REFINEMENT MONOIDS AND ADAPTABLE SEPARATED GRAPHS

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ABSTRACT. We define a subclass of separated graphs, the class of *adaptable separated graphs*, and study their associated monoids. We show that these monoids are primely generated conical refinement monoids, and we explicitly determine their associated *I*-systems. We also show that any finitely generated conical refinement monoid can be represented as the monoid of an adaptable separated graph. These results provide the first step toward an affirmative answer to the Realization Problem for von Neumann regular rings, in the finitely generated case.

INTRODUCTION.

The structure of commutative refinement monoids is generally very intricate, and it is difficult to rephrase their architecture in terms of combinatorial data. These monoids appear naturally in different contexts, such as non-stable K-theory of exchange rings and real rank zero C^* -algebras (see e.g. [9, 18]), classification of Boolean algebras (see e.g. [17, 20]), the realization problem for von Neumann regular rings (see below), and the theory of type semigroups (see e.g. [21, 22]). In this paper, based on the work developed in [11] and [12], we provide a concrete and useful description of a subclasss of all primely generated conical refinement monoids, which contains all the finitely generated ones, in terms of a specific type of separated graphs.

Recall that a separated graph [8] is a pair (E, C), where E is a directed graph and C is a partition of the set of edges of E which is finer than the partition induced by the source map $s: E^1 \to E^0$. Visually one may think of a separated graph as a directed graph where the edges have been given different colours. Several interesting algebras and C^* -algebras have been attached to these combinatorial objects, some of them having exotic behaviour (see for instance [7, 8]). Given a separated graph (E, C), one can naturally associate a monoid M(E, C) to it [8]. However, it is not always true that M(E, C) is a refinement monoid [8, Section 5].

Generalizing earlier work by Dobbertin [14] and Pierce [20], the first and third-named authors have completely determined in [11] the structure of primely generated conical refinement monoids. The main ingredient of this characterization is the notion of an I-system, which is a certain poset of semigroups generalizing the posets of groups used by Dobbertin

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in [14] (see Definition 1.1 below). Using this description, a characterization of the finitely generated conical refinement monoids which are isomorphic to a graph monoid M(E) for a (non-separated) directed graph E has been obtained in [12]. In particular, we stress the fact that *not* all such monoids are isomorphic to graph monoids. It is the purpose of this paper to show that a large class of primely generated conical refinement monoids, including all the finitely generated ones, can be obtained as monoids of the form M(E, C) for (E, C) belonging to a particularly well-behaved class of separated graphs, the *adaptable separated graphs* (see Definition 1.4 below).

Concretely, the main result of this paper (Theorem 2.1) is the following:

Theorem. The following two statements hold:

- (1) If (E, C) is an adaptable separated graph, then M(E, C) is a primely generated conical refinement monoid.
- (2) For any finitely generated conical refinement monoid M, there exists an adaptable separated graph (E, C) such that $M \cong M(E, C)$.

We now outline some of the applications of the results obtained in this note. Concretely, we use the structure of an adaptable separated graph in order to get two realization results. The first application is given in [5], where the authors, jointly with A. Sims, attach to each adaptable separated graph (E, C) an E^* -unitary inverse semigroup S(E, C). Moreover, using techniques developed by Paterson [19] and Exel [15], they build from this inverse semigroup S(E, C) an ample Hausdorff étale topological groupoid $\mathcal{G}(E, C)$ satisfying

$$\operatorname{Typ}(\mathcal{G}(E,C)) \cong M(E,C).$$

In particular, we see from Theorem 2.1(2) that all finitely generated conical refinement monoids arise as type semigroups of this well-behaved class of topological groupoids. The second application concerns the *Realization Problem for von Neumann regular rings*, posed by Goodearl in [16]. This wonders which refinement monoids appear as a $\mathcal{V}(R)$ for a von Neumann regular ring R, where the latter stands for the monoid of isomorphism classes of finitely generated projective (left, say) R-modules, with the operation induced from direct sum (see [2] for a survey on this problem). For an adaptable separated graph (E, C) and an arbitrary field K, we build in [4] a von Neumann regular K-algebra $Q_K(E, C)$, which is a certain universal localization of the Steinberg algebra $A_K(\mathcal{G}(E, C))$ of the above groupoid $\mathcal{G}(E, C)$, and which satisfies that

$$\mathcal{V}(Q_K(E,C)) \cong M(E,C).$$

Again, Theorem 2.1(2) gives that the realization problem for von Neumann regular K-algebras has a positive answer for any finitely generated conical refinement monoid. This construction extends at once the constructions given in [3] and [6].

The paper is organized as follows. In the first section we introduce background material needed for our results. We have splitted this in three subsections, concerning commutative monoids, primely generated refinement monoids, and separated graphs, respectively. In Section 2, we prove our results. We have divided this section into two subsections, in each of which we prove one of the statements of our Theorem.

1. Preliminaries

1.1. **Basics on commutative monoids.** All semigroups and monoids considered in this paper are commutative. We will denote by \mathbb{N} the semigroup of positive integers, and by \mathbb{Z}^+ the monoid of non-negative integers.

Given a commutative monoid M, we set $M^* := M \setminus \{0\}$. We say that M is *conical* if M^* is a semigroup, that is, if, for all x, y in M, x + y = 0 only when x = y = 0.

We say that a monoid M is separative provided 2x = 2y = x + y always implies x = y; there are a number of equivalent formulations of this property, see e.g. [9, Lemma 2.1]. We say that M is a refinement monoid if, for all a, b, c, d in M such that a+b=c+d, there exist w, x, y, z in M such that a = w + x, b = y + z, c = w + y and d = x + z. A basic example of refinement monoid is the monoid M(E) associated to a countable row-finite graph E [10, Proposition 4.4].

If $x, y \in M$, we write $x \leq y$ if there exists $z \in M$ such that x + z = y. Note that \leq is a translation-invariant pre-order on M, called the *algebraic pre-order* of M. All inequalities in commutative monoids will be with respect to this pre-order. An element p in a monoid M is a *prime element* if p is not invertible in M, and, whenever $p \leq a + b$ for $a, b \in M$, then either $p \leq a$ or $p \leq b$. The monoid M is *primely generated* if every non-invertible element of M can be written as a sum of prime elements.

An element $x \in M$ is regular if $2x \leq x$. An element $x \in M$ is an *idempotent* if 2x = x. An element $x \in M$ is free if $nx \leq mx$ implies $n \leq m$. Any element of a separative monoid is either free or regular. In particular, this is the case for any primely generated refinement monoid, by [13, Theorem 4.5]. Furthermore, every finitely generated refinement monoid is primely generated [13, Corollary 6.8].

A subset S of a monoid M is called an *order-ideal* if S is a subset of M containing 0, closed under taking sums and summands within M. An order-ideal can also be described as a submonoid I of M, which is hereditary with respect to the canonical pre-order \leq on M: $x \leq y$ and $y \in I$ imply $x \in I$. A non-trivial monoid is said to be *simple* if it has no non-trivial order-ideals.

