalisers, to recover a natural Γ -graded inverse semigroup related to \mathcal{G} , which will be denoted by S . In particular, we will show that a quotient S/\sim of this inverse semigroup acts on the Stone spectrum \widehat{D} , so that we can construct from it a Γ -graded groupoid of germs $(S/\sim) \times_\varphi \widehat{D}$.

We write $C_{\text{lc}}(\mathcal{G}, R^\times \cup \{0\})$ for the set $\{f \in A_R(\mathcal{G}) : f(\mathcal{G}) \subseteq R^\times \cup \{0\}\}$ of functions in $A_R(\mathcal{G})$ whose nonzero values are units. We define $*$: $C_{\text{lc}}(\mathcal{G}, R^\times \cup \{0\}) \rightarrow C_{\text{lc}}(\mathcal{G}, R^\times \cup \{0\})$ by

$$(3.2) \quad f^*(\gamma) = \begin{cases} f(\gamma^{-1})^{-1} & \text{if } f(\gamma^{-1}) \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

It will be convenient to write $r^* := r^{-1}$ for $r \in R^\times$ and $0^* := 0$. Under this notation, we have $f^*(\gamma) = f(\gamma^{-1})^*$ for all $f \in C_{\text{lc}}(\mathcal{G}, R^\times \cup \{0\})$ and $\gamma \in \mathcal{G}$.

We define

$$C_{\text{lc}}(\mathcal{G}, R^\times \cup \{0\})_\star := \{f \in C_{\text{lc}}(\mathcal{G}, R^\times \cup \{0\}) : f \text{ is homogeneous}\},$$

and for $g \in \Gamma$, we write $C_{\text{lc}}(\mathcal{G}, R^\times \cup \{0\})_g := C_{\text{lc}}(\mathcal{G}, R^\times \cup \{0\}) \cap A_R(\mathcal{G})_g$. We further define

$$(3.3) \quad S := \{f \in C_{\text{lc}}(\mathcal{G}, R^\times \cup \{0\})_\star : \text{supp}(f) \text{ is a bisection}\}.$$

We can reinterpret Proposition 3.4 as follows.

Corollary 3.6. *Let \mathcal{G} be an ample Hausdorff groupoid, and let $c : \mathcal{G} \rightarrow \Gamma$ be a grading of \mathcal{G} by a discrete group. Suppose that $c^{-1}(e)$ is topologically principal. Let R be a commutative integral domain with 1. The map $f \mapsto f^*$ on S is antimultiplicative, and for $f \in S$, we have $f^*f = 1_{s(\text{supp}(f))}$ and $ff^* = 1_{r(\text{supp}(f))}$. The map*

$$f \mapsto (f, f^*)$$

is a bijection from S to $N_\star(D)$.

Proof. Clearly $f^* \in S$ and $f^{**} = f$ for $f \in S$. Now, for $f, g \in S$, and $\gamma \in \mathcal{G}$, we have $\gamma \in \text{supp}(fg)$ if and only if $\gamma \in \text{supp}(f)\text{supp}(g)$ and in this case, there are unique elements $\alpha \in \text{supp}(f)$ and $\beta \in \text{supp}(g)$ such that $\gamma = \alpha\beta$. We thus have for $\gamma \in \text{supp}(g^*)\text{supp}(f^*)$ that

$$\begin{aligned} (fg)^*(\gamma) &= (fg)(\gamma^{-1})^{-1} = (f(\beta^{-1})g(\alpha^{-1}))^{-1} \\ &= g(\alpha^{-1})^{-1}f(\beta^{-1})^{-1} = g^*(\alpha)f^*(\beta) = (g^*f^*)(\gamma), \end{aligned}$$

where $\alpha \in \text{supp}(g^*)$ and $\beta \in \text{supp}(f^*)$ are the unique elements such that $\gamma = \alpha\beta$. If $\gamma \notin \text{supp}(g^*)\text{supp}(f^*)$, then $(fg)^*(\gamma) = 0 = (g^*)(f^*)(\gamma)$. So $*$ is antimultiplicative on S . (We do not need commutativity of R for this computation.)

If $f \in S$ and $\gamma \in \mathcal{G}$, then the displayed equation above shows that $(ff^*)(\gamma) = 0$ except when $\gamma \in \text{supp}(f)\text{supp}(f)^{-1} = r(\text{supp}(f))$, because $\text{supp}(f)$ is a bisection. Moreover, if $\gamma = \alpha\alpha^{-1} = r(\alpha)$ for $\alpha \in \text{supp}(f)$, then $(ff^*)(\gamma) = f(\alpha)f(\alpha)^{-1} = 1$. Hence, we conclude that $ff^* = 1_{r(\text{supp}(f))}$. An identical calculation shows that $f^*f = 1_{s(\text{supp}(f))}$. It follows immediately that $ff^*f = f$ and $f^*f^*f^* = f^*$, so that (f, f^*) satisfies (N2). For $K \subseteq \mathcal{G}^{(0)}$ compact open, we have $f1_K = f|_{\text{supp}(f)K} \in S$, and likewise $1_Kf \in S$; thus, $f1_Kf^* = (f1_K)(1_Kf^*)$ and $f^*1_Kf = (1_Kf)^*(1_Kf)$ belong to D by the reasoning just applied to ff^* and f^*f . Since D is the R -linear span of the 1_K , it follows that $fDf^* \cup f^*Df \subseteq D$. So we have established that (f, f^*) satisfies (N1) obtaining that the map $f \mapsto (f, f^*)$ carries S to $N(D)$.

The map $f \mapsto (f, f^*)$ is clearly injective, and Proposition 3.4 shows that every homogeneous normaliser has the form (f, f^*) ; therefore, $f \mapsto (f, f^*)$ is also surjective. \square

Define $\tilde{c} : S \rightarrow \Gamma$ by $\tilde{c}(f) = g$ if and only if $f \in A_R(\mathcal{G})_g$. We now show that S is an inverse semigroup and that \tilde{c} is a Γ -grading of S .

