THE NUCLEAR DIMENSION OF $\mathcal{O}_\infty$-STABLE C^*-ALGEBRAS

JOAN BOSA, JAMES GABE, AIDAN SIMS, AND STUART WHITE

Abstract. We show that every nuclear $\mathcal{O}_\infty$-stable *-homomorphism with a separable exact domain has nuclear dimension at most 1. In particular separable, nuclear, $\mathcal{O}_\infty$-stable C^*-algebras have nuclear dimension 1. We also characterise when $\mathcal{O}_\infty$-stable C^*-algebras have finite decomposition rank in terms of quasidiagonality and primitive-ideal structure, and determine when full $\mathcal{O}_2$-stable *-homomorphisms have nuclear dimension 0.

Introduction

Nuclear dimension for C^*-algebras, introduced in [59], provides a noncommutative analogue of Lebesgue covering dimension. Via the Gelfand transform, every commutative C^*-algebra A is isomorphic to the algebra $C_0(X)$ of continuous functions vanishing at infinity on some locally compact Hausdorff space X, and then the nuclear dimension of A is exactly the Lebesgue covering dimension of X. Simple C^*-algebras lie at the other extreme. These are highly non-commutative, and here nuclear dimension provides the dividing line between the tame and the wild. Indeed, separable, simple, unital, C^*-algebras of finite nuclear dimension satisfying the Universal Coefficient Theorem (UCT) of Rosenberg and Schochet [45] are now classified by their Elliott invariants ([25, 39, 23, 15, 50]; this precise statement can be found as [50, Corollary D]).

Accordingly there has been substantial interest in determining the nuclear dimension of C^*-algebras, with a heavy initial focus on Kirchberg algebras (the simple, separable, nuclear and purely infinite algebras classified by independently Kirchberg and Phillips using Kasparov's bivariant K-theory). There have been two complementary strands of development. Kirchberg algebras satisfying the UCT were shown to have nuclear dimension one in [46], building on the earlier work [47, 18, 59]. As it remains a major open problem whether all separable nuclear C^*-algebras (or equivalently all Kirchberg algebras) satisfy the UCT, Matui and Sato instigated a new approach to the nuclear dimension computation of Kirchberg algebras in [36] which does not rely on the UCT. A remarkable theorem of Kirchberg [26] (see also [29]), characterises these now eponymous algebras amongst simple separable nuclear C^*-algebras: A is Kirchberg if and only if it tensorially absorbs the

Research partially supported by an Alexander von Humboldt Foundation Fellowship (SW), Australian Research Council grant DP180100595 (AS), a Carlsberg Foundation Internationalisation Fellowship (JG), by the DGI-MINECO and European Regional Development Fund through grant MTM2017-83487-P (JB), EPSRC:EP/R025061/1 (SW), and the Spanish Ministry of Economy and Competitiveness, through the María de Maeztu Programme for Units of Excellence in R&D (MDM-2014-0445) (JB).

Cuntz algebra $\mathcal{O}_\infty$ in the sense that $A \cong A \otimes \mathcal{O}_\infty$. Peeling off a tensor factor of $\mathcal{O}_\infty$ creates extra space that makes the analysis in [36] possible. This approach ultimately led to a proof that all Kirchberg algebras have nuclear dimension 1 ([5]). Tensorial absorption of other strongly self-absorbing C^*-algebras has also been used to compute nuclear dimension for many finite simple C^*-algebras in [16, 48, 5, 10, 9].

Tensorial absorbtion of $\mathcal{O}_\infty$ also has profound implications for non-simple C^*-algebras, including Kirchberg and Rørdam's algebraic characterisation of $\mathcal{O}_\infty$-stable nuclear C^*-algebras ([32]) and Kirchberg's classification of separable nuclear $\mathcal{O}_\infty$-stable C^*-algebras via ideal-related bivariant K-theory ([25]) as outlined in [27]. This has played an important role in the classification of non-simple Cuntz–Krieger algebras and their automorphisms ([8, 14]) and other non-simple algebras with small ideal lattices ([1]). So it is natural to seek to compute the nuclear dimension of $\mathcal{O}_\infty$-stable C^*-algebras. In [49], Szabó showed that separable $\mathcal{O}_\infty$-stable algebras have nuclear dimension at most 3 (extending the special cases from [2]) but the precise value remained open. Our first main result settles this question:

Theorem A. *Let A be a separable, nuclear $\mathcal{O}_\infty$-stable C^*-algebra. Then A has nuclear dimension* 1.

Nuclear dimension was preceded by the earlier notion of decomposition rank from [33]. The difference between the two conditions appears small, but is significant. It is tied up with the notion of quasidiagonality, which is an external approximation property originating in work of Halmos [40]. Since quasidiagonality implies stable finiteness, Kirchberg algebras are never quasidiagonal. However every C^*-algebra with finite decomposition rank is quasidiagonal [33, Equation (3.3) and Proposition 5.1]. So all Kirchberg algebras have nuclear dimension 1, but infinite decomposition rank. Indeed, since finite decomposition rank passes to quotients, if A has finite decomposition rank, then every quotient of A is quasidiagonal. So, for example, while $C_0((0,1], \mathcal{O}_\infty)$ has finite nuclear dimension (this dates back to [59]) and is quasidiagonal by Voiculescu's famous homotopy invariance of quasidiagonality [53], it has infinite decomposition rank because it surjects onto $\mathcal{O}_\infty$. It is perhaps surprising at first sight that there can exist $\mathcal{O}_\infty$-stable C^*-algebras of finite decomposition rank, but Rørdam's $\mathcal{O}_\infty$-stable approximately subhomogeneous algebra from [44] provides an example.

Quasidiagonality has repeatedly played a major role in the structure and classification theory for C^*-algebras. For example, in hindsight Winter's use in [55] of finite decomposition rank to access tracial approximations, and thence Lin's classification results ([34]), for C^*-algebras of real rank zero, hinges on quasidiagonality. This became explicit in Matui and Sato's work [36], which planted the seeds for the use of quasidiagonality in the classification results of [15, 50]. We now know that for simple C^*-algebras, quasidiagonality delineates between finite nuclear dimension and finite decomposition rank; building on [36, 48, 5], it is shown in [10] that a simple unital C^*-algebra with finite nuclear dimension has finite decomposition rank precisely when it and all its traces are quasidiagonal (the latter in the sense of [7]).

Our second main result completely characterises when nuclear $\mathcal{O}_\infty$-stable C^*-algebras have finite decomposition rank. Although quasidiagonality need not pass to quotients, finite decomposition rank does, so a necessary condition for finite decomposition rank is that every quotient is quasidiagonal. We prove that this necessary condition is also sufficient. Moreover using [21], we can also describe when these algebras have finite decomposition rank in terms of the primitive ideal space.

Theorem B. *Let A be a separable, nuclear $\mathcal{O}_\infty$-stable C^*-algebra. Then the following are equivalent:*

(i) A has finite decomposition rank;
(ii) A has decomposition rank 1;
(iii) All quotients of A are quasidiagonal;
(iv) Every non-zero hereditary C^*-subalgebra of A is stable;
(v) The primitive ideal space of A has no locally closed, one point subsets.

In order to classify separable nuclear $\mathcal{O}_\infty$-stable C^*-algebras Kirchberg classified morphisms between these algebras. More generally, he classified nuclear *-homomorphisms from separable exact C^*-algebras into $\mathcal{O}_\infty$-stable C^*-algebras. Although originally defined for C^*-algebras, the definitions of both decomposition rank and nuclear dimension of a C^*-algebra A are given in terms of approximation properties for the identity map $\mathrm{id}_A\colon A \to A$, and make perfect sense for other *-homomorphisms between C^*-algebras (see [51]). Following the philosophy that we should detect classifiability through nuclear dimension, it is then reasonable to ask whether the maps classified by Kirchberg are of finite nuclear dimension.

Motivated by Kirchberg's transfer of the nuclearity hypothesis from C^*-algebras to morphisms in his classification of nuclear maps between non-nuclear C^*-algebras, one should seek to do likewise with other ingredients of classification theorems. In particular, in his new treatment [20] of Kirchberg's $\mathcal{O}_2$-stable classification theorem, the second named author introduces $\mathcal{O}_\infty$ and $\mathcal{O}_2$-stable *-homomorphisms. These include all *-homomorphisms whose domain or codomain is $\mathcal{O}_\infty$-stable or $\mathcal{O}_2$-stable respectively, as well as considerably more examples. He shows that nuclear $\mathcal{O}_2$-stable maps out of separable exact C^*-algebras are classified by their behaviour on ideals. The forthcoming work [22] will establish a version of Kirchberg's classification theorem for $\mathcal{O}_\infty$-stable maps. Our third main theorem establishes finite nuclear dimension for such maps; in particular, in answer to the question above, the maps classified by Kirchberg all have nuclear dimension at most 1.

Theorem C. *Let A and B be C^*-algebras with A separable and exact. Let $\theta\colon A \to B$ be a nuclear $\mathcal{O}_\infty$-stable *-homomorphism. Then θ has nuclear dimension at most 1. If every quotient of A is quasidiagonal, then the decomposition rank of θ is at most 1.*

In addition to recognising classifiability via nuclear dimension, we regard this theorem as a proof of concept for the development of regularity theory for *-homomorphisms in the spirit of the Toms–Winter conjecture (see [17, 56, 59]). In particular, in the setting of stably finite C^*-algebras, the Jiang–Su algebra, $\mathcal{Z}$, from [24] is the appropriate replacement for $\mathcal{O}_\infty$, and a key future goal is to develop a suitable notion of Jiang–Su stability for maps, and an analogue of Winter's $\mathcal{Z}$-stability theorem from [56, 57] in this context.

The results discussed above show that very large classes of maps have nuclear dimension either 0 or 1; but which ones have dimension 0? For spaces, it is more or less immediate that zero dimensionality coincides with total disconnectedness; and for C^*-algebras Winter established early in the theory that zero dimensionality coincides with approximate finite dimensionality (see [54], noting that this uses an earlier notion of non-commutative covering dimension). For maps, characterising zero dimensionality seems harder, but we do so for full $\mathcal{O}_2$-stable maps, in terms of quasidiagonality.

Theorem D. *Let A and B be C^*-algebras with A separable and exact, and suppose that $\theta\colon A \to B$ is a full, $\mathcal{O}_2$-stable *-homomorphism. Then the nuclear dimension of θ is given by*

$$\dim_{\mathrm{nuc}}\theta = \begin{cases} 0 & \text{if } \theta \text{ is nuclear and } A \text{ is quasidiagonal,} \\ 1 & \text{if } \theta \text{ is nuclear and } A \text{ is not quasidiagonal,} \\ \infty & \text{otherwise.} \end{cases} \tag{0.1}$$

Discussion of methods.

All existing techniques for passing from tensorial absorption to finite topological dimension rely on some aspect of classification. In [59], Winter and Zacharias estimate the nuclear dimension of Cuntz algebras by direct analysis. They combine this with Rørdam's models for UCT-Kirchberg algebras as inductive limits of algebras of the form $\bigoplus_{i=1}^k M_{m_i} \otimes \mathcal{O}_{n_i} \otimes C(\mathbb{T})$ (obtained using the Kirchberg–Phillips classification) to show that UCT-Kirchberg algebras have nuclear dimension at most 5. This approach was refined in [18] and [47], leading to the realisation in [46] of UCT-Kirchberg algebras as inductive limits of higher rank graph algebras with nuclear dimension 1.

Matui and Sato's use of $\mathcal{O}_\infty$-absorption to obtain finite nuclear dimension absent the UCT also uses classification, in the form of uniqueness results for maps. Since $\mathcal{O}_\infty$ is strongly self-absorbing, to compute the nuclear dimension of an $\mathcal{O}_\infty$-stable C^*-algebra A, it suffices to compute the nuclear dimension of the map $\mathrm{id}_A \otimes 1_{\mathcal{O}_\infty}\colon A \to A \otimes \mathcal{O}_\infty$. Matui and Sato do this for Kirchberg algebras in [36] using a trade off between $\mathcal{O}_\infty$-stability and Kirchberg's $\mathcal{O}_2$-embedding theorem. They approximate $\mathrm{id}_A \otimes 1_{\mathcal{O}_\infty}$ with two copies of an embedding $A \hookrightarrow \mathcal{O}_2 \hookrightarrow 1_A \otimes \mathcal{O}_\infty \subseteq A \otimes \mathcal{O}_\infty$; classification results enter implicitly through a 2×2 matrix trick à la Connes. This strategy was made both more general and more explicit in [2, Theorem 3.3]. Szabó's work [49] also uses a trade off of this nature; he uses Rørdam's strongly purely infinite approximately subhomogeneous algebra $\mathcal{A}_{[0,1]}$ in place of $\mathcal{O}_2$, obtaining a bound on nuclear dimension for arbitrary $\mathcal{O}_\infty$-stable nuclear C^*-algebras. These methods yield approximations which naturally decompose into 4 summands (via approximations for $\mathcal{O}_2$ and $\mathcal{A}_{[0,1]}$ that each decompose into two summands), so yield nuclear dimension at most 3.[1]

[1]Recall that approximations that decompose into $n+1$ summands witness nuclear dimension at most n; see Definition 1.5 below.

To obtain the exact nuclear dimension value of 1, we must avoid passing through intermediate one dimensional algebras like $\mathcal{O}_2$, and instead build factorisations through zero dimensional building blocks directly. The analysis of [5, Section 9] achieves this for Kirchberg algebras in four broad steps. First use a positive contraction $h \in \mathcal{O}_\infty$ of spectrum $[0,1]$ to decompose $\mathrm{id}_A \otimes 1_{\mathcal{O}_\infty}$ as the sum of two order zero maps $\mathrm{id}_A \otimes h$ and $\mathrm{id}_A \otimes (1_{\mathcal{O}_\infty} - h)$. Next use Voiculescu's quasidiagonality of the cone over A from [53] to obtain a sequence $(\phi_n \colon A \to F_n)_{n=1}^\infty$ of approximately order zero maps into finite dimensional algebras F_n. Next embed each F_n into $A \otimes \mathcal{O}_\infty$ using the second tensor factor, and thereby regard each ϕ_n as a map $A \to A \otimes \mathcal{O}_\infty$. Finally employ ingredients from Kirchberg's classification of simple purely infinite nuclear C^*-algebras to power a 2×2 matrix trick to see that the sequence $(\phi_n)_{n=1}^\infty$ is approximately unitarily equivalent to each of $\mathrm{id}_A \otimes h$ and $\mathrm{id}_A \otimes (1_{\mathcal{O}_\infty} - h)$. This gives the required finite-dimensional approximations.

The broad strategy of the previous paragraph underpins our proof of Theorem A. However when A is non-simple, both the construction of finite dimensional models and the appropriate classification vehicle for transferring these models to $\mathrm{id}_A \otimes h$ are more delicate.[2] For the latter, we must use classification results for non-simple algebras. The order zero map $\mathrm{id}_A \otimes h$, gives rise to a *-homomorphism $\pi \colon f \otimes a \mapsto a \otimes f(h)$ from the cone $C_0((0,1]) \otimes A$ into $A \otimes \mathcal{O}_\infty$. Since embeddings of cones into Kirchberg algebras are $\mathcal{O}_2$-stable (see Lemma 4.2), the second author's classification framework for $\mathcal{O}_2$-stable maps in [20] applies to show that π is determined up to approximate equivalence by its behaviour on ideals.

To obtain our finite dimensional models, we construct a sequence of diagrams

$$\begin{array}{ccc} A & \xrightarrow{\qquad\qquad} & A \otimes \mathcal{O}_\infty \\ & {\scriptstyle \tilde{\psi}_n} \searrow \quad \nearrow {\scriptstyle \eta_n} & \\ & F_n & \end{array} \tag{0.2}$$

in which the F_n are finite dimensional C^*-algebras, the η_n are contractive order zero maps, and the $\tilde{\psi}_n$ are completely positive contractions and approximately order zero (in a point-norm sense). The sequence $(\eta_n \circ \tilde{\psi}_n)_{n=1}^\infty$ induces an order zero map θ from A into the sequence algebra $(A \otimes \mathcal{O}_\infty)_{(\infty)}$ (for readers unfamiliar with sequence algebras, these are described just before Lemma 1.4). Through the duality ([58]) between order zero maps on A and *-homomorphisms out of $C_0((0,1]) \otimes A$, this θ is determined by a *-homomorphism $\rho \colon C_0((0,1]) \otimes A \to (A \otimes \mathcal{O}_\infty)_{(\infty)}$. Our goal is to choose the diagrams (0.2) so that ρ is $\mathcal{O}_2$-stable and induces the same map on ideals as π (viewed as a map into $(A \otimes \mathcal{O}_\infty)_{(\infty)}$). Classification will then ensure that ρ and π, and hence also the associated order zero maps, are approximately

[2] By a trick from [2], h and $1_{\mathcal{O}_\infty} - h$ are approximately unitarily equivalent in $\mathcal{O}_\infty$, so that $\mathrm{id}_A \otimes h$ and $\mathrm{id}_A \otimes (1_{\mathcal{O}_\infty} - h)$ are approximately unitarily equivalent. So it suffices to model $\mathrm{id}_A \otimes h$, and we suppress $\mathrm{id}_A \otimes (1_{\mathcal{O}_\infty} - h)$ henceforth.

equivalent.[3] This provides the required finite-dimensional approximations for $\mathrm{id}_A \otimes h$, and hence shows that A has nuclear dimension one.

The construction of the diagrams (0.2) occupies the bulk of the paper.[4] When A has exactly one nontrivial ideal I, the idea is to take the 'downward maps' $\tilde{\psi}_n$ as a direct sum $\tilde{\psi}_{n,1} \oplus \tilde{\psi}_{n,2} \colon A \to F_{n,1} \oplus F_{n,2}$ where $(\tilde{\psi}_{n,1})_{n=1}^\infty$ arises from order zero maps corresponding to the quasidiagonality of the cone over A and $(\tilde{\psi}_{n,2})_{n=1}^\infty$ by following the quotient $A \to A/I$ by maps obtained from the quasidiagonality of the cone over A/I. The 'upward maps' η_n are obtained by fixing embeddings $\iota_{n,i} \colon F_{n,i} \to \mathcal{O}_\infty$ and orthogonal[5] positive contractions h_I and h_A which are full in I and A respectively, and setting $\eta_n(x_1, x_2) = h_I \otimes \iota_1(x_1) + h_A \otimes \iota_2(x_2)$. Then elements $x \in I$ are annihilated under the sequence $(\tilde{\psi}_{n,2})_{n=1}^\infty$, and so $(\eta_n \circ \tilde{\psi}_n(x))_{n=1}^\infty$ sees only the first component, which generates the ideal I. Keeping track of exactly which ideal is generated in a sequence algebra turns out to be a little delicate; we do this in Lemma 3.5, where we also need an additional smearing across the interval to ensure that the resulting map ρ has the required behaviour on all ideals of $C_0((0,1]) \otimes A$, not just those of the form $C_0((0,1]) \otimes I$ for some $I \lhd A$.[6] It remains to arrange $\mathcal{O}_2$-stability, which we do in Section 4, using a similar smearing to merge the ρ from Section 3 with an embedding of a cone into $\mathcal{O}_\infty$.

To handle general ideal lattices, we simultaneously model the ideal lattice of A by finite sublattices, while performing a similar construction to that outlined above. The machinery we need to approximate the ideal lattice of A by finite sublattices is set up in Section 2, with technical difficulties arising throughout our main arguments because the approximating finite sublattice is typically not linearly ordered.

With more care, the strategy discussed above also takes care of the nuclear dimension part of Theorem C. As it turns out, the decomposition rank component of Theorem C is a little easier in that we can use the quasidiagonality hypothesis directly in place of quasidiagonality of cones in our main technical construction. We review nuclear dimension and decomposition rank in Section 1, setting out the criteria we use to recognise finite decomposition rank. This enables us to handle both the nuclear dimension and decomposition rank cases of Theorem C in tandem, keeping track of the required extra detail throughout Sections 3 and 4 and the proof of Theorem C in Section 5.

Section 6 turns to the characterisation of finite decomposition rank, using Theorem C and the developments of [21] to prove Theorem B. We end in

---

[3]The form of approximate equivalence is subtle; classification gives approximate Murray and von Neumann equivalence of ρ and π, which in turn gives approximate unitary equivalence after passing to a 2×2 matrix amplification. We perform the nuclear dimension calculations in this matrix amplification, and subsequently compress back to the original C^*-algebra A.

[4]We note for comparison with Lemmas 3.5 and 4.6 (which provide the meat of the construction) that for technical reasons we actually construct approximate *-homomorphisms $\psi_n \colon C_0((0,1]) \otimes A \to F_n$, and then the maps $\tilde{\psi}_n$ in (0.2) are given by $\tilde{\psi}_n(a) = \psi_n(\mathrm{id}_{(0,1]} \otimes a)$. In the main body of the paper, we make no explicit reference to $\tilde{\psi}_n$.

[5]$\mathcal{O}_\infty$-stability facilitates this orthogonality.

[6]This is the point behind the composition with the multiplication map m^* in (3.34).

Section 7 with the proof of Theorem D. With Theorem C in place, the last component of Theorem D is obtained by using quasidiagonality of A to produce finite dimensional models for a full map θ; the fullness assumption ensures that the map carries no ideal data, and then $\mathcal{O}_2$-stable classification ensures that these models can actually be used to approximate θ.

Acknowledgments. Portions of this research where undertaken during the research programme on the classification of Operator Algebras: Complexity, Rigidity and Dynamics at the Institut Mittag–Leffler in 2016, and the intensive research programme on Operator Algebras: Dynamics and Interactions at CRM, Barcelona in 2017. We thank the organisers and funders of these programmes.

1. Nuclear dimension and decomposition rank

In this section we recall the definitions of the nuclear dimension and decomposition rank from [59] and [33] respectively, and collect some relatively standard results for later use. Throughout the paper if a, b are elements of a C^*-algebra, and $\epsilon > 0$, we write $a \approx_\epsilon b$ to mean $\|a - b\| \leq \epsilon$.

Recall that a completely positive map $\phi\colon A \to B$ between C^*-algebras is said to be *order zero* if, for every $a, b \in A_+$ with $ab = 0$, one has $\phi(a)\phi(b) = 0$. Building on [60], the structure theory for these maps was developed in [58]. In particular, there is a *-homomorphism $\pi_\phi\colon A \to \mathcal{M}(C^*(\phi(A)))$ (called the *supporting *-homomorphism*), taking values in the multiplier algebra $\mathcal{M}(C^*(\phi(A)))$ of $C^*(\phi(A))$, and a positive contraction $h \in \mathcal{M}(C^*(\phi(A))) \cap \pi(A)'$ such that

$$\phi(a) = h\pi(a), \quad a \in A. \tag{1.1}$$

This then induces a *-homomorphism $\rho_\phi\colon C_0((0,1]) \otimes A \to C^*(\phi(A)) \subseteq B$ such that

$$\rho_\phi(f \otimes a) = f(h)\pi(a), \quad f \in C_0((0,1]),\ a \in A. \tag{1.2}$$

Conversely, each *-homomorphism from $C_0((0,1]) \otimes A$ to B determines an order zero map from A to B by composition with the completely positive and contractive (cpc) order zero inclusion $a \mapsto \mathrm{id}_{(0,1]} \otimes a$ of A into its cone.

Proposition 1.1 ([58, Corollary 3.1]). *Let A, B be C^*-algebras. There is a one-to-one correspondence between cpc, order zero maps $\phi\colon A \to B$ and *-homomorphisms $\rho\colon C_0((0,1]) \otimes A \to B$, where ϕ and ρ are related by the commuting diagram*

$$\begin{array}{ccc} A & \xrightarrow{a \mapsto \mathrm{id}_{(0,1]} \otimes a} & C_0((0,1]) \otimes A \\ & \searrow^{\phi} & \downarrow{\scriptstyle \rho} \\ & & B\,. \end{array} \tag{1.3}$$

The structural theorem yields a functional calculus on order zero maps (cf. [58, Corollary 4.2]): if $\phi\colon A \to B$ is a cpc order zero map, and $f \in C_0((0,1])_+$ is a contraction, then $f(\phi)\colon A \to B$ is the cpc order zero map such that

$$f(\phi)(a) = \rho_\phi(f \otimes a), \quad a \in A. \tag{1.4}$$

We record two facts regarding induced *-homomorphisms on the cones and functional calculus for later use. These are proved by routine verification of the defining properties of $\rho_{\eta\circ\psi}$ and $f(\phi)$ respectively.