If $(S_k)_{k\in\Lambda}$ is a family of (commutative) semigroups, $\bigoplus_{k\in\Lambda} S_k$ (resp. $\prod_{k\in\Lambda} S_k$) stands for the coproduct (resp. the product) of the semigroups S_k , $k \in \Lambda$, in the category of commutative semigroups. If the semigroups S_k are subsemigroups of a semigroup S, we will denote by $\sum_{k\in\Lambda} S_k$ the subsemigroup of S generated by $\bigcup_{k\in\Lambda} S_k$. Note that $\sum_{k\in\Lambda} S_k$ is the image of the canonical map $\bigoplus_{k\in\Lambda} S_k \to S$. We will use the notation $\langle X \rangle$ to denote the semigroup generated by a subset X of a semigroup S.

Given a semigroup M, we will denote by G(M) the Grothendieck group of M. There exists a semigroup homomorphism $\psi_M \colon M \to G(M)$ such that for any semigroup homomorphism $\eta \colon M \to H$ to a group H there is a unique group homomorphism $\tilde{\eta} \colon G(M) \to H$ such that $\tilde{\eta} \circ \psi_M = \eta$. G(M) is abelian and it is generated as a group by $\psi(M)$. If M is already a group then G(M) = M. If M is a semigroup of the form $\mathbb{N} \times G$, where G is an abelian group, then $G(M) = \mathbb{Z} \times G$. In this case, we will view G as a subgroup of $\mathbb{Z} \times G$ by means of the identification $g \leftrightarrow (0, g)$.

Let M be a conical commutative monoid, and let $x \in M$ be any element. The archimedean component of M generated by x is the subsemigroup

$$G_M[x] := \{a \in M : a \le nx \text{ and } x \le ma \text{ for some } n, m \in \mathbb{N}\}$$

For any $x \in M$, $G_M[x]$ is a simple semigroup. If M is separative, then $G_M[x]$ is a cancellative semigroup; if moreover x is a regular element, then $G_M[x]$ is an abelian group.

1.2. Primely generated refinement monoids. The structure of primely generated refinement monoids has been recently described in [11]. We recall here some basic facts.

Given a poset (I, \leq) , we say that a subset A of I is a *lower set* if $x \leq y$ in I and $y \in A$ implies $x \in A$. For any $i \in I$, we will denote by $I \downarrow i = \{x \in I : x \leq i\}$ the lower subset generated by i. We will write x < y if $x \leq y$ and $x \neq y$.

The following definition is crucial for this work:

Definition 1.1 ([11, Definition 1.1]). Let $I = (I, \leq)$ be a poset. An *I*-system

$$\mathcal{J} = (I, \leq, (G_i)_{i \in I}, \varphi_{ji} (i < j))$$

is given by the following data:

- (a) A partition $I = I_{free} \sqcup I_{reg}$ (we admit one of the two sets I_{free} or I_{reg} to be empty).
- (b) A family $\{G_i\}_{i \in I}$ of abelian groups. We adopt the following notation:
 - (1) For $i \in I_{reg}$, set $M_i = G_i$, and $\widehat{G}_i = G_i = M_i$.
 - (2) For $i \in I_{free}$, set $M_i = \mathbb{N} \times G_i$, and $\widehat{G}_i = \mathbb{Z} \times G_i$

Observe that, in any case, \hat{G}_i is the Grothendieck group of M_i .

(c) A family of semigroup homomorphisms $\varphi_{ji} \colon M_i \to G_j$ for all i < j, to which we associate, for all i < j, the unique extension $\widehat{\varphi}_{ji} \colon \widehat{G}_i \to G_j$ of φ_{ji} to a group homomorphism from the Grothendieck group of M_i to G_j (we look at these maps as maps from \widehat{G}_i to \widehat{G}_j). We require that the family $\{\varphi_{ji}\}$ satisfies the following conditions: (1) The assignment

$$\left\{\begin{array}{rrr} i & \mapsto & \widehat{G}_i \\ (i < j) & \mapsto & \widehat{\varphi}_{ji} \end{array}\right\}$$

defines a functor from the category I to the category of abelian groups (where we set $\hat{\varphi}_{ii} = \mathrm{id}_{\hat{G}_i}$ for all $i \in I$).

(2) For each $i \in I_{free}$ we have that the map

$$\bigoplus_{k < i} \varphi_{ik} \colon \bigoplus_{k < i} M_k \to G_i$$

is surjective.

We say that an *I*-system $\mathcal{J} = (I, \leq, (G_i)_{i \in I}, \varphi_{ji} (i < j))$ is *finitely generated* in case *I* is a finite poset and all the groups G_i are finitely generated.

To every *I*-system \mathcal{J} one can associate a primely generated conical refinement monoid $M(\mathcal{J})$, and conversely to any primely generated conical refinement monoid M, we can associate an *I*-system \mathcal{J} such that $M \cong M(\mathcal{J})$, see Sections 1 and 2 of [11] respectively.

1.3. Separated graphs. Here, we recall definitions and properties about separated graphs that will be needed in the sequel. In particular, we define the notion of *adaptable* separated graph, which is crucial for this paper. We refer the reader to [1] and [8] for more information and general notation about (separated) graphs.

Let E be a directed graph, and let \leq be the preorder on E^0 determined by $w \geq v$ if there is a path in E from w to v. Let I be the antisymmetrization of E^0 , with the partial order \leq induced by the order on E^0 . Thus, denoting by [v] the class of $v \in E^0$ in I, we have $[v] \leq [w]$ if and only if $v \leq w$.

For $v \in E^0$, we refer to the set [v] as the component of v, and we will denote by E[v] the restriction of E to [v], that is, the graph with $E[v]^0 = [v]$ and $E[v]^1 = \{e \in E^1 \mid s(e) \in [v] \}$ and $r(e) \in [v]\}$. If J is a lower subset of I, we will denote by $E|_J$ the restriction of the graph E to the set of vertices $\{v \in E^0 \mid [v] \in J\}$.

We now describe our graphs.

Definition 1.2 ([8, Definition 2.1]). A separated graph is a pair (E, C) where E is a directed graph, $C = \bigsqcup_{v \in E^0} C_v$, and C_v is a partition of $s^{-1}(v)$ (into pairwise disjoint nonempty subsets) for every vertex v. (In case v is a sink, we take C_v to be the empty family of subsets of $s^{-1}(v)$). If all the sets in C are finite, we shall say that (E, C) is a finitely separated graph.

From now on, we will assume that all our separated graphs are finitely separated graphs without any further comment.

Following [8], we associate the following monoid to any finitely separated graph.

Definition 1.3 ([8, Definition 4.1]). Given a finitely separated graph (E, C), we define the monoid of the separated graph (E, C), to be

(1.1)
$$M(E,C) = \left\langle a_v \ (v \in E^0) : a_v = \sum_{\{e \in X\}} a_{r(e)} \text{ for every } X \in C_v, v \in E^0 \right\rangle.$$

Recall that a directed graph is said to be *transitive* if any two vertices can be connected by a finite directed path.