Lemma 3.7. *Let \mathcal{G} be an ample Hausdorff groupoid, and let $c : \mathcal{G} \rightarrow \Gamma$ be a grading of \mathcal{G} by a discrete group. Suppose that $c^{-1}(e)$ is topologically principal. Let R be a commutative integral domain with 1. Then S is an inverse semigroup under multiplication in $A_R(\mathcal{G})$ and the operation $*$ described at (3.2), and \tilde{c} is a Γ -grading of S .*

Proof. Fix $f_1, f_2 \in S$, say $f \in C_{lc}(\mathcal{G}, R^\times \cup \{0\})_g$ and $f_2 \in C_{lc}(\mathcal{G}, R^\times \cup \{0\})_h$. Since R is an integral domain, $\text{supp}(f_1 f_2) = \text{supp}(f_1) \text{supp}(f_2) \subseteq c^{-1}(g)c^{-1}(h)$. Moreover, because $\text{supp}(f_1)$ and $\text{supp}(f_2)$ are compact open bisections, so is $\text{supp}(f_1) \text{supp}(f_2)$. Each $\gamma \in \text{supp}(f_1 f_2)$ can be factorised uniquely as $\gamma = \gamma_1 \gamma_2$ with $\gamma_1 \in \text{supp}(f_1)$ and $\gamma_2 \in \text{supp}(f_2)$, and, since R^\times is closed under multiplication, the value

$$(f_1 f_2)(\gamma) = f_1(\gamma_1) f_2(\gamma_2) \in R^\times \cdot R^\times$$

belongs to $R^\times \subseteq R^\times \cup \{0\}$. So S is closed under multiplication, implying that \tilde{c} carries multiplication in S to multiplication in Γ . The set S is clearly closed under $*$, and $*$ is antimultiplicative on S by Corollary 3.6; hence, Corollary 3.6 also implies that $f f^* f = f$ and $f^* f f^* = f^*$.

As the idempotents of S are of the form 1_K , where K is an open compact subset of $\mathcal{G}^{(0)}$, they commute implying that S is an inverse semigroup by [18, Theorem 1.3]. \square

3.3. From S to the inverse semigroup of compact open bisections. The collection $S_{\mathcal{G}}$ of homogeneous compact open bisections (including the empty bisection) of \mathcal{G} forms an inverse semigroup under composition and with involution $U \mapsto U^{-1}$. Our next task is to pass from the inverse semigroup S described in the preceding section to a smaller inverse semigroup S/\sim , which we prove is isomorphic to $S_{\mathcal{G}}$.

We write $E(D)$ for the boolean ring of idempotent elements of D .

Lemma 3.8. *Let \mathcal{G} be an ample Hausdorff groupoid, and let $c : \mathcal{G} \rightarrow \Gamma$ be a grading of \mathcal{G} by a discrete group. Suppose that $c^{-1}(e)$ is topologically principal. Let R be a commutative integral domain with 1. Let S be the inverse semigroup of Lemma 3.7, and $\tilde{c} : S \rightarrow \Gamma$ the accompanying Γ grading. There is an equivalence relation \sim on S such that $f \sim h$ if and only if all of the following three conditions are satisfied:*

- (1) $\tilde{c}(f) = \tilde{c}(h)$,
- (2) $f^* p f = h^* p h$ for every $p \in E(D)$, and
- (3) $f p f^* = h p h^*$ for every $p \in E(D)$.

The set S/\sim is an inverse semigroup under the operations $[f][g] = [fg]$ and $[f]^* = [f^*]$. The map $q : S \rightarrow S_{\mathcal{G}}$ given by $q(f) = \text{supp}(f)$ descends to an isomorphism $\tilde{q} : S/\sim \rightarrow S_{\mathcal{G}}$ of inverse semigroups.

Proof. It suffices to show that $q : S \rightarrow S_{\mathcal{G}}$ is a surjective semigroup homomorphism, and that $q(f) = q(h)$ if and only if $f \sim h$.

Since R is an integral domain, we have $\text{supp}(fh) = \text{supp}(f) \text{supp}(h)$ for all $f, h \in S$, and we have $\text{supp}(f^*) = \text{supp}(f)^{-1}$ by definition of $*$ on S . So the map q is a semigroup homomorphism. Note that it is surjective because each homogeneous compact open bisection V satisfies $V = q(1_V)$.

To see that $q(f) = q(h) \iff f \sim h$, first suppose that $q(f) = q(h)$. Then $\text{supp}(f) = \text{supp}(h)$. Since $\tilde{c}(f) = g$ if and only if $\text{supp}(f) \subseteq c^{-1}(g)$, it follows that $\tilde{c}(f) = \tilde{c}(h)$. If $p \in D$ is an idempotent, then $p = 1_K$ for some compact open $K \subseteq \mathcal{G}^{(0)}$. We then have $p f = f|_{K \text{supp}(f)}$ and $p h = h|_{K \text{supp}(h)} = h|_{K \text{supp}(f)}$. Thus, Corollary 3.6 applied to $p f, p h \in S$ shows that

$$\begin{aligned} f^* p f &= (p f)^*(p f) = 1_{s(K \text{supp}(f))} = 1_{s(K q(f))} \\ &= 1_{s(K q(h))} = 1_{s(K \text{supp}(h))} = (p h)^*(p h) = h^* p h, \end{aligned}$$

and similarly $f p f^* = h p h^*$.