Lemma 1.2. *Let $\psi\colon A \to F$ be a *-homomorphism, and let $\eta\colon F \to B$ be a cpc order zero map. The *-homomorphism $\rho_{\eta\circ\psi}\colon C_0((0,1])\otimes A \to B$ induced by the composition $\eta\circ\psi$ is exactly the composition*

$$C_0((0,1])\otimes A \xrightarrow{\mathrm{id}_{C_0((0,1])}\otimes\psi} C_0((0,1])\otimes F \xrightarrow{\rho_\eta} B, \tag{1.5}$$

where ρ_η is the *-homomorphism induced by η.

Lemma 1.3. *Let $\phi_1,\dots,\phi_n\colon A \to B$ be cpc order zero maps with pairwise orthogonal ranges, and let $f \in C_0((0,1])_+$ be a contraction. Then $\phi := \sum_{i=1}^n \phi_i\colon A \to B$ is a cpc order zero map, and*

$$f(\phi) = \sum_{i=1}^n f(\phi_i)\colon A \to B. \tag{1.6}$$

Throughout the paper we often use sequence algebras to encode approximate behaviour. Given a sequence $(B_n)_{n=1}^\infty$ of C^*-algebras, the *sequence algebra* is the quotient of the ℓ^∞-product by the c_0-sum: $\prod_{n=1}^\infty B_n/\bigoplus_{n=1}^\infty B_n$. In the special, but particularly relevant, case where $B_n = B$ for all n, we write $B_{(\infty)}$ for this sequence algebra.[7] In this situation B embeds into $B_{(\infty)}$ as constant sequences, and we will often implicitly regard B as a C^*-subalgebra of $B_{(\infty)}$. When needed, we will denote the embedding by $\iota_B\colon B \to B_{(\infty)}$.

We denote elements in a sequence algebra $\prod_{n=1}^\infty B_n/\bigoplus_{n=1}^\infty B_n$ by representatives in $\prod_{n=1}^\infty B_n$, rather than introducing additional notation for the class they represent. By the same principle, given cpc maps $\phi_n\colon A \to B_n$, we often write $(\phi_n)_{n=1}^\infty$ for the induced map $A_{(\infty)} \to \prod_{n=1}^\infty B_n/\bigoplus_{n=1}^\infty B_n$ and also for its restriction to A. For later use we record the following routine fact regarding functional calculus of sequences of order zero maps, which does not seem to have explicitly appeared in the literature.

Lemma 1.4. *Let $(\eta_n\colon F_n \to B_n)_{n=1}^\infty$ be a sequence of cpc order zero maps between C^*-algebras, and let*

$$\eta\colon \frac{\prod_{n=1}^\infty F_n}{\bigoplus_{n=1}^\infty F_n} \to \frac{\prod_{n=1}^\infty B_n}{\bigoplus_{n=1}^\infty B_n} \tag{1.7}$$

be the cpc order zero map induced by $(\eta_n)_{n=1}^\infty$. For a contraction $f \in C_0((0,1])_+$, the sequence $(f(\eta_n))_{n=1}^\infty$ induces the cpc order zero map $f(\eta)$.

Proof. Let $\rho_n\colon C_0((0,1])\otimes F_n \to B_n$ be the *-homomorphism induced by η_n as in Proposition 1.1. We obtain a *-homomorphism

$$\rho\colon C_0((0,1])\otimes \prod_{n=1}^\infty F_n/\bigoplus_{n=1}^\infty F_n \to \prod_{n=1}^\infty B_n/\bigoplus_{n=1}^\infty B_n, \tag{1.8}$$

[7] This notation is a little unusual; B_∞ is more normal, but since the Cuntz algebra $\mathcal{O}_\infty$ plays a major role in this paper, we prefer $B_{(\infty)}$ here.

such that $\rho(f \otimes (x_n)_{n=1}^\infty) = (\rho_n(f \otimes x_n))_{n=1}^\infty$ for all $f \in C_0((0,1])$ and $(x_n)_{n=1}^\infty \in \prod_{n=1}^\infty B_n / \bigoplus_{n=1}^\infty B_n$. So it suffices to check that ρ is the *-homomorphism associated to η by Proposition 1.1. This follows as, for any $(x_n)_{n=1}^\infty \in \prod_{n=1}^\infty B_n / \bigoplus_{n=1}^\infty B_n$, we have

$$\rho(\mathrm{id}_{(0,1]} \otimes (x_n))_{n=1}^\infty = (\eta_n(x_n))_{n=1}^\infty = \eta((x_n)_{n=1}^\infty). \tag{1.9}$$ □

We next recall the definitions of nuclear dimension and decomposition rank, from [59] and [33] respectively. We work at the level of morphisms, these definitions were first developed in [51].

Definition 1.5. Let A and B be C^*-algebras, let $\phi\colon A \to B$ be a *-homomorphism and let $n \in \mathbb{N}$. Then ϕ is said to have *nuclear dimension at most* n (written $\dim_{\mathrm{nuc}}\phi \leq n$) if for any finite subset $\mathcal{F} \subseteq A$ and $\epsilon > 0$, there exist finite dimensional C^*-algebras $F^{(0)}, \ldots, F^{(n)}$ and maps

$$A \xrightarrow{\ \psi\ } F := F^{(0)} \oplus \cdots \oplus F^{(n)} \xrightarrow{\ \eta\ } B \tag{1.10}$$

such that ψ is cpc, $\eta|_{F^{(i)}}$ is cpc order zero for $i = 0, \ldots, n$, and such that

$$\|\eta(\psi(x)) - \phi(x)\| \leq \epsilon, \quad x \in \mathcal{F}. \tag{1.11}$$

Such an approximation (F, ψ, η) is called an *n-decomposable approximation* of ϕ for $\mathcal{F}$ up to ϵ. So $\dim_{\mathrm{nuc}}\phi = \inf\{n : \dim_{\mathrm{nuc}}\phi \leq n\}$ with the convention that $\inf \emptyset = \infty$. If additionally the maps η in these n-decomposable approximations can always be taken contractive, then ϕ is said to have *decomposition rank at most* n, written $\mathrm{dr}\,\phi \leq n$, and again $\mathrm{dr}\,\phi = \inf\{n : \mathrm{dr}\,\phi \leq n\}$. The *nuclear dimension* and *decomposition rank* of a C^*-algebra A, written $\dim_{\mathrm{nuc}}A$ and $\mathrm{dr}\,A$ respectively, are defined to be the nuclear dimension and decomposition rank of the identity map id_A.

Note that the definitions of nuclear dimension 0 and decomposition rank 0 coincide, but, for $n \geq 1$, that $\mathrm{dr}\,\phi = n$ is a stronger condition than that $\dim_{\mathrm{nuc}}\phi = n$.

The next result is a modification to maps of the fact that finite nuclear dimension and decomposition rank are inherited by hereditary subalgebras (see [59, Proposition 2.5] and [33, Proposition 3.8] respectively). The proof is a minor modification of [59, Proposition 2.5] to this context.

Proposition 1.6. *Let $\phi\colon A \to B$ be a *-homomorphism, and let $B_0 \subseteq B$ be a hereditary C^*-subalgebra such that $\phi(A) \subseteq B_0$. Let $\phi_0\colon A \to B_0$ be the corestriction of ϕ. Then* $\dim_{\mathrm{nuc}}\phi = \dim_{\mathrm{nuc}}\phi_0$ *and* $\mathrm{dr}\,\phi = \mathrm{dr}\,\phi_0$.

Proof. Clearly $\dim_{\mathrm{nuc}}\phi_0 \geq \dim_{\mathrm{nuc}}\phi$ and $\mathrm{dr}\,\phi_0 \geq \mathrm{dr}\,\phi$, so we will prove the other inequalities. Assume first that $\dim_{\mathrm{nuc}}\phi \leq n < \infty$.

Let $a_1, \ldots, a_m \in A$ be positive contractions, and $\epsilon > 0$. By perturbing each a_j slightly, we may assume that there is a positive contraction $e \in A$, such that $ea_j = a_j$ for $j = 1, \ldots, m$. Let

$$\epsilon_0 := \min\left\{\frac{\epsilon^8}{(35(n+1))^8}, \frac{1}{2^{18}}\right\}. \tag{1.12}$$

Let $h \in B_0$ be a positive contraction such that

$$\|h\phi(e) - \phi(e)\| < \epsilon_0, \tag{1.13}$$

and pick an n-decomposable cpc approximation

$$\Big(F = \bigoplus_{i=0}^{n} F^{(i)},\ \psi\colon A \to F,\ \eta = \sum_{i=0}^{n} \eta^{(i)}\colon F \to B\Big) \tag{1.14}$$

such that

$$\|\eta(\psi(x)) - \phi(x)\| < \epsilon_0, \quad x \in \{e, a_1, \ldots, a_m\}. \tag{1.15}$$

Let $\chi_{[\epsilon_0^{1/2},1]}$ denote the characteristic function of $[\epsilon_0^{1/2}, 1]$, and let p be the projection $p \coloneqq \chi_{[\epsilon_0^{1/2},1]}(\psi(e)) \in F$. Note that

$$p \leq \epsilon_0^{-1/2}\psi(e), \quad \text{and} \quad (1_F - p)\psi(e) \leq \epsilon_0^{1/2} 1_F. \tag{1.16}$$

For each $i = 0, \ldots, n$, define $p^{(i)} \coloneqq 1_{F^{(i)}} p$. Then (working in the unitisation $\widetilde{B}$ of B for convenience),

$$\begin{aligned}
\|\eta^{(i)}(p^{(i)})(1_{\widetilde{B}} - h)\| &= \|(1_{\widetilde{B}} - h)\eta^{(i)}(p^{(i)})^2(1_{\widetilde{B}} - h)\|^{1/2} \\
&\leq \|(1_{\widetilde{B}} - h)\eta^{(i)}(p^{(i)})(1_{\widetilde{B}} - h)\|^{1/2} \\
&\leq \|(1_{\widetilde{B}} - h)\eta(p)(1_{\widetilde{B}} - h)\|^{1/2} \\
&\leq \epsilon_0^{-1/4}\|(1_{\widetilde{B}} - h)\eta(\psi(e))(1_{\widetilde{B}} - h)\|^{1/2} && \text{by (1.16)} \\
&\leq \epsilon_0^{-1/4}\|(1_{\widetilde{B}} - h)\eta(\psi(e))\|^{1/2} \\
&\leq \epsilon_0^{-1/4}(\|(1_{\widetilde{B}} - h)\phi(e)\| + \epsilon_0)^{1/2} && \text{by (1.15)} \\
&\leq 2^{1/2}\epsilon_0^{1/4} && \text{by (1.13)} \qquad (1.17) \\
&\leq 1/16 && \text{by (1.12).}
\end{aligned}$$

Let $\widehat{F} \coloneqq pFp$ and $\widehat{F}^{(i)} \coloneqq pF^{(i)}p$ for $i = 0, \ldots, n$, and let $\widehat{\psi}\colon A \to \widehat{F}$ be the cpc map such that

$$\widehat{\psi}(a) = p\psi(a)p, \quad a \in A. \tag{1.18}$$

By [33, Lemma 3.6] and (1.17), for each $i \in \{0, \ldots, n\}$ there exists a cpc order zero map

$$\hat{\eta}^{(i)}\colon \widehat{F}^{(i)} \to \overline{hBh} \subseteq B_0 \tag{1.19}$$

such that for all positive $x \in \widehat{F}^{(i)}$,

$$\|\widehat{\eta}^{(i)}(x) - \eta^{(i)}(x)\| \leq 8 \cdot 2^{1/4}\epsilon_0^{1/8}\|x\| \leq 16\epsilon_0^{1/8}\|x\|. \tag{1.20}$$

Let

$$\widehat{\eta} \coloneqq \sum_{i=0}^{n} \widehat{\eta}^{(i)}\colon \widehat{F} \to B_0, \tag{1.21}$$

which is a sum of $n + 1$ cpc order zero maps by construction.

Fix $j \leq m$. Since ψ is cpc and each a_j is a positive contraction, (1.20) gives

$$\|\widehat{\eta}(\widehat{\psi}(a_j)) - \eta(\widehat{\psi}(a_j))\| \leq 16\epsilon_0^{1/8}. \tag{1.22}$$

Using that $a_j \leq e$ by assumption at the third step, and using (1.16) at the final step, we calculate,

$$\begin{aligned} \|(1_F - p)\psi(a_j)\| &= \|(1_F - p)\psi(a_j)^2(1_F - p)\|^{1/2} \\ &\leq \|(1_F - p)\psi(a_j)(1_F - p)\|^{1/2} \\ &\leq \|(1_F - p)\psi(e)(1_F - p)\|^{1/2} \\ &\leq \|(1_F - p)\psi(e)\|^{1/2} \\ &\leq \epsilon_0^{1/4}. \end{aligned} \tag{1.23}$$

Hence, using (1.23) at the second step, we see that

$$\begin{aligned} \|\widehat{\psi}(a_j) - \psi(a_j)\| &\leq \|p\psi(a_j)(1_F - p)\| + \|(1_F - p)\psi(a_j)\| \\ &\leq 2\epsilon_0^{1/4}. \end{aligned} \tag{1.24}$$

Consequently, using (1.22) and (1.24) at the third step, and then (1.12) at the final step, we have

$$\begin{aligned} &\|\widehat{\eta}(\widehat{\psi}(a_j)) - \phi(a_j)\| \\ &\quad\leq \|\widehat{\eta}(\widehat{\psi}(a_j)) - \eta(\widehat{\psi}(a_j))\| + \|\eta(\widehat{\psi}(a_j) - \psi(a_j))\| \\ &\qquad + \|\eta(\psi(a_j)) - \phi(a_j)\| \\ &\quad\leq 16\epsilon_0^{1/8} + 2(n+1)\epsilon_0^{1/4} + \epsilon_0 \\ &\quad\leq 19(n+1)\epsilon_0^{1/8} \\ &\quad\leq \epsilon, \end{aligned} \tag{1.25}$$

and so $\dim_{\mathrm{nuc}}\phi_0 \leq n$.

Finally note that when $\mathrm{dr}\,\phi \leq n$, we can additionally assume that $\|\eta\| \leq 1$. Then (1.20) ensures that $\|\widehat{\eta}\| \leq 1 + 16(n+1)\epsilon_0^{1/8}$. Let $\widetilde{\eta} \coloneqq \widehat{\eta}/\|\widehat{\eta}\|$. Then for $j = 1, \ldots, m$, using (1.25) and (1.12) at the second and third steps respectively, we have

$$\begin{aligned} \|\widetilde{\eta}(\psi(a_j)) - \phi(a_j)\| &\leq \|\widehat{\eta}(\psi(a_j)) - \phi(a_j)\| + \|\widetilde{\eta} - \widehat{\eta}\| \\ &\leq 19(n+1)\epsilon_0^{1/8} + 16(n+1)\epsilon_0^{1/8} \\ &\leq \epsilon, \end{aligned} \tag{1.26}$$

and so $\mathrm{dr}\,\phi_0 \leq n$. □

We end this section by discussing the difference between finite nuclear dimension and decomposition rank, aiming for a criterion for recognising when a system of approximations which a priori witnesses finite nuclear dimension also gives finite decomposition rank. Firstly, if $(F_i, \psi_i, \eta_i)_i$ is a net of n-decomposable approximations for id_A showing that A has finite nuclear dimension, then [59, Proposition 3.2] shows that by removing certain full matrix summands from the F_i (and modifying ψ_i and η_i accordingly), one can additionally assume that the maps $\psi_i \colon A \to F_i$ are approximately order zero. The corresponding result for decomposition rank from [33, Proposition 5.1] shows that if each η_i is also contractive, a suitable restriction of the approximations can be found so that the maps $\psi_i\colon A\to F_i$ are approximately multiplicative; this is why finite decomposition rank entails quasidiagonality. Very similar proofs can be used to establish the same results for any *-homomorphism in place of the identity map. We sketch the details of the decomposition rank case, which we will use later in the paper.

Proposition 1.7. *Let $\theta\colon A\to B$ be a *-homomorphism between C^*-algebras with $\dim_{\mathrm{nuc}}(\theta)\leq n$. Then there exist a net of n-decomposable approximations $(F_i,\psi_i,\eta_i)_i$ for θ such that the maps ψ_i are approximately order zero, i.e. the induced map $\psi\colon A\to\prod_i F_i/\bigoplus_i F_i$ is cpc order zero. If additionally $\mathrm{dr}\,\theta\leq n$, then the maps ψ_i can be taken to be approximately multiplicative, i.e. the induced map $\psi\colon A\to\prod_i F_i/\bigoplus_i F_i$ is a *-homomorphism.*

Proof. We will prove the second assertion, about θ with decomposition rank at most n. The nuclear dimension statement follows from a very similar small modification to the corresponding statement for algebras of finite nuclear dimension [59, Proposition 3.2].[8]

Fix a finite subset $\mathcal{F}\subseteq A$ of positive elements of norm 1 and $0<\epsilon<1$. Assume that θ has decomposition rank at most n. Using the Stinespring calculation of [33, Lemma 3.4], it suffices to find an n-decomposable approximation (F,ψ,η) of θ for $\mathcal{F}$ up to ϵ with η contractive such that

$$\|\psi(x^2)-\psi(x)^2\|<\epsilon,\quad x\in\mathcal{F}. \tag{1.27}$$

We closely follow the proof of [33, Proposition 5.1]. Fix an n-decomposable approximation $(\tilde{F},\tilde{\psi},\tilde{\eta})$ of θ on $\mathcal{F}$ up to $\frac{\epsilon^4}{6(n+1)}$ with $\tilde{\eta}$ contractive. Decompose $\tilde{F}=M_{r_1}\oplus\ldots\oplus M_{r_s}$, and write $\tilde{\psi}_j$ and $\tilde{\eta}_j$ for the respective components of $\tilde{\psi}$ and $\tilde{\eta}$. Set

$$I\coloneqq\{i\in\{1,\ldots,s\}:\|\tilde{\psi}_i(x^2)-\tilde{\psi}_i(x)^2\|\geq\epsilon^2\text{ for some }x\in\mathcal{F}\}. \tag{1.28}$$

Then, calculating identically[9] to [33, Proposition 5.1], for $i\in I$, we have $\|\sum_{i\in I}\tilde{\eta}_i(1_{M_{r_i}})\|\leq\frac{\epsilon^2}{2}$. Now define $F\coloneqq\bigoplus_{i\in\{1,\ldots,s\}\setminus I}M_{r_i}$. Let $\psi\colon A\to F$ be the compression of $\tilde{\psi}$ by 1_F to F and let $\eta\colon F\to B$ be the restriction of $\tilde{\eta}$. Note that (1.27) holds by construction. Moreover, for any contraction $x\in A$, we have

$$\|\eta\psi(x)-\tilde{\eta}\tilde{\psi}(x)\|\leq\Big\|\sum_{i\in I}\tilde{\eta}_i(1_{M_{r_i}})\Big\|\leq\frac{\epsilon^2}{2}. \tag{1.29}$$

Thus

$$\|\theta(x)-\eta\psi(x)\|\leq\|\theta(x)-\tilde{\eta}\tilde{\psi}(x)\|+\frac{\epsilon^2}{2}<\epsilon,\quad x\in\mathcal{F}. \tag{1.30}$$

□

Corollary 1.8. *Let $\theta\colon A\to B$ be an injective *-homomorphism with finite decomposition rank. Then A is quasidiagonal.*

Proof. As quasidiagonality is a local property, we may assume that A is separable (cf. [6, Exercise 1.1]). Let $(F_n,\psi_n,\eta_n)_{n=1}^\infty$ be an approximation for θ as in Proposition 1.7, so the map $\psi\colon A\to\prod_{n=1}^\infty F_n/\bigoplus_{n=1}^\infty F_n$ induced

[8]Note that the second line of the proof of [59, Proposition 3.2] contains a typo; it should ask for $0<\epsilon<\frac{1}{(n+2)^{16}}$.

[9]This is where the tolarance $\frac{\epsilon^4}{6(n+1)}$ plays a role in the proof.

by $(\psi_n)_{n=1}^\infty$ is a *-homomorphism. The map $\eta\colon \prod_{n=1}^\infty F_n/\bigoplus_{n=1}^\infty F_n \to B_{(\infty)}$ induced by $(\eta_n)_{n=1}^\infty$ satisfies $\iota\circ\theta = \eta\circ\psi$, where $\iota\colon B \to B_{(\infty)}$ is the canonical inclusion. As $\iota\circ\theta$ is injective, so too is ψ, so $(\psi_n)_{n=1}^\infty$ witnesses quasidiagonality of A. □

A kind of converse to Proposition 1.7 provides a method for detecting when a finite nuclear dimensional approximation (F_i, ψ_i, η_i) also gives rise to finite decomposition rank. With the benefit of hindsight, we see that the criterion below is used to prove the decomposition rank case of [5, Theorem F] and [10, Theorem B], albeit in the case where A is unital (when Lemma 1.9 has an easier proof).

Lemma 1.9. *Let $\phi\colon A \to B$ be a *-homomorphism and let $n \in \mathbb{N}_0$. Suppose there is a net $(F_i, \psi_i, \eta_i)_i$ of n-decomposable approximations converging point norm to ϕ such that the induced map*

$$\Psi := (\psi_i)\colon A \to \prod_i F_i/\bigoplus_i F_i, \tag{1.31}$$

is a *-homomorphism. Then $\operatorname{dr}\phi \le n$.

The point of the lemma is that the η_i are not assumed to be contractive.

Proof. Let $(F_i, \psi_i, \eta_i)_i$ be as in the statement of the lemma, and fix a finite subset $\mathcal{F} \subseteq A$ of contractions and $0 < \epsilon < 1$. Define $\delta := (6n+15)^{-1}\epsilon$. Pick positive contractions $e_0, e_1 \in A$ such that $e_0e_1 = e_1$ and

$$\max\{\|ae_1 - a\|, \|e_1a - a\|\} \le \delta, \quad a \in \mathcal{F}. \tag{1.32}$$

Let $g\colon [0,1] \to [0,1]$ denote the continuous function which is 0 on $[0, 1-\delta]$, 1 on $[1-\delta/2, 1]$ and affine otherwise. By hypothesis Ψ is a *-homomorphism, so as $e_0e_1 = e_1$,

$$g(\Psi(e_0))\Psi(e_1) = \Psi(e_1). \tag{1.33}$$

Hence there exists i such that (F_i, ψ_i, η_i) approximates ϕ on $\mathcal{F} \cup \{e_0, e_1\}$ up to δ,

$$\psi_i(a) \approx_{4\delta} \psi_i(e_1)\psi_i(a)\psi_i(e_1), \quad a \in \mathcal{F}, \tag{1.34}$$

and

$$g(\psi_i(e_0))\psi_i(e_1) \approx_\delta \psi_i(e_1). \tag{1.35}$$

Let $\chi_{[1-\delta,1]}$ denote the characteristic function of $[1-\delta, 1]$, and let $p_i := \chi_{[1-\delta,1]}(\psi_i(e_0)) \in F_i$ be the corresponding spectral projection. Define $\widehat{F_i} := p_iF_ip_i$, define $\widehat{\psi_i}\colon A \to F_i$ by $\widehat{\psi_i}(\cdot) := p_i\psi_i(\cdot)p_i$ and define $\widehat{\eta_i}\colon F_i \to A$ by $\widehat{\eta_i} := \frac{1-\delta}{1+\delta}\eta_i|_{F_i}$. Note that $\widehat{\psi_i}$ is cpc and $\widehat{\eta_i}$ is cp. The algebra $\widehat{F_i}$ inherits the decomposition into $(n+1)$-summands on which $\widehat{\eta_i}$ restricts to a cpc order zero map from F_i and η_i.