Definition 1.4. Let (E, C) be a finitely separated graph and let (I, \leq) be the antisymmetrization of (E^0, \leq) . We say that (E, C) is *adaptable* if I is finite, and there exist a partition $I = I_{\text{free}} \sqcup I_{\text{reg}}$, and a family of subgraphs $\{E_p\}_{p \in I}$ of E such that the following conditions are satisfied:

- (1) $E^0 = \bigsqcup_{p \in I} E_p^0$, where E_p is a transitive row-finite graph if $p \in I_{\text{reg}}$ and $E_p^0 = \{v^p\}$ is a single vertex if $p \in I_{\text{free}}$.
- (2) For $p \in I_{\text{reg}}$ and $w \in E_p^0$, we have that $|C_w| = 1$ and $|s_{E_p}^{-1}(w)| \ge 2$. Moreover, all edges departing from w either belong to the graph E_p or connect w to a vertex $u \in E_q^0$, with q < p in I.
- (3) For $p \in I_{\text{free}}$, we have that $s^{-1}(v^p) = \emptyset$ if and only if p is minimal in I. If p is not minimal, then there is a positive integer k(p) such that $C_{v^p} = \{X_1^{(p)}, \ldots, X_{k(p)}^{(p)}\}$. Moreover, each $X_i^{(p)}$ is of the form

$$X_{i}^{(p)} = \{\alpha(p,i), \beta(p,i,1), \beta(p,i,2), \dots, \beta(p,i,g(p,i))\}$$

for some $g(p,i) \ge 1$, where $\alpha(p,i)$ is a loop, i.e., $s(\alpha(p,i)) = r(\alpha(p,i)) = v^p$, and $r(\beta(p,i,t)) \in E_q^0$ for q < p in I. Finally, we have $E_p^1 = \{\alpha(p,1), \ldots, \alpha(p,k(p))\}$.

The edges connecting a vertex $v \in E_p^0$ to a vertex $w \in E_q^0$ with q < p in I will be called *connectors*.

2. Adaptable separated graphs and their associated monoids.

In this section we show the main result of the paper:

Theorem 2.1. The following two statements hold:

- (1) If (E, C) is an adaptable separated graph, then M(E, C) is a primely generated conical refinement monoid.
- (2) For any finitely generated conical refinement monoid M, there exists an adaptable separated graph (E, C) such that $M \cong M(E, C)$.

We have divided the proof in two parts. First we show statement (1) (Proposition 2.6), and, subsequently, we show the realization result stated in (2) (Theorem 2.11).

2.1. The monoid of an adaptable separated graph. We show below that the monoid M(E, C) associated to an adaptable separated graph (E, C) is a primely generated conical refinement monoid. As a consequence, we obtain from [11, Theorem 2.7] that there is a poset \mathbb{P} , with a partition $\mathbb{P} = \mathbb{P}_{\text{free}} \sqcup \mathbb{P}_{\text{reg}}$, and a \mathbb{P} -system \mathcal{J} such that $M(E, C) \cong M(\mathcal{J})$. We will explicitly determine this system.

To show our results, we will need the "confluence" property of the congruence associated to our separated graphs (E, C). This was established for all graph monoids M(E) of ordinary row-finite graphs in [10, Lemma 4.3]. Amongst other things, this enables us to show the refinement property of the monoids M(E, C), when (E, C) is an adaptable separated graph.

Let (E, C) be an adaptable separated graph, and F be the free commutative monoid on the set E^0 . The nonzero elements of F can be written in a unique form up to permutation as $\sum_{i=1}^{n} v_i$, where $v_i \in E^0$. Now we will give a description of the congruence on F generated by the relations (1.1) (see Definition 1.3) on F.

It will be convenient to introduce the following notation. For $X \in C_v$ ($v \in E^0$), write

$$\mathbf{r}(X) := \sum_{e \in X} r(e) \in F.$$

With this new notation, the relations in (1.1) become $v = \mathbf{r}(X)$ for every $v \in E^0$ and every $X \in C_v$.

Definition 2.2. Define a binary relation \rightarrow_1 on $F \setminus \{0\}$ as follows. Let $\sum_{i=1}^n v_i \in F \setminus \{0\}$, and let $X \in C_{v_j}$ for some $j \in \{1, 2, ..., n\}$. Then $\sum_{i=1}^n v_i \rightarrow_1 \sum_{i \neq j} v_i + \mathbf{r}(X)$. Let \rightarrow be the transitive and reflexive closure of \rightarrow_1 on $F \setminus \{0\}$, that is, $\alpha \rightarrow \beta$ if and only if there is a finite string $\alpha = \alpha_0 \rightarrow_1 \alpha_1 \rightarrow_1 \cdots \rightarrow_1 \alpha_t = \beta$.

Let \sim be the congruence on F generated by the relation \rightarrow_1 (or, equivalently, by the relation \rightarrow). Namely $\alpha \sim \alpha$ for all $\alpha \in F$ and, for $\alpha, \beta \neq 0$, we have $\alpha \sim \beta$ if and only if there is a finite string $\alpha = \alpha_0, \alpha_1, \ldots, \alpha_n = \beta$, such that, for each $i = 0, \ldots, n-1$, either $\alpha_i \rightarrow_1 \alpha_{i+1}$ or $\alpha_{i+1} \rightarrow_1 \alpha_i$. The number n above will be called the *length* of the string. \Box

It is clear that \sim is the congruence on F generated by relations (1.1), and so $M(E, C) = F/\sim$.

The support of an element γ in F, denoted $\operatorname{supp}(\gamma) \subseteq E^0$, is the set of basis elements appearing in the canonical expression of γ .

The proof of the following easy lemma is similar to the one of [10, Lemma 4.2].

Lemma 2.3. (cf. [10, Lemma 4.2]) Let (E, C) be any finitely separated graph. Let \rightarrow be the binary relation on F defined above and $\alpha, \beta \in F \setminus \{0\}$. Assume that $\alpha = \alpha_1 + \alpha_2$ and $\alpha \rightarrow \beta$. Then β can be written as $\beta = \beta_1 + \beta_2$, with $\alpha_1 \rightarrow \beta_1$ and $\alpha_2 \rightarrow \beta_2$.

We are now ready to obtain the crucial lemma that gives the important "confluence" property of the congruence \sim on the free commutative monoid F.

Lemma 2.4. Let (E, C) be an adaptable separated graph. Let α and β be nonzero elements in F. Then $\alpha \sim \beta$ if and only if there is $\gamma \in F$ such that $\alpha \to \gamma$ and $\beta \to \gamma$.

Proof. The proof is similar to the proof of [10, Lemma 4.3]. We highlight the point in which both proofs differ.