Now suppose $f \sim h$. We must show that $\text{supp}(f) = \text{supp}(h)$. Since $\tilde{c}(f) = \tilde{c}(h)$, we have $\text{supp}(f)\text{supp}(h)^{-1} \subseteq c^{-1}(e)$ which is topologically principal. Putting $p = 1_{r(\text{supp}(f)) \cup r(\text{supp}(h))}$ in (2), we see that $f^*f = h^*h$, and so $s(\text{supp}(f)) = s(\text{supp}(h))$. We will show that $\text{supp}(f) \cap \text{supp}(h)$ is dense in $\text{supp}(f)$. Let U be a non-empty open subset of \mathcal{G} contained in $\text{supp}(f)$. Since $c^{-1}(e)$ is topologically principal and $r(U)$ is open, there is $\alpha \in U$ such that the isotropy at $r(\alpha)$ is trivial. Let $\beta \in \text{supp}(h)$ be the unique element with $s(\alpha) = s(\beta)$. We suppose that $\beta \neq \alpha$ to derive a contradiction. Since the isotropy at $r(\alpha)$ is trivial, and $\beta \neq \alpha$, we have $r(\beta) \neq r(\alpha)$; thus, there exist disjoint compact open sets $V, W \subseteq \mathcal{G}^{(0)}$ such that $r(\alpha) \in V$ and $r(\beta) \in W$. Let $X = s(V \text{supp}(f)) \cap s(W \text{supp}(h))$. Then $(f1_X f^*)(r(\alpha)) = 1$ and $(h1_X h^*)(r(\alpha)) = 0$, contradicting (3). This shows that $\text{supp}(f) \cap \text{supp}(h)$ is dense in $\text{supp}(f)$. Since \mathcal{G} is Hausdorff, the compact sets $\text{supp}(f)$ and $\text{supp}(h)$ are closed, and so $\text{supp}(f) \cap \text{supp}(h)$ is closed. Hence $\text{supp}(f) \cap \text{supp}(h) = \text{supp}(f)$. Similarly $\text{supp}(f) \cap \text{supp}(h) = \text{supp}(h)$ and thus $\text{supp}(f) = \text{supp}(h)$. \square

Since R has no nontrivial idempotent elements, the Boolean ring $E(D)$ is precisely the set $\{1_K : K \subseteq \mathcal{G}^{(0)} \text{ is compact open}\}$, and so corresponds to the Boolean algebra of compact open subsets of $\mathcal{G}^{(0)}$. We write \widehat{D} for the Stone spectrum of $E(D)$: that is, the space of Boolean-ring homomorphisms $\pi : E(D) \rightarrow \{0, 1\}$. By Stone duality, there is a homeomorphism $\varepsilon : \mathcal{G}^{(0)} \rightarrow \widehat{D}$ such that $\varepsilon_u(p) := p(u)$ for $p \in E(D)$; the inverse of this map takes a Boolean-ring homomorphism $\pi : E(D) \rightarrow \{0, 1\}$ to the unique point in $(\bigcap_{\pi(1_K)=1} K) \setminus (\bigcup_{\pi(1_K)=0} K)$.

Recall that there is an action θ of $S_{\mathcal{G}}$ on $\mathcal{G}^{(0)}$ such that $\text{dom}(\theta_V) = s(V)$, $\text{cod}(\theta_V) = r(V)$ and $\theta_V(s(\alpha)) = r(\alpha)$ for all $\alpha \in V$.

Lemma 3.9. *Let \mathcal{G} be an ample Hausdorff groupoid, $c : \mathcal{G} \rightarrow \Gamma$ be a grading of \mathcal{G} by a discrete group with $c^{-1}(e)$ topologically principal, R be a commutative integral domain with 1, S be the inverse semigroup described at (3.3), and S/\sim be the quotient described in Lemma 3.8. With $\varepsilon : \mathcal{G}^{(0)} \rightarrow \widehat{D}$ and $\theta : S_{\mathcal{G}} \curvearrowright \mathcal{G}^{(0)}$ as above, and $\tilde{q} : S/\sim \rightarrow S_{\mathcal{G}}$ as in Lemma 3.8, we have*

$$(3.4) \quad \varepsilon_{\theta_{q(f)}(u)}(p) = \varepsilon_u(f^*pf) \quad \text{for all } f \in S, u \in s(\text{supp}(f)) \text{ and } p \in E(D).$$

Proof. Fix $f \in S$, $u \in s(\text{supp}(f))$ and $p \in E(D)$. Then $p = 1_K$ for some compact open $K \subseteq \mathcal{G}^{(0)}$. Let $\alpha \in \text{supp}(f)$ be the unique element with $s(\alpha) = u$. Then

$$\varepsilon_{\theta_{q(f)}(u)}(p) = p(\theta_{q(f)}(u)) = 1_K(r(\alpha)).$$

Also,

$$\begin{aligned} \varepsilon_u(f^*pf) &= (f^*1_K f)(u) = \sum_{\beta\gamma=u} (1_K f)^*(\beta) f(\gamma) = (1_K f)^*(\alpha^{-1}) f(\alpha) \\ &= (1_K f)(\alpha)^* f(\alpha) = 1_K(r(\alpha)) f(\alpha)^* f(\alpha) = 1_K(r(\alpha)). \quad \square \end{aligned}$$

3.4. Groupoids of germs. If φ is an action of a countable inverse semigroup S on a locally compact Hausdorff space X , then the groupoid of germs $S \times_{\varphi} X$ is defined as follows (see for example [10, 22]). Define a relation \sim on $\{(s, x) \in S \times X : x \in \text{dom}(\varphi_s)\}$ by $(s, x) \sim (s', y)$ if $x = y$ and there is an idempotent $p \in E(S)$ such that $x \in \text{dom}(p)$ and $sp = s'p$. This is an equivalence relation, and the collection $S \times_{\varphi} X$ of equivalence classes for this relation is a locally compact étale groupoid with unit space X and structure

maps $r([s, x]) = \varphi_s(x)$, $s([s, x]) = x$, $[s, \varphi_t(x)][t, x] = [st, x]$, and $[s, x]^{-1} = [s^*, \varphi_s(x)]$. Moreover, if $\tilde{c} : S \rightarrow \Gamma$ is a grading, then, as idempotent elements $p \in S$ satisfy $\tilde{c}(p) = e$, there is a grading of $S \times_\varphi X$ given by $[s, x] \mapsto \tilde{c}(s)$.