By definition of p_i, we have $(1-\delta)p_i \le \psi_i(e_0)$, so that

$$\eta_i((1-\delta)p_i) \le \eta_i(\psi_i(e_0)) \approx_\delta \phi(e_0). \tag{1.36}$$

In particular $(1-\delta)\|\eta_i(p_i)\| \le 1+\delta$ and thus

$$\|\widehat{\eta_i}\| = \frac{1-\delta}{1+\delta}\|\eta_i(p_i)\| \le 1. \tag{1.37}$$

The definition of p_i also ensures that

$$g(\psi(e_0))p_i = p_i g(\psi(e_0)) = g(\psi(e_0)). \tag{1.38}$$

For $a \in \mathcal{F}$, applying (1.34), (1.35), and then (1.38), we have

$$\begin{aligned}\widehat{\eta}_i(\widehat{\psi}_i(a)) &\approx_{4\delta} \tfrac{1-\delta}{1+\delta}\eta_i(p_i\psi_i(e_1)\psi_i(a)\psi_i(e_1)p_i)\\ &\approx_{2\delta} \tfrac{1-\delta}{1+\delta}\eta_i(p_i g(\psi_i(e_0))\psi_i(e_1)\psi_i(a)\psi_i(e_1)g(\psi_i(e_0))p_i)\\ &= \tfrac{1-\delta}{1+\delta}\eta_i(g(\psi_i(e_0))\psi_i(e_1)\psi_i(a)\psi_i(e_1)g(\psi(e_0))). \end{aligned}\tag{1.39}$$

Now, using that $\|\eta_i\| \leq n+1$, we continue the calculation above, obtaining

$$\begin{aligned}\widehat{\eta}_i(\widehat{\psi}_i(a)) &\approx_{6\delta+2(n+1)\delta} \tfrac{1-\delta}{1+\delta}\eta_i(\psi_i(e_1)\psi_i(a)\psi_i(e_1)) && \text{by (1.35)}\\ &\approx_{4(n+1)\delta} \tfrac{1-\delta}{1+\delta}\eta_i(\psi_i(a))) && \text{by (1.34)}\\ &\approx_{\delta} \tfrac{1-\delta}{1+\delta}\phi(a). \end{aligned}\tag{1.40}$$

Since $1 - \frac{1-\delta}{1+\delta} \leq 2\delta$, this gives

$$\|\phi(a) - \widehat{\eta}_i\widehat{\psi}_i(a)\| \leq 9\delta + 6(n+1)\delta = \epsilon, \quad a \in \mathcal{F}. \tag{1.41}$$

Hence $\operatorname{dr}\phi \leq n$. □

2. The ideal lattices of C^*-algebras

In this section, we collect various facts regarding the ideal lattice of a C^*-algebra, which we need for our main existence argument.

Here, and throughout, by an ideal in a C^*-algebra A we always mean a closed, two-sided ideal. As is customary, $\overline{ASA}$ denotes the ideal generated by a non-empty subset $S \subseteq A$. So $\overline{ASA} = \overline{\operatorname{span}}\{axb : a, b \in A,\ x \in S\}$. If $S = \{a\}$ is a singleton, we write $\overline{AaA}$ rather than $\overline{A\{a\}A}$.

Given an ideal I in a C^*-algebra A, an element $a \in I$ is *full*, if it generates I as an ideal; that is, if $I = \overline{AaA}$. Every ideal in a separable C^*-algebra has a full element (for example, any strictly positive element is full).

The lattice $\mathcal{I}(A)$ of ideals of a C^*-algebra A is complete in the sense that every $\mathcal{S} \subseteq \mathcal{I}(A)$ has a supremum and an infimum. Indeed,

$$\sup \mathcal{S} = \overline{\sum_{I\in\mathcal{S}} I}, \qquad \text{and} \qquad \inf \mathcal{S} = \bigcap_{I\in\mathcal{S}} I \tag{2.1}$$

for every non-empty subset $\mathcal{S} \subseteq \mathcal{I}(A)$; by convention, we take $\sup\emptyset = 0$ and $\inf\emptyset = A$. There is a notion of compact containment in $\mathcal{I}(A)$: if $I, J \in \mathcal{I}(A)$, then I is *compactly contained* in J, written $I \Subset J$, if whenever $(I_\lambda)_{\lambda\in\Lambda}$ is a family in $\mathcal{I}(A)$ such that $J \subseteq \overline{\sum_{\lambda\in\Lambda} I_\lambda}$ then there are finitely many $\lambda_1, \dots, \lambda_n \in \Lambda$ such that $I \subseteq \sum_{i=1}^n I_{\lambda_i}$.

With suprema and compact containment, the ideal lattice of a separable C^*-algebra fits into the framework of the abstract Cuntz semigroup category Cu introduced in [11] (see [20, Proposition 2.5]). Given C^*-algebras A and B, a map between their ideal lattices $\Phi\colon \mathcal{I}(A) \to \mathcal{I}(B)$ is a Cu-*morphism* if it preserves suprema and compact containment. By [20, Lemma 2.12] any *-homomorphism $\phi\colon A \to B$ induces a Cu-morphism

$$\mathcal{I}(\phi)\colon \mathcal{I}(A) \to \mathcal{I}(B), \qquad I \mapsto \overline{B\phi(I)B}. \tag{2.2}$$

Then $\mathcal{I}(\cdot)$ is a covariant functor from C^*-algebras and *-homomorphisms to complete lattices and Cu-morphisms [20, Proposition 2.15].

The presence of a full element in an ideal is detected by the ideal lattice structure ([20, Corollary 2.3]), and so preserved by Cu-morphisms.

Lemma 2.1. *Let A, B be C^*-algebras with A separable, and let $\Theta\colon \mathcal{I}(A) \to \mathcal{I}(B)$ be a* Cu*-morphism. Then, for every non-zero ideal $J \in \mathcal{I}(A)$, the ideal $\Theta(J)$ has a full element.*

Proof. Since A is separable, J has a full element. So [20, Corollary 2.3] gives $J = \overline{\bigcup_{n=1}^\infty J_n}$ for some sequence of ideals $J_n \in \mathcal{I}(A)$ such that $J_n \Subset J_{n+1}$ for all n. As Θ preserves suprema and compact containment, $\Theta(J) = \overline{\bigcup_{n=1}^\infty \Theta(J_n)}$, and $\Theta(J_n) \Subset \Theta(J_{n+1})$ for all n. Hence $\Theta(J)$ has a full element by [20, Corollary 2.3]. □

It is well-known that the ideal lattice of a separable C^*-algebra is countably generated. The next lemma provides a slight strengthening of this statement.

Lemma 2.2. *Let A be a separable C^*-algebra. Then there is a sequence $(J_n)_{n=1}^\infty$ in $\mathcal{I}(A)$, such that for any $J \in \mathcal{I}(A)$ there is a strictly increasing sequence $(k_n)_{n=1}^\infty$ in $\mathbb{N}$, such that $J_{k_n} \Subset J_{k_{n+1}}$ for all n, and $\overline{\bigcup_{n=1}^\infty J_{k_n}} = J$.*

Proof. As A is separable, [38, Corollary 4.3.4] shows that $\mathcal{I}(A)$ has a countable basis $\mathcal{B} \subseteq \mathcal{I}(A)$; that is, for every $I \in \mathcal{I}(A)$, there exists $\mathcal{B}_I \subseteq \mathcal{B}$ such that $I = \sup \mathcal{B}_I$. Replacing $\mathcal{B}$ with $\{\sum_{I\in\mathcal{S}} I : \emptyset \neq \mathcal{S} \subseteq \mathcal{B} \text{ is finite}\}$, we may assume that $\mathcal{B}$ is upwards directed. Let $(J_n)_{n=1}^\infty$ be a sequence in $\mathcal{B}$ in which each $I \in \mathcal{B}$ appears infinitely often; that is, $\{n \in \mathbb{N} : J_n = I\}$ is infinite for each $I \in \mathcal{B}$.

Fix $J \in \mathcal{I}(A)$. As A is separable, J contains a full element, so by [20, Corollary 2.3] there is a sequence $I_1 \Subset I_2 \Subset \dots$ in $\mathcal{I}(A)$ such that $J = \overline{\bigcup_{n=1}^\infty I_n}$. As $\mathcal{B}$ is an upwards directed basis, and as each element in $(J_n)_{n=1}^\infty$ appears infinitely often, we may use compact containment to find $k_1 < k_2 < \dots$ such that $I_{2n} \subseteq J_{k_n} \subseteq I_{2n+1}$ for each $n \in \mathbb{N}$. Then $J_{k_1} \Subset J_{k_2} \Subset \dots$ and $J = \overline{\bigcup_{n=1}^\infty J_{k_n}}$. □

The previous lemma allows us to approximate the ideal lattice of a separable C^*-algebra by an increasing sequence of finite lattices. The definition and lemma below collect some notation and an easy fact regarding these finite lattices.

Definition 2.3. Let $\mathcal{I}$ be a finite lattice. Given $I \in \mathcal{I}$, we call $J \in \mathcal{I}$ a *predecessor* of I if $J < I$ and $\{K \in \mathcal{I} : J \leq K \leq I\} = \{J, I\}$. Note that an element $J \in \mathcal{I}$ is the unique predecessor of $I \in \mathcal{I}$ if $\{K \in \mathcal{I} : K < I\} = \{K \in \mathcal{I} : K \leq J\}$. We write $\mathcal{P}(\mathcal{I})$ for the set of elements in $\mathcal{I}$ that have a unique predecessor. For $I \in \mathcal{P}(\mathcal{I})$, we write $P(I)$ for the unique predecessor of I.

For the next lemma, note that every finite lattice $\mathcal{I}$ has a minimum element $\inf_{I\in\mathcal{I}} I$, which is by convention the supremum in $\mathcal{I}$ of the empty set.

Lemma 2.4. *Let $\mathcal{I}$ be a finite lattice. Then any $I \in \mathcal{I}$ satisfies*

$$I = \sup_{\substack{J \in \mathcal{P}(\mathcal{I}) \\ J \leq I}} J. \tag{2.3}$$

Proof. As the empty supremum defines the minimal element, (2.3) holds for the minimal element of $\mathcal{I}$. Fix $I \in \mathcal{I}$ and suppose inductively that (2.3) holds for all $J \in \mathcal{I}$ with $J \leq I$ and $J \neq I$. If I has a unique predecessor then (2.3) holds vacuously. Otherwise I is the join of its predecessors, each of which satisfies (2.3), so I satisfies (2.3). □

For nested ideals $J \subseteq I$ in a C^*-algebra A, let

$$I^J \coloneqq \{x \in A : xI \cup Ix \subseteq J\}. \tag{2.4}$$

This is readily seen to be an ideal in A.

The next lemma combines the above construction of ideals with the unique predecessors.

Lemma 2.5. *Let A be a C^*-algebra, and let $\mathcal{I} \subseteq \mathcal{I}(A)$ be a finite sublattice. Suppose that $J \in \mathcal{I}$ and $K \in \mathcal{P}(\mathcal{I})$ satisfy $K \not\subseteq J$. Then $J \subseteq K^{P(K)}$.*

Proof. Since $K \not\subseteq J$, we have $J \cap K \subsetneq K$. Since $P(K)$ is the unique predecessor of K in $\mathcal{I}$, we have $J \cap K \subseteq P(K)$. Since $J \cap K = KJ = JK$, we deduce that $J \subseteq K^{P(K)}$. □

The following 'lower bound' lemma plays a key role in our proof of Sublemma 3.7 below, which we use to control our 'upward maps'. In the following proof and elsewhere, given a positive element a of a C^*-algebra and a positive constant λ, we write $(a-\lambda)_+$ for the element obtained by applying functional calculus for a to the function $t \mapsto \max\{t-\lambda, 0\}$.

Lemma 2.6. *Let A be a C^*-algebra and let $J \Subset I$ be a compact containment of ideals in A. For any full, positive element $a \in I$, there exists a constant $\lambda > 0$ such that, for any ideals $K_1 \subsetneq K_2 \subseteq J$, we have*

$$\|a + K_2^{K_1}\|_{A/K_2^{K_1}} \geq \lambda. \tag{2.5}$$

Proof. As $I = \overline{\bigcup_{\lambda>0} A(a-\lambda)_+ A}$, the compact containment $J \Subset I$ provides $\lambda > 0$ such that $J \subseteq \overline{A(a-\lambda)_+ A}$.

Fix ideals $K_1 \subsetneq K_2 \subseteq J$. Suppose for contradiction that $(a-\lambda)_+ \in K_2^{K_1}$. Then

$$K_2 \subseteq \overline{A(a-\lambda)_+ A} \subseteq K_2^{K_1}. \tag{2.6}$$

Then $K_2 \subseteq K_2 \cap K_2^{K_1} = K_2 \cdot K_2^{K_1} \subseteq K_1$, by definition of $K_2^{K_1}$, contradicting $K_1 \subsetneq K_2$. Hence $(a-\lambda)_+ \notin K_2^{K_1}$, and so

$$\|a + K_2^{K_1}\|_{A/K_2^{K_1}} \geq \lambda. \tag{2.7}$$

□

Before turning to our main construction, we record a standard fact regarding the behaviour of ideals in $C_0((0,1])$ under the multiplication map.

Lemma 2.7. *Let $m\colon [0,1]\times[0,1]\to[0,1]$ be the multiplication map, and $m^*\colon C_0((0,1])\to C_0((0,1])\otimes C_0((0,1])$ be the induced *-homomorphism. For every non-zero, positive element $h\in C_0((0,1])$ there are positive, non-zero contractions $f,g\in C_0((0,1])$ such that $g(1)=1$ and $f\otimes g\le m^*(h)$.*

In particular, for any non-zero $I\in\mathcal{I}(C_0((0,1]))$, the ideal $\mathcal{I}(m^*)(I)$ contains a non-zero, positive contraction of the form $f\otimes g$, for which $g(1)=1$.

Proof. We may find $\delta>0$ and a non-empty, open subset $U\subseteq(0,1]$ such that $h(s)\ge\delta$ for $s\in U$. Fix $s\in U\setminus\{1\}$. As $(s,1)\in m^{-1}(U)$ we may pick $\epsilon>0$ such that $[s-\epsilon,s+\epsilon]\times[1-\epsilon,1]\subseteq m^{-1}(U)$. Let $f,g\in C_0((0,1])$ be non-zero, positive contractions supported in $[s-\epsilon,s+\epsilon]$ and $[1-\epsilon,1]$ respectively, and such that $\|f\|\le\delta$ and $g(1)=1$. Then $f\otimes g\le m^*(h)$. □

3. Main technical construction

In this section we provide the main technical construction of the paper, obtaining *-homomorphisms $C_0((0,1])\otimes A\to B_{(\infty)}$ induced by sequences of maps of nuclear dimension zero with specified behaviour on ideals. Our first lemma is the main tool for producing the 'downward' maps, using Voiculescu's quasidiagonality of cones [53] in the spirit of earlier nuclear dimension computations [36, 48, 5].

Lemma 3.1. *Let A be a separable C^*-algebra, and let $J\subsetneq I\subseteq A$ be ideals. Then there exist a sequence $(M_{k_n})_{n=1}^\infty$ of matrix algebras and a sequence $(\psi_n)_{n=1}^\infty$ of cpc maps $\psi_n\colon C_0((0,1])\otimes A\to M_{k_n}$ such that the induced map*

$$\psi := (\psi_n)_{n=1}^\infty\colon C_0((0,1])\otimes A\to\prod_n M_{k_n}/\bigoplus_n M_{k_n} \tag{3.1}$$

is a *-homomorphism such that for all $f\in C_0((0,1])$ and $a\in A$,

$$|f(1)|\,\|a+I^J\|_{A/I^J}\le\|\psi(f\otimes a)\|\le\|f\|\,\|a+I^J\|_{A/I^J}. \tag{3.2}$$

If A/I^J is quasidiagonal we can choose the ψ_n so that $\psi_n(C_0((0,1))\otimes A)=0$ for all $n\in\mathbb{N}$.

Proof. By Voiculescu's quasidiagonality of cones [53] there exist integers k_n and cpc maps

$$\widetilde{\psi}_n\colon C_0((0,1])\otimes(A/I^J)\to M_{k_n} \tag{3.3}$$

such that the map $\widetilde{\psi}\colon C_0((0,1])\otimes A/I^J\to\prod_n M_{k_n}/\bigoplus_n M_{k_n}$ induced by the $\widetilde{\psi}_n$ is an injective *-homomorphism. Let $q_{I^J}\colon A\to A/I^J$ denote the quotient map, and for each $n\in\mathbb{N}$, let

$$\psi_n := \widetilde{\psi}_n\circ(\mathrm{id}_{C_0((0,1])}\otimes q_{I^J})\colon C_0((0,1])\otimes A\to M_{k_n}. \tag{3.4}$$

Let $\psi\colon C((0,1])\otimes A\to\prod_{n=1}^\infty M_{k_n}/\bigoplus_{n=1}^\infty M_{k_n}$ be the induced map, so $\psi=\widetilde{\psi}\circ(\mathrm{id}_{C_0((0,1])}\otimes q_{I^J})$. Then for $f\in C_0((0,1])$ and $a\in A$, since $\widetilde{\psi}$ is isometric,

$$\|\psi(f\otimes a)\|=\|\widetilde{\psi}(f\otimes q_{I^J}(a))\|=\|f\|\,\|q_{I^J}(a)\|\ge|f(1)|\,\|q_{I^J}(a)\|. \tag{3.5}$$

If A/I^J is quasidiagonal, then there exist integers k_n and cpc maps $\overline{\psi}_n\colon A/I^J\to M_{k_n}$ such that the induced map $\overline{\psi}:=(\overline{\psi}_n)_{n=1}^\infty\colon A/I^J\to$

$\prod_n M_{k_n}/\bigoplus_n M_{k_n}$ is an injective *-homomorphism. Let $\mathrm{ev}_1\colon C_0((0,1])\otimes A\to A$ be evaluation at 1, and for each n, let

$$\psi_n := \overline{\psi}_n\circ q_{I^J}\circ \mathrm{ev}_1\colon C_0((0,1])\otimes A\to M_{k_n}, \tag{3.6}$$

so that $\psi_n(C_0((0,1))\otimes A)=0$. Let $\psi\colon A\to\prod_{n=1}^\infty M_{k_n}/\bigoplus_{n=1}^\infty M_{k_n}$ be the map induced by $(\psi_n)_{n=1}^\infty$. Then

$$\|\psi(f\otimes a)\| = |f(1)|\,\|\overline{\psi}(q_{I^J}(a))\| = |f(1)|\,\|q_{I^J}(a)\|\le \|f\|\,\|q_{I^J}(a)\| \tag{3.7}$$

for all $a\in A$ and $f\in C_0((0,1])$. □

Recall that if $a,b\in A$ are positive elements, then a is said to be *Cuntz dominated* by b, written $a\precsim b$, if there is a sequence of elements $(z_n)_{n=1}^\infty$ in A such that $z_n^*bz_n\to a$. In general it is not possible to choose $(z_n)_{n=1}^\infty$ to be bounded without additional assumptions. The next lemma identifies one such assumption.

Lemma 3.2. *Let A be a C^*-algebra, and let $a,b\in A_+$. Suppose that for every continuous function $f\colon[0,\|b\|]\to[0,1]$ with $f(0)=0$ and $f(\|b\|)\neq 0$, we have $a\precsim f(b)$. Then there is a sequence $(z_n)_{n=1}^\infty$ in A such that $z_n^*bz_n\to a$ and*

$$\|z_n\| = (\|a\|/\|b\|)^{1/2},\quad n\in\mathbb{N}. \tag{3.8}$$

Proof. We may assume that $\|a\|=\|b\|=1$. For each $n\in\mathbb{N}$, let $\delta_n:=1-1/n$. Our hypothesis implies that for each n we have $a\preceq(b-\delta_n)_+$, so there exists $y_n\in A$ such that $\|y_n^*(b-\delta_n)_+y_n-a\|<1/n$. Let $z_n':=(b-\delta_n)_+^{1/2}y_n$ so that $\|z_n'\|\to\|a\|^{1/2}=1$, and let $z_n:=z_n'/\|z_n'\|$ for each n. Then each $\|z_n\|=1$ and $z_n^*z_n\to a$.

Let $f_n\colon[0,1]\to[0,1]$ be the continuous function which is 0 on $[0,\delta_n-1/n]$, 1 on $[\delta_n,1]$ and affine otherwise. Then $f_n(b)z_n=z_n$, and

$$\|f_n(b)-f_n(b)b\|\le\max\Big\{\sup_{t\in[\delta_n,1]}(1-t),\ \sup_{t\in[\delta_n-1/n,\delta_n]}f_n(t)(1-t)\Big\}\le 2/n. \tag{3.9}$$

Hence

$$z_n^*bz_n = z_n^*f_n(b)bz_n \approx_{2/n} z_n^*f_n(b)z_n = z_n^*z_n. \tag{3.10}$$

Therefore $z_n^*bz_n\to a$. □

We now turn to the construction of the ‘upwards maps’. Here we need the additional space given by absorption of the Cuntz algebra $\mathcal{O}_\infty$ from [12]. Recall that a C^*-algebra B is $\mathcal{O}_\infty$-stable if $B\otimes\mathcal{O}_\infty\cong B$. By [31, Proposition 4.5], each such C^*-algebra B is *purely infinite* in the sense that B has no characters and for all $a,b\in B_+$ with $a\in\overline{BbB}$, we have $a\precsim b$. We begin by recording a by-now-standard consequence of $\mathcal{O}_\infty$-stability for repeated later use (and also as results of this type in the literature are often stated with separability hypotheses).

Lemma 3.3. *Let B be a C^*-algebra such that $B\cong B\otimes\mathcal{O}_\infty$. Then there exists an isomorphism $\kappa\colon B\otimes\mathcal{O}_\infty\to B$ such that $\mathcal{I}(\kappa)(K\otimes\mathcal{O}_\infty)=K$ for all $K\in\mathcal{I}(B)$.*

Proof. Fix any isomorphism $\theta\colon B \cong B \otimes \mathcal{O}_\infty$. There exists an isomorphism $\alpha\colon \mathcal{O}_\infty \to \mathcal{O}_\infty \otimes \mathcal{O}_\infty$ which is approximately unitarily equivalent to the first factor embedding $\mathrm{id}_{\mathcal{O}_\infty} \otimes 1_{\mathcal{O}_\infty}\colon \mathcal{O}_\infty \to \mathcal{O}_\infty \otimes \mathcal{O}_\infty$.[10] Define κ to be the composition

$$B \otimes \mathcal{O}_\infty \xrightarrow{\theta\otimes \mathrm{id}_{\mathcal{O}_\infty}} B \otimes \mathcal{O}_\infty \otimes \mathcal{O}_\infty \xrightarrow{\mathrm{id}_B \otimes \alpha^{-1}} B \otimes \mathcal{O}_\infty \xrightarrow{\theta^{-1}} B. \tag{3.11}$$

By construction κ is an isomorphism and κ^{-1} is approximately unitarily equivalent (via unitaries in $B^\sim \otimes \mathcal{O}_\infty$) to the first factor embedding $\mathrm{id}_B \otimes 1_{\mathcal{O}_\infty}\colon B \to B \otimes \mathcal{O}_\infty$. Therefore $\mathcal{I}(\kappa^{-1}) = \mathcal{I}(\mathrm{id}_B \otimes 1_{\mathcal{O}_\infty})$. Since $\mathcal{I}$ is functorial ([20, Proposition 2.15]), $\mathcal{I}(\kappa) = \mathcal{I}(\mathrm{id}_B \otimes 1_{\mathcal{O}_\infty})^{-1}$, so that $\mathcal{I}(\kappa)(K \otimes \mathcal{O}_\infty) = K$ for all $K \in \mathcal{I}(B)$. $\square$

The next lemma produces appropriate families of pairwise orthogonal positive elements $(h_I)_{I\in\mathcal{P}(I)}$. We shall produce our 'upwards maps' out of finite dimensional algebras by embedding the finite dimensional algebras into $\mathcal{O}_\infty$ and tensoring by these (h_I). The symbol $\mathcal{P}(\mathcal{I})$ is defined in Definition 2.3.

