THE CUNTZ SEMIGROUP OF A RING

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Abstract. For any ring R, we introduce an invariant in the form of a partially ordered abelian semigroup $\mathrm{S}(R)$ built from an equivalence relation on the class of countably generated projective modules. We call $\mathrm{S}(R)$ the Cuntz semigroup of the ring R. This construction is akin to the manufacture of the Cuntz semigroup of a C*-algebra using countably generated Hilbert modules. To circumvent the lack of a topology in a general ring R, we deepen our understanding of countably projective modules over R, thus uncovering new features in their direct limit decompositions, which in turn yields two equivalent descriptions of $\mathrm{S}(R)$. The Cuntz semigroup of R is part of a new invariant $\mathrm{SCu}(R)$ which includes an ambient semigroup in the category of abstract Cuntz semigroups that provides additional information. We provide computations for both $\mathrm{S}(R)$ and $\mathrm{SCu}(R)$ in a number of interesting situations, such as unit-regular rings, semilocal rings, and in the context of nearly simple domains. We also relate our construcion to the Cuntz semigroup of a C*-algebra.

1. Introduction

The study of a ring using the collection of its projective (right) modules is central in modern algebra. Much attention has been directed to finitely generated projective modules, mostly with K-Theory in mind, since for a unital ring R the Grothendieck group $\mathrm{K}_0(R)$ is constructed out of the monoid $\mathrm{V}(R)$ of isomorphism classes of such modules. There has also been an intensive use of countably generated projective modules. This may be justified keeping in mind that, by a well known theorem of Kaplansky, any projective module is a direct sum of countably generated projective ones. In this case, one might use the monoid $\mathrm{V}^*(R)$ of isomorphism classes of countably generated projective modules to analize the ring R. It is worth noticing that both monoids $\mathrm{V}(R)$ and $\mathrm{V}^*(R)$ are naturally equipped with the so-called algebraic order, given by complements.

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The structure of $V^*(R)$ has attracted considerable attention in the last years; see, for instance, [22] and [23]. For a semilocal ring, and following a result obtained in [31], one has that $V^*(R)$ can be viewed as a submonoid of $V^*(R/J(R))$, which in turn is isomorphic to $\overline{\mathbb{N}}^r$ for a suitable r, where $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ with the obvious operations. A relevant problem is then to determine which submonoids of $\overline{\mathbb{N}}^r$ are realized by semilocal rings. A full characterization of such submonoids in the noetherian case is obtained in [22, Theorem 2.6], as those submonoids defined by a system of equations in the sense of [22, Definition 2.5]. (We note that previous results for semilocal rings, but for the monoid $V(R)$ and full affine submonoids of $\mathbb{N}^r$, were already obtained in [17].) Further progress was carried out in [23, Theorem 1.6] for not necessarily noetherian semilocal rings. There, the authors studied countably generated projective modules that are finitely generated modulo the Jacobson radical and showed that they appear in a wide variety of situations.

Our aim here is to introduce an object $S(R)$ that can also be built out of countably generated projective modules, albeit using a relation weaker than isomorphism. Its construction is inspired by that of the Cuntz semigroup of a C*-algebra, as the latter possesses a rich ordered structure that extends the algebraic order, and has played an important role in the theory of C*-algebras in recent years. Let us review this construction in relation to the main theme of this paper.

For the class of C*-algebras, that is, self-adjoint, norm-closed subalgebras of the algebra of bounded operators on a Hilbert space, we encounter in countably generated Hilbert modules the analytic siblings of countably generated projective modules. Roughly speaking, a Hilbert module over a C*-algebra A is an A-module, together with an A-valued inner product, which is complete with respect to a suitable norm. The A-module $A^{(\mathbb{N})}$, with a natural inner product, gives rise to the standard Hilbert module H_A. It is a celebrated theorem due to Kasparov that any countably generated Hilbert A-module H is isometrically isomorphic to a complement of H_A; see [26]. From this point of view, countably generated Hilbert modules play a role akin to countably generated projective modules for C*-algebras. In fact, an algebraically finitely generated Hilbert module is a finitely generated projective module; see, e.g. [8, Theorem 3.12]. Further, the monoid of isomorphism classes of finitely generated Hilbert A-modules is isomorphic to the monoid $V(A)$ of the C*-algebra, as shown in [8, Proposition 3.10]. We also remark that, as proved recently by Brown and Lin in [11], over a separable C*-algebra every countably generated Hilbert module is projective (with bounded module maps as morphisms). This is a step forward in the direction of characterizing projective Hilbert modules over a C*-algebra.

An equivalence relation among countably generated Hilbert A-modules, weaker than isomorphism, was studied in [14]. In there, the authors proved that the monoid arising from said equivalence relation may be identified with the complete Cuntz semigroup invariant $\mathrm{Cu}(A)$ of the C*-algebra A. (The terminology 'semigroup' is used for historical reasons, although in fact $\mathrm{Cu}(A)$ is a monoid.) The original (uncomplete) Cuntz semigroup $W_C(A)$ was constructed by Cuntz in [15] using positive elements and a suitable comparison relation among them that, when restricted to idempotents, yields the usual Murray-von Neumann comparison. In short, we say that a is Cuntz subequivalent to b, and write $a \precsim_{\mathrm{Cu}} b$, provided a can be approximated arbitrarily well by elements of the form xby. Compared with the construction of the group K_0, this approach is advantageous since every C*-algebra has an abundance of positive elements but may have a complete lack of idempotents. The exact relation between $\mathrm{W}_{\mathrm{C}}(A)$ and $\mathrm{Cu}(A)$ may be expressed by the isomorphism $\mathrm{Cu}(A) \cong \mathrm{W}_{\mathrm{C}}(A \otimes \mathcal{K})$, where $\mathcal{K}$ is the algebra of compact operators on an infinite dimensional Hilbert space. Alternatively, $\mathrm{Cu}(A)$ may be thought of as the completion of $\mathrm{W}_{\mathrm{C}}(A)$; see [3, Theorem 3.2.8]. It was shown in [14] that $\mathrm{Cu}(A)$ sits in a well-behaved category of partially ordered monoids, termed Cu, in which each object admits suprema of increasing sequences, among other continuity properties. Furthermore, the assignment $A \mapsto \mathrm{Cu}(A)$ is a continuous functor; see [14, 3].

The Cuntz semigroup plays a prominent role in the classification programme of C*-algebras initiated by G. A. Elliott and is a key ingredient in delimiting the optimal class of such algebras amenable to classification by the Elliott invariant (that consists essentially of K_0, K_1, and the trace simplex). Indeed, the examples constructed by A. S. Toms in [34] can be distinguished by their Cuntz semigroups, but not by a swath of other well known topological invariants for C*-algebras that include, among others, the Elliott invariant and the stable rank.

When trying to adapt the ideas above to the purely algebraic setting one has to bear in mind that, in nature, the Cuntz semigroup is an analytic object. Thus one first needs to use an algebraic analogue of Cuntz comparison for general elements in a ring. We take advantage of the approach carried out in [9], in which one defines $x \precsim_1 y$, provided $x = rys$ for some elements $r, s \in R$, in order to construct a partially ordered monoid $\mathrm{W}(R)$ for any weakly s-unital ring R; see Paragraphs 2.4 and 2.5. By considering suitable equivalence classes of increasing sequences with respect to the relation $\precsim_1$, we obtain a monoid $\Lambda(R)$ in the category Cu that contains $\mathrm{W}(R)$. The object $\Lambda(R)$ can be conveniently identified with the monoid of intervals in $\mathrm{W}(R)$, but it is in general too large for our purposes. This differs fundamentally from what happens in the C*-algebraic case, and the reason may be found in the lack of a topology in R. To remedy this drawback, we restrict our attention to a well-behaved partially ordered submonoid $\mathrm{S}(R)$ of $\Lambda(R)$ which, for a C*-algebra A, is resemblant to $\mathrm{Cu}(A)$ and its role as the completion of $\mathrm{W}_{\mathrm{C}}(A)$. We will term $\mathrm{S}(R)$ the *Cuntz semigroup of the ring R* and this is the main object of study in this paper. At this point, we mention that the construction of the Cuntz semigroup $\mathrm{W}_{\mathrm{C}}(A)$ for a C*-algebra A served as inspiration to Hung and Li to introduce in [25] a semigroup for any unital ring R, termed the *Malcolmson semigroup*, and denoted by $\mathrm{W}_{\mathrm{M}}(R)$, in order to study Sylvester rank functions over the ring R. To construct this semigroup one uses a relation stronger than $\precsim_1$ and, as it turns out, for any C*-algebra A the semigroup $\mathrm{W}_{\mathrm{C}}(A)$ is a homomorphic image of $\mathrm{W}_{\mathrm{M}}(A)$ via an order-preserving map; see [25, Lemma 5.1, Proposition 5.2]. We shall review this construction and its relation to our semigroup $\mathrm{W}(R)$ in Section 3.

To explain how one constructs the Cuntz semigroup $\mathrm{S}(R)$ we take a slight detour that finally yields two equivalent pictures of the same object and spurs our motivation at the same time. More concretely, given countably generated projective modules P and Q over R, we combine the approach carried out for Hilbert modules in [14] with an abstraction of the above-mentioned relation $\precsim_1$ to write $P \precsim Q$ if the inclusion of any finitely generated module X of P may be factorized through Q; see Paragraph 4.4. By antisymmetrizating the above relation, we get an ordered abelian monoid $\mathrm{CP}(R)$ and a natural surjective morphism $\mathrm{V}^*(R) \to \mathrm{CP}(R)$. As we show in Section 6, if R is either unit-regular or unital and semilocal, this is an isomorphism of abelian monoids, but not of ordered monoids, as $\mathrm{V}^*(R)$ is algebraically ordered, but $\mathrm{CP}(R)$ is not, except in trivial situations. Further investigation on countably generated projective modules structure leads us to reformulate the proof, obtained in [29], that any such module can be written as a sequential inductive limit of free modules such that, for each n, the nth transition map consists of multiplication by a matrix x_n with the property that $x_n = y_{n+1}x_{n+1}x_n$ for a suitable matrix y_{n+1} (hence in particular $x_n \precsim_1 x_{n+1}$). Our arguments uncover additional and crucial information in such an inductive limit decomposition and in doing so we are able to relate the monoid $\mathrm{CP}(R)$ to the submonoid $\mathrm{S}(R)$ of $\Lambda(R)$ consisting of (suitable equivalence classes of) increasing sequences (x_n) with respect to $\precsim_1$ arising from inductive limits as above:

Theorem A (4.13, 4.3)**.** *Let R be any ring. Then $\mathrm{CP}(R)$ and the Cuntz semigroup $\mathrm{S}(R)$ of R are order-isomorphic monoids. Moreover, every increasing sequence in $\mathrm{CP}(R)$ (or $\mathrm{S}(R)$) has a supremum.*

Despite the analogy of the construction of the Cuntz semigroup $\mathrm{S}(R)$ of a ring R with that of a C*-algebra, it is unclear whether $\mathrm{S}(R)$ is a Cu-semigroup. We remedy this fact by considering the pair $\mathrm{SCu}(R) = (\Lambda(R), \mathrm{S}(R))$. This is an instance of an object in the category SCu of pairs (Λ, S), where Λ is a monoid in the category Cu and S is a submonoid of Λ closed under suprema of a certain type of sequences. The definition of this new abstract category balances the fact that $\mathrm{S}(R)$ might not be an object in Cu with an ambient object which does belong to the category and is still intimately related to $\mathrm{S}(R)$. More concretely, we prove:

Theorem B (5.7)**.** *Let Rings^{ws} denote the category of weakly s-unital rings. Then, the assignment*

$$\begin{array}{cccc} \mathrm{SCu}\colon & \mathrm{Rings}^{ws} & \longrightarrow & \mathrm{SCu} \\ & R & \mapsto & (\Lambda(R), \mathrm{S}(R)) \end{array}$$

is functorial.