Proposition 5.4 of [10] implies that for ample Hausdorff groupoids \mathcal{G} , the groupoid of germs for the action θ of $S_{\mathcal{G}}$ on $\mathcal{G}^{(0)}$ is canonically isomorphic to \mathcal{G} . Combining the bunch of results displayed along the preceding subsections with it, we recover \mathcal{G} from S and D .

Lemma 3.10. *Let \mathcal{G} be an ample Hausdorff groupoid, $c : \mathcal{G} \rightarrow \Gamma$ be a grading of \mathcal{G} by a discrete group with $c^{-1}(e)$ topologically principal, R be a commutative integral domain with 1, S be the inverse semigroup described at (3.3), and S/\sim be the quotient described in Lemma 3.8. Then, there is an action φ of S/\sim on \widehat{D} such that $\text{dom}([f]) = \{\pi : \pi(1_{s(\text{supp}(f))}) = 1\}$, $\text{cod}([f]) = \{\pi : \pi(1_{r(\text{supp}(f))}) = 1\}$, and*

$$(3.5) \quad \varphi_{[f]}(\pi)(p) = \pi(f^*pf) \quad \text{for all } f \in S, \pi \in \text{dom}([f]) \text{ and } p \in E(D).$$

Moreover, if $\tilde{q} : S/\sim \rightarrow S_{\mathcal{G}}$ is the isomorphism of Lemma 3.8, then the homeomorphism $\varepsilon : \mathcal{G}^{(0)} \rightarrow \widehat{D}$ intertwines $\theta_{\tilde{q}([f])}$ and $\varphi_{[f]}$ for every $f \in S$.

Proof. Lemma 3.9 shows that the formula given for φ satisfies

$$\varphi_{[f]}(\varepsilon_u) = \varepsilon_{\theta_{\tilde{q}([f])}(u)}$$

for all f and u , so the result follows by pulling the action θ back to an action of S/\sim via the isomorphism \tilde{q} . \square

We now obtain our key result.

Corollary 3.11. *Let \mathcal{G} be an ample Hausdorff groupoid, $c : \mathcal{G} \rightarrow \Gamma$ be a grading of \mathcal{G} by a discrete group with $c^{-1}(e)$ topologically principal, R be a commutative integral domain with 1, S be the inverse semigroup described at (3.3), and S/\sim be the quotient described in Lemma 3.8. Let $\varphi : S/\sim \curvearrowright \widehat{D}$ be the action of Lemma 3.10. Then, there is an isomorphism $\pi : (S/\sim) \times_\varphi \widehat{D} \rightarrow \mathcal{G}$ such that*

$$\pi([f], \varepsilon_{s(\alpha)}) = \alpha$$

for all $f \in S$ and $\alpha \in \text{supp}(f)$. Moreover, π intertwines the grading on $(S/\sim) \times_\varphi \widehat{D}$ induced by \tilde{c} and the grading c of \mathcal{G} .

Proof. Proposition 5.4 of [10] implies that there is an isomorphism

$$\pi_0 : S_{\mathcal{G}} \times_\theta \mathcal{G}^{(0)} \rightarrow \mathcal{G}$$

such that $\pi_0([V, s(\alpha)]) = \alpha$ for all $V \in S_{\mathcal{G}}$ and $\alpha \in V$. This π_0 clearly carries the grading on $S_{\mathcal{G}}$ induced by c to \mathcal{G} . The final statement of Lemma 3.10 shows that \tilde{q} and ε induce an isomorphism of the action of S/\sim on \widehat{D} with that of $S_{\mathcal{G}}$ on $\mathcal{G}^{(0)}$. Thus, there is an isomorphism $\pi' : (S/\sim) \times_\varphi \widehat{D} \rightarrow S_{\mathcal{G}} \times_\theta \mathcal{G}^{(0)}$ such that $\pi'([f], \varepsilon_u) = [\tilde{q}(f), u]$ for all f and u , and π' takes the grading \tilde{c} to the grading of $S_{\mathcal{G}}$ induced by c . Therefore, $\pi = \pi_0 \circ \pi'$ is the desired isomorphism. \square

Remark 3.12. As an alternative to Corollary 3.11, one might aim to employ Lawson's noncommutative generalisation of Stone duality [19] to recover \mathcal{G} from S/\sim . The idea is that the inverse semigroup S/\sim can be made into a boolean inverse monoid by adjoining a unit. Lawson's results [19, Proposition 2.13, Proposition 2.18(8) and Proposition 2.23] imply that we can recover an ample groupoid \mathcal{G} from this boolean inverse monoid as a

space of filters with respect to a suitable order structure, and that we can then pass back to S/\sim as the boolean inverse monoid of compact open bisections of \mathcal{G} . By removing from $\mathcal{G}^{(0)}$ the filter at infinity corresponding to the adjoined unit of S/\sim one would expect to recover \mathcal{G} , and one could pass back to \mathcal{G} by adding a point at infinity to $\mathcal{G}^{(0)}$.

3.5. Maximal abelian subalgebras and the proof of Theorem 3.1. The final step towards proving our main theorem is to show that $D_{\mathcal{G}}$, the set $A_R(\mathcal{G}^{(0)}) \subseteq A_R(\mathcal{G})$, is a maximal abelian subalgebra of $A_R(\mathcal{G})_e$.

Lemma 3.13. *Let \mathcal{G} be a topologically principal ample Hausdorff groupoid and R be a commutative integral domain with 1. Then, $D_{\mathcal{G}}$ is a maximal abelian subring of $A_R(\mathcal{G})$.*

Proof. Certainly $D_{\mathcal{G}}$ is an abelian subring of $A_R(\mathcal{G})$, so we just have to show that it is a maximal abelian subring. For this purpose, suppose that $f \in A_R(\mathcal{G}) \setminus D_{\mathcal{G}}$; we must find $a \in D_{\mathcal{G}}$ such that $fa \neq af$. Fix $\alpha \in \text{supp}(f) \setminus \mathcal{G}^{(0)}$. Because $\text{supp}(f) \subseteq \mathcal{G}$ is open and \mathcal{G} is topologically principal, we may assume that the isotropy at $s(\alpha)$ is trivial; therefore, $s(\alpha) \neq r(\alpha)$. So we can choose disjoint compact open neighbourhoods $V, W \subseteq \mathcal{G}^{(0)}$ of $r(\alpha)$ and $s(\alpha)$, respectively. We now have $(1_V f 1_W)(\alpha) = f(\alpha) \neq 0$ whereas $(1_V 1_W f)(\alpha) = 0$. In particular, $f 1_W \neq 1_W f$. \square

We can now prove our main theorem.