Assume that $\alpha \sim \beta$. Then there exists a finite string $\alpha = \alpha_0, \alpha_1, \ldots, \alpha_n = \beta$ such that, for each $i = 0, \ldots, n-1$, either $\alpha_i \to_1 \alpha_{i+1}$ or $\alpha_{i+1} \to_1 \alpha_i$. We proceed by induction on n. If n = 0, then $\alpha = \beta$ and there is nothing to prove. Assume the result is true for strings of length n-1, and let $\alpha = \alpha_0, \alpha_1, \ldots, \alpha_n = \beta$ be a string of length n. By induction hypothesis, there is $\lambda \in F$ such that $\alpha \to \lambda$ and $\alpha_{n-1} \to \lambda$. Now there are two cases to consider. If $\beta \to_1 \alpha_{n-1}$, then $\beta \to \lambda$ and we are done. Assume that $\alpha_{n-1} \to_1 \beta$. By definition of \to_1 , there is a basis element $v \in E^0$ in the support of α_{n-1} and $X \in C_v$ such that $\alpha_{n-1} = v + \alpha'_{n-1}$ and $\beta = \mathbf{r}(X) + \alpha'_{n-1}$. By Lemma 2.3, we have $\lambda = \lambda(v) + \lambda'$, where $v \to \lambda(v)$ and $\alpha'_{n-1} \to \lambda'$. If the length of the string from v to $\lambda(v)$ is positive, then we have $\mathbf{r}(Y) \to \lambda(v)$ for some $Y \in C_v$. If $[v] \in I_{\text{reg}}$, then X = Y and the proof continues as in [10, Lemma 4.3]. If $[v] \in I_{\text{free}}$, then X may be distinct from Y, but in this case, write $\lambda'' := \lambda + (\mathbf{r}(X) - v)$. Then we have

$$\beta = \mathbf{r}(X) + \alpha'_{n-1} = v + (\mathbf{r}(X) - v) + \alpha'_{n-1}$$

$$\rightarrow_1 \mathbf{r}(Y) + (\mathbf{r}(X) - v) + \alpha'_{n-1}$$

$$\rightarrow \lambda(v) + \alpha'_{n-1} + (\mathbf{r}(X) - v)$$

$$\rightarrow \lambda(v) + \lambda' + (\mathbf{r}(X) - v)$$

$$= \lambda + (\mathbf{r}(X) - v) = \lambda''.$$

On the other hand, since $v + \alpha'_{n-1} \to \lambda$ and since $[v] \in I_{\text{free}}$, it follows easily by induction on the length of this string that $v \in \text{supp}(\lambda)$ and thus $\lambda \to_1 \lambda + (\mathbf{r}(X) - v) = \lambda''$. Hence $\alpha \to \lambda \to \lambda''$ and $\beta \to \lambda''$, as desired.

In the remaining case that $v = \lambda(v)$, set $\gamma = \mathbf{r}(X) + \lambda'$. Then we have $\lambda \to_1 \gamma$ and so $\alpha \to \gamma$, and also $\beta = \mathbf{r}(X) + \alpha'_{n-1} \to \mathbf{r}(X) + \lambda' = \gamma$. This concludes the proof. \Box

Now, exactly the same proof as in [10, Proposition 4.4] (using Lemmas 2.3 and 2.4) gives the following result.

Proposition 2.5. Let (E, C) be an adaptable separated graph. Then the monoid M(E, C) is a refinement monoid.

We now show that, for any adaptable separated graph, the monoid M(E, C) is a primely generated monoid.

Proposition 2.6. Let (E, C) be an adaptable separated graph and let (I, \leq) be the antisymmetrization of E^0 with respect to the path-way pre-order. Then M(E, C) is a primely generated conical refinement monoid. *Proof.* By [8, Lemma 4.2], M(E, C) is a nonzero, conical monoid whenever (E, C) is an arbitrary finitely separated graph such that E^0 is non-empty.

Suppose now that (E, C) is an adaptable separated graph. By Proposition 2.5, M(E, C) is a refinement monoid. We now show that M(E, C) is primely generated. For this, it is enough to observe that each generator a_v , with $v \in E^0$ is prime in M(E, C). For this purpose, we work in the free monoid F generated by E^0 and we use the notation introduced above. We have to show that if we have a relation $[v] + [\delta] = [\alpha_1] + [\alpha_2]$ in $F/\sim = M(E, C)$, then there is $i \in \{1, 2\}$ such that $[v] \leq [\alpha_i]$. Now since $v + \delta \sim \alpha_1 + \alpha_2$ in F, we have by Lemma 2.4 that there is $\gamma \in F$ such that $v + \delta \to \gamma$ and $\alpha_1 + \alpha_2 \to \gamma$. By Lemma 2.3, we can write $\gamma = \gamma_1 + \gamma_2$ with $\alpha_i \to \gamma_i$ for i = 1, 2. Since $v + \delta \to \gamma$ and, by the definition of an adaptable separated graph, each $X \in C$ contains at least a loop, we see that v belongs to the support of γ . Therefore, it belongs to the support of γ_i for some $i \in \{1, 2\}$. We can thus assume that $\gamma_1 = v + \gamma'_1$ and therefore

$$[\alpha_1] = [\gamma_1] = [v] + [\gamma_1']$$

showing that $[v] \leq [\alpha_1]$, as desired.

It follows from Proposition 2.6 and [11, Theorem 2.7] that for any adaptable separated graph (E, C) there exists a poset \mathbb{P} , a partition $\mathbb{P} = \mathbb{P}_{\text{free}} \sqcup \mathbb{P}_{\text{reg}}$, and a \mathbb{P} -system \mathcal{J} such that $M(E, C) \cong M(\mathcal{J})$. We close this subsection by explicitly computing this system. Together with our main result in the next subsection (Theorem 2.11), this allows us to express all the structure of a finitely generated conical refinement monoid in terms of the information contained in a representing adaptable separated graph.

Let (E, C) be an adaptable separated graph and let (I, \leq) be the antisymmetrization of E^0 with respect to the path-way pre-order. In order to neatly express our result, we first define a certain *I*-system and then we will show it is isomorphic to the system corresponding to M(E, C).

Definition 2.7. Let (E, C) be an adaptable separated graph, let (I, \leq) be the antisymmetrization of E^0 , and let $I = I_{\text{free}} \sqcup I_{\text{reg}}$ be the canonical partition of $I = E^0/\sim$ (see Definition 1.4). Define an *I*-system $\mathcal{J}'' = (I, \leq, (G''_p)_{p \in I}, \varphi''_{p,q} (q < p))$ as follows:

(1) For each $p \in I_{\text{free}}$ minimal, define $G''_p := \{0\}$ (*i.e.* $M_p = \mathbb{N}$). Now for each non-minimal $p \in I_{\text{free}}$, consider the abelian group G''_p generated by elements x^p_w , where w is a vertex in E such that [w] , subject to the relations

(2.1)
$$x_w^p = \sum_{e \in s_E^{-1}(w)} x_{r(e)}^p, \quad [w] \in I_{\text{reg}},$$

and

(2.2)
$$\sum_{j=1}^{g(q,i)} x_{r(\beta(q,i,j))}^p = 0, \qquad (i = 1, \dots, k(q)) \quad \text{for } q \in I_{\text{free}}, q \le p.$$

(2) For $p \in I_{\text{reg}}$, we let G''_p be the abelian group with generators x^p_w , where w is a vertex in E such that $[w] \leq p$, and with relations (2.1) for every $w \in E^0$ such that $[w] \in I_{\text{reg}}$ and $[w] \leq p$, and (2.2) for every $q \in I_{\text{free}}$ (note that in the latter case, q < p for any $q \in I_{\text{free}}$, because $p \in I_{\text{reg}}$).