Lemma 3.4. *Let A and B be C^*-algebras such that A is separable and $B \otimes \mathcal{O}_\infty \cong B$. Suppose that $\Theta\colon \mathcal{I}(A) \to \mathcal{I}(B)$ is a* Cu*-morphism. Let $\mathcal{I}_0 \subseteq \mathcal{I}(A)$ be a finite sublattice. Then there exists a collection $\{h_I : I \in \mathcal{P}(\mathcal{I}_0)\} \subseteq B$ of pairwise orthogonal positive elements, each with spectrum $[0,1]$ such that*

- (a) *for each $I \in \mathcal{P}(\mathcal{I}_0)$, the element h_I is full in $\Theta(I)$, and*
- (b) *for any pair of ideals $J_1 \subseteq J_2$ in $\mathcal{I}_0$, such that $J_1 \Subset J_2$ in $\mathcal{I}(A)$, and any non-zero, continuous function $f\colon [0,1] \to [0,1]$ with $f(0) = 0$, $\Theta(J_1)$ is contained in the ideal of B generated by*

$$\sum_{\substack{I\in\mathcal{P}(\mathcal{I}_0)\\ I\subseteq J_2}} f(h_I). \tag{3.12}$$

Proof. By Lemma 2.1, for each non-zero $I \in \mathcal{P}(\mathcal{I}_0)$, the ideal $\Theta(I)$ contains a full positive element k_I of norm 1. Use Lemma 3.3 to fix an isomorphism $\kappa\colon B \otimes \mathcal{O}_\infty \to B$ such that $\mathcal{I}(\kappa)(K \otimes \mathcal{O}_\infty) = K$ for all $K \in \mathcal{I}(B)$. Choose non-zero pairwise orthogonal projections $(p_I)_{I\in\mathcal{I}_0}$ in $\mathcal{O}_\infty$, and define pairwise orthogonal positive elements of norm 1 by $h'_I := \kappa(k_I \otimes p_I)$ for $I \in \mathcal{P}(\mathcal{I}_0)$. Since $\mathcal{O}_\infty$ is simple and each k_I is full in I, each $k_I \otimes p_I$ is full in $\Theta(I) \otimes \mathcal{O}_\infty$. Hence, by the definition of κ, each h'_I is full in $\Theta(I)$.

For $n \in \mathbb{N}$, let $g_n\colon [0,1] \to [0,1]$ be the continuous function which is 0 on $[0, \frac{1}{2n}]$, 1 on $[\frac{1}{n}, 1]$ and affine on $[\frac{1}{2n}, \frac{1}{n}]$. Fix $J_1 \subseteq J_2$ in $\mathcal{I}_0$ such that $J_1 \Subset J_2$ in $\mathcal{I}(A)$. Lemma 2.4 implies that

$$J_2 = \sum_{\substack{I\in\mathcal{P}(\mathcal{I}_0)\\ I\subseteq J_2}} I. \tag{3.13}$$

[10] By [29, Theorem 3.15], one has $\mathcal{O}_\infty \cong \mathcal{O}_\infty \otimes \mathcal{O}_\infty$, and by [35, Theorem 3.3], any two unital *-homomorphisms $\phi, \psi\colon \mathcal{O}_\infty \to \mathcal{O}_\infty \otimes \mathcal{O}_\infty$ are approximately unitarily equivalent.

Since ideals are hereditary, for positive elements $a_1, \dots, a_n$ of a C^*-algebra D, we have $\overline{D\big(\sum_i a_i\big)D} = \sum_i \overline{Da_iD}$. Thus the positive element

$$h''_{J_2} := \sum_{\substack{I \in \mathcal{P}(\mathcal{I}_0) \\ I \subseteq J_2}} h'_I \tag{3.14}$$

is full in $\Theta(J_2)$. As $(g_n(h''_{J_2}))_{n=1}^\infty$ is an increasing approximate identity for $C^*(h''_{J_2})$, it follows that

$$\overline{\bigcup_{n=1}^{\infty} \overline{Bg_n(h''_{J_2})B}} = \overline{Bh''_{J_2}B} = \Theta(J_2). \tag{3.15}$$

As Θ is a Cu-morphism, and $J_1 \Subset J_2$, it follows that $\Theta(J_1) \Subset \Theta(J_2)$. Hence there exists $n_{J_1,J_2} \in \mathbb{N}$, such that every $n \geq n_{J_1,J_2}$ satisfies

$$\Theta(J_1) \subseteq \overline{Bg_n(h''_{J_2})B}. \tag{3.16}$$

Since $\mathcal{I}_0$ is finite, we may define $n_0 := \max\{n_{J_1,J_2} : J_1, J_2 \in \mathcal{I}_0 \text{ and } J_1 \Subset J_2 \text{ in } \mathcal{I}(A)\}$. Let $r\colon [0,1] \to [0,1]$ be the continuous function satisfying $r(t) = (2n_0)t$ for $t \in [0, \frac{1}{2n_0}]$ and $r(t) = 1$ for $t \geq \frac{1}{2n_0}$. So $r(t)g_{n_0}(t) = g_{n_0}(t)$ for all t. Fix $h \in (\mathcal{O}_\infty)_+$ with spectrum $\sigma(h) = [0,1]$, and for each $I \in \mathcal{P}(\mathcal{I}_0)$, let $h_I := \kappa(r(h'_I) \otimes h)$. Each h_I is positive with spectrum $[0,1]$ since both $r(h'_I)$ and h are positive elements with spectrum $[0,1]$.

By choice of κ and by simplicity of $\mathcal{O}_\infty$, we have $\overline{Bh_IB} = \overline{Br(h'_I)B}$ for each $I \in \mathcal{P}(\mathcal{I}_0)$. For each $I \in \mathcal{P}(\mathcal{I}_0)$, the element $r(h'_I)$ is a positive contraction in $\Theta(I)$ with $r(h'_I) \geq h'_I$, and so $r(h'_I)$ has norm one and is full in $\Theta(I)$. Hence h_I is full in $\Theta(I)$. Since the h'_I are pairwise orthogonal positive contractions so are the h_I. It remains to check (b).

Take $J_1 \subseteq J_2$ in $\mathcal{I}_0$ with $J_1 \Subset J_2$ in $\mathcal{I}(A)$. Let $f\colon [0,1] \to [0,1]$ be non-zero and continuous with $f(0) = 0$. Let K be the ideal generated by

$$\sum_{\substack{I \in \mathcal{P}(\mathcal{I}_0) \\ I \subseteq J_2}} f(h_I). \tag{3.17}$$

We must show that $\Theta(J_1) \subseteq K$.

Let $\Phi_I\colon C_0((0,1]) \otimes C_0((0,1]) \to B \otimes \mathcal{O}_\infty$ be the $*$-homomorphism given on elementary tensors by $\Phi_I(f_1 \otimes f_2) = f_1(r(h'_I)) \otimes f_2(h)$. Let $m\colon (0,1] \times (0,1] \to (0,1]$ be multiplication, and let $m^*\colon C_0((0,1]) \to C_0((0,1]) \otimes C_0((0,1])$ be the induced homomorphism. Lemma 2.7 yields positive functions $f_1, f_2 \in C_0((0,1])$ such that $f_1(1) = 1$ and

$$f_1 \otimes f_2 \leq m^*(f) \in C_0((0,1]) \otimes C_0((0,1]). \tag{3.18}$$

As $\Phi_I(m^*(f)) = f(r(h'_I) \otimes h)$, it follows that

$$f(h_I) = \kappa(\Phi_I(m^*(f))) \geq \kappa(\Phi_I(f_1 \otimes f_2)) = \kappa(f_1(r(h'_I)) \otimes f_2(h)), \tag{3.19}$$

for $I \in \mathcal{P}(\mathcal{I}_0)$. As $(f_1 \circ r) \cdot g_{n_0} = g_{n_0}$ it follows that K contains the element

$$\sum_{\substack{I \in \mathcal{P}(\mathcal{I}_0) \\ I \subseteq J_2}} \kappa(g_{n_0}(h'_I) \otimes f_2(h)). \tag{3.20}$$

As h has spectrum $[0,1]$, the element $f_2(h) \in \mathcal{O}_\infty$ is non-zero and thus full. Hence by the defining property of κ, the element (3.20) generates the same ideal as $\sum_{I\in\mathcal{P}(\mathcal{I}_0),\, I\subseteq J_2} g_{n_0}(h'_I)$. Using first that the h'_I are pairwise orthogonal, and then the definition (3.14) of h''_{J_2} we calculate:

$$\sum_{\substack{I\in\mathcal{P}(\mathcal{I}_0)\\ I\subseteq J_2}} g_{n_0}(h'_I) = g_{n_0}\Big(\sum_{\substack{I\in\mathcal{P}(\mathcal{I}_0)\\ I\subseteq J_2}} h'_I\Big) = g_{n_0}(h''_{J_2}). \tag{3.21}$$

In particular, $g_{n_0}(h''_{J_2}) \in K$. As $n_0 \geq n_{J_1,J_2}$, it follows from (3.16) that

$$\Theta(J_1) \subseteq \overline{B g_{n_0}(h''_{J_2}) B} \subseteq K. \tag{3.22}$$ □

We now give our main technical existence result realising Cu-morphisms by *-homomorphisms from cones. In the proof, we will use that a finite lattice has a unique minimum element (in our case 0), and again take the convention that the supremum of the empty set is this minimum element. See Section 2 for further details.

Lemma 3.5. *Let A and B be C^*-algebras with A separable and $B\otimes\mathcal{O}_\infty \cong B$, and let $\Theta\colon \mathcal{I}(A) \to \mathcal{I}(B)$ be a* Cu*-morphism. Then for $n \in \mathbb{N}$, there are finite dimensional C^*-algebras F_n and cpc maps $\psi_n\colon C_0((0,1])\otimes A \to F_n$ and $\eta_n\colon F_n \to B$ such that, writing*

$$\psi := (\psi_n)_{n=1}^\infty \colon C_0((0,1])\otimes A \to \frac{\prod_{n=1}^\infty F_n}{\bigoplus_{n=1}^\infty F_n}, \quad \text{and} \tag{3.23}$$

$$\eta := (\eta_n)_{n=1}^\infty \colon \frac{\prod_{n=1}^\infty F_n}{\bigoplus_{n=1}^\infty F_n} \to B_{(\infty)} \tag{3.24}$$

for the induced maps, the following are satisfied:

- (*a*) *ψ is a *-homomorphism;*
- (*b*) *each η_n is order zero; and*
- (*c*) *the *-homomorphism $\rho\colon C_0((0,1])\otimes A \to B_{(\infty)}$ induced by the cpc order zero map $(\eta\circ\psi)(\mathrm{id}_{(0,1]}\otimes\cdot)$ (see Proposition 1.1) satisfies*

$$\mathcal{I}(\rho)(I\otimes J) = \overline{B_{(\infty)}\Theta(J)B_{(\infty)}} \tag{3.25}$$

for any $J\in\mathcal{I}(A)$ and any non-zero $I\in\mathcal{I}(C_0((0,1]))$.

If every quotient of A is quasidiagonal, we can additionally arrange that $\psi_n(C_0((0,1))\otimes A) = 0$ for each $n\in\mathbb{N}$.

Proof. Lemma 2.2 yields a countable basis $(J_n)_{n=1}^\infty$ for $\mathcal{I}(A)$ such that for each $J\in\mathcal{I}(A)$ there exist $k_1 < k_2 < \dots$ satisfying $J_{k_1} \Subset J_{k_2} \Subset \dots$ and $J = \overline{\bigcup_{n=1}^\infty J_{k_n}}$. For each n, let $\mathcal{I}_n$ be the sublattice of $\mathcal{I}(A)$ generated by $0, A, J_1, \dots, J_n$. Since the ideal lattice of a C^*-algebra is distributive (it is isomorphic to the lattice of open subsets of its primitive-ideal space; see [38, Theorem 4.1.3]), each $\mathcal{I}_n$ is finite. Clearly $\mathcal{I}_n \subseteq \mathcal{I}_{n+1}$ for all n.

Let $\mathrm{id}_{(0,1]} \in \mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \dots \subseteq C_0((0,1])$ be a sequence of finite subsets with $\overline{\bigcup_i \mathcal{F}_i} = C_0((0,1])$. Let $\mathcal{G}_1 \subseteq \mathcal{G}_2 \subseteq \dots \subseteq A$ be a sequence of finite sets of positive contractions such that for each k,

$$J_k \cap \Big(\bigcup_{n=1}^\infty \mathcal{G}_n\Big) \tag{3.26}$$

is dense in the set of positive contractions in J_k.

Fix $n \in \mathbb{N}$. Write $\mathcal{F}_n \otimes \mathcal{G}_n := \{f \otimes a : f \in \mathcal{F}_n \text{ and } a \in \mathcal{G}_n\}$. Using the notation of Definition 2.3, for $I \in \mathcal{P}(\mathcal{I}_n)$ let $P_n(I)$ denote the unique predecessor of I in $\mathcal{I}_n$. For such an I, use Lemma 3.1 to pick an integer $N(n,I)$ and a cpc map $\psi_{n,I} \colon C_0((0,1]) \otimes A \to M_{N(n,I)}$ that is $(\mathcal{F}_n \otimes \mathcal{G}_n, 1/n)$-multiplicative and has the property that for all $f \in \mathcal{F}_n$ and $a \in \mathcal{G}_n$,

$$
\begin{aligned}
|f(1)|\,\|a + I^{P_n(I)}\|_{A/I^{P_n(I)}} - \tfrac{1}{n} &\leq \|\psi_{n,I}(f \otimes a)\| \\
&\leq \|f\|\,\|a + I^{P_n(I)}\|_{A/I^{P_n(I)}} + \tfrac{1}{n}.
\end{aligned}
\tag{3.27}
$$

If every quotient of A is quasidiagonal, Lemma 3.1 allows us to choose each $\psi_{n,I}$ so that additionally $\psi_{n,I}(C_0((0,1)) \otimes A) = 0$.

Let $F_n := \bigoplus_{I \in \mathcal{P}(\mathcal{I}_n)} M_{N(n,I)}$ and let $\psi_n \colon C_0((0,1]) \otimes A \to F_n$ be the cpc map such that

$$
\psi_n(x) = \bigoplus_{I \in \mathcal{P}(\mathcal{I}_n)} \psi_{n,I}(x), \quad x \in C_0((0,1]) \otimes A. \tag{3.28}
$$

If every quotient of A is quasidiagonal, then $\psi_n(C_0((0,1)) \otimes A) = 0$.

Let $(h_{n,I})_{I \in \mathcal{P}(\mathcal{I}_n)}$ be a collection of pairwise orthogonal, positive elements of norm 1 in B, satisfying conditions (a) and (b) of Lemma 3.4. Choose embeddings $\iota_{n,I} \colon M_{N(n,I)} \to \mathcal{O}_\infty$, and for each I, let $\eta'_{n,I} \colon M_{N(n,I)} \to B \otimes \mathcal{O}_\infty$ be the cpc order zero map such that

$$
\eta'_{n,I}(x) = h_{n,I} \otimes \iota_{n,I}(x), \quad x \in M_{N(n,I)}. \tag{3.29}
$$

For a positive contraction $f \in C_0((0,1])$, we have

$$
f(\eta'_{n,I})(x) = f(h_{n,I}) \otimes \iota_{n,I}(x), \quad x \in M_{N(n,I)}. \tag{3.30}
$$

By Lemma 3.3 there exists an isomorphism $\kappa \colon B \otimes \mathcal{O}_\infty \to B$ such that $\mathcal{I}(\kappa)(K \otimes \mathcal{O}_\infty) = K$ for all $K \in \mathcal{I}(B)$. Define $\eta_{n,I} := \kappa \circ \eta'_{n,I} \colon M_{N(n,I)} \to B$ and

$$
\eta_n := \bigoplus_{I \in \mathcal{P}(\mathcal{I}_n)} \eta_{n,I} = \kappa \circ \Big(\bigoplus_{I \in \mathcal{P}(\mathcal{I}_n)} \eta'_{n,I} \Big) \colon F_n \to B. \tag{3.31}
$$

Since, if every quotient of A is quasidiagonal, we have chosen the ψ_n such that $\psi_n(C_0((0,1)) \otimes A) = 0$, it now suffices to show that the ψ_n and η_n satisfy (a)–(c). Since the $\eta_{n,I}$ are cpc order zero maps with pairwise orthogonal ranges, each η_n is cpc order zero, which is (b).

Let $\psi \colon C_0((0,1]) \otimes A \to \frac{\prod_{n=1}^\infty F_n}{\bigoplus_{n=1}^\infty F_n}$ and $\eta \colon \frac{\prod_{n=1}^\infty F_n}{\bigoplus_{n=1}^\infty F_n} \to B_{(\infty)}$ be as in (3.23) and (3.24). Then ψ is a cpc map, and η is a cpc map of order zero. As each ψ_n is $(\mathcal{F}_n \otimes \mathcal{G}_n, 1/n)$-multiplicative, it follows that $\bigcup_{n=1}^\infty \mathcal{F}_n \otimes \mathcal{G}_n$ is contained in the multiplicative domain of ψ, and hence ψ is a *-homomorphism. This proves (a), so it remains to prove (c).

As $\eta \circ \psi \colon C_0((0,1]) \otimes A \to B_{(\infty)}$ is a cpc map of order zero, we let

$$
\rho_{\eta \circ \psi} \colon C_0((0,1]) \otimes C_0((0,1]) \otimes A \to B_{(\infty)} \tag{3.32}
$$

be the induced *-homomorphism given by Proposition 1.1. Then

$$
\rho_{\eta \circ \psi}(\mathrm{id}_{(0,1]} \otimes x) = \eta \circ \psi(x), \quad x \in C_0((0,1]) \otimes A. \tag{3.33}
$$

Let $m^* \colon C_0((0,1]) \to C_0((0,1]) \otimes C_0((0,1])$ be the *-homomorphism induced by the multiplication map $m \colon (0,1] \times (0,1] \to (0,1]$, and let ρ denote the composition

$$C_0((0,1])\otimes A \xrightarrow{m^*\otimes \mathrm{id}_A} C_0((0,1])\otimes C_0((0,1])\otimes A \xrightarrow{\rho_{\eta\circ\psi}} B_{(\infty)}. \tag{3.34}$$

Since $m^*(\mathrm{id}_{(0,1]}) = \mathrm{id}_{(0,1]}\otimes \mathrm{id}_{(0,1]}$, for each $a\in A$, we have

$$\rho(\mathrm{id}_{(0,1]}\otimes a) = \rho_{\eta\circ\psi}(\mathrm{id}_{(0,1]}\otimes \mathrm{id}_{(0,1]}\otimes a) = (\eta\circ\psi)(\mathrm{id}_{(0,1]}\otimes a). \tag{3.35}$$

Hence ρ is the *-homomorphism induced by the cpc order zero map $(\eta\circ\psi)(\mathrm{id}_{(0,1]}\otimes\cdot)$ as in Proposition 1.1.

Now fix a non-zero $I\in\mathcal{I}(C_0((0,1]))$, and $J\in\mathcal{I}(A)$. To prove (c), it suffices to prove the following two sublemmas.

Sublemma 3.6. *With notation as in the proof of Lemma 3.5, we have* $\mathcal{I}(\rho)(I\otimes J)\subseteq\overline{B_{(\infty)}\Theta(J)B_{(\infty)}}$.

Sublemma 3.7. *With notation as in the proof of Lemma 3.5, we have* $\overline{B_{(\infty)}\Theta(J)B_{(\infty)}}\subseteq\mathcal{I}(\rho)(I\otimes J)$.

Proof of Sublemma 3.6. Since $\mathcal{I}(\rho)(I\otimes J)\subseteq\mathcal{I}(\rho)(C_0((0,1])\otimes J)$, it is enough to show that $\mathcal{I}(\rho)(C_0((0,1])\otimes J)\subseteq\overline{B_{(\infty)}\Theta(J)B_{(\infty)}}$.

By choice of $(J_n)_{n=1}^\infty$ there exist $k_1<k_2<\dots$ such that $J_{k_1}\Subset J_{k_2}\Subset\dots$ and $J=\overline{\bigcup_{n=1}^\infty J_{k_n}}$. As $\mathcal{I}(\rho)$ preserves suprema, it suffices to show that each $\mathcal{I}(\rho)(C_0((0,1])\otimes J_{k_n})\subseteq\overline{B_{(\infty)}\Theta(J)B_{(\infty)}}$. Fix $n\in\mathbb{N}$ and let $K:=J_{k_n}$.

As $K\Subset J$ and Θ is a Cu-morphism, $\Theta(K)\Subset\Theta(J)$. As B is $\mathcal{O}_\infty$-stable it is strongly purely infinite, and hence weakly purely infinite by [32, Theorem 9.1]. Thus [20, Proposition 6.5] gives $\Theta(K)_{(\infty)}\subseteq\overline{B_{(\infty)}\Theta(J)B_{(\infty)}}$. So it suffices to show that

$$\mathcal{I}(\rho)(C_0((0,1])\otimes K)\subseteq\Theta(K)_{(\infty)}. \tag{3.36}$$

Since $K=J_{k_n}$, our choice of the $(\mathcal{G}_m)_{m=1}^\infty$ ensures that $K\cap(\bigcup_{m=1}^\infty\mathcal{G}_m)$ is dense in the set of positive contractions in K. Since $\mathrm{id}_{(0,1]}\in C_0((0,1])$ is a full element, it therefore suffices to show that for each $m\geq k_n$ and $a\in K\cap\mathcal{G}_m$,

$$\rho(\mathrm{id}_{(0,1]}\otimes a)\in\Theta(K)_{(\infty)}. \tag{3.37}$$

So fix such m and a.

By (3.35), the element $\rho(\mathrm{id}_{(0,1]}\otimes a)$ is represented by the sequence $(\eta_r\circ\psi_r(\mathrm{id}_{(0,1]}\otimes a))_{r=1}^\infty\in\prod_{r=1}^\infty B$. For $r\geq m$, let

$$x_r := \sum_{\substack{L\in\mathcal{P}(\mathcal{I}_r)\\ L\subseteq K}} \eta_{r,L}\circ\psi_{r,L}(\mathrm{id}_{(0,1]}\otimes a),\ \text{and}$$

$$y_r := \sum_{\substack{L\in\mathcal{P}(\mathcal{I}_r)\\ L\not\subseteq K}} \eta_{r,L}\circ\psi_{r,L}(\mathrm{id}_{(0,1]}\otimes a). \tag{3.38}$$

Then

$$\eta_r\circ\psi_r(\mathrm{id}_{(0,1]}\otimes a) = x_r+y_r. \tag{3.39}$$

Since $(\eta_{r,L})_{L\in\mathcal{P}(\mathcal{I}_r)}$ is a family of contractive maps with pairwise orthogonal images, x_r and y_r are contractions. For $L\in\mathcal{P}(\mathcal{I}_r)$ such that $L\not\subseteq K$, contractivity of $\eta_{r,L}$ and (3.27) show that $\|\eta_{r,L}\circ\psi_{r,L}(\mathrm{id}_{(0,1]}\otimes a)\| \le \frac{1}{r} + \|a + L^{P_r(L)}\|_{A/L^{P_r(L)}}$, and so

$$\begin{aligned}\|y_r\| &= \max\{\|\eta_{r,L}\circ\psi_{r,L}(\mathrm{id}_{(0,1]}\otimes a)\| : L\in\mathcal{P}(\mathcal{I}_r),\ L\not\subseteq K\}\\ &\le \frac{1}{r} + \max\{\|a+L^{P_r(L)}\|_{A/L^{P_r(L)}} : L\in\mathcal{P}(\mathcal{I}_r),\ L\not\subseteq K\}.\end{aligned} \tag{3.40}$$

As $K = J_{k_n}$ and $r\ge m\ge k_n$, we have $K\in\mathcal{I}_r$. Lemma 2.5 gives $K\subseteq L^{P_r(L)}$ for any $L\in\mathcal{P}(\mathcal{I}_r)$ satisfying $L\not\subseteq K$. Hence $\|a+L^{P_r(L)}\|_{A/L^{P_r(L)}} = 0$ for any such L. Thus $\|y_r\|\le 1/r$. Consequently $\rho(\mathrm{id}_{(0,1]}\otimes a)$ is represented by the sequence $(x_r)_{r=1}^\infty$.