Proof of Theorem 3.1. Clearly if $\bar{\rho} : \mathcal{G} \rightarrow \mathcal{H}$ is a graded isomorphism of groupoids, then there is a graded isomorphism $\rho : A_R(\mathcal{G}) \rightarrow A_R(\mathcal{H})$ given by $\rho(f) = f \circ \bar{\rho}^{-1}$. Moreover, this isomorphism carries $D_{\mathcal{G}}$ to $D_{\mathcal{H}}$ because $\bar{\rho}$ carries $\mathcal{G}^{(0)}$ to $\mathcal{H}^{(0)}$.

For the reverse implication, suppose that $\rho : A_R(\mathcal{G}) \rightarrow A_R(\mathcal{H})$ is a graded ring isomorphism with $\rho(D_{\mathcal{G}}) \subseteq D_{\mathcal{H}}$. Lemma 3.13 applied to $c^{-1}(e)$ implies that $D_{\mathcal{G}}$ is maximal abelian subring of $A_R(\mathcal{G})_e$, which induces that $\rho(D_{\mathcal{G}})$ is a maximal abelian subring of $A_R(\mathcal{H})_e$ because ρ is a graded ring isomorphism. Since $\rho(D_{\mathcal{G}})$ is contained in the abelian subring $D_{\mathcal{H}}$ of $A_R(\mathcal{H})_e$, we deduce that $\rho(D_{\mathcal{G}}) = D_{\mathcal{H}}$. Thus, ρ restricts to an isomorphism of boolean rings $E(D_{\mathcal{G}}) \cong E(D_{\mathcal{H}})$ inducing a homeomorphism $\rho^* : \widehat{D}_{\mathcal{H}} \cong \widehat{D}_{\mathcal{G}}$ given by $\rho^*(\pi) = \pi \circ \rho$.

If (n, m) is a graded normaliser of $D_{\mathcal{G}}$ in $A_R(\mathcal{G})$, then $(\rho(n), \rho(m))$ is a graded normaliser of $D_{\mathcal{H}}$ in $A_R(\mathcal{H})$. Indeed, the claim follows because the conditions defining a normaliser involve only the ring structure. Corollary 3.6 shows that the inverse semigroup $S \subseteq C_{lc}(\mathcal{G}, R^\times \cup \{0\})_\star$ of functions supported on compact open bisections satisfies $S = \{n : (n, m) \in N_\star(D_{\mathcal{G}})\}$ and the corresponding inverse semigroup $T \subseteq C_{lc}(\mathcal{H}, R^\times \cup \{0\})_\star$ is equal to $\{a : (a, b) \in N_\star(D_{\mathcal{H}})\}$. Thus, ρ restricts to a graded isomorphism $S \cong T$ of inverse semigroups. Since ρ carries $E(D_{\mathcal{G}})$ to $E(D_{\mathcal{H}})$, the equivalence relations \sim_S on S and \sim_T on T defined in Lemma 3.8 satisfy $f \sim_S h$ if and only if $\rho(f) \sim_T \rho(h)$, and so ρ descends to a graded isomorphism $\tilde{\rho} : S/\sim_S \rightarrow T/\sim_T$. The definition of ρ^* in the preceding paragraph and the formula (3.5) in Lemma 3.10 show that $\rho^*(\varphi_{\tilde{\rho}([f])}(\pi)) = \varphi_{[f]}(\rho^*(\pi))$ for all $f \in S$ and $\pi \in \widehat{D}_{\mathcal{H}}$. So there exists a graded isomorphism

$$\hat{\rho} : (S/\sim_S) \times_\varphi \widehat{D}_{\mathcal{G}} \cong (T/\sim_T) \times_\varphi \widehat{D}_{\mathcal{H}}$$

that carries $[[f], \rho^*(\pi)]$ to $[[\rho(f)], \pi]$ for $f \in S$ and $\pi \in \widehat{D}_{\mathcal{H}}$. Now the graded isomorphisms $\pi_{\mathcal{G}} : (S/\sim_S) \times_\varphi \widehat{D}_{\mathcal{G}} \rightarrow \mathcal{G}$ and $\pi_{\mathcal{H}} : (T/\sim_T) \times_\varphi \widehat{D}_{\mathcal{H}} \rightarrow \mathcal{H}$ yield a graded isomorphism $\bar{\rho} := \pi_{\mathcal{H}} \circ \hat{\rho} \circ \pi_{\mathcal{G}}^{-1} : \mathcal{G} \rightarrow \mathcal{H}$. \square

Remark 3.14. It is worth discussing the extent to which the hypotheses on our main theorem are necessary.

- If $c^{-1}(\mathcal{G})_e$ is not topologically principal, then $D_{\mathcal{G}}$ is not necessarily maximal abelian in $A_R(\mathcal{G})_e$; and, in addition, key steps in our analysis of the normalisers of $D_{\mathcal{G}}$ and of the quotient S/\sim break down. It is not clear, however, that this hypothesis is necessary to our main results: Brown–Clark–an Huef [6] show that it holds for arbitrary graph groupoids with the trivial grading; and in the special case that \mathcal{G} has one unit (is a discrete group) and c is the trivial grading. More concretely, our result reduces to the classical fact that the group-ring construction and the group-of-units construction are adjoint functors.
- It is unclear whether it is necessary to assume that R is an integral domain or that it is unital. These hypotheses are used heavily in our analysis, but we do not have a counterexample to our main result in their absence.
- It is, however, necessary to make some assumptions on R : Let $R := C_{\text{lc}}(K)$, the ring of locally-constant complex-valued functions on the Cantor set. Since $K \cong K \sqcup K$, we have $R \cong R \oplus R$. Hence, for any ample Hausdorff groupoid \mathcal{G} , there exists a diagonal-preserving isomorphism $A_R(\mathcal{G}) \cong A_R(\mathcal{G}) \oplus A_R(\mathcal{G}) = A_R(\mathcal{G} \sqcup \mathcal{G})$, whereas \mathcal{G} and $\mathcal{G} \sqcup \mathcal{G}$ are not usually isomorphic.