Recalling that $M''_p = G''_p$ if $p \in I_{\text{reg}}$ and $M''_p = \mathbb{N} \times G''_p$ is $p \in I_{\text{free}}$, we now define the connecting homomorphisms $\varphi''_{p,q} \colon M''_q \to G''_p$, for q < p, as follows:

$$\varphi_{p,q}''(x_w^q) = x_w^p, \qquad \text{if } q \in I_{\text{reg}},$$

and

$$\varphi_{p,q}^{\prime\prime}(n, \sum_{w < v^q} c_w x_w^q) = n x_{v^q}^p + \sum_{w < v^p} c_w x_w^p, \quad \text{if } q \in I_{\text{free}}$$

where $n \in \mathbb{N}$, and $c_w \in \mathbb{Z}$ are almost all 0. It is straightforward to show that \mathcal{J}'' is an *I*-system.

Remark 2.8. Note that, in case $p \in I_{\text{reg}}$, the relations in G''_p can be expressed in the form $x^p_w = \sum_{e \in X} x^p_{r(e)}$, for each $X \in C_w$ and each $w \in E^0$ such that $[w] \leq p$. The resulting group is therefore the Grothendieck group of the monoid $M(E_H, C^H)$, where (E_H, C^H) is the restriction of the separated graph (E, C) to the hereditary set $H := \{w \in E^0 : [w] \leq p\}$. However, this is not the case when $p \in I_{\text{free}}$, due to the fact that, in that case, we are only considering generators x^p_w for $w \in E^0$ such that [w] < p.

Proposition 2.9. Let (E, C) be an adaptable separated graph, let $I = I_{\text{free}} \sqcup I_{\text{reg}}$ be the canonical partition of $I = E^0/\sim$, and let \mathcal{J}'' be the I-system of Definition 2.7. Let $\mathbb{P} = \mathbb{P}_{\text{free}} \sqcup \mathbb{P}_{\text{reg}}$ be the poset associated to M(E, C), and $\mathcal{J} = (\mathbb{P}, \leq, (G_p)_{p \in \mathbb{P}}, \varphi_{p,q} (q < p))$ be the corresponding \mathbb{P} -system. Then there is an isomorphism of systems $\mathcal{J}'' \cong \mathcal{J}$. In particular

$$M(E,C) \cong M(\mathcal{J}) \cong M(\mathcal{J}'').$$

Proof. Since M(E, C) is a primely generated conical refinement monoid, there is a \mathbb{P} -system \mathcal{J} such that $M(E, C) \cong M(\mathcal{J})$. This system is described in detail in [11, Section 2]. We are going to follow that reference in order to identify the \mathbb{P} -system \mathcal{J} with the *I*-system \mathcal{J}'' . The first thing we do is to identify \mathbb{P} with *I*.

Let us define a relation \triangleleft on I as follows. For $p, q \in I$, set $p \triangleleft q$ if p < q or $p = q \in I_{\text{reg.}}$ Observe that \triangleleft is an antisymmetric and transitive relation on I. Now define the monoid $M(I, \triangleleft)$ as the commutative monoid with family of generators I and with relations q = q + pif $p \triangleleft q$. The monoid $M(I, \triangleleft)$ is an antisymmetric finitely generated refinement monoid, and its set of primes is precisely I. Moreover, the regular (resp. free) primes of $M(I, \triangleleft)$ are exactly the elements in I_{reg} (resp. I_{free}). Now, it is straightforward, using the defining properties of an adaptable separated graph, to show that the antisymmetrization $\overline{M(E, C)}$ of M(E, C) is isomorphic to $M(I, \triangleleft)$, sending $\overline{a_v} \in \overline{M(E, C)}$ to $[v] \in M(I, \triangleleft)$. It follows from the description of the poset \mathbb{P} associated to M(E, C) given in [11, p. 390, (1)] and the above observations that \mathbb{P} , with its canonical partition $\mathbb{P} = \mathbb{P}_{\text{free}} \sqcup \mathbb{P}_{\text{reg}}$, can be identified with I, and its partition $I = I_{\text{free}} \sqcup I_{\text{reg}}$. Hence, the construction in [11, Section 2] gives rise to an I-system $\mathcal{J} = (I, \leq, \{G_p\}_{p \in I}, \varphi_{pq}(q < p))$.

It remains to identify the groups G_p , for $p \in I$, and the maps $\varphi_{pq}: M_q \to G_p$ for q < p (see [11, Section 2]). First, we observe that every hereditary subset of E^0 is C-saturated, because each $X \in C$ contains at least one loop. Therefore it follows from [8, Corollary 6.10] that the order-ideal of M(E, C) generated by a hereditary subset H of E^0 is generated as a monoid by $\{a_v : v \in H\}$.

Following [11, Section 2], the group G'_p is defined for each $p \in I_{\text{free}}$ to be the set

 $\{a_{v^p} + \alpha : \alpha \in M(E, C) \text{ and } a_{v^p} + \alpha \leq a_{v^p}\},\$

endowed with the product $(a_{v^p} + \alpha) \circ (a_{v^p} + \beta) = a_{v^p} + (\alpha + \beta)$. We want to show that $G''_p \cong G'_p$. To this end we define a map $\lambda_p: G''_p \to G'_p$ by $\lambda_p(x^p_w) = a_{v^p} + a_w$ for $w \in E^0$ with [w] < p. Clearly, the defining relations of G''_p are preserved by λ_p , so this assignment defines a group homomorphism. Now if $\alpha \in M(E, C)$ and $a_{v^p} + \alpha \leq a_{v^p}$, then α belongs to the order-ideal of M(E,C) generated by a_{v^p} , and by the previous remark, it follows that α must be a sum of elements of the form a_w with $w \leq v$. Now, it follows from the fact that a_{v^p} is free that α is a sum of elements of the form a_w with $w < v^p$. This shows that λ_p is surjective. In order to show that λ_p is injective, let $\sum_{w \in A} n_w x_w^p - \sum_{w' \in B} m_{w'} x_{w'}^p$ be an element in the kernel of λ_p , where $A \cap B = \emptyset$, and $n_w, m_{w'} > 0$. It then follows that $a_{v^p} + \sum_{w \in A} n_w a_w = a_{v_p} + \sum_{w' \in B} m_{w'} a_{w'}$ in M(E,C). Let F be the free commutative monoid generated by E^0 . It follows from Lemma 2.4 that there is $\gamma \in F$ such that $v^p + \sum_{w \in A} n_w w \to \gamma$ and $v^p + \sum_{w' \in B} m_{w'} w' \to \gamma$ in F. Note that $\gamma = v^p + \gamma'$, where $\gamma' = \sum_{w < v^p} l_w w$ for some $l_w \ge 0$. Now we transform $\sum_{w \in A} n_w x_w^p$ using corresponding steps to the ones used in the transformation $v^p + \sum_{w \in A} n_w w \to \gamma$, replacing each occurrence of a step $v^q \to_1 v^q + \sum_{j=1}^{g(q,i)} r(\beta(q,i,j))$ for some $i = 1, \ldots, k(q)$ by the identity $0 = \sum_{j=1}^{g(q,i)} x_{r(\beta(q,i,j))}^p$ in G''_p , for each $i = 1, \ldots, k(q)$, if $[w] = q \in I_{\text{free}}$ and $q \leq p$, and each occurrence of a step $w \to_1 \sum_{e \in s^{-1}(w)} r(e)$ by the identity $x_w^p = \sum_{e \in s_E^{-1}(w)} x_{r(e)}^p$ if $[w] \in I_{\text{reg}}$ and [w] < p. By using this process, we arrive at the identity $\sum_{w \in A} n_w x_w^p = \sum_{w < v^p} l_w x_w^p$ in G_p'' . With the same reasoning, we obtain $\sum_{w' \in B} m_{w'} x_{w'}^p = \sum_{w < v^p} l_w x_w^p$. So we get that $\sum_{w \in A}^{w} n_w x_w^p - \sum_{w' \in B}^{w} m_{w'} x_{w'}^p = 0$, as desired. Finally the group G_p is naturally isomorphic to G'_p through the map $G'_p \to G_p$, $a_{v^p} + \alpha \mapsto (a_{v^p} + \alpha) - a_{v^p}$ ([11, Remark 2.5]), so we get the isomorphism $G''_p \cong G_p$, which sends x^p_w to $(a_{v^p} + a_w) - a_{v^p}$.