In a subsequent paper ([1]) we examine other structural properties of the object $\mathrm{SCu}(R)$, such as a natural notion of ideal and quotient, and how these notions parametrize the ideal lattice of a ring. There, we also show that the category SCu admits inductive limits and analyse when the assignment $R \mapsto \mathrm{SCu}(R)$ is continuous.

We analyse the construction of this new Cuntz semigroup in a variety of situations. Firstly, since the original motivation of this paper comes from C*-algebra theory, we relate $\mathrm{Cu}(A)$ to $\mathrm{S}(A)$ for any C*-algebra A, by showing the former is a retract of the latter, as follows:

Theorem C (7.6)**.** *Given a C*-algebra A, there exist ordered monoid morphisms $\varphi\colon \mathrm{Cu}(A) \to \mathrm{S}(A)$ and $\phi\colon \mathrm{S}(A) \to \mathrm{Cu}(A)$ that preserve suprema of increasing sequences and such that $\phi \circ \varphi = \mathrm{id}_{\mathrm{Cu}(A)}$.*

Secondly, we show that in a number of interesting examples outside the class of C*-algebras the monoids $\mathrm{W}(R)$ and $\mathrm{S}(R)$, together with their order structure, can be identified:

Theorem D (6.11, 6.13, 8.4)**.** *Let R be a unital ring, and let P and Q be countably generated projective R-modules. Then:*

(i) *If R is unit-regular, we have $[P] \leq [Q]$ in $\mathrm{CP}(R)$ precisely when P is isomorphic to a submodule of Q. It follows that $\mathrm{W}(R) \cong \mathrm{V}(R)$ and $\mathrm{S}(R) \cong \mathrm{CP}(R) \cong \Lambda(R)$. Thus $\mathrm{S}(R)$ is a Cu-semigroup.*

(ii) *If R semilocal, we have $[P] \leq [Q]$ in $\mathrm{CP}(R)$ precisely when P is isomorphic to a pure submodule of Q. In this case, as abelian monoids, we have $\mathrm{S}(R) \cong \mathrm{CP}(R) \cong \mathrm{V}^*(R)$.*

(iii) *If R is a nearly simple domain, then $\mathrm{W}(R) \cong \mathbb{N} \times \mathbb{N}$, and $(r,s) \leq (r',s')$ precisely when $r \leq r'$ and $r+s \leq r'+s'$. Moreover, $\mathrm{SCu}(J(R)) \cong (\overline{\mathbb{N}}, 0)$.*

The article is organized as follows. In Section 2 we review the definition of the Cuntz semigroup $\mathrm{W_C}(A)$ for a C*-algebra A and the category its completion naturally belongs to, and we define its algebraic counterpart $\mathrm{W}(R)$ together with the natural construction $\Lambda(R)$, which will conveniently serve as an ambient monoid later on. In Section 3 we relate the semigroup $\mathrm{W}(R)$ with the Malcolmson semigroup introduced in [25], and show both semigroups may be identified for unital von Neumann regular rings. Section 4 constitutes the heart of this paper, where we construct the Cuntz semigroup $\mathrm{S}(R)$ for a ring R and prove Theorem A. This is technically demanding as we need to split the proof into the unital and non-unital case. In Section 5 we introduce the category SCu and establish Theorem B. In Section 6 we study compact elements in $\mathrm{S}(R)$ and prove parts (i) and (ii) of Theorem D. We revisit C*-algebras in Section 7 to relate $\mathrm{S}(A)$ and $\mathrm{Cu}(A)$, thus proving Theorem C. Section 8 is devoted to the analysis of the class of nearly simple domains and to prove part (iii) of Theorem D.

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2. The Cuntz semigroup of a C*-algebra and the semigroup $\mathrm{W}(R)$

In this section we recall the definition of the Cuntz semigroup of a C*-algebra and its most natural adaptation to a purely algebraic framework.

2.1 (Diagonal sum in $M_\infty(R)$)**.** Given a ring R, we denote by $M_\infty(R)$ the ring of infinite matrices with only a finite number of nonzero entries. That is, given an element $x \in M_\infty(R)$ there exist $n, m \geq 1$ and a finite matrix $z \in M_{n\times m}(R)$ such that

$$x = \begin{pmatrix} z & 0 & \cdots \\ 0 & 0 & \\ \vdots & & \ddots \end{pmatrix}.$$

We will call z a *finite representative* of x, and we will say that x is the infinite matrix represented by z. We will tacitly identify x and z when no confusion can arise.

Given two finite rectangular matrices $x \in M_{n_1\times m_1}(R)$ and $y \in M_{n_2\times m_2}(R)$, we will denote by $x \oplus y$ the infinite matrix

$$\begin{pmatrix} x & 0 & 0 & \cdots \\ 0 & y & 0 & \\ 0 & 0 & 0 & \\ \vdots & & & \ddots \end{pmatrix}.$$

In other words, $x\oplus y$ is the infinite matrix represented by the rectangular matrix

$$\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \in M_{(n_1+n_2)\times(m_1+m_2)}(R).$$

2.2 (Cuntz subequivalence and the Cuntz semigroup)**.** We shall denote by $\mathcal{K}$ the algebra of compact operators over an infinite-dimensional Hilbert space. Let A be a C*-algebra. Given positive elements $a, b \in A$, we say that a is *Cuntz subequivalent to* b, in symbols $a \precsim_{\mathrm{Cu}} b$, provided that there is a sequence (x_n) in A such that $a = \lim_n x_n b x_n^*$. Equivalently, there are sequences (x_n), (y_n) in A such that $a = \lim_n x_n b y_n$ (see [15]). We say that a and b are *Cuntz equivalent*, in symbols $a \sim_{\mathrm{Cu}} b$, if both $a \precsim_{\mathrm{Cu}} b$ and $b \precsim_{\mathrm{Cu}} a$ occur. One can use the second equivalent definition of Cuntz subequivalence to extend $\sim_{\mathrm{Cu}}$ to arbitrary elements. That does not have any effect on the theory since, as it happens, $a^*a \sim_{\mathrm{Cu}} aa^* \in A_+$ for any $a \in A$.

By extending this relation in the natural way to $M_\infty(A)_+$, one can define a partially ordered set

$$\mathrm{W_C}(A) = M_\infty(A)_+/\sim,$$

with order given by $[a]_{\mathrm{Cu}} \leq [b]_{\mathrm{Cu}}$ whenever $a \precsim_{\mathrm{Cu}} b$ (and where $[a]_{\mathrm{Cu}}$ denotes the equivalence class of a). It becomes a positively ordered semigroup by defining $[a]_{\mathrm{Cu}} + [b]_{\mathrm{Cu}} = [a \oplus b]_{\mathrm{Cu}}$. The semigroup $\mathrm{W_C}(A)$, originally defined in [15], is most currently referred to as the *classical Cuntz semigroup*. The *complete Cuntz semigroup* of a C*-algebra A is $\mathrm{Cu}(A) = \mathrm{W_C}(A \otimes \mathcal{K})$. (See [8], and [18] for survey articles on the Cuntz semigroup.)

Coward, Elliott, and Ivanescu introduced in [14] the category Cu, which captures continuity properties of the semigroup $\mathrm{Cu}(A)$.

2.3 (The category Cu and abstract Cu-semigroups)**.** Given a positively ordered monoid S, we write $x \ll y$ (and say that x is *compactly contained* in y, or that

x is *way-below* y), if whenever (y_n) is an increasing sequence in S for which the supremum $\sup_n y_n$ exists, then $y \leq \sup_n y_n$ implies that there exists k such that $x \leq y_k$. (See [19, I-1].)

Using it, we consider the following axioms for S:

(O1) Every increasing sequence (x_n) in S has a supremum $\sup_n x_n \in S$.
(O2) Every element $x \in S$ is the supremum of a sequence (x_n) such that $x_n \ll x_{n+1}$ for all n. We say that (x_n) is a *rapidly increasing sequence.*
(O3) If $x', x, y', y \in S$ satisfy $x' \ll x$ and $y' \ll y$ then $x' + y' \ll x + y$.
(O4) If (x_n) and (y_n) are increasing sequences in S, then $\sup_n(x_n + y_n) = \sup_n x_n + \sup_n y_n$.

An *abstract Cuntz semigroup* (or just a Cu-*semigroup*) is a positively ordered monoid satisfying axioms (O1)-(O4). A Cu-morphism between two Cu-semigroups S and T is a positively ordered monoid morphism $f\colon S \to T$ that preserves compact containment and suprema of increasing sequences. The category Cu has as objects the Cu-semigroups and as morphisms the Cu-morphisms. It was shown in [14] that the natural models of Cu-semigroups are the complete Cuntz semigroups of C*-algebras.

Also, the natural models of Cu-morphisms are the *-homomorphisms of C*-algebras. More specifically, given C*-algebras A and B and a *-homomorphism $\varphi\colon A \to B$, we may define $\mathrm{Cu}(\varphi)\colon \mathrm{Cu}(A) \to \mathrm{Cu}(B)$ by $\mathrm{Cu}(\varphi)([a]) = [(\varphi \otimes \mathrm{id})(a)]$, for any $a \in (A \otimes \mathcal{K})_+$, which is a Cu-morphism. In this way, the assignment $A \mapsto \mathrm{Cu}(A)$ determines a functor from the category of C*-algebras to the category Cu, which turns out to be continuous (see [14] and also [3]). It was shown in [3, Theorem 3.28] that $\mathrm{Cu}(A)$ is, suitably interpreted, a completion of $\mathrm{W_C}(A)$.

Elements that will become relevant in the theory are the so-called *compact* elements. By definition, an element x in a Cu-semigroup S is termed compact provided $x \ll x$. The natural sources of compact elements in Cuntz semigroups of C*-algebras are the classes of projections, i.e. self-adjoint idempotents. In significant cases, these are the only ones (see, e.g. [12]). A Cuntz semigroup S is said to be *algebraic* provided every element is the supremum of a sequence of compact elements ([3, Definition 5.5.1]).

A *sub*-Cu-*semigroup* of a Cu-semigroup T is a submonoid S of T such that the inclusion $\iota\colon S \to T$ is a Cu-morphism; see, for example, [3]. For example, let $\mathbb{N}$ be the set of natural numbers with 0. Then, $\overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$ is a Cu-semigroup and a submonoid of the Cu-semigroup $[0, \infty]$, but it is not a Cu-subsemigroup of $[0, \infty]$, because, for instance, $2 \ll 2$ in $\overline{\mathbb{N}}$ but 2 is not compactly contained in itself in $[0, \infty]$.