Suppose that $r\ge m$ and that $L\in\mathcal{P}(\mathcal{I}_r)$ satisfies $L\subseteq K$. Recall from (3.29) that $\eta'_{r,L}(z) = h_{r,L}\otimes\iota_{r,L}(z)$ for every $z\in M_{N(r,L)}$, where $h_{r,L}\in\Theta(L)\subseteq\Theta(K)$. Thus

$$\eta'_{r,L}\circ\psi_{r,L}(\mathrm{id}_{(0,1]}\otimes a)\in\Theta(K)\otimes\mathcal{O}_\infty. \tag{3.41}$$

Recall that $\kappa\colon B\otimes\mathcal{O}_\infty\xrightarrow{\cong} B$ satisfies $\mathcal{I}(\kappa)(K'\otimes\mathcal{O}_\infty) = K'$ for $K'\in\mathcal{I}(B)$. Since $\eta_{r,L} = \kappa\circ\eta'_{r,L}$ it follows from (3.41) that

$$\eta_{r,L}\circ\psi_{r,L}(\mathrm{id}_{(0,1]}\otimes a)\in\Theta(K). \tag{3.42}$$

Hence for $r\ge m$, we have $x_r\in\Theta(K)$ (see (3.38)). Since $\rho(\mathrm{id}_{(0,1]}\otimes a)$ is represented by the sequence $(x_r)_{r=1}^\infty$ it follows that $\rho(\mathrm{id}_{(0,1]}\otimes a)$ belongs to $\Theta(K)_{(\infty)}$, as required. □ (Sublemma 3.6).

Proof of Sublemma 3.7. Recall that we have fixed $I\in\mathcal{I}(C_0((0,1]))$ non-zero, and $J\in\mathcal{I}(A)$. Let $\iota\colon B\to B_{(\infty)}$ be the canonical inclusion. Then

$$\mathcal{I}(\iota)\circ\Theta(K) = \overline{B_{(\infty)}\Theta(K)B_{(\infty)}},\quad K\in\mathcal{I}(A). \tag{3.43}$$

As A is separable, $J = \sup\{J_0\in\mathcal{I}(A) : J_0\Subset J\}$ by [20, Corollary 2.3]. Fix $J_0\in\mathcal{I}(A)$ such that $J_0\Subset J$. Since $\mathcal{I}(\iota)\circ\Theta$ preserves suprema, it suffices to show that

$$\overline{B_{(\infty)}\Theta(J_0)B_{(\infty)}}\subseteq\mathcal{I}(\rho)(I\otimes J). \tag{3.44}$$

By (3.34), it follows from [20, Proposition 2.15] that

$$\mathcal{I}(\rho) = \mathcal{I}(\rho_{\eta\circ\psi})\circ\mathcal{I}(m^*\otimes\mathrm{id}_A). \tag{3.45}$$

Hence

$$\mathcal{I}(\rho)(I\otimes J) = \mathcal{I}(\rho_{\eta\circ\psi})(\mathcal{I}(m^*)(I)\otimes J). \tag{3.46}$$

As $I\neq 0$ by assumption, Lemma 2.7 yields positive, non-zero contractions $f,g\in C_0((0,1])$ such that $f\otimes g\in\mathcal{I}(m^*)(I)$, and $g(1)=1$. We may assume without loss of generality that $\|f\|=1$. Let $a\in J$ be a full, positive contraction. Equation (3.46) gives

$$\rho_{\eta\circ\psi}(f\otimes g\otimes a)\in\mathcal{I}(\rho_{\eta\circ\psi})(\mathcal{I}(m^*)(I)\otimes J) = \mathcal{I}(\rho)(I\otimes J), \tag{3.47}$$

so it suffices to show that

$$\overline{B_{(\infty)}\Theta(J_0)B_{(\infty)}}\subseteq\overline{B_{(\infty)}\rho_{\eta\circ\psi}(f\otimes g\otimes a)B_{(\infty)}}. \tag{3.48}$$

Recall that ψ and η are the maps defined in (3.23) and (3.24), and that $\rho_{\eta\circ\psi}$ is the *-homomorphism induced by the composition $\eta\circ\psi$. Let ρ_η be the *-homomorphism induced by η as in Proposition 1.1. Since ψ is a *-homomorphism and η is a cpc order zero map, Lemma 1.2 implies that $\rho_{\eta\circ\psi}$ is equal to the composition

(3.49)
$$C_0((0,1])\otimes C_0((0,1])\otimes A \xrightarrow{\mathrm{id}_{C_0((0,1])}\otimes\psi} C_0((0,1])\otimes \frac{\prod_{n=1}^\infty F_n}{\bigoplus_{n=1}^\infty F_n} \xrightarrow{\rho_\eta} B_{(\infty)}.$$

Hence, with $f(\eta)$ as in (1.4),

$$\rho_{\eta\circ\psi}(f\otimes g\otimes a)=\rho_\eta(f\otimes\psi(g\otimes a))=f(\eta)(\psi(g\otimes a)). \tag{3.50}$$

As η is induced by the sequence $(\eta_n)_{n=1}^\infty$ of cpc order zero maps, Lemma 1.4 implies that $f(\eta)$ is induced by the sequence $(f(\eta_n))_{n=1}^\infty$ of cpc order zero maps. As ψ is represented by the sequence $(\psi_n)_{n=1}^\infty$, it follows that $\rho_{\eta\circ\psi}(f\otimes g\otimes a)$ is represented by the sequence

$$(f(\eta_n)(\psi_n(g\otimes a)))_{n=1}^\infty. \tag{3.51}$$

Fix $n\in\mathbb{N}$ and $x=(x_K)_{K\in\mathcal{P}(\mathcal{I}_n)}\in F_n=\bigoplus_{K\in\mathcal{P}(\mathcal{I}_n)}M_{N(n,K)}$. By (3.31), we have

$$f(\eta_n)(x)=f\Big(\kappa\circ\bigoplus_{K\in\mathcal{P}(\mathcal{I}_n)}\eta'_{n,K}\Big)(x). \tag{3.52}$$

As κ is an isomorphism, we obtain $f(\eta_n)(x)=\kappa\big(f\big(\bigoplus_{K\in\mathcal{P}(\mathcal{I}_n)}\eta'_{n,K}\big)(x_K)\big)$. For fixed n, the maps $(\eta_{n,K})_{K\in\mathcal{P}(\mathcal{I}_n)}$ have mutually orthogonal ranges, so Lemma 1.3 implies that $f(\eta_n)(x)=\sum_{K\in\mathcal{P}(\mathcal{I}_n)}\kappa(f(\eta'_{n,K})(x_K))$. Applying (3.30) now yields

$$f(\eta_n)(x)=\sum_{K\in\mathcal{P}(\mathcal{I}_n)}\kappa(f(h_{n,K})\otimes\iota_{n,K}(x_K)). \tag{3.53}$$

Combining (3.53), (3.51), and (3.28), we deduce that $\rho_{\eta\circ\psi}(f\otimes g\otimes a)$ is represented by the sequence

$$\Big(\sum_{K\in\mathcal{P}(\mathcal{I}_n)}\kappa(f(h_{n,K})\otimes\iota_{n,K}(\psi_{n,K}(g\otimes a)))\Big)_{n=1}^\infty\in\prod_{n=1}^\infty B. \tag{3.54}$$

By choice of $(J_k)_{k=1}^\infty$, there exist $k_1<k_2<\dots$ such that $J_{k_{l-1}}\Subset J_{k_l}$ for all l, and $J=\overline{\bigcup J_{k_l}}$. Since $J_0\Subset J$, there exists $l\in\mathbb{N}$ such that

$$J_0\subseteq J_{k_{l-1}}. \tag{3.55}$$

As $a\in J$ is a full, positive contraction, Lemma 2.6 applied to $J_{k_l}\Subset J$ provides $\lambda>0$, such that for any ideals $K_1\subsetneq K_2\subseteq J_{k_l}$, we have

$$\|a+K_2^{K_1}\|_{A/K_2^{K_1}}\geq\lambda. \tag{3.56}$$

Pick $k_0\geq\max\{k_l,10/\lambda\}$ such that there exist $g_0\in\mathcal{F}_{k_0}$ and $a_0\in\mathcal{G}_{k_0}$ satisfying

$$\|g_0-g\|\leq\lambda/10 \quad\text{and}\quad \|a_0-a\|\leq\lambda/10. \tag{3.57}$$

We chose g satisfying $g(1)=1$, so $|g_0(1)-1|\leq\lambda/10$. For $n\geq k_0$ and $K\in\mathcal{P}(\mathcal{I}_n)$, we calculate, using the fact that a_0 is a contraction and $\psi_{n,K}$ is contractive at the first step,

$$\begin{aligned}\|\psi_{n,K}(g\otimes a)\| &\geq \|\psi_{n,K}(g_0\otimes a_0)\| - \frac{2\lambda}{10} && \text{by (3.57)}\\ &\geq |g_0(1)|\,\|a_0+K^{P_n(K)}\|_{A/K^{P_n(K)}} - \frac{3\lambda}{10} && \text{by (3.27)}\\ &\geq \|a+K^{P_n(K)}\|_{A/K^{P_n(K)}} - \frac{\lambda}{2} && \text{by (3.57).}\end{aligned}\tag{3.58}$$

As $k_0 \geq k_l$, it follows that $J_{k_{l-1}}, J_{k_l} \in \mathcal{I}_n$ for every $n \geq k_0$. Hence, whenever $n \geq k_0$ and $K \in \mathcal{P}(\mathcal{I}_n)$ satisfies $K \subseteq J_{k_l}$, applying (3.58) and then (3.56), we obtain

$$\begin{aligned}\|\psi_{n,K}(g\otimes a)\| &\geq \|a+K^{P_n(K)}\|_{A/K^{P_n(K)}} - \lambda/2\\ &\geq \lambda - \lambda/2\\ &\geq \lambda/2.\end{aligned}\tag{3.59}$$

Recall that $\psi_{n,K}$ takes values in the matrix algebra $M_{N(n,K)}$. Thus, by (3.59), for any $n \geq k_0$ and $K \in \mathcal{P}(\mathcal{I}_n)$ satisfying $K \subseteq J_{k_l}$ there exists a non-zero projection $p_{n,K} \in M_{N(n,K)}$, such that

$$\psi_{n,K}(g\otimes a) \geq p_{n,K}\cdot \lambda/2.\tag{3.60}$$

Recall from (3.54) that $\rho_{\eta\circ\psi}(f\otimes g\otimes a)$ is represented by the sequence $\big(\sum_{K\in\mathcal{P}(\mathcal{I}_n)} \kappa(f(h_{n,K})\otimes \iota_{n,K}(\psi_{n,K}(g\otimes a)))\big)_{n=1}^{\infty}$. For each $n \geq k_0$, Equation (3.60) implies that

$$\begin{aligned}&\sum_{\substack{K\in\mathcal{P}(\mathcal{I}_n)\\ K\subseteq J_{k_l}}} \kappa(f(h_{n,K})\otimes \iota_{n,K}(p_{n,K}))\\ &\qquad\leq \frac{2}{\lambda}\sum_{K\in\mathcal{P}(\mathcal{I}_n)} \kappa(f(h_{n,K})\otimes \iota_{n,K}(\psi_{n,K}(g\otimes a))).\end{aligned}\tag{3.61}$$

So the element in $B_{(\infty)}$ represented by[11]

$$\Big(\sum_{\substack{K\in\mathcal{P}(\mathcal{I}_n)\\ K\subseteq J_{k_l}}} \kappa(f(h_{n,K})\otimes \iota_{n,K}(p_{n,K}))\Big)_{n=k_0}^{\infty}\tag{3.62}$$

belongs to the ideal generated by $\rho_{\eta\circ\psi}(f\otimes g\otimes a)$. Thus, to establish (3.48) and complete the proof of the sublemma, it suffices to show that $\Theta(J_0)$ is contained in the ideal of $B_{(\infty)}$ generated by the element represented by (3.62).

Recall that $f \in C_0((0,1])$ is a positive contraction with $\|f\| = 1$. Fix $f_0\colon [0,1] \to [0,1]$ continuous with $f_0(0) = 0$ and $f_0(1) = 1$. Then $f_0 \circ f\colon [0,1] \to [0,1]$ is a non-zero, continuous function such that $f_0 \circ f(0) = 0$. Fix $n \geq k_0$. Using first that the $f(h_{n,K}) \otimes \iota_{n,K}(p_{n,K})$ are mutually

[11]We only define the sequence for $n \geq k_0$, as such sequences still determine a unique element in $B_{(\infty)}$.

orthogonal, and then that $\iota_{n,K}(p_{n,K})$ is a projection, we calculate:

$$f_0\Big(\sum_{\substack{K\in\mathcal{P}(\mathcal{I}_n)\\ K\subseteq J_{k_l}}}\kappa(f(h_{n,K})\otimes\iota_{n,K}(p_{n,K}))\Big) \tag{3.63}$$

$$= \sum_{\substack{K\in\mathcal{P}(\mathcal{I}_n)\\ K\subseteq J_{k_l}}}\kappa(f_0(f(h_{n,K})\otimes\iota_{n,K}(p_{n,K})))$$

$$= \sum_{\substack{K\in\mathcal{P}(\mathcal{I}_n)\\ K\subseteq J_{k_l}}}\kappa((f_0\circ f)(h_{n,K})\otimes\iota_{n,K}(p_{n,K})). \tag{3.64}$$

As the $\iota_{n,K}(p_{n,K})\in\mathcal{O}_\infty$ are non-zero projections, and as $\mathcal{O}_\infty$ is simple, the defining property of κ ensures that the element in (3.64) generates the same ideal as

$$\sum_{\substack{K\in\mathcal{P}(\mathcal{I}_n)\\ K\subseteq J_{k_l}}}(f_0\circ f)(h_{n,K}). \tag{3.65}$$

Since $f_0\circ f\colon[0,1]\to[0,1]$ is non-zero, continuous and maps 0 to 0, and as $J_{k_{l-1}}\Subset J_{k_l}$ by (3.55), Lemma 3.4(b) implies that $\Theta(J_{k_{l-1}})$ is contained in the ideal generated by the element of (3.65) by choice of $h_{n,K}$ (see the text just above (3.29)). Hence $\Theta(J_{k_{l-1}})$ is contained in the ideal generated by the element of (3.63).

Now fix a positive $b\in\Theta(J_0)$ with $\|b\|=1$. To show that $\Theta(J_0)$ is contained in the ideal generated by the element c of $B_{(\infty)}$ represented by (3.62), it suffices to show that b belongs to $\overline{B_{(\infty)}cB_{(\infty)}}$. As B is $\mathcal{O}_\infty$-stable, it is purely infinite[12] by [31, Proposition 4.5]. Since b is a positive contraction in $\Theta(J_0)\Subset\Theta(J_{k_{l-1}})$, it is Cuntz dominated by the element in (3.63). The element

$$\sum_{\substack{K\in\mathcal{P}(\mathcal{I}_n)\\ K\subseteq J_{k_l}}}\kappa(f(h_{n,K})\otimes\iota_{n,K}(p_{n,K})) \tag{3.66}$$

is a positive contraction of norm 1 by Lemma 3.4, since all $h_{n,K}$ are pairwise orthogonal, positive elements with spectrum $[0,1]$, and since $f\in C_0((0,1])$ is a positive contraction with $\|f\|=1$ (see definition of f right after (3.46)). Since $f_0\colon[0,1]\to[0,1]$ is continuous and satisfies $f_0(0)=0$ and $f_0(1)=1$, Lemma 3.2 gives a contraction $z_n\in B$ such that

$$\Big\|b-z_n^*\Big(\sum_{\substack{K\in\mathcal{P}(\mathcal{I}_n)\\ K\subseteq J_{k_l}}}\kappa(f(h_{n,K})\otimes\iota_{n,K}(p_{n,K}))\Big)z_n\Big\|<\frac{1}{n}. \tag{3.67}$$

Let $z\in B_{(\infty)}$ be the element induced by $(z_n)_{n=k_0}^\infty$. By (3.67), we have $\iota(b)=z^*cz$ as required. □ (Sublemma 3.7).

With the two sublemmas in place, the proof of Lemma 3.5 is complete. □

[12]Recall that a C^*-algebra D is *purely infinite* if it has no characters, and if whenever $d_1,d_2\in D$ are positive such that $d_1\in\overline{Dd_2D}$ then $d_1\precsim d_2$.

4. $\mathcal{O}_2$- and $\mathcal{O}_\infty$-stable *-homomorphisms

Our main objective in this section is to upgrade the technical construction of the previous section to additionally ensure that ρ is $\mathcal{O}_2$-stable. This is Lemma 4.6 below.

We begin with a review of $\mathcal{O}_2$ and $\mathcal{O}_\infty$-stability for maps. We use a sequential version of a relative commutant construction developed by Kirchberg in the setting of ultrapowers [28]. If $\theta\colon A \to B$ is a *-homomorphism, we denote the relative commutant of $\theta(A) \subseteq B \subseteq B_{(\infty)}$ by $B_{(\infty)} \cap \theta(A)'$. The annihilator,

$$\operatorname{Ann}\theta(A) := \{x \in B_{(\infty)} : x\theta(A) = \theta(A)x = \{0\}\}, \tag{4.1}$$

of $\theta(A)$ in $B_{(\infty)}$ is an ideal in $B_{(\infty)} \cap \theta(A)'$. If A is separable, then any sequential approximate identity $(e_n)_{n=1}^\infty$ for A represents a unit for $(B_{(\infty)} \cap \theta(A)')/\operatorname{Ann}\theta(A)$. With this notation, we recall the definitions of $\mathcal{O}_2$ and $\mathcal{O}_\infty$-stability of maps from [20, Definition 3.16].

Definition 4.1. Let A and B be C^*-algebras with A separable, and let $\theta\colon A \to B$ be a *-homomorphism. Then θ is *$\mathcal{O}_2$-stable* (respectively *$\mathcal{O}_\infty$-stable*) if $\mathcal{O}_2$ (respectively $\mathcal{O}_\infty$) embeds unitally into

$$\frac{B_{(\infty)} \cap \theta(A)'}{\operatorname{Ann}\theta(A)}. \tag{4.2}$$

In this framework, it goes back to work of Kirchberg, Lin, Rørdam and Phillips (abstracted to strongly self-absorbing algebras in [52]), that a separable C^*-algebra A is $\mathcal{O}_2$-stable (resp. $\mathcal{O}_\infty$-stable) if and only if the identity map id_A is $\mathcal{O}_2$-stable (resp. $\mathcal{O}_\infty$-stable), (see [20, Proposition 3.19] for this exact statement).

The next lemma is extracted from the proof of [2, Theorem 4.3] to show that embeddings of cones into simple purely infinite algebras are $\mathcal{O}_2$-stable.

Lemma 4.2. *Let A be a simple, purely infinite C^*-algebra, and let $h \in A$ be a positive element with spectrum $[0,1]$. Then the *-homomorphism $\phi\colon C_0((0,1]) \to A$ given by functional calculus on h is $\mathcal{O}_2$-stable.*

Proof. Fix $n \in \mathbb{N}$. As A has real rank zero ([61]), find $0 < \lambda_1 \leq \cdots \leq \lambda_k \leq 1$ and non-zero pairwise orthogonal projections $\tilde{p}_1, \ldots, \tilde{p}_k \in A$, such that

$$\|h - \sum_{i=1}^k \lambda_i \tilde{p}_i\| < 1/(2n). \tag{4.3}$$

Set $\lambda_0 = 0$, and notice that $\lambda_i - \lambda_{i-1} < 1/(2n)$ for $i = 1, \ldots, k$.

As A is simple and purely infinite, find a non-zero subprojection p_k of $\tilde{p}_k$ such that $[p_k]_0 = 0$ in $K_0(A)$. Now find a non-zero subprojection q_{k-1} of $\tilde{p}_{k-1}$ so that $[q_{k-1}]_0 = -[\tilde{p}_k - p_k]_0$ in $K_0(A)$, and set $p_{k-1} = q_{k-1} + (\tilde{p}_k - p_k)$. Next find a non-zero subprojection q_{k-2} of $\tilde{p}_{k-2}$ so that $[q_{k-2}]_0 = -[\tilde{p}_{k-1} - q_{k-1}]_0$ in $K_0(A)$ and set $p_{k-2} = q_{k-2} + (\tilde{p}_{k-1} - q_{k-1})$. Carry on in this way to find $p_{k-3}, \ldots, p_1$ so that each $[p_i]_0 = 0$. Then

$$\|\sum_{i=1}^k \lambda_i p_i - \sum_{i=1}^k \lambda_i \tilde{p}_i\| \leq \max_{i=1,\ldots,k} (\lambda_i - \lambda_{i-1}) < \frac{1}{2n}, \tag{4.4}$$

so that

$$\|h - \sum_{i=1}^{k} \lambda_i p_i\| < 1/n. \tag{4.5}$$

As each p_i is properly infinite with $[p_i]_0 = 0$, fix unital embeddings $\theta_{n,i}\colon \mathcal{O}_2 \to p_iAp_i$. Let $\theta_n = \bigoplus_{i=1}^k \theta_{n,i}\colon \mathcal{O}_2 \to \bigoplus_{i=1}^k p_iAp_i \subseteq A$. Then $\|[\theta_n(x), h]\| \to 0$ for any $x \in \mathcal{O}_2$, and $\theta_n(1_{\mathcal{O}_2})h \to h$. Thus $(\theta_n)_{n=1}^\infty$ induces a unital embedding of $\mathcal{O}_2$ into $(B_{(\infty)} \cap \phi(C_0((0,1]))')/\mathrm{Ann}\,\phi(C_0((0,1]))$. □

As expected, the tensor product of an $\mathcal{O}_2$-stable (respectively $\mathcal{O}_\infty$-stable) *-homomorphism with another *-homomorphism is again $\mathcal{O}_2$-stable (respectively $\mathcal{O}_\infty$-stable).

Lemma 4.3. *Let $\phi\colon A \to B$ and $\psi\colon C \to D$ be *-homomorphisms with A and C separable, and suppose that ψ is $\mathcal{O}_2$-stable (or $\mathcal{O}_\infty$-stable). Then $\phi \otimes \psi\colon A \otimes_{\max} C \to B \otimes_{\max} D$ is $\mathcal{O}_2$-stable (or $\mathcal{O}_\infty$-stable).*

Proof. The canonical *-homomorphism $B_{(\infty)} \otimes_{\max} D_{(\infty)} \to (B \otimes_{\max} D)_{(\infty)}$ induces a unital *-homomorphism from the maximal tensor product

$$\Big(\frac{B_{(\infty)} \cap \phi(A)'}{\mathrm{Ann}\,\phi(A)}\Big) \otimes_{\max} \Big(\frac{D_{(\infty)} \cap \psi(C)'}{\mathrm{Ann}\,\psi(C)}\Big) \tag{4.6}$$

to the quotient

$$\frac{(B \otimes_{\max} D)_{(\infty)} \cap (\phi \otimes \psi)(A \otimes_{\max} C)'}{\mathrm{Ann}\,(\phi \otimes \psi)(A \otimes_{\max} C)}. \tag{4.7}$$

Since $(B_{(\infty)} \cap \phi(A)')/\mathrm{Ann}\,(\phi(A))$ is unital, a unital embedding $\mathcal{O}_2 \to (D_{(\infty)} \cap \psi(C)')/\mathrm{Ann}\,(\psi(C))$ given by $\mathcal{O}_2$-stability of ψ gives rise to a unital embedding of $\mathcal{O}_2$ into the algebra of (4.7). So $\phi \otimes \psi$ is $\mathcal{O}_2$-stable. The same argument works with $\mathcal{O}_\infty$ in place of $\mathcal{O}_2$. □

Following [31], a non-zero, positive element $a \in A$ is called *properly infinite* if $a \oplus a \precsim a \oplus 0$ in $M_2(A)$. We record the following lemma for later use.