4. APPLICATIONS

4.1. Topologically principal groupoids. Here we record what our results say for ungraded ample groupoids \mathcal{G} . Given any groupoid \mathcal{G} we can endow it with the trivial grading $\Gamma : \mathcal{G} \rightarrow \{e\}$, and then apply our main theorem. In this instance, we have $N_{\star}(D) = N(D)$ and $C_{\text{lc}}(\mathcal{G}, R^{\times} \cup \{0\})_{\star} = C_{\text{lc}}(\mathcal{G}, R^{\times} \cup \{0\})$, so that the inverse semigroup S is just the collection of all $(R^{\times} \cup \{0\})$ -valued elements of $A_R(\mathcal{G})$ supported on bisections.

Theorem 4.1. *Let \mathcal{G} be an ample Hausdorff groupoid and suppose that \mathcal{G} is topologically principal. Let R be a commutative integral domain with 1, and let*

$$S := \{n \in A_R(\mathcal{G}) : \text{there exists } m \in A_R(\mathcal{G}) \text{ such that } nDm \cup mDn \subseteq D, nmn = n, \text{ and } mnm = m\}.$$

Then, the following hold:

- (1) $S = \{n \in A_R(\mathcal{G}) : \text{supp}(n) \text{ is a bisection and } n(\mathcal{G}) \subseteq R^{\times} \cup \{0\}\}$.
- (2) *For each $n \in S$, there is a unique $n^* \in A_R(\mathcal{G})$ such that $nDn^* \cup n^*Dn \subseteq D$, $nn^*n = n$ and $n^*nn^* = n^*$. Moreover, the element n^* belongs to S , and S is an inverse semigroup.*
- (3) *There is an equivalence relation on S given by $n_1 \sim n_2$ if and only if $n_1^*pn_1 = n_2^*pn_2$ and $n_1pn_1^* = n_2pn_2^*$ for every idempotent $p \in D$. Furthermore, S/\sim is an inverse semigroup under the operations inherited from S .*
- (4) *There is an action φ of S/\sim on the Stone spectrum \widehat{D} of the collection of idempotent elements of D such that $\varphi_{[n]}(\pi)(p) = \pi(n^*pn)$ for all $n \in S$, $\pi \in \widehat{D}$ and every idempotent $p \in D$.*
- (5) *There is an isomorphism $\mathcal{G} \cong (S/\sim) \times_{\varphi} \widehat{D}$ that carries $\alpha \in \mathcal{G}$ to $[[1_V], \varepsilon_s(\alpha)]$ for any compact open bisection V containing α .*

Proof. Let $\Gamma : \mathcal{G} \rightarrow \{e\}$ be the trivial grading. Corollary 3.6 applied to this grading Γ gives (1) and (2). Part (3) follows from Lemma 3.8 with this Γ , and (4) follows from

Lemma 3.10. Finally, the inverse of the isomorphism π obtained from Corollary 3.11 satisfies the formula described in (5). \square

Remark 4.2. The action of S on \widehat{D} given by $(n \cdot \pi)(e) = \pi(n^*en)$, which descends to the action of S/\sim in Theorem 4.1(4) is usually called the *spectral action*, and it enjoys a universal property (see [26] for more details). In particular, as described in [26, Example 5.9], this action is the dual of the (right) Munn representation.

Moreover, the isomorphism described in Theorem 4.1(4) induces the isomorphism of Steinberg algebras $A_R(N(D)/\sim \times_{\varphi} \widehat{D}) \cong A_R(\mathcal{G})$ described in [26, Theorem 6.3]; thus, Theorem 4.1 is, in a sense, the dual of Steinberg's result.

Corollary 4.3. *Let \mathcal{G} and \mathcal{H} be ample Hausdorff topologically principal groupoids and R be a commutative integral domain with 1. The following are equivalent:*

- (1) *The groupoids \mathcal{G} and \mathcal{H} are isomorphic as topological groupoids.*
- (2) *There is ring-isomorphism $\rho : A_R(\mathcal{G}) \rightarrow A_R(\mathcal{H})$ such that $\rho(D_{\mathcal{G}}) \subseteq D_{\mathcal{H}}$.*

Proof. Apply Theorem 3.1 to the trivial grading $\Gamma : \mathcal{G} \rightarrow \{e\}$. \square

4.2. Ring-isomorphisms of Leavitt path algebras. In this short section, we make an observation about the implications of our results for Leavitt path algebras. For background on Leavitt path algebras and on Abrams and Tomforde's isomorphism conjecture, see [1, 2, 5, 27, 28].

Abrams and Tomforde conjectured that if E and F are graphs for which there is a ring isomorphism $L_R(E) \cong L_R(F)$ for some ring R , then $C^*(E) \cong C^*(F)$ as C^* -algebras. This conjecture remains open, but we make some headway (see also [6] and Remark 4.5 below).