We now introduce an algebraic analog of the classical Cuntz semigroup. For this, we first need to consider a class of rings suitable to our needs. The relation $\precsim_1$ below was already considered in [9].

2.4 (s-unital rings)**.** We recall that a ring R is said to be s-*unital* if, for every element $a \in R$, there is $b \in R$ such that $a = ba = ab$. Evidently this includes all unital rings, σ-unital rings, and rings with local units.

We will say that a ring R is *weakly s-unital* if for every $n \geq 1$ and every element $a \in M_n(R)$, there are $b, c \in M_n(R)$ such that $a = bac$.

By [4, Lemma 2.2], given a finite family $a_1, \ldots, a_n$ of elements of an s-unital ring R, there is $b \in R$ such that $ba_i = a_i = a_i b$ for $i = 1, \ldots, n$. From this, one can show that if R is s-unital then so is the ring $M_\infty(R)$. It also follows that any s-unital ring is weakly s-unital.

2.5 (The semigroup $\mathrm{W}(R)$)**.** Let R be any ring. Given elements $a, b \in R$, we write $a \precsim_1 b$ if there exist elements $r, t \in R$ such that

$$a = rbt.$$

The relation $\precsim_1$ is clearly transitive by construction. Assume further that R is weakly s-unital, and then $\precsim_1$ is also reflexive. We write $a \sim_1 b$ provided $a \precsim_1 b$ and $b \precsim_1 a$.

If e, f are idempotents in R, then an easy argument shows that $e \precsim_1 f$ if and only if $e \sim f'$ and $f' \leq f$ in the sense that $e = xy$ whilst $yx = f'$, for elements $x \in eRf', y \in f'Re$. That is, the relation $\precsim_1$ agrees with the usual Murray-von Neumann subequivalence $\precsim_{\mathrm{MvN}}$ for idempotents. Therefore, if $e \sim f$, then $e \sim_1 f$, but the converse does not necessarily hold – it will if all idempotents are *finite*, in the sense that they do not contain proper equivalent copies of themselves.

In case A is a C*-algebra and $p, q \in A$ are projections, then it is known that $p \precsim_{\mathrm{Cu}} q$ if and only if $p = vv^*$ and $v^*v \leq q$. In other words, Cuntz subequivalence, when restricted to projections, agrees with the usual Murray-von Neumann subequivalence. It follows from this that $p \precsim_{\mathrm{Cu}} q$ precisely when $p \precsim_1 q$. However, this will not hold for general positive elements, and thus one cannot expect that our algebraic construction below coincides with the C*-algebraic one. Notice also that C*-algebras are in general neither weakly s-unital nor s-unital.

We now extend the relation $\precsim_1$ to $M_\infty(R)$ and define

$$\mathrm{W}(R) = M_\infty(R)/{\sim_1}\ .$$

Denote the class of $a \in M_\infty(R)$ by $[a]$. As we show below, this partially ordered set becomes an abelian semigroup by defining $[a]+[b] = [\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)]$, for any $a, b \in M_\infty(R)$.

Lemma 2.6. *For any weakly s-unital ring R, the poset $\mathrm{W}(R)$, equipped with the addition defined above, is a positively ordered commutative monoid.*

Proof. We first have to show that addition is well-defined.

Note the following fact. Let $u \in M_{k\times l}(R)$ and $v \in M_{t\times s}(R)$ be finite matrices with coefficients in R, and let x, y be the infinite matrices represented by u and v respectively. Then $x \precsim_1 y$ in $M_\infty(R)$ if and only if there are matrices $a \in M_{k\times t}(R)$ and $b \in M_{s\times l}(R)$ such that $u = avb$.

For the addition, let $w, w' \in \mathrm{W}(R)$ and suppose that u and v are finite representatives of w, and that u' and v' are finite representatives of w'. Using the above observation we find finite matrices a, b, a', b' of suitable sizes such that

$$u = avb \text{ and } u' = a'v'b'.$$

We then have that $u\oplus u' = (a\oplus a')(v\oplus v')(b\oplus b')$. This shows that $(u\oplus u') \precsim_1 (v\oplus v')$, and similarly we have that $(v \oplus v') \precsim_1 (u \oplus u')$, and thus $[(u \oplus u')] = [(v \oplus v')]$.

The same argument shows that addition is compatible with the order in $\mathrm{W}(R)$. Further, it is clear that the class $[0]$ is the zero element and that addition is associative.

To see that it is also commutative, let $w, w' \in \mathrm{W}(R)$, and let u, u' be finite representatives of w, w', respectively. Since R is weakly s-unital, we may choose finite matrices v, z, v', z' of suitable sizes such that $vuz = u$ and $v'u'z' = u'$.

Then, we have

$$\begin{pmatrix} u' & 0 \\ 0 & u \end{pmatrix} = \begin{pmatrix} 0 & v' \\ v & 0 \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & u' \end{pmatrix} \begin{pmatrix} 0 & z \\ z' & 0 \end{pmatrix} \precsim_1 \begin{pmatrix} u & 0 \\ 0 & u' \end{pmatrix}.$$

Hence $u' \oplus u \precsim_1 u \oplus u'$. Thus, one gets

$$w' + w = [u' \oplus u] \leq [u \oplus u'] = w + w',$$

and by symmetry $w + w' \leq w' + w$, showing that $w' + w = w + w'$, as desired. □

2.7 (The semigroup $\mathrm{V}(R)$)**.** We shall denote as customary by $\mathrm{V}(R)$ the semigroup of Murray-von Neumann equivalence classes of idempotents in $M_\infty(R)$, and we denote the class of an idempotent $e \in M_\infty(R)$ by $[e]_{\mathrm{MvN}}$. Our observations above mean that there is a an order-embedding $\iota : \mathrm{V}(R) \to \mathrm{W}(R)$, given by $[e]_{\mathrm{MvN}} \mapsto [e]$. This map is injective if R is *stably finite*, in the sense that $x + y = x$ in $\mathrm{V}(R)$ precisely when $y = 0$.

In particular, the next result shows how the different orders behave via ι. Indeed, it is shown that every element of $\iota(V(R))$ can be complemented in $\mathrm{W}(R)$. It is worth noticing the converse does not always hold; see Remark 8.5.

Lemma 2.8. *Let R be a weakly s-unital ring, and let $x \in \iota(V(R))$ and $y \in \mathrm{W}(R)$. If $x \leq y$, then there exists $z \in \mathrm{W}(R)$ such that $x + z = y$.*

Proof. Let e be an idempotent in $M_\infty(R)$ such that $x = [e]$, and let $v \in M_\infty(R)$ satisfy $y = [v]$. Using that $e \precsim_1 v$, we can find elements r, s such that $e = rvs$. Since e is idempotent, we may also assume that $r = er$ and $s = se$. Thus, the element $f := vsr = vser$ is an idempotent in vR satisfying $[f] = [e] = x$.

Now set $w := v - fv \in M_\infty(R)$. Since R is weakly s-unital, there exist $a, b \in M_\infty(R)$ such that $awb = w$. Thus, we have

$$v = \begin{pmatrix} f & a \end{pmatrix} \begin{pmatrix} f & 0 \\ 0 & v - fv \end{pmatrix} \begin{pmatrix} v \\ b \end{pmatrix}$$

and, consequently, $y = [v] \leq [f] + [w] = x + [w]$.

Using once again that R is weakly s-unital, let $c, d \in M_\infty(R)$ satisfy $v = cvd$. One gets

$$\begin{pmatrix} f & 0 \\ 0 & v - fv \end{pmatrix} = \begin{pmatrix} fc \\ c - fc \end{pmatrix} v \begin{pmatrix} dsr & d - dsrv \end{pmatrix}.$$

This shows that $[f] + [w] \leq [v] = y$. Setting $z := [w]$, we obtain

$$x + z = [f] + [w] = y,$$

as desired. □

In a more concrete settting, recall that an element a in a ring R is said to be a *von Neumann regular element* provided there is $x \in R$ such that $a = axa$. Upon replacing x by $x' = xax$, we may also assume that $x = xax$. The ring R is said to be a von Neumann regular ring if every element is von Neumann regular. (See [20].)

Lemma 2.9. *Let R be a stably finite von Neumann regular ring. Then, the natural map $\mathrm{V}(R) \to \mathrm{W}(R)$ is an order-isomorphism.*

Proof. Let $a \in R$, and write $a = axa$ with $x = xax$. It is then an easy exercise to verify that $a \sim_1 ax =: e$, which is an idempotent. Since matrices over a von Neumann regular ring are also von Neumann regular, this shows that the map $\mathrm{V}(R) \to \mathrm{W}(R)$ is surjective. □

The relation between $\precsim_1$ and $\precsim_{\mathrm{Cu}}$ for general positive elements in a C*-algebra is examined with some more detail in the lemma below.

Lemma 2.10. *Let A be a C^*-algebra, and let $a, b \in A$. If $a \precsim_1 b$, then $a^*a \precsim_{\mathrm{Cu}} b^*b$. Therefore, there is a positively ordered monoid morphism $\iota_{\mathrm{C}} \colon \mathrm{W}(A) \to \mathrm{W}_{\mathrm{C}}(A)$, given by $\iota_{\mathrm{C}}([a]) = [a^*a]_{\mathrm{Cu}}$.*

Proof. Suppose that $a \precsim_1 b$, and write $a = sbr$ for some $s, r \in A$. Then

$$a^*a = r^*b^*s^*sbr \precsim_{\mathrm{Cu}} b^*s^*sb \leq \|s\|^2 b^*b \precsim_{\mathrm{Cu}} b^*b.$$

Notice that, in particular, since $x \sim_{\mathrm{Cu}} x^2$ for any $x \in A_+$, we have $a \precsim_{\mathrm{Cu}} b$ whenever $a, b \in A_+$ and $a \precsim_1 b$. □

2.11 (The semigroup $\Lambda(R)$)**.** We now proceed to construct an object in the category Cu for any weakly s-unital ring R. In the C*-setting this can be done by simply considering $\mathrm{W}_C(A \otimes \mathcal{K})$, but there is no algebraic analogue of the compact operators $\mathcal{K}$. Another approach is through a completion of $\mathrm{W}(R)$ by a so called *auxiliary relation* in $\mathrm{W}(R)$; see [19, Definition I-1.11], [3, Definition 2.2.4] and also Remark 6.4 below. But again, there is no clear algebraic analogue of such an auxiliary relation for our needs.

The approach below is partly inspired by the description of $\mathrm{Cu}(A)$ in [14] using certain increasing sequences. We will first make a general construction that works for a general ring, and then specialize to the weakly s-unital setting.

Let R be a ring. Set

$$T(R) = \{(a_n)_n \mid a_n \in M_\infty(R) \text{ and } a_n \precsim_1 a_{n+1} \text{ for all } n\}.$$

Given $(a_n), (b_n) \in T(R)$, write $(a_n) \precsim (b_n)$ if for any n, there exists m with $a_n \precsim_1 b_m$. We also write $(a_n) \sim (b_n)$ provided $(a_n) \precsim (b_n)$ and $(b_n) \precsim (a_n)$. Notice that, since for $(a_n) \in T(R)$ we have $a_n \precsim_1 a_{n+1}$, we see that $(a_n) \precsim (a_n)$ and thus the relation $\precsim$ is reflexive even if R is not unital. It is also clearly transitive. Let

$$\Lambda(R) = T(R)/\sim,$$

which is a partially ordered set with the order induced by $\precsim$. We denote by $[(a_n)]$ the $\sim$-equivalence class of a sequence (a_n) in $T(R)$. We now equip $\Lambda(R)$ with a semigroup structure, and to this end we need to be careful with the choices of representatives.