If $p = [v] \in I_{\text{reg}}$, then the archimedian component of a_v in M(E, C) is a group, and G_p is defined to be this group, see [11, Section 2]. Let e_p be the neutral element of G_p . Then one may check as before that the map $\lambda_p \colon G''_p \to G_p$ given by $x^p_w \mapsto e_p + a_w$ for $[w] \leq p$, is a group isomorphism.

Finally it is straightforward to show that $\varphi_{p,q} \circ \widetilde{\lambda}_q = \lambda_p \circ \varphi_{p,q}''$ whenever q < p in I, where $\widetilde{\lambda}_q \colon M_q'' \to M_q$ is the map induced by λ_q . Hence we get an isomorphism of I-systems $\mathcal{J}'' \cong \mathcal{J}$. Since $M(\mathcal{J}) \cong M(E, C)$ ([11, Theorem 2.7]), we get the last assertion in the statement. \Box

2.2. Representing finitely generated refinement monoids. In this subsection, given any finitely generated conical refinement monoid M, we build an adaptable separated graph (E, C) such that its associated monoid is isomorphic to M.

To this end recall from Section 1 and [11, Sections 1 and 2] that, given any finitely generated conical refinement monoid M, one canonically associates to it an I-system

$$\mathcal{J} = (I, \leq, (G_i)_{i \in I}, \varphi_{ji} \ (i < j))$$

such that $M \cong M(\mathcal{J})$ ([11, Theorem 2.7]). Moreover, the *I*-system \mathcal{J} is finitely generated (meaning that *I* is finite and all the abelian groups G_i are finitely generated, [11, Proposition 2.9]).

We now remind some terminology and facts concerning our monoids before proving the main result of this section (see [14, 13, 11, 12] for background material).

For $i \in I$, we define the *lower cover* L(I, i) of i in I as

$$L(I, i) := \{ j \in I \mid j < i \text{ and } [j, i] = \{ j, i \} \}.$$

Let $p \in I_{\text{free}}$ and let $L(I, p) = \{q_1, \ldots, q_n\}$ be its lower cover. The archimedian component M_p of p has the form $M_p = \mathbb{N} \times G_p$ for the finitely generated abelian group G_p .

Using the notation established in [12, Section 2], we denote by J_p the lower subset of I generated by q_1, \ldots, q_n , and let M_{J_p} be the associated semigroup (cf. [12, Corollary 2.4]). Then, by [12, Lemma 5.1], there is a surjective semigroup homomorphism

$$\varphi_p \colon M_{J_p} \to G_p$$

which is induced by the various maps φ_{pq} for q < p. Consequently, we obtain a surjective group homomorphism $G(\varphi_p): G(M_{J_p}) \to G_p$. We say that an element x in $G(M_{J_p})$ is strictly positive if it belongs to the image of the canonical map $\iota_{J_p}: M_{J_p} \to G(M_{J_p})$. We write $G(M_{J_p})^{++} = \iota_{J_p}(M_{J_p})$ for the set of strictly positive elements.

With the notation above, we provide the last proposition needed for Theorem 2.11.

Proposition 2.10. With the above notation and caveats, we have that the kernel of $G(\varphi_p)$ is generated by a finite number x_1, \ldots, x_k of strictly positive elements.

Proof. Since $G(M_{J_p})$ is a finitely generated abelian group, we have that the kernel of $G(\varphi_p)$ is generated by a finite number of elements y_1, \ldots, y_m . So, it is enough to show that the subgroup generated by an element y in the kernel of $G(\varphi_p)$ is contained in the subgroup generated by two strictly positive elements in the kernel of $G(\varphi_p)$.

Recall that $L(I, p) = \{q_1, \ldots, q_n\}$ is the lower cover of p. We assume that q_1, \ldots, q_r are free and that q_{r+1}, \ldots, q_n are regular. Now, let $y \in \ker(G(\varphi_p))$. Using that the element y can be expressed as a difference of two elements from $G(M_{J_p})^{++}$ and [12, Lemma 5.3], we see that there exist positive integers $n_i, m_i, i = 1, \ldots, r$, and elements $g_i \in G_{q_i}, i = 1, \ldots, n, h_j \in G_{q_j},$ $j = 1, \ldots, r$, such that

$$y = \iota_{J_p} \left(\sum_{i=1}^r \chi_{q_i}(n_i, g_i) + \sum_{i=r+1}^n \chi_{q_i}(g_i) \right) - \iota_{J_p} \left(\sum_{j=1}^r \chi_{q_j}(m_j, h_j) \right)$$

Since φ_p is surjective and $G(\varphi_p)(y) = 0$, there exists $z \in M_{J_p}$ such that

$$-\varphi_p\Big(\sum_{i=1}^r \chi_{q_i}(n_i, g_i) + \sum_{i=r+1}^n \chi_{q_i}(g_i)\Big) = -\varphi_p\Big(\sum_{j=1}^r \chi_{q_j}(m_j, h_j)\Big) = \varphi_p(z).$$

Therefore, if we define the elements $x_1 = (\sum_{i=1}^r \chi_{q_i}(n_i, g_i) + \sum_{i=r+1}^n \chi_{q_i}(g_i)) + z \in M_{J_p}$ and $x_2 = (\sum_{j=1}^r \chi_{q_j}(m_j, h_j)) + z \in M_{J_p}$, then we have $\iota_{J_p}(x_1), \iota_{J_p}(x_2) \in \ker(\varphi_p) \cap G(M_{J_p})^{++}$, and $y = \iota_{J_p}(x_1) - \iota_{J_p}(x_2)$. This shows the result.