Corollary 4.4. *Suppose that E and F are graphs in which every cycle has an exit. Then the following are equivalent:*

- (1) *There exists a commutative integral domain R with 1 for which there is an isomorphism $\pi : L_R(E) \rightarrow L_R(F)$ satisfying $\pi(s_{\mu}s_{\mu}^*) \in \text{span}_R\{s_{\eta}s_{\eta}^* : \eta \in F^*\}$ for all $\mu \in E^*$.*
- (2) *There exists a commutative integral domain R with 1 for which there is an isomorphism $\pi : L_R(E) \rightarrow L_R(F)$ satisfying $\pi(s_{\mu}s_{\mu}^*)s_{\eta}s_{\eta}^* = s_{\eta}s_{\eta}^*\pi(s_{\mu}s_{\mu}^*)$ for all $\mu \in E^*$ and $\eta \in F^*$.*
- (3) *For every $*$ -ring S there exists a $*$ -isomorphism of $L_S(E)$ onto $L_S(F)$ that carries $\text{span}_S\{s_{\mu}s_{\mu}^* : \mu \in E^*\}$ to $\text{span}_S\{s_{\eta}s_{\eta}^* : \eta \in F^*\}$.*
- (4) *There is an isomorphism $\psi : C^*(E) \rightarrow C^*(F)$ such that $\psi(\overline{\text{span}}\{s_{\mu}s_{\mu}^* : \mu \in E^*\}) = \overline{\text{span}}\{s_{\eta}s_{\eta}^* : \eta \in F^*\}$.*
- (5) *There is an isomorphism $\psi : C^*(E) \rightarrow C^*(F)$ such that $\psi(s_{\mu}s_{\mu}^*)s_{\eta}s_{\eta}^* = s_{\eta}s_{\eta}^*\psi(s_{\mu}s_{\mu}^*)$ for all $\mu \in E^*$ and $\eta \in F^*$.*

Recall from [9, Example 3.2] that if E is a directed graph, then there is an isomorphism

$$\alpha_E : L_R(E) \cong A_R(\mathcal{G}_E)$$

that carries $\text{span}_S\{s_{\mu}s_{\mu}^* : \mu \in E^*\}$ to $D_{\mathcal{G}_E}$. We will use this isomorphism at a number of points in the proof of Corollary 4.4.

Proof of Corollary 4.4. It is well known (see [17]) that the groupoid \mathcal{G}_E of a directed graph E is topologically principal provided that every cycle in E has an exit. So \mathcal{G}_E and \mathcal{G}_F are topologically principal.

We first prove (1) \iff (2). The implication (1) implies (2) is trivial. For the reverse, observe that if π is as in (2), then each $\pi(s_\mu s_{\mu^*})$ commutes with every element of $\text{span}_R\{s_\eta s_{\eta^*} : \eta \in F^*\}$. Since the latter is a maximal abelian subring by Lemma 3.13, it follows that each $\pi(s_\mu s_{\mu^*}) \in \text{span}_R\{s_\eta s_{\eta^*} : \eta \in F^*\}$.

Next we prove that (1) implies (3) and (5). Suppose that (1) holds. Corollary 4.3 implies that the graph groupoids \mathcal{G}_E and \mathcal{G}_F are isomorphic; say $\rho : \mathcal{G}_F \rightarrow \mathcal{G}_E$ is an isomorphism. Then ρ restricts to a homeomorphism $\mathcal{G}_F^{(0)} \rightarrow \mathcal{G}_E^{(0)}$. For each $*$ -ring S , ρ induces a $*$ -isomorphism $\rho^* : A_S(\mathcal{G}_E) \rightarrow A_S(\mathcal{G}_F)$ satisfying $\rho^*(f) = f \circ \rho$. In particular, ρ carries $D_{\mathcal{G}_E} \subseteq A_S(\mathcal{G}_E)$ to $D_{\mathcal{G}_F} \subseteq A_S(\mathcal{G}_F)$. So $\alpha_F^{-1} \circ \rho^* \circ \alpha_E$ is a $*$ -isomorphism $L_S(E) \rightarrow L_S(F)$ as required in (3). Similarly the isomorphism ρ induces a C^* -algebra isomorphism $\rho^* : C^*(\mathcal{G}_E) \rightarrow C^*(\mathcal{G}_F)$ satisfying $\rho^*(f) = f \circ \rho$ for $f \in C_c(\mathcal{G}_E)$. In particular $\rho^*(C_0(\mathcal{G}_E^{(0)})) = C_0(\mathcal{G}_F^{(0)})$. It is standard that there is an isomorphism $\phi_E : C^*(E) \rightarrow C^*(\mathcal{G}_E)$ that carries $\overline{\text{span}}\{s_\mu s_\mu^* : \mu \in E^*\}$ to $C_0(\mathcal{G}_E^{(0)})$, and similarly a diagonal-preserving isomorphism $\phi_F : C^*(F) \rightarrow C^*(\mathcal{G}_F)$. So $\psi := \phi_F^{-1} \circ \rho^* \circ \phi_E$ is the isomorphism in (5).

Now we prove (3) implies (1). Suppose that (3) holds. Taking $S = \mathbb{F}_2$ (the field of two elements), for example, trivially gives (1).

For (5) implies (1), suppose that (5) holds. With ϕ_E and ϕ_F as above, the map $\phi_F \circ \psi \circ \phi_E^{-1}$ is an isomorphism $C^*(\mathcal{G}_E) \rightarrow C^*(\mathcal{G}_F)$ that carries the Cartan subalgebra $C_0(\mathcal{G}_E^{(0)})$ to the Cartan subalgebra $C_0(\mathcal{G}_F^{(0)})$. So [25, Proposition 4.13] implies that there is an isomorphism $\rho : \mathcal{G}_F \cong \mathcal{G}_E$ as in the first paragraph. This induces an isomorphism $\rho^* : A_R(\mathcal{G}_E) \rightarrow A_R(\mathcal{G}_F)$ that takes $D_{\mathcal{G}_E}$ to $D_{\mathcal{G}_F}$. So $\alpha_F^{-1} \circ \rho^* \circ \alpha_E$ is the desired isomorphism of Leavitt path algebras.