Thus, in analogy to the terminology introduced in Paragraph 2.1, given an element $w \in \Lambda(R)$, a *finite matricial representative* of w is any sequence (u_n) such that $u_n \in M_{k_{n+1}\times k_n}(R)$, where (k_n) is a sequence of positive integers, for which there exist matrices $v_{n+1} \in M_{k_{n+1}\times k_{n+2}}(R)$ and $z_{n+1} \in M_{k_{n+1}\times k_n}(R)$ such that $u_n = v_{n+1}u_{n+1}z_{n+1}$ for all n, and with $w = [(x_n)]$, where x_n is the infinite matrix represented by u_n for each $n \in \mathbb{N}$.

For $w, w' \in \Lambda(R)$, let (u_n) and (u'_n) be finite matricial representatives of w and w' respectively. The sum $w + w'$ is then defined as

$$w + w' = [(u_n \oplus u'_n)] \in \Lambda(R).$$

Lemma 2.12. *For any ring R, the poset $\Lambda(R)$, equipped with the addition defined above, is a positively ordered commutative monoid.*

Proof. The argument is similar to Lemma 2.6. We sketch the main steps in the proof.

If $w, w' \in \Lambda(R)$, let $(u_n), (v_n)$ be two finite matricial representatives of w, and let (u'_n) and (v'_n) be finite matricial representatives of w'. Given $n \in \mathbb{N}$, we see, using the first observation in the proof of Lemma 2.6, that there is $m \in \mathbb{N}$ and finite matrices a, b, a', b' of suitable sizes such that

$$u_n = av_mb, \qquad u'_n = a'v'_mb'.$$

Then $u_n \oplus u'_n = (a\oplus a')(v_m \oplus v'_m)(b\oplus b')$, and thus $(u_n \oplus u'_n) \precsim (v_n \oplus v'_n)$. Likewise, $(v_n \oplus v'_n) \precsim (u_n \oplus u'_n)$, whence $[(u_n \oplus u'_n)] = [(v_n \oplus v'_n)]$.

The rest of the argument follows the lines of Lemma 2.6. □

Proposition 2.13. *Let R be any ring. Then every increasing sequence in $\Lambda(R)$ has a supremum. If, further, R is weakly s-unital, then $\Lambda(R)$ is a* Cu*-semigroup.*

Proof. This is fundamentally contained in the arguments of [3, Proposition 3.1.6]. We offer some details for the convenience of the reader.

Let $([x_k])_k$ in $\Lambda(R)$ be an increasing sequence. We thus have that $x_k \precsim x_{k+1}$ for all k. Write $x_k = (x_n^{(k)})_n$, with $x_n^{(k)} \precsim_1 x_{n+1}^{(k)}$ for all n and all k. By an inductive process, we find an increasing sequence n_k such that $x_{n_i+j}^{(i)} \precsim_1 x_{n_k}^{(k)}$ if $i + j \leq k$.

To see this, set $n_1 = 0$ and using that $x_1 \precsim x_2$, find n_2 such that $x_{n_1+1}^{(1)} = x_1^{(1)} \precsim_1 x_{n_2}^{(2)}$. If n_i is constructed for $i \leq k$, we use that $x_1, \dots, x_k \precsim x_{k+1}$ to find n_{k+1} such that $x_{n_i+k}^{(1)}, x_{n_2+k-1}^{(2)}, \dots, x_{n_k+1}^{(k)} \precsim_1 x_{n_{k+1}}^{(k+1)}$, and thus the induction is complete.

After reindexing we may assume that $n_i = i$ and therefore $x_{i+j}^{(i)} \precsim_1 x_{i+j}^{(i+j)}$ for all i, j. Setting $y_n = x_n^{(n)}$, we have that $y := (y_n)$ satisfies $[y] = \sup_n [x_k]$. This shows the first part of the statement.

Assume now that R is weakly s-unital, hence $x \precsim_1 x$ for each $x \in M_\infty(R)$. Then, if $[(x_n)] \in \Lambda(R)$, the sequence $x^{(k)} := (x_1, \dots, x_k, x_k, \dots)$ belongs to $T(R)$, and it is not difficult to show that $[(x_n)] = \sup_k [x^{(k)}]$.

Using this fact, one may check that, given $[(x_n)], [(y_n)] \in \Lambda(R)$, we have $[(x_n)] \ll [(y_n)]$ in $\Lambda(R)$ if and only if there is m such that $x_n \precsim_1 y_m$ for all n.

From this, one gets that axioms (O2)-(O4) are satisfied in $\Lambda(R)$ and, combined with the first part of the proof, we obtain that $\Lambda(R)$ is a Cu-semigroup. □

In the weakly s-unital setting, the semigroup $\Lambda(R)$ can be conveniently identified with the monoid of intervals in the semigroup $\mathrm{W}(R)$. We make this connection explicit below, and in the sequel we will use both pictures interchangeably. Intervals have been used in many places, in connection with C*-algebras and other algebraic structures; see, for example, [35], [30], [2], or [3]. Our discussion below consists of well-known facts on intervals.

2.14 (Intervals)**.** Let M be a positively ordered monoid. Recall that an *interval* in M is a subset $I \subseteq M$ which is upward directed and downward hereditary. The set of intervals is denoted by $\Lambda(M)$, and it becomes a positively ordered monoid by defining

$$I + J = \{z \in M \mid z \leq x + y \text{ where } x \in I \text{ and } y \in J\},$$

and where order is given by set inclusion.

We say that an interval I in M is *countably generated* provided I has a countable, cofinal subset. Equivalently, there is an increasing sequence (x_n) in I such that $I = \{x \in M \mid x \leq x_n \text{ for some } n\}$. The set of countably generated intervals in M is denoted by $\Lambda_\sigma(M)$, which is clearly a positively ordered submonoid of $\Lambda(M)$. Indeed, if I and J have countable cofinal subsets (x_n) and (y_n), respectively, then $(x_n + y_n)$ is a countable cofinal subset for $I + J$.

We have a positively ordered monoid morphism $\phi\colon M \to \Lambda_\sigma(M)$ given by $\phi(x) = [0, x]$, which is an order-embedding. Notice that every increasing sequence (I_n) in $\Lambda_\sigma(M)$ has a supremum, simply given by $\cup_n I_n$. From this, and writing every interval $J \in \Lambda_\sigma(M)$ as $J = \cup_n [0, y_n]$ for an increasing cofinal sequence (y_n), it follows easily that $I \ll J$ if and only if there is $y \in J$ such that $I \subseteq [0, y]$.

It is then clear that, for each $I \in \Lambda_\sigma(M)$ and $x \in I$, we have $[0, x] \ll I$ and, if (x_n) is an increasing sequence in I which is a countable cofinal subset, then $I = \sup[0, x_n]$ with $[0, x_n] \ll [0, x_n] \ll [0, x_{n+1}]$.

It is also easy to verify that addition in $\Lambda_\sigma(M)$ is compatible with suprema and the compact containment relation.

Lemma 2.15. *Let M be a positively ordered monoid. Then, its set of countably generated intervals $\Lambda_\sigma(M)$ is always an algebraic* Cu*-semigroup.*

Proof. This follows directly from the discussion carried out in Paragraph 2.14. □

2.16 (The semigroup $\Lambda_{\mathrm{W}}(R)$)**.** Let R be a weakly s-unital ring. We let

$$\Lambda_{\mathrm{W}}(R) = \Lambda_\sigma(\mathrm{W}(R)).$$

Notice that, if for example D is any division ring, one has $\mathrm{W}(D) \cong \mathbb{N}$, where the isomorphism is given by assigning to each matrix its rank. It follows that $\Lambda_{\mathrm{W}}(D) \cong \overline{\mathbb{N}}$.

Another example is given by purely infinite simple rings. Recall that a unital simple ring R is said to be *purely infinite* provided R is not a division ring and, for every non-zero element $a \in R$, there are elements $x, y \in R$ such that $xay = 1$

(see [6, Theorem 1.6]) . This implies in particular that $a \sim_1 b$ for any non-zero elements $a, b \in R$, and thus $\mathrm{W}(R) \cong \{0, \infty\}$ and also $\Lambda_{\mathrm{W}}(R) \cong \{0, \infty\}$.

Proposition 2.17. *Let R be a weakly s-unital ring. Then there is an ordered monoid isomorphism*

$$\Lambda_{\mathrm{W}}(R) \cong \Lambda(R).$$

Proof. Given $I \in \Lambda_{\mathrm{W}}(R)$, let $([a_n])$ be an increasing sequence in $\mathrm{W}(R)$ such that it is a cofinal subset for I. Define $\varphi\colon \Lambda_{\mathrm{W}}(R) \to \Lambda(R)$ by $\varphi(I) = [(a_n)]$.

If $I \subseteq J$ and $([a_n])$, $([b_n])$ are increasing, countable cofinal subsets for I and J, respectively, then for each n, there is m such that $[a_n] \leq [b_m]$. Therefore $(a_n) \precsim (b_n)$ and thus φ is well defined and order-preserving. It is easy to verify that φ also preserves addition. Evidently, φ is surjective.

Let us check that φ is an order-embedding. Suppose that $I, J \in \Lambda_{\mathrm{W}}(R)$ satisfy $\varphi(I) \leq \varphi(J)$. Let $([a_n]), ([b_n])$ be increasing, countable cofinal sequences for I and J, respectively. Then by definition of the order in $\Lambda(R)$ we have that, for each n, there is m with $a_n \precsim_1 b_m$, which clearly implies that $I \subseteq J$.

Therefore φ is an ordered monoid isomorphism. □

Remark 2.18. We note that Proposition 2.17 offers an alternative proof that $\Lambda(R)$ is an object in Cu in the weakly s-unital setting.

3. The Malcomlson semigroup, Sylvester rank functions, and dimension functions

In this section, we briefly recall the construction of the Malcomlson semigroup as introduced in [25] and its relation to $\mathrm{W}(R)$.

3.1 (The Malcomlson semigroup)**.** Let R be a unital ring. Following [25] with a slight change of notation, we define a relation $\precsim_0$ in $M_\infty(R)$ by $a \precsim_0 b$ if either $a \precsim_1 b$ or $a = \left(\begin{smallmatrix} c & 0 \\ 0 & d \end{smallmatrix}\right)$ and $b = \left(\begin{smallmatrix} c & e \\ 0 & d \end{smallmatrix}\right)$ for suitable $c, d, e \in M_\infty(R)$.

Define $\precsim_{\mathrm{M}}$ to be the transitive closure of $\precsim_0$, so that $a \precsim_{\mathrm{M}} b$ if and only if there exist $a_1, \ldots, a_n \in M_\infty(R)$ with $a = a_1 \precsim_0 a_2 \precsim_0 \cdots \precsim_0 a_n = b$. Set $\sim_{\mathrm{M}}$ as the antisymmetrization of $\precsim_{\mathrm{M}}$ and define

$$\mathrm{W}_{\mathrm{M}}(R) = M_\infty(R)/{\sim_{\mathrm{M}}} .$$

Denote the elements in $\mathrm{W}_{\mathrm{M}}(R)$ by $[a]_{\mathrm{M}}$. It follows that $\mathrm{W}_{\mathrm{M}}(R)$ is a positively ordered abelian semigroup with addition given by $[a] + [b] = [\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)]$ and order induced by $\precsim_{\mathrm{M}}$.