Theorem 2.11. Let M be a finitely generated refinement monoid, and let \mathcal{J} be the associated I-system, so that $M \cong M(\mathcal{J})$. Then there is an adaptable separated graph (E, C) such that

$$M(E,C) \cong M(\mathcal{J}) \cong M.$$

Proof. The proof follows the lines of the proof of [12, Proposition 5.13]. This result says that, if the natural map $G(\varphi_p): G(M_{J_p}) \to G_p$ is an almost isomorphism for every free prime p, then there is a row-finite directed graph E such that $M \cong M(E)$. (In particular, this holds if every prime in M is regular). We will only outline the point in which the proof has to be adapted, recalling some of the relevant notation.

The proof works by induction. Assume that J is a lower subset of I and that an adaptable separated graph (E_J, C^J) of the desired form has been constructed so that there is a monoid isomorphism

$$\gamma_J \colon M(J) \to M(E_J, C^J),$$

where M(J) is the order-ideal of M generated by J, sending the canonical semigroup generators to the corresponding sets of vertices, as specified in [12, p. 113]. In case $J \neq I$, let p be a minimal element of $I \setminus J$, and write $J' = J \cup \{p\}$. If p is a regular prime or p is minimal, proceed as in the proof of [12, Proposition 5.13].

Assume that p is a non-minimal free prime. By Proposition 2.10, there are a finite number of strictly positive elements x_1, \ldots, x_k which generate the kernel of the map $G(\varphi_p)$. Now, using the same arguments as in the proof of [12, Proposition 5.13], we may find elements $\hat{x}_i \in M(E_J, C^J)$, $i = 1, \ldots, k$, which are non-negative integer combinations of the vertices of E_J such that $\gamma_J(x_i) = \hat{x}_i$ for $i = 1, \ldots, k$. Observe that $\hat{x}_i \in H_{\gamma_J(J_p)}$, so that we may consider its class (denoted in the same way) in $M_{\gamma_J(J_p)}$. Now, we introduce the adaptable separated graph $(E_{J'}, C^{J'})$. We define $E_{J'}^0 = E_J^0 \sqcup \{v^p\}$, and $C^{J'} \setminus C_{v^p}^{J'} = C^J$, that is, the structure of $(E_{J'}, C^{J'})$ is the same as the structure of (E_J, C^J) when restricted to the vertices of E_J . For the new vertex v^p we define $C_{v^p}^{J'} = \{X_1^{(p)}, \ldots, X_k^{(p)}\}$, where each $X_i^{(p)}$ has the form described in Definition 1.4(3), and the edges $\alpha(p, i), \beta(p, i, t), t = 1, \ldots, g(p, i)$ are chosen in such a way that the relations

(2.3)
$$v^p = v^p + \hat{x}_i$$

are satisfied in the graph monoid $M(E_{J'}, C^{J'})$, for i = 1, ..., k. (Here we set k(p) = k).

By Proposition 2.6, $M(E_{J'}, C^{J'})$ is a primely generated conical refinement monoid. Its corresponding system has been determined in Proposition 2.9. In particular, we know that the set of primes of $M(E_{J'}, C^{J'})$ is $\mathbb{P}(M(E_J, C^J)) \cup \{v^p\}$ and that v^p is a free prime in $M(E_{J'}, C^{J'})$. Consequently, we have that the archimedian component $M(E_{J'}, C^{J'})[v^p]$ of $M(E_{J'}, C^{J'})$ at v^p satisfies

$$M(E_{J'}, C^{J'})[v^p] = \mathbb{N} \times G'_{v^p}$$

for some abelian group G'_{v^p} , and that the map $\phi_p : M(E_{J'}, C^{J'})_{\gamma_J(J_p)} \to G'_{v^p}$ induced by the various semigroup homomorphisms

$$\phi_q^p \colon M(E_{J'}, C^{J'})_{\gamma_J(q)} \to G'_{v^p} \\
y \mapsto (v^p + y) - v^p$$

for q < p is surjective. So, we obtain a surjective group homomorphism

$$G(\phi_p)\colon G(M(E_{J'}, C^{J'})_{\gamma_J(J_p)}) \to G'_{v^p}.$$

In order to simplify the notation, we will write $M(E_{J'}, C^{J'})_{J_p}$ instead of $M(E_{J'}, C^{J'})_{\gamma_J(J_p)}$.

It is readily seen that the natural map $M(E_J, C^J) \to M(E_{J'}, C^{J'})$ defines a monoid isomorphism from $M(E_J, C^J)$ onto an order-ideal of $M(E_{J'}, C^{J'})$; hence, we will identify $M(E_J, C^J)$ with its image without further comment. Moreover, the component $M(E_{J'}, C^{J'})_{J_p}$ clearly co-incides with the component $M(E_J, C^J)_{J_p}$.

Now, the monoid isomorphism $\gamma_J \colon M(J) \to M(E_J, C^J)$ restricts to a semigroup isomorphism $M_{J_p} \to M(E_J, C^J)_{J_p}$, which induces a group isomorphism

$$\widetilde{\gamma}_{J_p} \colon G(M_{J_p}) \to G(M(E_J, C^J)_{J_p})$$

of the Grothendieck groups. Set $K := \ker(G(\phi_p))$, and notice that the relation (2.3) implies that $\widetilde{\gamma}_{J_p}(x_i) = \widehat{x}_i \in K$ for $i = 1, \ldots, k$.

Hence, there is a commutative diagram with exact rows

$$(2.4) \qquad \begin{array}{cccc} 0 & \longrightarrow & \langle x_1, \dots, x_k \rangle & \longrightarrow & G(M_{J_p}) & \xrightarrow{G(\varphi_p)} & G_p & \longrightarrow & 0 \\ & & & & & & & & & \\ 0 & \longrightarrow & K & \longrightarrow & G(M(E_J, C^J)_{J_p}) & \xrightarrow{G(\phi_p)} & G'_{v^p} & \longrightarrow & 0 \\ \end{array}$$

where $\gamma_p: G_p \to G'_{v_p}$ is the map induced from the cokernel of the inclusion $\langle x_1, \ldots, x_k \rangle \hookrightarrow G(M_J)$ to the cokernel of the inclusion $K \hookrightarrow G(M(E_J, C^J)_{J_p})$. Notice that γ_p is an onto map.