Lemma 4.4. *Any non-zero, positive element in the image of an $\mathcal{O}_\infty$-stable *-homomorphism, is properly infinite.*

Proof. Let $\theta\colon A \to B$ be an $\mathcal{O}_\infty$-stable *-homomorphism, and let $a \in A$ be a positive element such that $\theta(a)$ is non-zero. Let $s_1, s_2 \in B_{(\infty)} \cap \theta(A)'$ be elements so that $s_1 + \mathrm{Ann}\,\theta(A), s_2 + \mathrm{Ann}\,\theta(A)$ are isometries in $(B_{(\infty)} \cap \theta(A)')/\mathrm{Ann}\,\theta(A)$ with mutually orthogonal range projections. Let $(s_i^{(n)})_{n=1}^\infty \in \prod_{n=1}^\infty B$ be a lift of s_i for $i = 1, 2$. Let $x_n \coloneqq \theta(a)^{1/4} s_1^{(n)} \theta(a)^{1/4}$ and $y_n \coloneqq \theta(a)^{1/4} s_2^{(n)} \theta(a)^{1/4}$. Then $x_n, y_n \in \overline{\theta(a)B\theta(a)}$, $x_n^* x_n \to \theta(a)$, $y_n^* y_n \to \theta(a)$ and $x_n^* y_n \to 0$. By [31, Proposition 3.3(iv)], $\theta(a)$ is properly infinite. □

We are now able to upgrade our main technical lemma (Lemma 3.5) to additionally insist that the *-homomorphism constructed is $\mathcal{O}_2$-stable.

Lemma 4.5. *Let A and B be C^*-algebras with A separable and $B \otimes \mathcal{O}_\infty \cong B$, and let $\Theta\colon \mathcal{I}(A) \to \mathcal{I}(B)$ be a* Cu*-morphism. Then for $n \in \mathbb{N}$, there are* *finite dimensional C^*-algebras F_n and cpc maps $\psi_n\colon C_0((0,1])\otimes A\to F_n$ and $\eta_n\colon F_n\to B$ such that, writing*

$$\psi := (\psi_n)_{n=1}^\infty\colon C_0((0,1])\otimes A\to \frac{\prod_{n=1}^\infty F_n}{\bigoplus_{n=1}^\infty F_n},\quad \textit{and} \tag{4.8}$$

$$\eta := (\eta_n)_{n=1}^\infty\colon \frac{\prod_{n=1}^\infty F_n}{\bigoplus_{n=1}^\infty F_n}\to B_{(\infty)} \tag{4.9}$$

for the induced maps, the following are satisfied:

(*a*) *ψ is a *-homomorphism;*
(*b*) *each η_n is order zero;*
(*c*) *if $\rho\colon C_0((0,1])\otimes A\to B_{(\infty)}$ is the *-homomorphism induced by the cpc order zero map $(\eta\circ\psi)(\mathrm{id}_{(0,1]}\otimes\cdot)$ (see Proposition 1.1), then*

$$\mathcal{I}(\rho)(I\otimes J)=\overline{B_{(\infty)}\Theta(J)B_{(\infty)}} \tag{4.10}$$

for any $J\in\mathcal{I}(A)$ and any non-zero $I\in\mathcal{I}(C_0((0,1]))$; and
(*d*) *the *-homomorphism ρ is $\mathcal{O}_2$-stable.*

If every quotient of A is quasidiagonal, we may additionally arrange that $\psi_n(C_0((0,1))\otimes A)=0$ for each $n\in\mathbb{N}$.

Proof. Apply Lemma 3.5 to obtain $(F_n,\psi_n,\eta_n')_{n=1}^\infty$ satisfying (*a*)–(*c*) and satisfying $\psi_n(C_0((0,1])\otimes A)=0$ for all n if every quotient of A is quasidiagonal. Fix a positive contraction $h\in\mathcal{O}_\infty$ with spectrum $[0,1]$. By Lemma 3.3, there exists an isomorphism $\kappa\colon\mathcal{O}_\infty\otimes B\to B$ such that $\mathcal{I}(\kappa)(\mathcal{O}_\infty\otimes K)=K$ for each $K\in\mathcal{I}(B)$. Define $\eta_n:=\kappa(h\otimes\eta_n'(\cdot))\colon F_n\to B$. We will show that $(F_n,\psi_n,\eta_n)_{n=1}^\infty$ satisfies conditions (*a*)–(*d*) above.

Statement (*a*) and the final assertion are satisfied by choice of the ψ_n. Each η_n' is cpc order zero, so each η_n is cpc order zero as well, giving (*b*). It remains to check (*c*) and (*d*). We first verify (*d*).

The map κ induces a *-homomorphism

$$\kappa_0\colon\mathcal{O}_\infty\otimes B_{(\infty)}\to B_{(\infty)}, \tag{4.11}$$

such that, for $x\in\mathcal{O}_\infty$ and $(b_n)_{n=1}^\infty\in B_{(\infty)}$,

$$\kappa_0(x\otimes(b_n)_{n=1}^\infty)=(\kappa(x\otimes b_n))_{n=1}^\infty. \tag{4.12}$$

Let

$$\eta' := (\eta_n')_{n=1}^\infty\colon \frac{\prod_{n=1}^\infty F_n}{\bigoplus_{n=1}^\infty F_n}\to B_{(\infty)}. \tag{4.13}$$

Then $\kappa_0\circ(h\otimes\eta'(\cdot))=\eta$.

Let $\rho'\colon C_0((0,1])\otimes A\to B_{(\infty)}$ be the unique *-homomorphism such that

$$\rho'(\mathrm{id}_{(0,1]}\otimes a)=(\eta'\circ\psi)(\mathrm{id}_{(0,1]}\otimes a),\quad a\in A.\text{[13]} \tag{4.14}$$

Let $\iota\colon C_0((0,1])\to\mathcal{O}_\infty$ be the embedding given by $\iota(f)=f(h)$, and let $m^*\colon C_0((0,1])\to C_0((0,1])\otimes C_0((0,1])$ be the *-homomorphism induced by multiplication. Define ρ_0 as the composition

$$C_0((0,1])\otimes A\xrightarrow{m^*\otimes\mathrm{id}_A}C_0((0,1])\otimes C_0((0,1])\otimes A\xrightarrow{\iota\otimes\rho'}\mathcal{O}_\infty\otimes B_{(\infty)}\xrightarrow{\kappa_0}B_{(\infty)}. \tag{4.15}$$

[13]This does *not* imply that $\rho'=\eta'\circ\psi$ since $\eta'\circ\psi$ is not necessarily a *-homomorphism.

As $m^*(\mathrm{id}_{(0,1]}) = \mathrm{id}_{(0,1]} \otimes \mathrm{id}_{(0,1]}$ and $\iota(\mathrm{id}_{(0,1]}) = h$, for each $a \in A$,

$$\begin{aligned} \rho_0(\mathrm{id}_{(0,1]} \otimes a) &= (\kappa_0 \circ (\iota \otimes \rho'))(\mathrm{id}_{(0,1]} \otimes \mathrm{id}_{(0,1]} \otimes a) \\ &= \kappa_0(h \otimes \eta'(\psi(\mathrm{id}_{(0,1]} \otimes a))) \quad \text{by (4.14)} \\ &= (\eta \circ \psi)(\mathrm{id}_{(0,1]} \otimes a). \end{aligned} \tag{4.16}$$

Hence ρ_0 is the *-homomorphism induced by the cpc order zero map $(\eta \circ \psi)(\mathrm{id}_{(0,1]} \otimes \cdot)$ (see Proposition 1.1), and thus $\rho = \rho_0$.

The *-homomorphism

$$\iota \otimes \rho' \colon C_0((0,1]) \otimes C_0((0,1]) \otimes A \to \mathcal{O}_\infty \otimes B_{(\infty)} \tag{4.17}$$

is $\mathcal{O}_2$-stable by Lemmas 4.2 and 4.3. As ρ factors through this $\mathcal{O}_2$-stable map by construction of ρ_0, it follows that ρ is $\mathcal{O}_2$-stable by [20, Lemma 3.20]. Hence (d) is confirmed, and it remains to check (c).

For (c), take non-zero ideals $I, K \in \mathcal{I}(C_0((0,1]))$ and $J \in \mathcal{I}(A)$. Since $\mathcal{O}_\infty$ is simple and since ρ' satisfies Lemma 3.5(c),

$$\begin{aligned} \mathcal{I}(\iota \otimes \rho')(K \otimes I \otimes J) &= \mathcal{I}(\iota)(K) \otimes \mathcal{I}(\rho')(I \otimes J) \\ &= \mathcal{O}_\infty \otimes \overline{B_{(\infty)}\Theta(J)B_{(\infty)}}. \end{aligned} \tag{4.18}$$

As $\mathcal{I}(\iota \otimes \rho')$ is a Cu-morphism, it preserves suprema. Since ideals of the form $K \otimes I$ form a basis for $\mathcal{I}(C_0((0,1]) \otimes C_0((0,1]))$, it follows that for any non-zero ideals $I_0 \in \mathcal{I}(C_0((0,1]) \otimes C_0((0,1]))$ and $J \in \mathcal{I}(A)$,

$$\mathcal{I}(\iota \otimes \rho')(I_0 \otimes J) = \mathcal{O}_\infty \otimes \overline{B_{(\infty)}\Theta(J)B_{(\infty)}}. \tag{4.19}$$

By [20, Proposition 2.15], $\mathcal{I}((\iota \otimes \rho') \circ (m^* \otimes \mathrm{id}_A)) = \mathcal{I}(\iota \otimes \rho') \circ \mathcal{I}(m^* \otimes \mathrm{id}_A)$. Therefore, for any non-zero ideals $I \in \mathcal{I}(C_0((0,1]))$ and $J \in \mathcal{I}(A)$, using (4.19) at (4.20), we have

$$\begin{aligned} \mathcal{I}((\iota \otimes \rho') \circ (m^* \otimes \mathrm{id}_A))(I \otimes J) &= \mathcal{I}(\iota \otimes \rho')(\mathcal{I}(m^*)(I) \otimes J) \\ &= \mathcal{O}_\infty \otimes \overline{B_{(\infty)}\Theta(J)B_{(\infty)}}. \end{aligned} \tag{4.20}$$

By Lemma 2.1, the ideal $\Theta(J)$ contains a full element, say b_J. It follows that $\mathcal{I}((\iota \otimes \rho') \circ (m^* \otimes \mathrm{id}_A))(I \otimes J)$ is the ideal in $\mathcal{O}_\infty \otimes B_{(\infty)}$ generated by

$$1_{\mathcal{O}_\infty} \otimes b_J \in \mathcal{O}_\infty \otimes \Theta(J) \subseteq \mathcal{O}_\infty \otimes B_{(\infty)}. \tag{4.21}$$

Hence $\mathcal{I}(\kappa_0)(\mathcal{O}_\infty \otimes \overline{B_{(\infty)}\Theta(J)B_{(\infty)}})$ is the ideal generated by

$$\kappa_0(1_{\mathcal{O}_\infty} \otimes (b_J)_{n=1}^\infty) = (\kappa(1_{\mathcal{O}_\infty} \otimes b_J))_{n=1}^\infty. \tag{4.22}$$

Since $\kappa(1_{\mathcal{O}_\infty} \otimes b_J)$ is a full element in $\Theta(J)$ by choice of κ, the ideal generated by the element (4.22) is precisely $\overline{B_{(\infty)}\Theta(J)B_{(\infty)}}$. Hence, for any nonzero $I \in \mathcal{I}(C_0((0,1]))$ and $J \in \mathcal{I}(A)$,

$$\begin{aligned} \mathcal{I}(\rho)(I \otimes J) &= \mathcal{I}(\kappa_0) \circ \mathcal{I}((\iota \otimes \rho') \circ (m^* \otimes \mathrm{id}_A))(I \otimes J) \\ &= \mathcal{I}(\kappa_0)(\mathcal{O}_\infty \otimes \overline{B_{(\infty)}\Theta(J)B_{(\infty)}}) \qquad \text{by (4.20)} \\ &= \overline{B_{(\infty)}\Theta(J)B_{(\infty)}}, \end{aligned} \tag{4.23}$$

as required. □

We now turn to realising the ideal-lattice behaviour of $\mathcal{O}_\infty$-stable morphisms as opposed to morphisms into $\mathcal{O}_\infty$-stable algebras. We make use of $(B_{(\infty)})_{(\infty)}$ — that is, the sequence algebra of the sequence algebra $B_{(\infty)}$ — as a technical device. We eliminate it in the following section.

Lemma 4.6. *Let A and B be C^*-algebras with A separable, and suppose that $\theta\colon A \to B$ is an $\mathcal{O}_\infty$-stable *-homomorphism. Then for $n \in \mathbb{N}$, there are finite dimensional C^*-algebras F_n and cpc maps $\psi_n\colon C_0((0,1]) \otimes A \to F_n$ and $\eta_n\colon F_n \to B_{(\infty)}$ such that, writing*

$$\psi := (\psi_n)_{n=1}^\infty \colon C_0((0,1]) \otimes A \to \frac{\prod_{n=1}^\infty F_n}{\bigoplus_{n=1}^\infty F_n} \quad \text{and} \tag{4.24}$$

$$\eta := (\eta_n)_{n=1}^\infty \colon \frac{\prod_{n=1}^\infty F_n}{\bigoplus_{n=1}^\infty F_n} \to (B_{(\infty)})_{(\infty)} \tag{4.25}$$

for the induced maps, the following are satisfied:

(a) *ψ is a *-homomorphism;*
(b) *each η_n is order zero;*
(c) *if $\rho\colon C_0((0,1]) \otimes A \to (B_{(\infty)})_{(\infty)}$ is the *-homomorphism induced by the cpc order zero map $(\eta \circ \psi)(\mathrm{id}_{(0,1]} \otimes \cdot)$ (see Proposition 1.1), then*

$$\mathcal{I}(\rho)(I \otimes J) = \overline{(B_{(\infty)})_{(\infty)}\theta(J)(B_{(\infty)})_{(\infty)}} \tag{4.26}$$

for any $J \in \mathcal{I}(A)$ and any non-zero $I \in \mathcal{I}(C_0((0,1]))$; and
(d) *the *-homomorphism ρ is $\mathcal{O}_2$-stable.*

If every quotient of A is quasidiagonal, we may additionally arrange that $\psi_n(C_0((0,1)) \otimes A) = 0$ for each $n \in \mathbb{N}$.

Proof. As $\theta\colon A \to B$ is $\mathcal{O}_\infty$-stable, there is a unital embedding

$$j\colon \mathcal{O}_\infty \to \frac{B_{(\infty)} \cap \theta(A)'}{\mathrm{Ann}\,(\theta(A))}. \tag{4.27}$$

Thus there is an induced *-homomorphism $\theta_0\colon A \otimes \mathcal{O}_\infty \to B_{(\infty)}$ such that for all $a \in A$ and $x \in \mathcal{O}_\infty$, for any lift $\overline{j(x)} \in B_{(\infty)} \cap \Theta(A)'$ of $j(x)$, we have[14]

$$\theta_0(a \otimes x) = \theta(a)\overline{j(x)}. \tag{4.28}$$

In particular, $\theta_0(a \otimes 1_{\mathcal{O}_\infty}) = \theta(a)$ for all $a \in A$.

Let $D := \theta_0(A \otimes \mathcal{O}_\infty)$. Then D is $\mathcal{O}_\infty$-stable by, for example, [52, Corollary 3.3], and θ corestricts to a *-homomorphism $\theta|^D\colon A \to D$. Let $\Theta := \mathcal{I}(\theta|^D)\colon \mathcal{I}(A) \to \mathcal{I}(D)$, be the Cu-morphism induced by $\theta|^D$. Apply Lemma 4.5 to obtain $(F_n, \psi_n, \eta_n')_{n=1}^\infty$ and $\rho'\colon C_0((0,1]) \otimes A \to D_{(\infty)}$ satisfying (a)–(d) of that lemma and satisfying $\psi_n(C_0((0,1]) \otimes A) = 0$ for all n if every quotient of A is quasidiagonal. Let $\iota\colon D \hookrightarrow B_{(\infty)}$ be the inclusion, and $\eta_n := \iota \circ \eta_n'\colon F_n \to B_{(\infty)}$. We will show that $(F_n, \psi_n, \eta_n)_{n=1}^\infty$ satisfy conditions (a)–(d).

Clearly (a) is satisfied, and also the final assertion of the lemma. As each η_n' is order zero, so too is each η_n, so (b) is satisfied.

[14]Note that the definition of the annihilator ensures that the expression (4.28) does not depend on the choice of lift $\overline{j(x)}$.

Let $\iota_{(\infty)}\colon D_{(\infty)} \to (B_{(\infty)})_{(\infty)}$ be the *-homomorphism on sequence algebras induced by ι. Clearly $\iota_{(\infty)} \circ \eta' = \eta$ where $\eta' = (\eta'_n)_{n=1}^\infty$. So for each $a \in A$,

$$(\iota_{(\infty)} \circ \rho')(\mathrm{id}_{(0,1]} \otimes a) = (\iota_{(\infty)} \circ \eta' \circ \psi)(\mathrm{id}_{(0,1]} \otimes a) = (\eta \circ \psi)(\mathrm{id}_{(0,1]} \otimes a). \tag{4.29}$$

Hence $\iota_{(\infty)} \circ \rho'$ is the unique *-homomorphism induced by the cpc order zero map $(\eta \circ \psi)(\mathrm{id}_{(0,1]} \otimes \cdot)$ (see Proposition 1.1), so $\rho = \iota_{(\infty)} \circ \rho'$ by uniqueness. As the composition of an $\mathcal{O}_2$-stable *-homomorphism with any *-homomorphism is again $\mathcal{O}_2$-stable by [20, Lemma 3.20], the $\mathcal{O}_2$-stability of ρ' yields $\mathcal{O}_2$-stability of ρ. Hence (d) is satisfied and it remains to check (c).

To show (c), note that, writing $\iota_B\colon B \to B_{(\infty)}$ for the canonical embedding,

$$\iota \circ \theta|^D = \iota_B \circ \theta. \tag{4.30}$$

Functoriality of $\mathcal{I}$ gives

$$\mathcal{I}(\iota) \circ \Theta = \mathcal{I}(\iota \circ \theta|^D) = \mathcal{I}(\iota_B) \circ \mathcal{I}(\theta). \tag{4.31}$$

Let $\iota_D\colon D \to D_{(\infty)}$ be the canonical embedding. Fix non-zero ideals $I \in \mathcal{I}(C_0((0,1]))$ and $J \in \mathcal{I}(A)$. Using functoriality of $\mathcal{I}$ at the first step and that ρ satisfies Lemma 4.5(c) at the second, we calculate:

$$\begin{aligned}\mathcal{I}(\rho)(I \otimes J) &= \mathcal{I}(\iota_{(\infty)}) \circ \mathcal{I}(\rho')(I \otimes J)\\ &= \mathcal{I}(\iota_{(\infty)})(\overline{D_{(\infty)}\Theta(J)D_{(\infty)}})\\ &= \mathcal{I}(\iota_{(\infty)}) \circ \mathcal{I}(\iota_D) \circ \mathcal{I}(\theta|^D)(J).\end{aligned} \tag{4.32}$$

As $\iota_{(\infty)} \circ \iota_D \circ \theta|^D$ is equal to the composition of θ with the canonical inclusion $B \to (B_{(\infty)})_{(\infty)}$, it follows that

$$\mathcal{I}(\iota_{(\infty)}) \circ \mathcal{I}(\iota_D) \circ \mathcal{I}(\theta|^D)(J) = \overline{(B_{(\infty)})_{(\infty)}\theta(J)(B_{(\infty)})_{(\infty)}} \tag{4.33}$$

which verifies (c). □

5. Nuclear dimension of $\mathcal{O}_\infty$-stable maps

We now turn to the main results of the paper, combining the existence results of the previous sections with classification theorems to compute the nuclear dimension of $\mathcal{O}_\infty$-stable maps. The appropriate notion of equivalence in this context is approximate Murray–von Neumann equivalence:

Definition 5.1. Let A and B be C^*-algebras with A separable, and let $\phi, \psi\colon A \to B$ be *-homomorphisms. We say that ϕ and ψ are *approximately Murray–von Neumann equivalent* if there exists $v \in B_{(\infty)}$ such that $v^*\phi(a)v = \psi(a)$ and $v\psi(a)v^* = \phi(a)$ for all $a \in A$.

The precise classification ingredient we need is the following uniqueness theorem in the spirit of Kirchberg's classification results [25, 27].

Theorem 5.2 ([20, Theorem 3.23])**.** *Let A and B be C^*-algebras with A separable and exact. Suppose that $\phi, \psi\colon A \to B$ are nuclear, $\mathcal{O}_2$-stable *-homomorphisms. Then ϕ and ψ are approximately Murray–von Neumann equivalent if and only if $\mathcal{I}(\phi) = \mathcal{I}(\psi)$.*

We can now prove Theorem C, which asserts that every $\mathcal{O}_\infty$-stable homomorphism θ out of a separable exact C^*-algebra A has nuclear dimension at most 1, and that if moreover every quotient of A is quasidiagonal, then θ also has decomposition rank at most 1.

Proof of Theorem C. Let A and B be C^*-algebras with A separable and exact and let $\theta\colon A \to B$ be a nuclear $\mathcal{O}_\infty$-stable *-homomorphism. Fix a positive element $h_0 \in \mathcal{O}_\infty$ with spectrum $[0,1]$, and let $h_1 = 1_{\mathcal{O}_\infty} - h_0$. For $i \in \{0,1\}$, we write $\iota_i\colon C_0((0,1]) \to \mathcal{O}_\infty$ for the injective *-homomorphism that sends the generator $\mathrm{id}_{(0,1]} \in C_0((0,1])$ to h_i. These ι_i are $\mathcal{O}_2$-stable by Lemma 4.2, and Lemma 4.3 implies that the *-homomorphisms $\iota_i \otimes \mathrm{id}_A\colon C_0((0,1]) \otimes A \to \mathcal{O}_\infty \otimes A$ are also $\mathcal{O}_2$-stable.