Since clearly $\precsim_{\mathrm{M}}$ is weaker than $\precsim_1$, we have a positively ordered monoid morphism $\iota_{\mathrm{M}}\colon \mathrm{W}(R) \to \mathrm{W}_{\mathrm{M}}(R)$, given by $[a] \mapsto [a]_{\mathrm{M}}$.

As shown in [25, Lemma 5.1], if A is a C*-algebra and $a, b \in M_\infty(A)$, one has that $a \precsim_{\mathrm{M}} b$ implies $a \precsim_{\mathrm{Cu}} b$. There is then a positively ordered monoid morphism $\tilde{\iota}_{\mathrm{C}}\colon \mathrm{W}_{\mathrm{M}}(A) \to \mathrm{W}_{\mathrm{C}}(A)$. Combining this with Lemma 2.10, we have the following commutative diagram

$$\begin{array}{ccc} W(A) & \xrightarrow{\iota_M} & W_M(A) \\ & \searrow^{\iota_C} & \downarrow{\scriptstyle \tilde{\iota}_C} \\ & & W_C(A) \end{array} .$$

3.2 (States, Sylvester rank functions and dimension functions)**.** Given a positively ordered monoid S with an order-unit u, recall that a *state* on S normalized at u is a positively ordered semigroup map $s\colon S \to [0,\infty)$ such that $s(u)=1$. The set of states is customarily denoted by $\mathrm{St}(S,u)$.

Let R be a unital ring. A map $d\colon M_\infty(R) \to [0,\infty)$ such that the following conditions hold:

(i) $d(0)=0$ and $d(1)=1$;
(ii) $d(ab) \leq d(a), d(b)$ or, equivalently, $d(a) \leq d(b)$ whenever $a \precsim_1 b$;
(iii) $d(a)+d(b) = d\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)$,

will be called a *dimension function.* If, furthermore, d satisfies

(iv) $d(a)+d(b) \leq d\left(\begin{smallmatrix} a & c \\ 0 & b \end{smallmatrix}\right)$,

then we say that d is a *Sylvester matrix rank function* for R.

Denote the set of dimension functions by $\mathrm{DF}(R)$, and the subset of Sylvester matrix rank functions by $\mathbb{P}(R)$. Note that $\mathbb{P}(R)$ may be identified with the states on $W_M(R)$ normalized at $[1]$. Indeed, given $d \in \mathbb{P}(R)$, set $s_d([a]) = d(a)$. Conversely, if $s \in \mathrm{St}(W_M(R),[1])$, define $d_s(a) = s([a])$. Likewise, one may check that the set of dimension functions may be identified with $\mathrm{St}(W(R),[1])$.

Lemma 3.3. *Let R be an s-unital ring. Let $a,b,c \in M_\infty(R)$. If a is a von Neumann regular element, then*

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \precsim_1 \begin{pmatrix} a & c \\ 0 & b \end{pmatrix}.$$

In particular, if R is a unital von Neumann regular ring, then $\precsim_1$ is equivalent to $\precsim_M$ and thus the natural map $\iota_M\colon W(R) \to W_M(R)$ is an order-isomorphism, whence $\mathrm{DF}(R) = \mathbb{P}(R)$.

Proof. Let $S = M_\infty(R)$. Since a is von Neumann regular and R is s-unital, there are elements $a', b' \in S$ such that $a = aa'a$ and $b = b'b = bb'$ and $c = cb' = b'c$. Therefore

$$\begin{pmatrix} aa' & 0 \\ 0 & b' \end{pmatrix} \cdot \begin{pmatrix} a & c \\ 0 & b \end{pmatrix} \cdot \begin{pmatrix} a'a & -a'c \\ 0 & b' \end{pmatrix} = \begin{pmatrix} a & -aa'c + aa'cb' \\ 0 & b'bb' \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}.$$

The second part of the statement follows by definition of $\precsim_M$: if $a \precsim_M b$, then there are elements $a_1, \ldots, a_n \in M_\infty(R)$ such that $a = a_1 \precsim_0 \cdots \precsim_0 a_n = b$. For each i, we have by definition of $\precsim_0$ and the first part of the proof that $a_i \precsim_1 a_{i+1}$, whence $a \precsim_1 b$. □

3.4 (Functionals)**.** Let S be a Cu-semigroup. Recall that a *functional* on S is a positively ordered monoid morphism $\lambda\colon S \to [0,\infty]$ that respects suprema of increasing sequences. The set of functionals on S is denoted by $F(S)$.

Proposition 3.5. *Let R be a unital ring. Then, any dimension function d on R induces a unique functional $\lambda_d \in F(\Lambda_{\mathrm{W}}(R))$ such that $\lambda_d(\phi([x])) = d(x)$ for all $x \in M_\infty(R)$, where $\phi\colon \mathrm{W}(R) \to \Lambda_{\mathrm{W}}(R)$ is the canonical homomorphism. In particular, this is the case for any Sylvester matrix rank function.*

Proof. By Proposition 2.17, $\Lambda_{\mathrm{W}}(R)$ is order-isomorphic to $\Lambda(R)$. Let $x \in \Lambda(R)$ and let (x_n) be a representative for its class, which by definition satisfies $x_n \precsim_1 x_{n+1}$ for each n. Given a dimension function $d \in DF(R)$, set $\lambda_d(x) = \sup_n d(x_n)$. By construction, if $(x_n) \precsim (y_n)$, then for each n there is m with $x_n \precsim_1 y_m$, and thus $\sup_n d(x_n) \leq \sup_m d(y_m)$. This yields a well defined, order-preserving map $\lambda_d\colon \Lambda(R) \to [0, \infty]$.

Since addition in $\Lambda(R)$ is defined componentwise, we see that λ_d is additive. It remains to check that λ_d preserves suprema of increasing sequences. To do so, we recall how suprema of increasing sequences are constructed in $\Lambda(R)$; see Proposition 2.13. Given a sequence (x_k) in $\Lambda(R)$ such that $x_k \precsim x_{k+1}$ for all k, we write $x_k = (x_n^{(k)})_n$ and assume, reindexing if necessary, that $x_{i+j}^{(i)} \precsim_1 x_{i+j}^{(i+j)}$ for all i, j. Then $y = (y_n)$, where $y_n := x_n^{(n)}$, is the supremum of the sequence $(x_k)_k$.

Since $x_n^{(k)} = x_{n-k+k}^{(k)} \precsim_1 x_n^{(n)}$ whenever $n \geq k$, we have

$$\sup_k \lambda_d(x_k) = \sup_k \sup_n d(x_n^{(k)}) \leq \sup_n d(x_n^{(n)}) = \lambda_d(y).$$

Since the reverse inequality $\lambda_d(y) \leq \sup_k \lambda_d(x_k)$ is obvious, this shows that $\lambda_d(y) = \sup_k \lambda_d(x_k)$.

Uniqueness of λ_d follows from the fact that $[(x_n)] = \sup_n \phi([x_n])$ for $[(x_n)] \in \Lambda(R)$. ☐

4. The Cuntz semigroup of a ring

As motivated in the introduction, in this section we pursue a construction of a semigroup using countably generated projective modules over a ring R. We first define a subsemigroup $\mathrm{S}(R)$ of $\Lambda(R)$ consisting of certain classes of sequences that will yield a presentation of a countably generated module. Secondly, we will use a direct approach building another semigroup $\mathrm{CP}(R)$ as equivalence classes of countably generated projective right R-modules using a relation weaker than isomorphism and related to the one used in [14].

The main result of the section is that the semigroups $\mathrm{S}(R)$ and $\mathrm{CP}(R)$ happen to be isomorphic in complete generality (see Theorem 4.13). The semigroup $\mathrm{S}(R)$ is closed under suprema of increasing sequences, and it is the invariant that best resembles the Cuntz semigroup for rings. However, it is not obviously an object in the category Cu. We remedy this by considering the pair $(\Lambda(R), \mathrm{S}(R))$, of which the first component belongs to Cu; see Section 5.

4.1 (The semigroup $\mathrm{S}(R)$)**.** Let R be a ring. Let $\mathcal{S}(R)$ be the subset of $T(R)$ consisting of those sequences $(x_n)_n$, with $x_n \in M_\infty(R)$ for all $n \in \mathbb{N}$, such that for every n there exists $y_{n+1} \in M_\infty(R)$ with $y_{n+1}x_{n+1}x_n = x_n$.

We define $\mathrm{S}(R) = \mathcal{S}(R)/{\sim}$, which by construction is a subset of $\Lambda(R)$; see Paragraph 2.11. We call $\mathrm{S}(R)$ the *Cuntz semigroup* of R. Let us check it is a subsemigroup of $\Lambda(R)$. We continue to use the terminology introduced in Section 2, and thus given an element $w \in \mathrm{S}(R)$, a *finite matricial representative* of w is any sequence (u_n) such that $u_n \in M_{k_{n+1}\times k_n}(R)$, where (k_n) is a sequence of positive integers, for which there exists $v_{n+1} \in M_{k_{n+1}\times k_{n+2}}(R)$ such that $u_n = v_{n+1}u_{n+1}u_n$ for all n, and with $w = [(x_n)]$, where x_n is the infinite matrix represented by u_n for each $n \in \mathbb{N}$. Therefore, for $w, w' \in \mathrm{S}(R)$, we let (u_n) and (u'_n) be finite matricial representatives of w, w', respectively, and define $w + w' = [(u_n \oplus u'_n)]$.

That addition is well defined, associative, commutative, and that the class of the zero sequence is the identity follows as in the arguments in Lemmas 2.6 and 2.12.

The definition of $\mathrm{S}(R)$ is admittedly one-sided, and one can define a left version of the semigroup $\mathrm{S}_l(R)$ as the semigroup whose elements are equivalence classes under $\sim$ of sequences $(x_n)_n$ in $M_\infty(R)$ such that

$$x_n = x_n x_{n+1} y_{n+1}$$

for every $n \in \mathbb{N}$.

It follows that $\mathrm{S}_l(R) \cong \mathrm{S}(R^{\mathrm{op}})$. However, $\mathrm{S}_l(R)$ is not always isomorphic to $\mathrm{S}(R)$, as the example below testifies.

Example 4.2. There exists a ring R such that $\mathrm{S}(R) \ncong \mathrm{S}_l(R)$.

Proof. Let K be a field and let $K[x_0, x_1, \ldots]$ be the free algebra on countably many variables subject to the (non-commutative) relations $x_{i+1}x_i = x_i$, and take the subring R of polynomials with zero constant term.

Given a monomial $p = \prod_{i\leq j\leq n} x_j^{t_j}$ with $t_i, t_n > 0$, we will write $\mathrm{st}(p) = i$ and $\mathrm{end}(p) = n$.

Then, note that given any other monomial $q = x_l^{s_l} \ldots x_m^{s_m}$ with $s_l, s_m > 0$, we either have $pq = q$ if $\mathrm{st}(p) > \mathrm{st}(q)$ or $pq > p$ in the lexicographic order otherwise. Note that we always have $\mathrm{st}(p) \geq \mathrm{st}(pq)$.