We now define the map

$$\gamma_{J'} \colon M(J') \to M(E_{J'}, C^{J'})$$

extending the monoid isomorphism $\gamma_J : M(J) \to M(E_J, C^J)$, and defining $\gamma_{J'}$ on the component $M_p \cong \mathbb{N} \times G_p$ of M(J') by the formula

$$\gamma_{J'}(mp+g) = mv^p + \gamma_p(g)$$

for $m \in \mathbb{N}$ and $g \in G_p$. By [11, Corollary 1.8], to show that $\gamma_{J'}$ is a well-defined monoid homomorphism, it suffices to show that if q < p and $y \in G_M[q] = M_q$ then $\gamma_{J'}(y) + \gamma_{J'}(p) = \gamma_{J'}(\varphi_{p,q}(y) + p)$, that is, $\gamma_J(y) + v^p = \gamma_p(\varphi_{p,q}(y)) + v^p$. For $x \in M_{J_p}$, we may define a map $\tau_q \colon M_q \to G(M_{J_p})$

by $\tau_q(y) = (x + y) - x \in G(M_{J_p})$. The map τ_q is a semigroup homomorphism and does not depend on the particular choice of $x \in M_{J_p}$. Moreover, we have $\varphi_{p,q} = G(\varphi_p) \circ \tau_q$. Analogously, we have a map

$$\tau_{\gamma_J(q)} \colon M(E_{J'}, C^{J'})_{\gamma_J(q)} \to G(M(E_{J'}, C^{J'})_{J_p}) = G(M(E_J, C^J)_{J_p})$$

such that $\phi_{\gamma_J(q)}^{v^p} = G(\phi_p) \circ \tau_{\gamma_J(q)}$, and clearly $\widetilde{\gamma}_{J_p} \circ \tau_q = \tau_{\gamma_J(q)} \circ \gamma_J|_{M_q}$. Using this fact, and the commutativity of (2.4), we have that

$$\gamma_p(\varphi_{p,q}(y)) + v^p = \gamma_p(G(\varphi_p)(\tau_q(y))) + v^p$$

= $G(\phi_p)(\widetilde{\gamma}_{J_p}(\tau_q(y))) + v^p$
= $G(\phi_p)(\tau_{\gamma_J(q)}(\gamma_J(y))) + v^p$
= $\phi_{\gamma_J(q)}^{v^p}(\gamma_J(y)) + v^p$
= $((v^p + \gamma_J(y)) - v^p) + v^p$
= $v^p + \gamma_J(y)$,

as desired.

This shows that there is a well-defined monoid homomorphism

$$\gamma_{J'} \colon M(J') \to M(E_{J'}, C^{J'})$$

sending the canonical semigroup generators of M(J') to the corresponding canonical sets of vertices seen in $M(E_{J'}, C^{J'})$. In particular, $\gamma_{J'}$ is an onto map.

In order to prove the injectivity of $\gamma_{J'}$, we can build an inverse map $\delta_{J'} : M(E_{J'}, C^{J'}) \to M(J')$, as follows: on $M(E_J, C^J)$ we define $\delta_{J'}$ to be γ_J^{-1} , while $\delta_{J'}(v^p) := p$. Notice that

the only relations on $M(E_{J'}, C^{J'})$ not occurring already in $M(E_J, C^J)$ are $v^p = v^p + \hat{x}_i$, $i = 1, \ldots, k$, where $\gamma_J(x_i) = \hat{x}_i$. Thus, $\delta_{J'}(\hat{x}_i) = x_i$. But x_1, \ldots, x_k generate the kernel of the map

$$G(\varphi_p): G(M_{J_p}) \to G_p \hookrightarrow \widehat{G}_p = \mathbb{Z} \times G_p,$$

so that $(p + x_i) - p$ equals 0 in \widehat{G}_p . Hence, the relations $p = p + x_i$ hold in M(J'), for $i = 1, \ldots, k$. Thus, $\delta_{J'}$ is a well-defined monoid homomorphism, and it is the inverse of $\gamma_{J'}$. This completes the proof of the inductive step.

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References

- G. ABRAMS, P. ARA, M. SILES MOLINA, Leavitt Path Algebras, Springer Lecture Notes in Mathematics, vol. 2191, 2017.
- [2] P. ARA, The realization problem for von Neumann regular rings, in *Ring Theory 2007. Proceedings of the Fifth China-Japan-Korea Conference*, (H. Marubayashi, K. Masaike, K. Oshiro, M. Sato, Eds.), Hackensack, NJ (2009) World Scientific, pp. 21–37.
- [3] P. ARA, The regular algebra of a poset, Trans. Amer. Math. Soc. 362 (2010), no. 3, 1505–1546.
- [4] P. ARA, J. BOSA, E. PARDO, The realization problem for finitely generated refinement monoids, *In preparation*.
- [5] P. ARA, J. BOSA, E. PARDO, A. SIMS, The groupoids of adaptable separated graphs and their type semigroups, *Preprint*.
- [6] P. ARA, M. BRUSTENGA, The regular algebra of a quiver, J. Algebra 309 (2007), 207–235.
- [7] P. ARA, R. EXEL, Dynamical systems associated to separated graphs, graph algebras, and paradoxical decompositions, Adv. Math. 252 (2014), 748–804.
- [8] P. ARA, K.R. GOODEARL, Leavitt path algebras of separated graphs, J. Reine Angew. Math. 669 (2012), 165–224.
- [9] P. ARA, K.R. GOODEARL, K.C. O'MEARA, E. PARDO, Separative cancellation for projective modules over exchange rings, *Israel J. Math.* **105** (1998), 105–137.
- [10] P. ARA, M.A. MORENO, E. PARDO, Nonstable K-Theory for graph algebras, Algebra Rep. Th. 10 (2007), 157-178.
- [11] P. ARA, E. PARDO, Primely generated refinement monoids, Israel J. Math. 214 (2016), 379–419.
- [12] P. ARA, E. PARDO, Representing finitely generated refinement monoids as graph monoids, J. Algebra 480 (2017), 79–123.
- [13] G. BROOKFIELD, Cancellation in primely generated refinement monoids, Algebra Universalis 46 (2001), 343–371.
- [14] H. DOBBERTIN, Primely generated regular refinement monoids, J. Algebra 91 (1984), 166–175.
- [15] R. EXEL, Inverse semigroups and combinatorial C*-algebras, Bull. Braz. Math. Soc. (N.S.) 39 (2008), 191–313.
- [16] K.R. GOODEARL, Von Neumann regular rings and direct sum decomposition problems, in "Abelian groups and modules" (Padova,1994), Math. and its Applics. 343, pp. 249–255, Kluwer Acad. Publ., Dordrecht (1995).
- [17] J. KETONEN, The structure of countable Boolean algebras, Annals of Math. 108 (1978), 41–89.
- [18] E. ORTEGA, F. PERERA, M. RØRDAM, The corona factorization property and refinement monoids, *Trans. Amer. Math. Soc.* 363 (2011), 4505–4525.

- [19] A.L.T. PATERSON, "Groupoids, inverse semigroups, and their operator algebras", Progress in Mathematics, 170. Birkhäuser Boston, Inc., Boston, MA, 1999.
- [20] R. S. PIERCE, Countable Boolean algebras, in *Handbook of Boolean Algebras*, Vol. 3 (J. D. Monk and R. Bonnet, Eds.), Amsterdam (1989) North-Holland, pp. 775–876.
- [21] T. RAINONE, A. SIMS, A dichotomy for groupoid C*-algebras, arXiv:1707.04516 [math.OA].
- [22] F. WEHRUNG, Refinement monoids, equidecomposable types, and Boolean inverse semigroups, *Springer Lecture Notes in Math.* 2188 (2017).

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