As θ is $\mathcal{O}_\infty$-stable, there exists a unital embedding

$$j\colon \mathcal{O}_\infty \to \frac{B_{(\infty)} \cap \theta(A)'}{\mathrm{Ann}\,\theta(A)}. \tag{5.1}$$

Thus there exists a *-homomorphism $j \times \theta\colon \mathcal{O}_\infty \otimes A \to B_{(\infty)}$ such that for $x \in \mathcal{O}_\infty$ and $a \in A$, for any lift $\overline{j(x)} \in B_{(\infty)} \cap \theta(A)'$ of $j(x)$, we have $(j \times \theta)(x \otimes a) = \overline{j(x)}\theta(a)$. For $i \in \{0,1\}$, let $\mu^{(i)}$ denote the composition

$$C_0((0,1]) \otimes A \xrightarrow{\iota_i \otimes \mathrm{id}_A} \mathcal{O}_\infty \otimes A \xrightarrow{j \times \theta} B_{(\infty)} \subseteq (B_{(\infty)})_{(\infty)}. \tag{5.2}$$

Since the $\iota_i \otimes \mathrm{id}_A$ are $\mathcal{O}_2$-stable, [20, Lemma 3.20] implies that the compositions $\mu^{(i)} := (j \times \theta) \circ (\iota_i \otimes \mathrm{id}_A)$ are $\mathcal{O}_2$-stable. To see that each $\mu^{(i)}$ is nuclear, it suffices by [19, Theorem 2.9] to show that the order zero map

$$\theta^{(i)} := \mu^{(i)}(\mathrm{id}_{(0,1]} \otimes \cdot)\colon A \to (B_{(\infty)})_{(\infty)} \tag{5.3}$$

is nuclear. Let $\overline{h}_i \in B_{(\infty)} \cap \theta(A)' \subseteq (B_{(\infty)})_{(\infty)} \cap \theta(A)'$ be a positive lift of $j(h_i) = j(\iota_i(\mathrm{id}_{(0,1]}))$. Then

$$\theta^{(i)}(a) = \mu^{(i)}(\mathrm{id}_{(0,1]} \otimes a) = \overline{h}_i\theta(a) = \overline{h}_i^{1/2}\theta(a)\overline{h}_i^{1/2}, \quad a \in A. \tag{5.4}$$

As $\theta^{(i)}$ is the composition of the nuclear map $\theta\colon A \to B \subseteq (B_{(\infty)})_{(\infty)}$ and the cp map $\overline{h}_i^{1/2}(\cdot)\overline{h}_i^{1/2}\colon (B_{(\infty)})_{(\infty)} \to (B_{(\infty)})_{(\infty)}$, it is nuclear. Thus, each $\mu^{(i)}$ is nuclear.

Since $h_0 + h_1 = 1_{\mathcal{O}_\infty}$, we have $1_{B_{(\infty)}} - (\overline{h}_0 + \overline{h}_1) \in \mathrm{Ann}\,\theta(A)$. Hence (5.4) yields

$$\theta^{(0)}(a) + \theta^{(1)}(a) = \theta(a)(\overline{h}_0 + \overline{h}_1) = \theta(a), \quad a \in A. \tag{5.5}$$

Thus

$$\theta^{(0)} + \theta^{(1)} = \theta\colon A \to (B_{(\infty)})_{(\infty)}. \tag{5.6}$$

Let $(F_n, \psi_n, \eta_n)_{n=1}^\infty$ and $\rho\colon C_0((0,1]) \otimes A \to (B_{(\infty)})_{(\infty)}$ be as provided by Lemma 4.6 for the $\mathcal{O}_\infty$-stable map θ. If all quotients of A are quasidiagonal, we choose ψ_n such that $\psi_n(C_0((0,1)) \otimes A) = 0$. By construction, the order zero map $\rho(\mathrm{id}_{(0,1]} \otimes \cdot)\colon A \to (B_{(\infty)})_{(\infty)}$ is represented by the sequence

$$(\eta_n \circ \psi_n(\mathrm{id}_{(0,1]} \otimes \cdot)\colon A \to B_{(\infty)})_{n=1}^\infty. \tag{5.7}$$

As each $\eta_n \circ \psi_n(\mathrm{id}_{(0,1]} \otimes \cdot)$ is nuclear (they factor through the finite dimensional C^*-algebra F_n) and A is exact, $\rho(\mathrm{id}_{(0,1]} \otimes \cdot)$ is nuclear by [13, Proposition 3.3]. So [19, Theorem 2.9] implies that ρ is nuclear.

We will show that $\mathcal{I}(\rho) = \mathcal{I}(\mu^{(i)})$ for $i = 0, 1$. For this, fix $i \in \{0, 1\}$, and fix non-zero $I \in \mathcal{I}(C_0((0, 1]))$ and $J \in \mathcal{I}(A)$. As ρ and $\mu^{(i)}$ are Cu-morphisms and thus preserve suprema, and as $\{I \otimes J : I \in \mathcal{I}(C_0((0, 1]))$ and $J \in \mathcal{I}(A)\}$ is a basis for $\mathcal{I}(C_0((0, 1]) \otimes A)$ it suffices to check that $\mathcal{I}(\rho)(I \otimes J) = \mathcal{I}(\mu^{(i)})(I \otimes J)$.

By Lemma 4.6(c), we have

$$\mathcal{I}(\rho)(I \otimes J) = \overline{(B_{(\infty)})_{(\infty)}\theta(J)(B_{(\infty)})_{(\infty)}}. \tag{5.8}$$

As I is non-zero, and as $\mathcal{I}(\cdot)$ is a functor ([20, Proposition 2.5]), the definition (5.2) of $\mu^{(i)}$ gives

$$\begin{aligned} \mathcal{I}(\mu^{(i)})(I \otimes J) &= \mathcal{I}(j \times \theta)(\mathcal{I}(\iota_i \otimes \mathrm{id}_A)(I \otimes J)) \\ &= \mathcal{I}(j \times \theta)(\mathcal{O}_\infty \otimes J). \end{aligned} \tag{5.9}$$

By the definition of $j \times \theta\colon \mathcal{O}_\infty \otimes A \to B_{(\infty)} \subseteq (B_{(\infty)})_{(\infty)}$, the ideal $(j \times \theta)(\mathcal{O}_\infty \otimes J)$ is contained in the ideal generated by $\theta(J)$. Conversely, the defining property of $j \times \theta$, and then (5.9) give

$$\theta(J) = (j \times \theta)(1_{\mathcal{O}_\infty} \otimes J) \subseteq \mathcal{I}(\mu^{(i)})(I \otimes J). \tag{5.10}$$

Hence $\mathcal{I}(\mu^{(i)})(I \otimes J)$ is the ideal generated by $\theta(J)$. Thus (5.8) gives $\mathcal{I}(\mu^{(i)})(I \otimes J) = \mathcal{I}(\rho)(I \otimes J)$ as required.

Since ρ and $\mu^{(i)}$ are nuclear and $\mathcal{O}_2$-stable, Theorem 5.2 implies that ρ and $\mu^{(i)}$ are approximately Murray–von Neumann equivalent. Hence [20, Proposition 3.10] shows that

$$\rho \oplus 0 \text{ and } \mu^{(i)} \oplus 0\colon C_0((0, 1]) \otimes A \to M_2((B_{(\infty)})_{(\infty)}) \tag{5.11}$$

are approximately unitary equivalent (via unitaries in the minimal unitisation, though we work in the slightly larger unitisation $M_2((\widetilde{B}_{(\infty)})_{(\infty)})$). As A is separable, a standard reindexing argument for sequence algebras shows that $\rho \oplus 0$ and $\mu^{(i)} \oplus 0$ are unitary equivalent. Choose unitaries

$$u^{(i)} \in M_2((\widetilde{B}_{(\infty)})_{(\infty)}) = M_2(\widetilde{B}_{(\infty)})_{(\infty)} \tag{5.12}$$

such that

$$u^{(i)*}(\rho(x) \oplus 0)u^{(i)} = \mu^{(i)}(x) \oplus 0, \quad x \in C_0((0, 1]) \otimes A,\ i = 0, 1. \tag{5.13}$$

Choose a unitary lift $(u_n^{(i)})_{n=1}^\infty \in \prod_{n=1}^\infty M_2(\widetilde{B}_{(\infty)})$ of each $u^{(i)}$.

Define cpc maps

$$\widehat{\psi}_n := \psi_n(\mathrm{id}_{(0,1]} \otimes \cdot)\colon A \to F_n, \tag{5.14}$$

and, for $i = 0, 1$,

$$\widehat{\eta}_n^{(i)} := u_n^{(i)*}(\eta_n(\cdot) \oplus 0)u_n^{(i)}\colon F_n \to M_2(B_{(\infty)}). \tag{5.15}$$

Then $\widehat{\psi}_n \oplus \widehat{\psi}_n\colon A \to F_n \oplus F_n$ is cpc for each n, and, as each η_n is cpc order zero by Lemma 4.6(b), each $\widehat{\eta}_n^{(i)}$ is cpc order zero. Computing in $M_2(\tilde{B}_{(\infty)})_{(\infty)}$, for each $a \in A$, we have

$$
\begin{aligned}
\theta(a) \oplus 0 &= (\theta^{(0)}(a) \oplus 0) + (\theta^{(1)}(a) \oplus 0) && \text{by (5.6)} \\
&= (\mu^{(0)}(\mathrm{id}_{(0,1]} \otimes a) \oplus 0) + (\mu^{(1)}(\mathrm{id}_{(0,1]} \otimes a) \oplus 0) && \text{by (5.3)} \\
&= \sum_{i=0}^{1} u^{(i)*}(\rho(\mathrm{id}_{(0,1]} \otimes a) \oplus 0)u^{(i)} && \text{by (5.13)} \\
&= \left(\sum_{i=0}^{1} u_n^{(i)*}((\eta_n(\psi_n(\mathrm{id}_{(0,1]} \otimes a))) \oplus 0)u_n^{(i)} \right)_{n=1}^{\infty} && \text{by (5.7)} \\
&= \left(\sum_{i=0}^{1} (\widehat{\eta}_n^{(i)} \circ \widehat{\psi}_n)(a) \right)_{n=1}^{\infty} \\
&= \Big(((\widehat{\eta}_n^{(0)} + \widehat{\eta}_n^{(1)}) \circ (\widehat{\psi}_n \oplus \widehat{\psi}_n))(a) \Big)_{n=1}^{\infty}. && (5.16)
\end{aligned}
$$

Thus $\big(F_n \oplus F_n, \widehat{\psi}_n \oplus \widehat{\psi}_n, \widehat{\eta}^{(0)} + \widehat{\eta}^{(1)}\big)_{n=1}^{\infty}$ is a sequence of 1-decomposable approximations witnessing that $\theta \oplus 0\colon A \to M_2(B_{(\infty)})$ has nuclear dimension at most 1. By Proposition 1.6 it follows that $\theta\colon A \to B_{(\infty)}$ also has nuclear dimension at most 1. Hence $\dim_{\mathrm{nuc}}(\theta) \leq 1$ by [51, Proposition 2.5].

If all quotients of A are quasidiagonal, then we can additionally choose the ψ_n in Lemma 4.6 so that $\psi_n(C_0((0,1)) \otimes A) = 0$. Then Lemma 4.6($a$) implies that

$$
(\widehat{\psi}_n \oplus \widehat{\psi}_n)_{n=1}^{\infty}\colon A \to \frac{\prod_{n=1}^{\infty}(F_n \oplus F_n)}{\bigoplus_{n=1}^{\infty}(F_n \oplus F_n)} \tag{5.17}
$$

is a *-homomorphism. Now Lemma 1.9 shows that $\theta \oplus 0\colon A \to M_2(B_{(\infty)})$ has decomposition rank at most 1. Cutting to the hereditary subalgebra $B_{(\infty)}$ by Proposition 1.6, and removing the sequence algebra by [51, Proposition 2.5] just as above, we obtain $\mathrm{dr}\,\theta \leq 1$. □

Theorem A, which says that separable, nuclear, $\mathcal{O}_\infty$-stable C^*-algebras have nuclear dimension 1 is an immediate consequence of Theorem C.

Proof of Theorem A. Let A be a separable, nuclear, $\mathcal{O}_\infty$-stable C^*-algebra. By [20, Proposition 3.19], id_A is $\mathcal{O}_\infty$-stable, so

$$
\dim_{\mathrm{nuc}} A = \dim_{\mathrm{nuc}} \mathrm{id}_A \leq 1 \tag{5.18}
$$

by Theorem C. As $\mathcal{O}_\infty$-stable C^*-algebras are not approximately finite dimensional, it follows from [54, Theorem 3.4] that $\dim_{\mathrm{nuc}} A > 0$ so $\dim_{\mathrm{nuc}} A = 1$. □

6. Finite decomposition rank

In [44], Rørdam constructs a separable, nuclear, $\mathcal{O}_\infty$-stable C^*-algebra $\mathcal{A}_{[0,1]}$ that is an inductive limit of C^*-algebras of the form $C_0((0,1]) \otimes M_{2^n}$. This provides an example of a separable, nuclear, $\mathcal{O}_\infty$-stable C^*-algebra with decomposition rank one. In this section we characterise exactly when separable, nuclear, $\mathcal{O}_\infty$-stable C^*-algebras have finite decomposition rank.

In [21] it was established that a separable, nuclear, $\mathcal{O}_\infty$-stable C^*-algebra is quasidiagonal if and only if its primitive ideal space[15] has no non-empty, compact, open subsets.[16] This will be used to give a characterisation of when separable, nuclear, $\mathcal{O}_\infty$-stable C^*-algebras have finite decomposition rank in terms of their primitive ideal space.

Recall that a subset C of a topological space X is called *locally closed* if there are open subsets $U \subseteq V \subseteq X$ such that $C = V \setminus U$. Equivalently, C is locally closed if and only if there is an open subset $V \subseteq X$ such that $C = V \cap \overline{C}$, where $\overline{C}$ denotes the closure of C.

Remark 6.1. Recall from [38, Theorem 4.1.3] that there is an order isomorphism from $\mathcal{I}(A)$ to the collection of open subsets of $\operatorname{Prim} A$ that carries $I \in \mathcal{I}(A)$ to $U_I \coloneqq \{J \in \operatorname{Prim} A : I \not\subseteq J\}$. Since $I \mapsto U_I$ is an isomorphism of lattices, we have $U_I \cap U_J = U_{I\cap J}$ and $U_I \cup U_J = U_{I+J}$ for $I, J \in \mathcal{I}(A)$.

By a subquotient of a C^*-algebra A, we mean the quotient of an ideal in A; that is, a C^*-algebra of the form J/I where $I \subseteq J$ are ideals in A. The primitive ideal space $\operatorname{Prim}(J/I)$ is homeomorphic to the locally closed subset $U_J \setminus U_I$ of $\operatorname{Prim} A$ by [38, Theorem 4.1.11(ii)]. If $I_i \subseteq J_i$ are ideals in A for $i = 1, 2$ such that $U_{J_1} \setminus U_{I_1} = U_{J_2} \setminus U_{I_2}$, then the inclusions $I_i \hookrightarrow I_1 + I_2$ and $J_i \hookrightarrow J_1 + J_2$ induce isomorphisms $J_1/I_1 \cong (J_1 + J_2)/(I_1 + I_2) \cong J_2/I_2$.[17] It follows that there is an equivalence relation on subquotients of A given by $J_1/I_1 \sim J_2/I_2$ if and only if the inclusions $I_i \hookrightarrow I_1 + I_2$ and $J_i \hookrightarrow J_1 + J_2$ induce isomorphisms $J_1/I_1 \cong (J_1 + J_2)/(I_1 + I_2) \cong J_2/I_2$. It also follows that the map $J/I \mapsto U_J \setminus U_I$ descends to a bijection between equivalence classes of subquotients of A and locally closed subsets of $\operatorname{Prim} A$.

By [43, Proposition 4.1.1] a simple C^*-algebra A is purely infinite if and only if every non-zero hereditary C^*-subalgebra contains a non-zero, σ-unital, stable C^*-subalgebra. The following is a similar characterisation.

Proposition 6.2. *Let A be a C^*-algebra. Then every non-zero, σ-unital hereditary C^*-subalgebra of A is stable if and only if A is purely infinite and* $\operatorname{Prim} A$ *has no locally closed, one-point subsets.*

Proof. Suppose that A is purely infinite and that $\operatorname{Prim} A$ has no locally closed, one-point subsets, and let $B \subseteq A$ be a non-zero, σ-unital hereditary C^*-subalgebra. Then B is also purely infinite by [31, Proposition 4.17]. By [31, Theorem 4.24] B is stable if (and only if) B has no non-zero, unital quotients. Suppose for contradiction that B has a unital quotient. As any unital C^*-algebra has a simple quotient, B has a simple quotient, say B/I_B.

[15] Recall that the *primitive ideal space* $\operatorname{Prim} A$ (sometimes denoted $\check{A}$) of a C^*-algebra A is the set of all primitive ideals equipped with the Jacobson topology, see [38, Section 4.1]. The primitive ideal space is often non-Hausdorff, but is always T_0 [38, 4.1.4].

[16] More generally, this holds for any separable, exact C^*-algebra that is traceless in the sense of [32]. That $\mathcal{O}_\infty$-stable C^*-algebras are traceless follows from [32, Theorem 9.1].

[17] In fact, since $U_{J_1} \cup U_{J_2} = U_{J_1} \cup (U_{J_2} \setminus U_{I_2}) \cup U_{I_2} = U_{J_1} \cup (U_{J_1} \setminus U_{I_1}) \cup U_{I_2} = U_{J_1} \cup U_{I_2}$, one has $J_1 + J_2 = J_1 + I_2$. Similarly, $(U_{I_1} \cup U_{I_2}) \cap U_{J_1} = U_{I_1} \cup (U_{I_2} \cap (U_{J_1} \setminus U_{I_1})) = U_{I_1} \cup (U_{I_2} \cap (U_{J_2} \setminus U_{I_2})) = U_{I_1}$, and thus $I_1 = (I_1 + I_2) \cap J_1$. Hence

$$\frac{J_1}{I_1} = \frac{J_1}{(I_1 + I_2) \cap J_1} \cong \frac{J_1 + I_1 + I_2}{I_1 + I_2} = \frac{J_1 + J_2}{I_1 + I_2}$$

where the isomorphism is induced by inclusions. By symmetry the result follows.

Let $I_A := \overline{AI_BA}$ and $J_A := \overline{ABA}$ be the ideals in A generated by I_B and B respectively. As B is a full, hereditary C^*-subalgebra of J_A, it follows that $B/I_B \cong (B + I_A)/I_A$ is a full, hereditary C^*-subalgebra of J_A/I_A. Then B/I_B and J_A/I_A are strongly Morita equivalent,[18] and thus J_A/I_A is simple by the Rieffel correspondence.[19] Hence $\mathrm{Prim}\, A$ has a locally closed, one-point set, namely $U_{J_A} \setminus U_{I_A}$, by Remark 6.1. This is a contradiction, and so B has no unital quotients. Hence B is stable.

Conversely, suppose that all non-zero, σ-unital hereditary C^*-subalgebras of A are stable. Then every non-zero, positive element $a \in A$ is stable in the sense that $\overline{aAa}$ is stable. By [31, Proposition 3.7 and Theorem 4.16] it follows that A is purely infinite. Suppose for contradiction that $\mathrm{Prim}\, A$ has a locally closed, one-point subset $\{x\}$. By Remark 6.1 there exist ideals $I \subseteq J$ of A such that $\{x\} = U_J \setminus U_I$, and hence J/I is simple. As pure infiniteness passes to ideals and quotients by [31, Theorem 4.19], it follows that J/I is purely infinite and simple. Hence J/I contains a non-zero projection p. Let $a \in J$ be a positive lift of p. Then $\overline{aAa}$ has a unital quotient, namely $p(J/I)p$, contradicting stability of $\overline{aAa}$. Hence $\mathrm{Prim}\, A$ has no locally closed, one-point subsets. □

We do not know if the C^*-algebras considered in Proposition 6.2 are actually strongly purely infinite, and thus $\mathcal{O}_\infty$-stable under separability and nuclearity assumptions by [32].

We can now prove Theorem B, which states that for separable, nuclear, $\mathcal{O}_\infty$-stable C^*-algebras A, the following are equivalent:

(*i*) the decomposition rank of A is finite;
(*ii*) the decomposition rank of A is 1;
(*iii*) every quotient of A is quasidiagonal;
(*iv*) every non-zero, hereditary C^*-subalgebra of A is stable;
(*v*) the primitive ideal space of A has no locally closed, one-point subsets.

Proof of Theorem B. (*iii*) ⇒ (*ii*): Theorem C shows that when all quotients of A are quasidiagonal, then $\mathrm{dr}\, \mathrm{id}_A \leq 1$ (as the identity map inherits $\mathcal{O}_\infty$-stability from A). Accordingly $\mathrm{dr}\, A = 1$, as A is not approximately finite dimensional.

(*ii*) ⇒ (*i*): This is obvious.

(*i*) ⇒ (*v*): Suppose that A has finite decomposition rank, and assume for contradiction that $\mathrm{Prim}\, A$ has a locally closed, one-point subset. By Remark 6.1, A has a simple subquotient. As both pure infiniteness and finite decomposition rank pass to ideals and to quotients by [31, Theorem 4.19] and [33, (3.3) and Proposition 3.8], this subquotient is purely infinite, simple and has finite decomposition rank. This contradicts that C^*-algebras of finite decomposition rank are quasidiagonal [33, Proposition 5.1], and hence stably finite (see for example [6, Proposition 7.1.15]).

[18] If $A \subseteq B$ is a full, hereditary C^*-subalgebra, then $\overline{AB}$ is an imprimitivity A–B-bimodule, so A and B are strongly Morita equivalent. See [41, Chapter 3] for more details.

[19] The Rieffel correspondence asserts that strongly Morita equivalent C^*-algebras have order isomorphic ideal lattices, see for instance [41, Theorem 3.22]. In particular, if one is simple then so is the other.

$(v) \Rightarrow (iii)$: We prove the contrapositive statement, so suppose that I is an ideal in A such that A/I is not quasidiagonal. Let $U_I \subseteq \operatorname{Prim} A$ be the open subset corresponding to I. By [21, Corollary C], $\operatorname{Prim}(A/I)$ has a non-empty, compact, open subset $V \subseteq \operatorname{Prim}(A/I)$. By compactness of V and Zorn's lemma, V contains a maximal, open, proper subset W.[20] As primitive ideal spaces of C^*-algebras are T_0 (see [38, 4.1.4]), it follows that $V \setminus W$ is a singleton.[21] Let $W', V' \subseteq (\operatorname{Prim} A) \setminus U_I$ be the open subsets corresponding to V and W under the canonical identification of $\operatorname{Prim}(A/I)$ with $(\operatorname{Prim} A) \setminus U_I$ of [38, Theorem 4.1.11]. Then $W_0 := W' \cup U_I$ and $V_0 := V' \cup U_I$ are open subsets of $\operatorname{Prim} A$, and $V_0 \setminus W_0$ is a singleton, so $\operatorname{Prim} A$ has a locally closed, one-point subset.

$(iv) \Leftrightarrow (v)$: This follows from Proposition 6.2 as A is $\mathcal{O}_\infty$-stable and thus purely infinite. □

In particular for separable nuclear $\mathcal{O}_\infty$-stable C^*-algebras, finiteness of the decomposition rank is determined by the primitive ideal space.

Corollary 6.3. *Let A and B be separable, nuclear, $\mathcal{O}_\infty$-stable C^*-algebras such that $\operatorname{Prim} A$ and $\operatorname{Prim} B$ are homeomorphic. If A has finite decomposition rank then so does B.*

Example 6.4. The C^*-algebra $\mathcal{A}_{[0,1]}$ of [44] has primitive ideal space homeomorphic to $[0,1)$ equipped with the topology $\{\emptyset\} \cup \{[0,t) : 0 < t \leq 1\}$. In this space, a non-empty intersection of an open and a closed set has the form (a,b), so the space has no locally closed, one-point sets. Therefore, any separable, nuclear, $\mathcal{O}_\infty$-stable C^*-algebra B with $\operatorname{Prim} B \cong [0,1)$ with topology as above has decomposition rank 1.

A consequence of [30, Proposition 6.2] is that $\mathcal{A}_{[0,1]} \otimes B$ has decomposition rank 1 for any non-zero, separable, nuclear C^*-algebra B. This will be generalised in Corollary 6.6 below. We will need the following lemma.

Lemma 6.5. *Let X and Y be topological spaces. If $\{(x,y)\} \subseteq X \times Y$ is a locally closed, one-point subset, then so is $\{x\} \subseteq X$.*

Proof. Pick an open neighbourhood U of (x,y) such that $\{(x,y)\} = U \cap \overline{\{(x,y)\}}$. Let $U_X \subseteq X$ and $U_Y \subseteq Y$ be open subsets such that $(x,y) \in U_X \times U_Y \subseteq U$. Then $\{(x,y)\} = (U_X \times U_Y) \cap \overline{\{(x,y)\}} = (U_X \times U_Y) \cap \big(\overline{\{x\}} \times \overline{\{y\}}\big) = \big(U_X \cap \overline{\{x\}}\big) \times \big(U_Y \cap \overline{\{y\}}\big)$. In particular, $\{x\} = U_X \cap \overline{\{x\}}$. □

If a simple C^*-algebra A has finite decomposition rank, then $A \otimes \mathcal{O}_\infty$ is purely infinite and simple, and thus has infinite decomposition rank.[22] However, in the non-simple case we obtain the following permanence property.