Thus, let $P = A_1p_1 + \ldots + A_np_n \in M_\infty(R)$ be a non-zero polynomial with $p_1, \ldots, p_n$ distinct monomials and $A_i \in M_\infty(K)\backslash\{0\}$ for all i. We may assume that $\mathrm{st}(p_1) \geq \mathrm{st}(p_i)$ for each i and that p_1 is the smallest monomial in the lexicographic order among all monomials p_i with $\mathrm{st}(p_1) = \mathrm{st}(p_i)$.

Now let $Q = B_1q_1 + \ldots + B_mq_m \in M_\infty(R)$ be another polynomial with $q_1, \ldots, q_m$ monomials and $B_j \in M_\infty(K)$. If $PQ = P$, A_1p_1 must be equal to a combination of the form $M_1p_{i_1}q_{j_1} + \ldots + M_rp_{i_r}q_{j_r}$, where $M_k \in M_\infty(K)$. In particular, $p_{i_k}q_{j_k} = p_1$ for each k.

However, this would imply

$$\mathrm{st}(p_1) \geq \mathrm{st}(p_{i_k}) \geq \mathrm{st}(p_{i_k}q_{j_k}) = \mathrm{st}(p_1),$$

and so $\mathrm{st}(p_{i_k}) = \mathrm{st}(p_1)$ for all k. This implies $p_1 = p_{i_k}q_{j_k} > p_{i_k} \geq p_1$, a contradiction. Therefore, we must have $P = 0$, which shows that $\mathrm{S}_l(R) = 0$.

This is not the case for $\mathrm{S}(R)$, since the sequence $(x_0, x_1, x_2, \ldots)$ is in $\mathcal{S}(R)$. □

The lemma below shows one of the main properties of $\mathrm{S}(R)$: suprema exist for increasing sequences.

Lemma 4.3. *Let R be any ring. Then every increasing sequence in $\mathrm{S}(R)$ has a supremum.*

Proof. Let $([x_k])_k$ be an increasing sequence in $\mathrm{S}(R)$. Write $x_k = (x_n^{(k)})_n$ for each k, and find $y_{n+1}^{(k)} \in M_\infty(R)$ such that $x_n^{(k)} = y_{n+1}^{(k)} x_{n+1}^{(k)} x_n^{(k)}$.

Note that $([x_k])_k$ also belongs to $\Lambda(R)$ and that the order in $\mathrm{S}(R)$ and $\Lambda(R)$ is the same. By Proposition 2.13, the sequence $([x_k])_k$ has a supremum $z = [(z_n)]$ in $\Lambda(R)$. It is enough to check that $z \in \mathrm{S}(R)$.

To this end, recall from Proposition 2.13 that, after a possible reindexing, one may assume that $x_{i+j}^{(i)} \precsim_1 x_{i+j}^{(i+j)}$ for all i, j and then we take $z_n = x_n^{(n)}$. Since $x_n^{(n)} \precsim_1 x_{n+1}^{(n+1)}$, there are a_{n+1}, b_{n+1} such that $x_n^{(n)} = a_{n+1} x_{n+1}^{(n+1)} b_{n+1}$. Thus

$$x_n^{(n)} = y_{n+1}^{(n)} x_{n+1}^{(n)} x_n^{(n)} = y_{n+1}^{(n)} a_{n+1} x_{n+1}^{(n+1)} b_{n+1} x_n^{(n)}.$$

If we let $u_n = x_n^{(n)} b_n$, it follows that $u_n = (y_{n+1}^{(n)} a_{n+1}) u_{n+1} u_n$, whence $(u_n) \in \mathcal{S}(R)$. By construction we have $z = [(z_n)] = [(u_n)] \in \mathrm{S}(R)$, as was to be shown. □

We now proceed to introduce a semigroup steming directly from the class of countably generated projective R-modules, although with a new equivalence relation inspired by the construction in [14].

4.4 (The semigroup $\mathrm{CP}(R)$ for a unital ring R)**.** Let R be a unital ring. Let us denote by $\mathcal{CP}(R)$ the class of all countably generated projective right R-modules. The first natural relation between (countably generated) projective modules is given by isomorphism. This yields the semigroup $\mathrm{V}^*(R)$ of isomorphism classes of countably generated projective modules, with addition given by direct sum. This semigroup has been successfully considered in [22] and [24]. Below, we weaken the above relation to another relation $\sim$ compatible with direct sum, thereby constructing an abelian semigroup $\mathrm{CP}(R)$. Thus, in particular, there is a natural surjective semigroup homomorphism $\Phi_R \colon \mathrm{V}^*(R) \to \mathrm{CP}(R)$, which will be an isomorphism in some cases, but not always; see Section 6.

Given $P, Q \in \mathcal{CP}(R)$, we will write $P \precsim Q$ if and only if, for every finitely generated submodule X of P, there exists a factorization of the inclusion of X in P by Q, that is, there are module morphisms $\phi \colon X \to Q$ and $\phi \colon Q \to P$ such that $\psi \circ \phi = \mathrm{id}_X$. Namely, the diagram below is commutative:

$$\begin{tikzcd} X \arrow[r, "\phi"] \arrow[rr, bend right, "\mathrm{id}_X"'] & Q \arrow[r, "\psi"] & P \end{tikzcd}.$$

We define the partially ordered set $\mathrm{CP}(R)$ to be

$$\mathrm{CP}(R) := \mathcal{CP}(R)/{\sim},$$

where $\sim$ is the antisymmetrization of $\precsim$. Given an element P in $\mathcal{CP}(R)$, we will denote its equivalence class by $[P]$. For modules $P, Q \in \mathcal{CP}(R)$, we define $[P] + [Q] = [P \oplus Q]$.

Lemma 4.5. *The relation $\precsim$ is reflexive, transitive, and compatible with the direct sum of projective modules. Therefore,* $\mathrm{CP}(R)$ *is a commutative semigroup.*

Proof. That $\precsim$ is reflexive is trivial to verify. Let us show that it s transitive.

To this end, let P_1, P_2, $P_3 \in \mathcal{CP}(R)$ and assume that $P_1 \precsim P_2$ and $P_2 \precsim P_3$. Let $X \subseteq P_1$ be a finitely generated submodule. Then, by definition of $\precsim$, there is a commutative diagram

$$\underbrace{X \xrightarrow{\phi_1} P_2 \xrightarrow{\psi_1} P_1}_{\mathrm{id}_X} .$$

Note that $\phi_1(X)$ is a finitely generated submodule of P_2. Thus, since $P_2 \precsim P_3$, there is a commutative diagram

$$\underbrace{\phi_1(X) \xrightarrow{\phi_2} P_3 \xrightarrow{\psi_2} P_2}_{\mathrm{id}_{\phi_1(X)}}$$

Combining both diagrams, we define $\phi_3 = \phi_2 \circ \phi_1$ and $\psi_3 = \psi_1 \circ \psi_2$ to obtain:

$$\underbrace{X \xrightarrow{\phi_1} \phi_1(X) \xrightarrow{\phi_2} P_3}_{\phi_3} \underbrace{\xrightarrow{\psi_2} P_2 \xrightarrow{\psi_1} P_1}_{\psi_3} ,$$

which satisfies $\psi_3 \circ \phi_3 = \psi_1 \circ \psi_2 \circ \phi_2 \circ \phi_1 = \psi_1 \circ \mathrm{id}_{\phi_1(X)} \circ \phi_1 = \psi_1 \circ \phi_1 = \mathrm{id}_X$.

To show that $\precsim$ is compatible with the direct sum of modules, let P, Q, P', $Q' \in \mathcal{CP}(R)$ and suppose that $P \precsim P'$, $Q \precsim Q'$. Let $X \subseteq P \oplus Q$ be a finitely generated submodule. Then, there exist finitely generated submodules $X_P \subseteq P$ and $X_Q \subseteq Q$ such that $X \subseteq X_P \oplus X_Q$. By assumption, we have module maps $\phi_1 \colon X_P \to P'$, $\psi_1 \colon P' \to P$, $\phi_2 \colon X_Q \to Q'$, $\psi_2 \colon Q' \to Q$, and commutative diagrams

$$\underbrace{X_P \xrightarrow{\phi_1} P' \xrightarrow{\psi_1} P}_{\mathrm{id}_{X_P}} \qquad\qquad \underbrace{X_Q \xrightarrow{\phi_2} Q' \xrightarrow{\psi_2} Q}_{\mathrm{id}_{X_Q}} ,$$

which yields the following commutative diagram

$$X \subseteq \underbrace{X_P \oplus X_Q \xrightarrow{\phi_1 \oplus \phi_2} P' \oplus Q' \xrightarrow{\psi_1 \oplus \psi_2} P \oplus Q}_{\mathrm{id}_{X_P \oplus X_Q}} .$$

□

Remark 4.6. Given a finitely generated projective module F and a countably generated projective module P, it follows from our definition that $F \precsim P$ if and only if F is isomorphic to a direct summand of P.

We are now going to show that $\mathrm{CP}(R)$ is order-isomorphic to $\mathrm{S}(R)$. Instrumental ingredients in our proof will be the facts, proved in [29, Lemma 4.1], that every countably generated projective R-module P over a unital ring R is isomorphic to a direct limit of the form

$$R^{n_0} \xrightarrow{x_0 \cdot} R^{n_1} \xrightarrow{x_1 \cdot} \dots \longrightarrow P,$$

for some sequence of positive integers n_i, where for each $x_i \in M_{n_{i+1} \times n_i}(R)$ there exists $y_{i+1} \in M_{n_{i+1} \times n_{i+2}}(R)$ such that $y_{i+1} x_{i+1} x_i = x_i$, and that, conversely, any such direct limit is always projective. We will give below independent proofs of these facts, since our arguments offer additional information that will be useful later on.

For any ring R, unital or not, we will denote by $\mathrm{FCM}(R)$ the ring of those $\mathbb{N} \times \mathbb{N}$ matrices A with coefficients in R such that each column of A has only a finite number of nonzero entries. We refer to $\mathrm{FCM}(R)$ as the ring of *finite-column matrices* over R. When R is unital this ring can be identified with the ring $\mathrm{End}_R(R^{(\mathbb{N})})$ of R-module endomorphisms of the free R-module $R^{(\mathbb{N})}$.