It remains to prove (4) \iff (5). The implication (4) implies (5) is trivial; and (5) implies (4) by the same argument as we used for (2) implies (1) because Renault's theorems prove that $\overline{\text{span}}\{s_\eta s_\eta^* : \eta \in F^*\}$ is a maximal abelian subalgebra of $C^*(F)$. \square

Remark 4.5. We learned of the paper [6] in the later stages of the preparation of this manuscript. Our Corollary 4.4 is related to the main theorem [6, Theorem 5.3], though neither strictly generalises the other. There are two differences between the two results:

- Theorem 5.3 of [6] applies to row-finite graphs E and F with no sources, whereas our result applies to arbitrary graphs E and F in which every cycle has an exit.
- The hypotheses of [6, Theorem 5.3] demand that the isomorphism $\pi : L_R(E) \rightarrow L_R(F)$ should be a $*$ -isomorphism and that it should restrict to an isomorphism $\pi : D_E \rightarrow D_F$ that implements a homeomorphism $\kappa : E^\infty \rightarrow F^\infty$; whereas Corollary 4.4 requires only a ring isomorphism $L_R(E) \rightarrow L_R(F)$ that carries D_E into the commutant of D_F .

We use our results to obtain an improvement of [13, Theorem 3.6]. For its statement, we need some standard graph-theoretical definitions, as follows.

Definition 4.6. A graph E is said to be:

- (1) *strongly connected* if there is a path between any two vertices.
- (2) *essential* if it has no sinks or sources, and
- (3) *trivial* if it is a single cycle with no other edges or vertices.

Corollary 4.7. *Let E, F be finite, essential, non-trivial, strongly connected graphs, and let R be any commutative integral domain with 1. If there is an isomorphism $\phi : L_R(E) \rightarrow$*

$L_R(F)$ such that $\phi(\mathcal{D}(L_R(E))) \subseteq \mathcal{D}(L_R(F))$, then

$$\operatorname{sgn}(\det(I - A_E)) = \operatorname{sgn}(\det(I - A_F)).$$

Proof. It is straightforward to check that the conditions on E and F imply that both graphs have the property that every cycle has an exit. By Corollary 4.4, we obtain a C^* -algebra isomorphism $\bar{\phi}: C^*(E) \rightarrow C^*(F)$ such that $\bar{\phi}(\mathcal{D}(C^*(E))) = \mathcal{D}(C^*(F))$. It follows from [13, Theorem 3.3] (cf. [21, Theorem 3.6]) that $\operatorname{sgn}(\det(I - A_E)) = \operatorname{sgn}(\det(I - A_F))$. \square

This result can be applied to give a partial answer to one of the most intriguing open questions in the theory of Leavitt path algebras; namely, whether, for a commutative coefficient ring R with 1, the algebras $L_{2,R}$ and $L_{2-,R}$ are isomorphic. Johansen and Sørensen have recently shown that there is no $*$ -isomorphism between $L_{2,\mathbb{Z}}$ and $L_{2-,\mathbb{Z}}$ ([13]). Recall from e.g. [13] that $L_{2,R}$ denotes the classical Leavitt algebra of type $(1, 2)$ with coefficients in R . It is the Leavitt path R -algebra of the graph E_2 with one vertex and two arrows. The algebra $L_{2-,R}$ is the Leavitt path R -algebra associated to a graph E_{2-} depicted in the introduction to [13]. Over any regular supercoherent coefficient ring R , both algebras $L_{2,R}$ and $L_{2-,R}$ have trivial algebraic K -theory ([3]). However, they are distinguished by the numbers appearing in Corollary 4.7.

Corollary 4.8. *Let R be a commutative integral domain with 1. Then there is no isomorphism $\phi: L_{2,R} \rightarrow L_{2-,R}$ such that $\phi(\mathcal{D}(L_{2,R})) \subseteq \mathcal{D}(L_{2-,R})$ or $\phi^{-1}(\mathcal{D}(L_{2-,R})) \subseteq \mathcal{D}(L_{2,R})$.*

Proof. Assume there is an isomorphism $\phi: L_{2,R} \rightarrow L_{2-,R}$ such that $\phi(\mathcal{D}(L_{2,R})) \subseteq \mathcal{D}(L_{2-,R})$ or $\phi^{-1}(\mathcal{D}(L_{2-,R})) \subseteq \mathcal{D}(L_{2,R})$. The graphs E_2 and E_{2-} are finite, essential, non-trivial and strongly connected. Therefore, it follows from Corollary 4.7 that

$$\operatorname{sgn}(\det(I - A_{E_2})) = \operatorname{sgn}(\det(I - A_{E_{2-}})).$$

However $\det(I - A_{E_2}) = -1$ and $\det(I - A_{E_{2-}}) = +1$, so we obtain a contradiction. \square

4.3. Graded ring-isomorphisms of Kumjian–Pask algebras. In this section, we emphasise what extra information we obtain by keeping track of the graded structure in Section 3. Recall that for every k -graph Λ , the associated k -graph groupoid \mathcal{G}_Λ (see [15] or [12]) is \mathbb{Z}^k -graded, and $c^{-1}(0)$ is a principal groupoid. So our main theorem yields the following:

Corollary 4.9. *Suppose that Λ and Γ are k -graphs and that R is a commutative integral domain with 1. There is a graded ring-isomorphism $\phi: \operatorname{KP}_R(\Lambda) \cong \operatorname{KP}_R(\Gamma)$ such that $\phi(s_\mu s_{\mu^*}) s_\eta s_{\eta^*} = s_\eta s_{\eta^*} \phi(s_\mu s_{\mu^*})$ for all $\mu \in \Lambda$ and $\eta \in \Gamma$ if and only if the groupoids \mathcal{G}_Λ and \mathcal{G}_Γ are isomorphic, in which case there is a diagonal preserving isomorphism $\operatorname{KP}_S(\Lambda) \cong \operatorname{KP}_S(\Gamma)$ for every ring S , and there is a diagonal-preserving isomorphism $C^*(\Lambda) \cong C^*(\Gamma)$.*

Proof. The argument is essentially the same as the corresponding implications in Corollary 4.4, except that we apply Theorem 3.1 instead of Corollary 4.3. \square

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