[20]Consider the collection $\mathcal{U}$ of proper open subsets of V. Let $\mathcal{C} \subseteq \mathcal{U}$ be a chain and let $U := \bigcup \mathcal{C}$ which is an open subset of V. As V is compact it follows that $U \neq V$ (otherwise $U' = V$ for some $U' \in \mathcal{C}$, a contradiction), and thus $U \in \mathcal{U}$ is an upper bound for $\mathcal{C}$. By Zorn's lemma, $\mathcal{U}$ has a maximal element.

[21]If $x, y \in V \setminus W$ are distinct, then that $\operatorname{Prim}(A)$ is T_0 implies the existence of an open neighbourhood X of one that does not contain the other; but then $W \subsetneq W \cup (X \cap V) \subsetneq V$, contradicting maximality of W.

[22]If $A \otimes \mathcal{O}_\infty$ has finite decomposition rank, then it is quasidiagonal by [33, Theorem 5.1], and so is stably finite (see [6, Proposition 7.1.15], for example), a contradiction.

Corollary 6.6. *Let A be a separable, nuclear, $\mathcal{O}_\infty$-stable C^*-algebra with finite decomposition rank. Then $A \otimes B$ has decomposition rank one for any non-zero, separable, nuclear C^*-algebra B.*

Proof. The tensor product $A \otimes B$ is separable, nuclear and $\mathcal{O}_\infty$-stable. As A and B are separable and nuclear we have $\mathrm{Prim}(A \otimes B) \cong \mathrm{Prim}\, A \times \mathrm{Prim}\, B$ (see for instance [3, Theorem IV.3.4.25]). By Theorem B, $\mathrm{Prim}\, A$ has no locally closed, one-point subsets, so Lemma 6.5 implies that $\mathrm{Prim}(A \otimes B)$ has no locally closed, one-point subsets. So Theorem B implies that $A \otimes B$ has decomposition rank one. □

Problem 6.7. Characterise when nuclear, $\mathcal{O}_\infty$-stable *-homomorphisms have finite decomposition rank.

7. Zero dimensional $\mathcal{O}_2$-stable *-homomorphisms

A C^*-algebra A has nuclear dimension zero (equivalently decomposition rank zero) if and only if it is AF, in the sense of being able to approximate finite dimensional subsets by finite dimensional subalgebras; see [33]. Here we begin the investigation of when *-homomorphisms are zero dimensional, resolving the question for full, $\mathcal{O}_2$-stable maps with exact domains. Recall that fullness is a simplicity criterion for *-homomorphisms: $\theta\colon A \to B$ is called *full* if the image under θ of every non-zero $a \in A$ is full in B. Equivalently, θ is full if, for every $I \in \mathcal{I}(A)$,

$$\mathcal{I}(\theta)(I) = \begin{cases} 0 & \text{if } I = 0, \\ B & \text{otherwise.} \end{cases} \tag{7.1}$$

We begin with the special case of embeddings into $\mathcal{O}_2$, where Kirchberg famously showed that every separable and exact C^*-algebra A admits such an embedding [29].

Proposition 7.1. *Let A be a separable, exact C^*-algebra, and let $\phi\colon A \to \mathcal{O}_2$ be an injective *-homomorphism. Then*

$$\dim_{\mathrm{nuc}}\phi = \begin{cases} 0 & \textit{if } A \textit{ is quasidiagonal,} \\ 1 & \textit{otherwise.} \end{cases} \tag{7.2}$$

Proof. As $\mathcal{O}_2$ is nuclear and $\mathcal{O}_\infty$-stable, so is ϕ by [20, Proposition 3.18]. Hence $\dim_{\mathrm{nuc}}\phi \leq 1$ by Theorem C.

First suppose that $\dim_{\mathrm{nuc}}\phi = 0$. Since the definitions of nuclear dimension zero and decomposition rank zero coincide, $\mathrm{dr}\,\phi = 0$ and thus A is quasidiagonal by Corollary 1.8.

Now suppose that A is quasidiagonal. Let ω be a free ultrafilter on $\mathbb{N}$, and let $(\mathcal{O}_2)_\omega$ denote the norm ultraproduct of $\mathcal{O}_2$. Let $\iota\colon \mathcal{O}_2 \hookrightarrow (\mathcal{O}_2)_\omega$ be the canonical (unital) embedding. Then $\iota \circ \phi\colon A \to (\mathcal{O}_2)_\omega$ is nuclear and $\mathcal{O}_2$-stable.

Fix an isomorphism $\kappa\colon \mathcal{O}_2 \otimes \mathcal{O}_2 \to \mathcal{O}_2$ (the existence of which is due to Elliott, but was first recorded in [42]; see [43, Theorem 5.2.1]). By quasidiagonality of A there exist integers $(k_n)_{n=1}^\infty$ and cpc maps $\psi_n\colon A \to M_{k_n}$ such that

$$\psi := (\psi_n)_{n=1}^\infty\colon A \to \prod_{n=1}^\infty M_{k_n} / \bigoplus_{n=1}^\infty M_{k_n} \tag{7.3}$$

is an injective *-homomorphism. Let $\rho_n\colon M_{k_n} \to \mathcal{O}_2$ be an embedding for each n, and let

$$\eta_n := \kappa \circ (\rho_n \otimes 1_{\mathcal{O}_2})\colon M_{k_n} \to \mathcal{O}_2. \tag{7.4}$$

Then $(\eta_n \circ \psi_n)_\omega\colon A \to (\mathcal{O}_2)_\omega$ is an injective *-homomorphism with a cpc lift $(\eta_n \circ \psi_n)_{n=1}^\infty\colon A \to \prod_{n=1}^\infty \mathcal{O}_2$. As A is exact and $\mathcal{O}_2$ is nuclear, this cpc lift is nuclear by [13, Proposition 3.3], and thus $(\eta_n \circ \psi_n)_\omega$ is nuclear. To see that $(\eta_n \circ \psi_n)_\omega$ is $\mathcal{O}_2$-stable, observe that the unital C^*-subalgebra

$$\kappa(1_{\mathcal{O}_2} \otimes \mathcal{O}_2) \subseteq (\mathcal{O}_2)_\omega \tag{7.5}$$

is isomorphic to $\mathcal{O}_2$, and commutes with the image of $(\eta_n \circ \psi_n)_\omega$. Hence both $\phi\colon A \to (\mathcal{O}_2)_\omega$ and $(\eta_n \circ \psi_n)_\omega$ are nuclear, $\mathcal{O}_2$-stable homomorphisms of A into $(\mathcal{O}_2)_\omega$. Since $(\mathcal{O}_2)_\omega$ is simple by [43, Proposition 6.2.6], and since ϕ and $(\eta_n \circ \psi_n)_\omega$ are injective, we trivially have $\mathcal{I}(\phi) = \mathcal{I}\big((\eta_n \circ \psi_n)_\omega\big)$. Therefore Theorem 5.2 implies that ϕ and $(\eta_n \circ \psi_n)_\omega$ are approximately Murray–von Neumann equivalent.

By [20, Proposition 3.10], the maps

$$\phi \oplus 0,\ (\eta_n \circ \psi_n)_\omega \oplus 0\colon A \to M_2(\mathcal{O}_2)_\omega \tag{7.6}$$

are approximately unitary equivalent. Hence for $\mathcal{F} \subseteq A$ finite and $\epsilon > 0$, there exist a unitary $u \in M_2(\mathcal{O}_2)$ and an $n \in \mathbb{N}$ such that

$$\|\phi(a) \oplus 0 - \operatorname{Ad} u(\eta_n(\psi_n(a)) \oplus 0)\| < \epsilon, \qquad a \in \mathcal{F}. \tag{7.7}$$

Let $\eta := \operatorname{Ad} u \circ (\eta_n \oplus 0)\colon M_{k_n} \to M_2(\mathcal{O}_2)$ for each n. Then $\eta \circ \psi_n$ approximates $\phi \oplus 0$ up to ϵ on $\mathcal{F}$. As η is a *-homomorphism and ψ_n is a cpc map, it follows that $\phi \oplus 0$ has nuclear dimension zero. By Proposition 1.6, so too does ϕ. □

The following is essentially contained in [37], and uses tricks of Blackadar and Cuntz from [4]. Recall the definition of properly infinite, positive elements given above Lemma 4.4.

Lemma 7.2. *Let $b \in B_+$ and $\epsilon > 0$. Suppose that b and $(b-\epsilon)_+$ are properly infinite and generate the same ideal I in B. Then I contains a properly infinite, full projection.*

Proof. In the following, we use the Cuntz relation on positive elements, and write $a \sim b$ if and only if $a \precsim b$ and $b \precsim a$. Arguing as in the last part of the proof of [37, Proposition 2.7 (i) $\Rightarrow$ (ii)], there is a positive element $c_2 \in I$ for which $b \sim c_2$, and a projection $p \in I$ satisfying $pc_2 = c_2$.[23] In particular, $b \sim c_2 \precsim p$. As b is properly infinite and $p \in I = \overline{BbB}$, it follows that $p \precsim b$ by [31, Proposition 3.5(ii)]. Hence p and b are Cuntz equivalent, so p is properly infinite and full in I. □

[23]The following argument is essentially that of [37, Proposition 2.7 (i) $\Rightarrow$ (ii)]: Fix $0 < \epsilon_1 < \epsilon$. As b and $(b-\epsilon)_+$ are properly infinite and generate the same ideal, it follows that $b \sim (b-\epsilon)_+ \sim (b-\epsilon_1)_+$ by [31, Proposition 3.5(ii)]. By [31, Lemma 2.5(i) and Proposition 3.3] we can find mutually orthogonal, positive elements c_1 and c_2 in $\overline{(b-\epsilon_1)_+ I (b-\epsilon_1)_+}$ such that $(b-\epsilon)_+ \precsim c_1$ and $(b-\epsilon)_+ \precsim c_2$. In particular, $b \precsim c_1$. By [37, Remark 2.5] there is a contraction $x \in I$ such that $x^*x(b-\epsilon_1)_+ = (b-\epsilon_1)_+$ and $xx^* \in \overline{c_1 I c_1} \subseteq \overline{(b-\epsilon_1)_+ I (b-\epsilon_1)_+}$. Hence x is a scaling element in the sense of [4], i.e. x is a contraction for which $x^*xxx^* = xx^*$, and this x satisfies $x^*xc_2 = c_2$ and $xx^*c_2 = 0$. By [37, Remark 2.4] there is a projection $p \in I$ such that $pc_2 = c_2$.

We can now prove Theorem D: if θ is a full, $\mathcal{O}_2$-stable *-homomorphism out of a separable and exact C^*-algebra, then the nuclear dimension of θ is 0 if θ is nuclear and A is quasidiagonal, 1 if θ is nuclear and A is not quasidiagonal, and ∞ if θ is not nuclear.

Proof of Theorem D. If θ is not nuclear, then certainly $\dim_{\mathrm{nuc}}\theta = \infty$. So suppose θ is nuclear. Then by Theorem C we have $\dim_{\mathrm{nuc}}\theta \leq 1$. So it suffices to show that $\dim_{\mathrm{nuc}}\theta = 0$ if and only if A is quasidiagonal.

First suppose that $\dim_{\mathrm{nuc}}\theta = 0$. Then $\mathrm{dr}\,\theta = 0$ and thus A is quasidiagonal by Corollary 1.8.

Now suppose that A is quasidiagonal. Let $a \in A$ be positive and nonzero, and $0 < \epsilon < \|a\|$. As θ is $\mathcal{O}_2$-stable it is $\mathcal{O}_\infty$-stable. So as θ is also full, $\theta(a)$ and $(\theta(a) - \epsilon)_+$ are both properly infinite by Lemma 4.4, and they both generate B as an ideal. By Lemma 7.2, B contains a full, properly infinite projection. Hence there exists an embedding $\iota\colon \mathcal{O}_2 \to B$ such that $\iota(1_{\mathcal{O}_2})$ is a full projection in B. Let $\phi\colon A \to \mathcal{O}_2$ be an injective *-homomorphism as given by Kirchberg's embedding theorem. Then both θ and $\iota \circ \phi$ are nuclear, full, $\mathcal{O}_2$-stable *-homomorphisms, so they are approximately Murray–von Neumann equivalent by Theorem 5.2. By [20, Proposition 3.10], $\theta \oplus 0, (\iota \circ \phi) \oplus 0\colon A \to M_2(B)$ are approximately unitary equivalent, and in particular have the same nuclear dimension. By Proposition 7.1, ϕ has nuclear dimension zero, and thus so does $\theta \oplus 0$. Hence θ has nuclear dimension zero by Proposition 1.6. □

Problem 7.3. Characterise when *-homomorphisms have nuclear dimension zero.

References

[1] S. Arklint, G. Restorff, and E. Ruiz. Classification of real rank zero, purely infinite C^*-algebras with at most four primitive ideals. *J. Funct. Anal.*, 271(7):1921–1947, 2016.

[2] S. Barlak, D. Enders, H. Matui, G. Szabó, and W. Winter. The Rokhlin property vs. Rokhlin dimension 1 on unital Kirchberg algebras. *J. Noncommut. Geom.*, 9(4):1383–1393, 2015.

[3] B. Blackadar. *Operator algebras*, volume 122 of *Encyclopaedia of Mathematical Sciences*. Springer–Verlag, Berlin, 2006. Theory of C^*-algebras and von Neumann algebras, Operator Algebras and Non-commutative Geometry, III.

[4] B. Blackadar and J. Cuntz. The structure of stable algebraically simple C^*-algebras. *Amer. J. Math.*, 104(4):813–822, 1982.

[5] J. Bosa, N. Brown, Y. Sato, A. Tikuisis, S. White, and W. Winter. Covering dimension of C^*-algebras and 2-coloured classification. *Mem. Amer. Math. Soc.*, 257(1233):viii+97, 2019.

[6] N. Brown and N. Ozawa. *C^*-algebras and finite-dimensional approximations*, volume 88 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2008.

[7] N. Brown. Invariant means and finite representation theory of C^*-algebras. *Mem. Amer. Math. Soc.*, 184(865):viii+105, 2006.

[8] T. Carlsen, G. Restorff, and E. Ruiz. Strong classification of purely infinite Cuntz–Krieger algebras. *Trans. Amer. Math. Soc. Ser. B* 4:1–30, 2017.

[9] J. Castillejos and S. Evington. Nuclear dimension of simple stably projectionless C^*-algebras. arXiv:1901.11441.

[10] J. Castillejos, S. Evington, A. Tikuisis, S. White, and W. Winter. The nuclear dimension of simple C^*-algebras. arXiv:1901.05853.

[11] K. Coward, G. Elliott, and C. Ivanescu. The Cuntz semigroup as an invariant for C^*-algebras. *J. Reine Angew. Math.*, 623:161–193, 2008.
[12] J. Cuntz. Simple C^*-algebras generated by isometries. *Comm. Math. Phys.*, 57(2):173–185, 1977.
[13] M. Dadarlat. Quasidiagonal morphisms and homotopy. *J. Funct. Anal.*, 151(1):213–233, 1997.
[14] S. Eilers, G. Restorff, and E. Ruiz. Automorphisms of Cuntz–Krieger algebras. *J. Noncommut. Geom.*, 12(1):217–254, 2018.
[15] G. Elliott, G. Gong, H. Lin, and Z. Niu. On the classification of simple amenable C^*-algebras with finite decomposition rank, II. arXiv:1507.03437.
[16] G. Elliott, Z. Niu, L. Santiago, and A. Tikuisis. Decomposition rank of approximately subhomogeneous C^*-algebras. arXiv:1505.06100.
[17] G. Elliott and A. Toms. Regularity properties in the classification program for separable amenable C^*-algebras. *Bull. Amer. Math. Soc. (N.S.)*, 45(2):229–245, 2008.
[18] D. Enders. On the nuclear dimension of certain UCT-Kirchberg algebras. *J. Funct. Anal.*, 268(9):2695–2706, 2015.
[19] J. Gabe. Quasidiagonal traces on exact C^*-algebras. *J. Funct. Anal.*, 272(3):1104–1120, 2017.
[20] J. Gabe. A new proof of Kirchberg's $\mathcal{O}_2$-stable classification. *J. Reine Angew. Math.*, to appear. arXiv:1706.03690.
[21] J. Gabe. Traceless AF embeddings and unsuspended E-theory. arXiv:1804.080951.
[22] J. Gabe. Classification of O_∞-stable C^*-algebras. Manuscript in preparation.
[23] G. Gong, H. Lin, and Z. Niu. Classification of finite simple amenable $\mathcal{Z}$-stable C^*-algebras. arXiv:1501.00135.
[24] X. Jiang and H. Su. On a simple unital projectionless C^*-algebra. *Amer. J. Math.*, 121(2):359–413, 1999.
[25] E. Kirchberg. The classification of purely infinite C^*-algebras using Kasparov's theory. Manuscript in preparation.
[26] E. Kirchberg. Exact C^*-algebras, tensor products, and the classification of purely infinite algebras. In *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994)*, pages 943–954. Birkhäuser, Basel, 1995.
[27] E. Kirchberg. Das nicht-kommutative Michael-Auswahlprinzip und die Klassifikation nicht-einfacher Algebren. In *C^*-algebras (Münster, 1999)*, pages 92–141, Berlin, 2000. Springer.
[28] E. Kirchberg. Central sequences in C^*-algebras and strongly purely infinite algebras. In *Operator Algebras: The Abel Symposium 2004*, volume 1 of *Abel Symp.*, pages 175–231. Springer, Berlin, 2006.
[29] E. Kirchberg and N. C. Phillips. Embedding of exact C^*-algebras in the Cuntz algebra $\mathcal{O}_2$. *J. Reine Angew. Math.*, 525:17–53, 2000.
[30] E. Kirchberg and M. Rørdam. Purely infinite C^*-algebras: ideal-preserving zero homotopies. *Geom. Funct. Anal.*, 15(2):377–415, 2005.
[31] E. Kirchberg and M. Rørdam. Non-simple purely infinite C^*-algebras. *Amer. J. Math.*, 122(3):637–666, 2000.
[32] E. Kirchberg and M. Rørdam. Infinite non-simple C^*-algebras: absorbing the Cuntz algebras $\mathcal{O}_\infty$. *Adv. Math.*, 167(2):195–264, 2002.
[33] E. Kirchberg and W. Winter. Covering dimension and quasidiagonality. *Internat. J. Math.*, 15(1):63–85, 2004.
[34] H. Lin. Classification of simple C^*-algebras of tracial topological rank zero. *Duke Math. J.*, 125(1):91–119, 2004.
[35] H. Lin and N. C. Phillips. Approximate unitary equivalence of homomorphisms from $\mathcal{O}_\infty$. *J. Reine Angew. Math.*, 464:173–186, 1995.
[36] H. Matui and Y. Sato. Decomposition rank of UHF-absorbing C^*-algebras. *Duke Math. J.*, 163(14):2687–2708, 2014.
[37] C. Pasnicu and M. Rørdam. Purely infinite C^*-algebras of real rank zero. *J. Reine Angew. Math.*, 613:51–73, 2007.

[38] G. Pedersen. *C^*-algebras and their automorphism groups*, volume 14 of *London Mathematical Society Monographs*. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], London–New York, 1979.
[39] N. C. Phillips. A classification theorem for nuclear purely infinite simple C^*-algebras. *Doc. Math.*, 5:49–114, 2000.
[40] S. Popa. On local finite-dimensional approximations of C^*-algebras. *Pacific J. Math.*, 181(1):141–158, 1997.
[41] I. Raeburn and D. Williams. *Morita equivalence and continuous-trace C^*-algebras*, volume 60 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1998.
[42] M. Rørdam. A short proof of Elliott's theorem: $\mathcal{O}_2 \otimes \mathcal{O}_2 \cong \mathcal{O}_2$. *C. R. Math. Rep. Acad. Sci. Canada*, 16(1):31–36, 1994.
[43] M. Rørdam. *Classification of nuclear C^*-algebras.*, volume 126 of *Encyclopaedia of Mathematical Sciences*. Springer–Verlag, Berlin, 2002. Operator Algebras and Noncommutative Geometry, 7.
[44] M. Rørdam. A purely infinite AH-algebra and an application to AF-embeddability. *Israel J. Math.*, 141:61–82, 2004.
[45] J. Rosenberg and C. Schochet. The Künneth theorem and the universal coefficient theorem for Kasparov's generalized K-functor. *Duke Math. J.*, 55(2):431–474, 1987.
[46] E. Ruiz, A. Sims, and A. Sørensen. UCT-Kirchberg algebras have nuclear dimension one. *Adv. Math.*, 279:1–28, 2015.
[47] E. Ruiz, A. Sims, and M. Tomforde. The nuclear dimension of graph C^*-algebras. *Adv. Math.*, 272:96–123, 2015.
[48] Y. Sato, S. White, and W. Winter. Nuclear dimension and $\mathcal{Z}$-stability. *Invent. Math.*, 202(2):893–921, 2015.
[49] G. Szabó. On the nuclear dimension of strongly purely infinite C*-algebras. *Adv. Math.*, 306:1262–1268, 2017.
[50] A. Tikuisis, S. White, and W. Winter. Quasidiagonality of nuclear C^*-algebras. *Ann. of Math. (2)*, 185:229–284, 2017.
[51] A. Tikuisis and W. Winter. Decomposition rank of $\mathcal{Z}$-stable C^*-algebras. *Anal. PDE*, 7(3), 2014.
[52] A. Toms and W. Winter. Strongly self-absorbing C^*-algebras. *Trans. Amer. Math. Soc.*, 359(8):3999–4029, 2007.
[53] D. Voiculescu. A note on quasi-diagonal C^*-algebras and homotopy. *Duke Math. J.*, 62(2):267–271, 1991.
[54] W. Winter. Covering dimension for nuclear C^*-algebras. *J. Funct. Anal.*, 199(2):535–556, 2003.
[55] W. Winter. On the classification of simple $\mathcal{Z}$-stable C^*-algebras with real rank zero and finite decomposition rank. *J. London Math. Soc. (2)*, 74(1):167–183, 2006.
[56] W. Winter. Decomposition rank and $\mathcal{Z}$-stability. *Invent. Math.*, 179(2):229–301, 2010.
[57] W. Winter. Nuclear dimension and $\mathcal{Z}$-stability of pure C^*-algebras. *Invent. Math.*, 187(2):259–342, 2012.
[58] W. Winter and J. Zacharias. Completely positive maps of order zero. *Münster J. Math.*, 2:311–324, 2009.
[59] W. Winter and J. Zacharias. The nuclear dimension of C^*-algebras. *Adv. Math.*, 224(2):461–498, 2010.
[60] M. Wolff. Disjointness preserving operators on C^*-algebras *Arch. Math. (Basel)*, 62(3):248–253, 1994.
[61] S. Zhang. A property of purely infinite simple C^*-algebras. *Proc. Amer. Math. Soc.*, 109(3):717–720, 1990.

Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra (Barcelona), Spain, and Barcelona Graduate School of Mathematics (BGSMath).

E-mail address: jbosa@mat.uab.cat

James Gabe, School of Mathematics and Applied Statistics, University of Wollongong, Wollongong NSW 2522, Australia.

E-mail address: jamiegabe123@hotmail.com

Aidan Sims, School of Mathematics and Applied Statistics, University of Wollongong, Wollongong NSW 2522, Australia.

E-mail address: asims@uow.edu.au

Stuart White, School of Mathematics and Statistics, University of Glasgow, Glasgow, G12 8QW, Scotland and Mathematisches Institut der WWU Münster, Einsteinstrasse 62, 48149 Münster, Germany.

E-mail address: stuart.white@glasgow.ac.uk