Lemma 4.7. *Let R be a unital ring, and let $P \in \mathcal{CP}(R)$. Then P is isomorphic to a projective module of the form $\varinjlim_i (R^{k_i}, Z_i\cdot)$, where $Z_i \in M_{k_{i+1}\times k_i}(R^+)$ for an increasing sequence of positive integers (k_i), and moreover $Z_{i+1}Z_i = \begin{pmatrix} Z_i \\ 0_{(k_{i+2}-k_{i+1})\times k_i} \end{pmatrix}$ for all $i \in \mathbb{N}$. In particular, P can be written in the form $P \cong \varinjlim_i (R^{n_i}, x_i\cdot)$ for some sequence of positive integers n_i, where for each $x_i \in M_{n_{i+1}\times n_i}(R)$ there exists $y_{i+1} \in M_{n_{i+1}\times n_{i+2}}(R)$ such that $y_{i+1}x_{i+1}x_i = x_i$.*

Proof. We can assume that $P = E(R^{(\mathbb{N})})$ for an idempotent $E = (e_{ij}) \in \mathrm{FCM}(R)$. For each $n \geq 1$, we identify R^n with $R^n \times \{0\} \times \{0\} \times \cdots$ in $R^{(\mathbb{N})}$. Take an arbitrary positive integer k_0. Let $k_1 > k_0$ be an integer such that $e_{k,l} = 0$ for all (k,l) such that $k > k_1$ and $l \leq k_0$. In particular, one has $E(R^{k_0}) \subseteq R^{k_1}$. Proceeding inductively we may find an increasing sequence of positive integers (k_i) such that $e_{k,l} = 0$ for all (k,l) such that $k > k_{i+1}$ and $l < k_i$. Then we have $E(R^{k_i}) \subseteq R^{k_{i+1}}$ for all $i \in \mathbb{N}$. Let Z_i be the $k_{i+1} \times k_i$ upper left corner of E. Thus, we get $Z_i \in M_{k_{i+1}\times k_i}(R)$ and $Z_{i+1}Z_i = \begin{pmatrix} Z_i \\ 0_{(k_{i+2}-k_{i+1})\times k_i} \end{pmatrix}$ for all $i \in \mathbb{N}$. Moreover $P = \bigcup_{i=0}^{\infty} Z_i R^{k_i}$ and $P \cong \varinjlim (R^{k_i}, Z_i\cdot)$, as claimed. □

4.8 (Splittings)**.** Let (n_i) be a sequence of positive integers, and let (x_i) be a sequence, with $x_i \in M_{n_{i+1}\times n_i}(R)$, such that there exists $y_{i+1} \in M_{n_{i+1}\times n_{i+2}}(R)$ satisfying $y_{i+1}x_{i+1}x_i = x_i$. We show that $P := \varinjlim_i (R^{n_i}, x_i\cdot)$ is a countably generated projective R-module by exhibiting a concrete splitting of P into $R^{(\mathbb{N})}$.

Let $\phi_i \colon R^{n_i} \to P$ be the canonical morphisms into the direct limit, and denote by $P_i \subseteq P$ the image of ϕ_i. Note that P_i is a finitely generated submodule of P, with generators $z_i^j := \phi_i(e_j)$. Setting $\mathbf{z}_i = (z_i^1, \ldots, z_i^{n_i})$, which is a row matrix with coefficients in P, it follows that P is described by the generators $(z_i^j)_{i,j}$, subject to the relations $\mathbf{z}_{i+1}x_i = \mathbf{z}_i$ for all $i \in \mathbb{N}$.

Further, note that for every pair $j \geq i+1$ the following diagram is commutative

$$\begin{array}{ccc} R^{n_j} & \xrightarrow{\;y_{i+1}\cdots y_j x_j\cdot\;} & R^{n_{i+1}} \\ {\scriptstyle x_j\cdot}\downarrow & \nearrow & \\ R^{n_{j+1}} & \xrightarrow[\;y_{i+1}\cdots y_{j+1}x_{j+1}\cdot\;]{} & R^{n_{i+1}} \end{array}$$

In particular, for every fixed i the previous diagrams induce a morphism $g_i \colon P \to R^{n_{i+1}}$. Restricting to each component, we define $g_i^j := \pi_j \circ g_i \colon P \to R$, for $j \in \{1, \ldots, n_{i+1}\}$, where $\pi_j \colon R^{n_{i+1}} \to R$ is the projection onto the j-th component.

Lemma 4.9. *Following the above notation, a concrete splitting of P into $R^{(\mathbb{N})}$ is given by the formulas*

$$\pi\colon R^{(\mathbb{N})} = R^{n_1} \oplus R^{n_2} \oplus \cdots \to P, \quad \iota\colon P \to R^{(\mathbb{N})} = R^{n_1} \oplus R^{n_2} \oplus \cdots ,$$

where

$$\pi(a_1, a_2, \dots) = \sum_{i=1}^{\infty} \phi_i(a_i), \quad \iota(x) = (g_0(x), g_1(x) - x_1 g_0(x), g_2(x) - x_2 g_1(x), \dots)$$

for $a_i \in R^{n_i}$ and $x \in P$. Then one has $\pi \circ \iota = \mathrm{id}_P$. In particular, P is a countably generated projective right R-module.

Proof. Suppose that $x \in P_i$, and write $x = \phi_i(a)$ for $a \in R^{n_i}$. Then we have

$$\begin{aligned}\phi_{i+1}(g_i(x)) &= \phi_{i+1}(g_i(\phi_i(a))) = \phi_{i+1}(g_i(\phi_{i+1}(x_i a))) \\ &= \phi_{i+1}(y_{i+1} x_{i+1} x_i a) = \phi_{i+1}(x_i a) \\ &= \phi_i(a) = x.\end{aligned}$$

Hence $(\phi_{i+1} \circ g_i)|_{P_i} = \mathrm{id}_{P_i}$, or equivalently $\phi_{i+1} \circ g_i \circ \phi_i = \phi_i$. Using this, the identity $\pi \circ \iota = \mathrm{id}_P$ is easily checked. □

Theorem 4.10. *Let R be a unital ring. Then*

$$\mathrm{CP}(R) \cong \mathrm{S}(R).$$

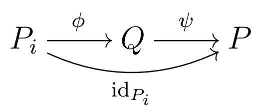
Proof. Given a countably generated projective module P, it follows from Lemma 4.7 (see also [29, Lemma 4.1(2)]) that $P \cong \lim(R^{n_i}, x_i \cdot)$, where $x_i \in M_{n_{i+1} \times n_i}(R)$ are such that, for each i there exists $y_{i+1} \in M_{n_{i+1} \times n_{i+2}}(R)$ such that $x_i = y_{i+1} x_{i+1} x_i$. In particular, the sequence (x_i) determines an element in $\mathcal{S}(R)$, through the usual identification of x_i with the matrix $\mathrm{diag}(x_i, 0, 0, \dots)$ in $M_\infty(R)$.

We will show that the map $[P] \mapsto [(x_i)]$ defines an isomorphism between $\mathrm{CP}(R)$ and $\mathrm{S}(R)$.

First, note that this map is surjective by Lemma 4.9 (see also [29, Lemma 4.1(1)]). Moreover, it is also additive by how addition is defined in both $\mathrm{CP}(R)$ and $\mathrm{S}(R)$; see 4.1, 4.4.

Hence, we will conclude the proof showing that $P \precsim Q$ if and only if $(x_i) \precsim (x'_i)$, where $P \cong \lim(R^{n_i}, x_i \cdot)$ and $Q \cong \lim(R^{n'_i}, x'_i \cdot)$ are the corresponding representations as direct limits.

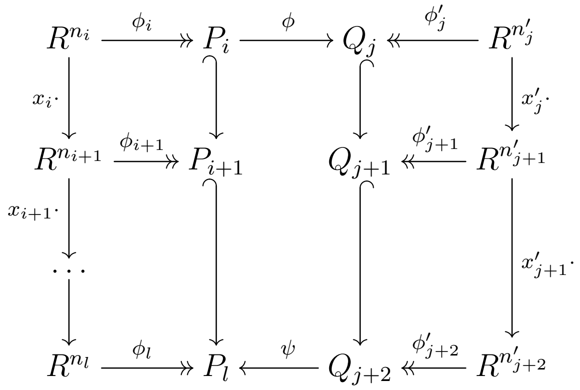
To prove the forward implication, let P, $Q \in \mathcal{CP}(R)$ and suppose that $P \precsim Q$. Write $P = \bigcup_{i=1}^{\infty} P_i$ and $Q = \bigcup_{i=1}^{\infty} Q_i$ as in Paragraph 4.8. Then, given i, there are module homomorphisms $\phi\colon P_i \to Q$ and $\psi\colon Q \to P$ such that the diagram



is commutative.

Since P_i is finitely generated, so is $\phi(P_i)$. In particular, $\phi(P_i) \subseteq Q_j$ for some j. By the same reasoning, one has that $\psi(Q_{j+2}) \subseteq P_l$ for some l.

Using the definition of the maps g_i, ϕ_i from Paragraph 4.8, one gets the following commutative diagram.



Now note that, given $r \in R^{n_i}$ and $r' \in R^{n'_j}$ such that $q := \phi\phi_i(r) = \phi'_j(r')$, we have

$$
\begin{aligned}
q = \phi'_j(r') &= \phi'_{j+1}(x'_j r') = \phi'_{j+1}(y'_{j+1}x'_{j+1}x'_j r') \\
&= \phi'_{j+1}g'_j\phi'_{j+1}(x'_j r') = \phi'_{j+1}g'_j(q) = \phi'_{j+2}x'_{j+1}g'_j(q),
\end{aligned}
$$

where in the fourth step we have used that $g'_j\phi'_{j+1} = y'_{j+1}x'_{j+1}$.

Thus, one gets that

$$
\begin{aligned}
x_{l-1}\dots x_i(r) &= y_l x_l x_{l-1}\dots x_i(r) = g_{l-1}\phi_l(x_{l-1}\dots x_i(r)) \\
&= g_{l-1}\phi_{l-1}(x_{l-2}\dots x_i(r)) = \dots = g_{l-1}\phi_i(r) \\
&= g_{l-1}\psi\phi\phi_i(r) = g_{l-1}\psi(q) = (g_{l-1}\psi\phi'_{j+2})x'_{j+1}(g'_j\phi\phi_i)(r).
\end{aligned}
$$

Since this holds for every $r \in R^{n_i}$, we can multiply by $y_{i+1}\dots y_{l-1}$ to obtain

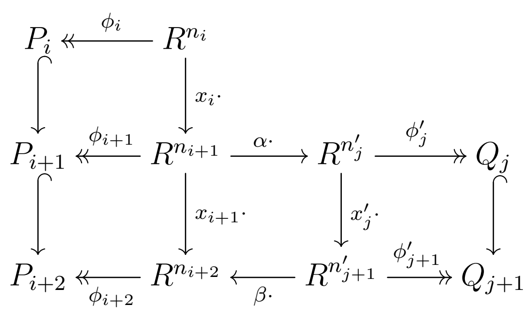
$$x_i = (y_{i+1}\dots y_{l-1}g_{l-1}\psi\phi'_{j+2})x'_{j+1}(g'_j\phi\phi_i).$$

It follows that $(x_i) \precsim (x'_i)$, as required.

For the converse, let $(x_i) \precsim (x'_i)$. For any fixed i we need to construct a commutative diagram

$$P_i \xrightarrow{\phi} Q \xrightarrow{\psi} P_i, \qquad \psi\phi = \mathrm{id}_{P_i}.$$

We know that, for every fixed i, there exist $j \in \mathbb{N}$, $\alpha \in M_{n'_j \times n_{i+1}}(R)$ and $\beta \in M_{n_{i+2}\times n'_{j+1}}(R)$ such that $x_{i+1} = \beta x'_j \alpha$. Thus, one gets the following commutative diagram:



Define $\phi := \phi'_j \alpha g_{i|_{P_i}}$ and $\psi = \phi_{i+2}\beta g'_j$. We have

$$\psi\phi = \phi_{i+2}\beta(g'_j\phi'_j)\alpha g_{i|_{P_i}} = \phi_{i+2}\beta x'_j \alpha g_{i|_{P_i}} = \phi_{i+2}x_{i+1}g_{i|_{P_i}} = \phi_{i+1}g_{i|_{P_i}} = \mathrm{id}_{P_i},$$

as desired. □

4.11 (The semigroup $\mathrm{CP}(R)$ for non-unital R)**.** Let R be an arbitrary ring, and let $R^+ = \mathbb{Z} \oplus R$ be the unitization of R. Observe that R sits as a two-sided ideal of R^+. We will denote by Mod-R^+ the category of unital right R^+-modules, and by AMod-R the category of arbitrary R-modules. Note that we have an isomorphism of categories AMod-$R \cong$ Mod-R^+ sending an arbitrary R-module M to the unique unital R^+-module whose underlying additive group is $(M, +)$ and whose multiplication is given by $x(n, r) = nx + xr$ for $x \in M$, $n \in \mathbb{Z}$ and $r \in R$.

Recall that, for a ring R, we denote by $\mathrm{FCM}(R)$ the ring of finite-column matrices over R. For an arbitrary ring R, we will denote by $\mathcal{CP}(R)$ the class of all countably generated unital projective right R^+-modules P such that $P = PR$. The class $\mathcal{CP}(R)$ agrees with the previously defined class $\mathcal{CP}(R)$ whenever R is a unital ring. Given such module P, there exists an idempotent $E \in \mathrm{FCM}(R^+)$ such that $E((R^+)^{(\mathbb{N})}) \cong P$. Since $P = PR$, it follows that $E \in \mathrm{FCM}(R)$ and $E((R^+)^{(\mathbb{N})}) = E(R^{(\mathbb{N})})$. Conversely, given an idempotent $E \in \mathrm{FCM}(R)$, the unital R^+-module $P = E(R^{(\mathbb{N})}) = E((R^+)^{(\mathbb{N})})$ is countably generated and projective, and $P = PR$. Moreover if Q is also in $\mathcal{CP}(R)$, then $P \cong Q$ if and only if E and F are Murray-von Neuman equivalent idempotents in $\mathrm{FCM}(R)$ (see Paragraph 2.5). This extends the well-known relation between isomorphism classes of finitely generated unital projective R^+-modules P such that $P = PR$ and idempotent matrices in $M_\infty(R)$, see e.g. [21, § 5.1].

Observe that, with the above notation, we have that $\mathcal{CP}(R)$ is a subclass of $\mathcal{CP}(R^+)$. Moreover $\mathcal{CP}(R)$ is closed in $\mathcal{CP}(R^+)$ under direct summands and countable direct sums. We consider the relation $\precsim$ inherited from the relation $\precsim$ which we have defined in $\mathcal{CP}(R^+)$, that is, for $P, Q \in \mathcal{CP}(R)$, we set $P \precsim Q$ if and only if $P \precsim Q$ in $\mathcal{CP}(R^+)$. We denote by $\mathrm{CP}(R)$ the monoid of equivalence classes of objects in $\mathcal{CP}(R)$ with respect to the relation $\precsim$. Note that $\mathrm{CP}(R)$ order-embeds in $\mathrm{CP}(R^+)$. We further define $\mathrm{V}^*(R)$ as the monoid of isomorphism classes of modules from $\mathcal{CP}(R)$. As in the case of unital rings, we have a canonical surjective homomorphism $\Phi_R\colon \mathrm{V}^*(R) \to \mathrm{CP}(R)$.

It is also easily checked that $\mathrm{S}(R)$ order-embeds in $\mathrm{S}(R^+)$. We will show that the isomorphism $\psi\colon \mathrm{CP}(R^+) \to \mathrm{S}(R^+)$, displayed in Theorem 4.10, restricts to an isomorphism from $\mathrm{CP}(R)$ onto $\mathrm{S}(R)$.

In order to obtain this result, we find a concrete realization of the idempotent matrix $E \in \mathrm{FCM}(R)$ corresponding to a sequence (x_i) in $\mathcal{S}(R)$.

Lemma 4.12. *Let R be a ring, and let $(x_i) \in \mathcal{S}(R)$, with $x_i \in M_{n_{i+1}\times n_i}(R)$, where all n_i's are positive integers. Then the countably generated projective R^+-module $P = \varinjlim_i ((R^+)^{n_i}, x_i\cdot)$ is isomorphic to a module of the form $Q = \bigcup_i z_i(R^{(\mathbb{N})})$, where $z_i \in M_\infty(R)$ and $z_{i+1}z_i = z_i$ for all $i \in \mathbb{N}$. More precisely, there is a sequence of positive integers (k_i), with $k_{i+1} > k_i$ for all $i \in \mathbb{N}$, such that each z_i is represented* *by a matrix* $Z_i \in M_{k_{i+1}\times k_i}(R)$ *and* $Z_{i+1}Z_i = \begin{pmatrix} Z_i \\ 0_{(k_{i+2}-k_{i+1})\times k_i}\end{pmatrix}$ *for all* $i \in \mathbb{N}$*, so that* $P \cong \varinjlim_i ((R^+)^{k_i}, Z_i\cdot)$.

Proof. By Lemma 4.9, one can explicitly compute the idempotent matrix $E \in \mathrm{End}(R^{(\mathbb{N})})$ such that $E(R^{(\mathbb{N})}) \cong P$, where $P \cong \lim((R^+)^{n_i}, x_i\cdot)$.

Using the notation of Paragraph 4.8, the splitting found in Lemma 4.9 gives the idempotent $E = \iota \circ \pi \in \mathrm{End}(R^{(\mathbb{N})})$. For $i \geq 1$, the column E_{i-1} of E (with respect to the decomposition $(R^+)^{(\mathbb{N})} = (R^+)^{n_1} \oplus (R^+)^{n_2} \oplus \cdots$), is given by

$$E_{i-1} = \begin{pmatrix} y_1y_2\cdots y_ix_i \\ y_2y_3\cdots y_ix_i - x_1y_1y_2\cdots y_ix_i \\ y_3y_4\cdots y_ix_i - x_2y_2y_3\cdots y_ix_i \\ \vdots \\ y_ix_i - x_{i-1}y_{i-1}y_ix_i \\ x_i - x_iy_ix_i \\ \mathbf{0} \end{pmatrix}.$$

Note that the i-th column E_i of E has (at most) $i+1$ nonzero coefficients. Let $Z_i \in M_{(n_1+\cdots+n_{i+2})\times(n_1+\cdots+n_{i+1})}(R)$ be the matrix consisting of the upper left $(n_1 + \cdots + n_{i+2}) \times (n_1 + \cdots + n_{i+1})$ corner of E. One has

$$Z_{i+1}Z_i = \begin{pmatrix} Z_i \\ 0_{n_{i+3}\times(n_1+\cdots+n_{i+1})}\end{pmatrix}.$$

Let $z_i \in M_\infty(R)$ be the infinite matrix represented by Z_i. Then $(z_i) \in \mathcal{S}(R)$ and $z_{i+1}z_i = z_i$ for all $i \geq 1$. Moreover, we have

$$P \cong E(R^{(\mathbb{N})}) = \bigcup_{i=0}^{\infty} z_i((R^+)^{(\mathbb{N})}) \cong \varinjlim_i((R^+)^{k_i}, Z_i\cdot),$$

where $k_i = n_1 + \cdots + n_{i+1}$ for all $i \geq 0$. ☐

We can now obtain the following result, generalizing Theorem 4.10.

Theorem 4.13. *Let* R *be a ring. Then there is an isomorphism* $\mathrm{S}(R) \cong \mathrm{CP}(R)$ *such that the following diagram*

$$\begin{array}{ccc} \mathrm{CP}(R) & \xrightarrow{\psi|_{\mathrm{CP}(R)}} & \mathrm{S}(R) \\ \big\downarrow & & \big\downarrow \\ \mathrm{CP}(R^+) & \xrightarrow{\psi} & \mathrm{S}(R^+) \end{array}$$

is commutative, where $\psi\colon \mathrm{CP}(R^+) \to \mathrm{S}(R^+)$ *is the isomorphism defined in Theorem 4.10.*

Proof. We need to show that the restriction of the map $\psi\colon \mathrm{CP}(R^+) \to \mathrm{S}(R^+)$ defined in the proof of Theorem 4.10 sends $\mathrm{CP}(R)$ onto $\mathrm{S}(R)$.

Given $P \in \mathcal{CP}(R)$, we can assume that $P = E((R^+)^{(\mathbb{N})})$ for an idempotent $E = (e_{ij}) \in FCM(R)$. The procedure given in the proof of Lemma 4.7

gives us an increasing sequence of positive integers (k_i) and a sequence (Z_i), with $Z_i \in M_{k_{i+1}\times k_i}(R)$ and $Z_{i+1}Z_i = \begin{pmatrix} Z_i \\ 0_{(k_{i+2}-k_{i+1})\times k_i}\end{pmatrix}$ for all $i \in \mathbb{N}$, such that $P \cong \varinjlim((R^+)^{k_i}, Z_i\cdot)$. By the definition of the map ψ we have that $\psi([P]) = [(z_i)_i] \in \mathrm{S}(R)$, where $z_i \in M_\infty(R)$ are represented by Z_i for all $i \in \mathbb{N}$. Hence $\psi([P]) \in \mathrm{S}(R)$, as desired.

Now if $w \in \mathrm{S}(R)$, it follows from Lemma 4.12 that $\psi^{-1}(w)$ can be represented by a countably generated unital projective right R^+-module of the form $P = E(R^{(\mathbb{N})})$, where $E \in FCM(R)$. In particular, $P = PR$, so that $P \in \mathcal{CP}(R)$, and hence $w = \psi(\psi^{-1}(w)) = \psi([P]) \in \psi(\mathrm{CP}(R))$. This shows that $\psi(\mathrm{CP}(R)) = \mathrm{S}(R)$, completing the proof. □

The following corollary is a useful consequence of the above proof.

Corollary 4.14. *Let R be a ring. Then every element in $\mathrm{S}(R)$ has a representative of the form (z_i), where $z_i \in M_\infty(R)$ satisfy $z_{i+1}z_i = z_i$ for all $i \in \mathbb{N}$. More precisely there is a non-decreasing sequence of positive integers (k_i) such that each z_i is represented by a matrix $Z_i \in M_{k_{i+1}\times k_i}(R)$ and $Z_{i+1}Z_i = \begin{pmatrix} Z_i \\ 0_{(k_{i+2}-k_{i+1})\times k_i}\end{pmatrix}$ for all $i \in \mathbb{N}$.*

5. The category SCu and the pair SCu(R)

As mentioned above, we will consider in this section the pair $(\Lambda(R), \mathrm{S}(R))$ and show that it sits naturally in a category that we term SCu consisting of pairs (S, W) where S is a Cu-semigroup and W is a subsemigroup closed under suprema of certain sequences. We also show that the assignment $R \mapsto (\Lambda(R), \mathrm{S}(R))$ is functorial.

5.1 (Weakly increasing sequences)**.** Let S be a Cu-semigroup. A sequence (x_n) of elements in S is said to be *weakly increasing* if, for every n and for every $x \ll x_n$, there exists m_0 such that $x \ll x_m$ whenever $m \geq m_0$.

It is evident that every increasing sequence in S is also weakly increasing. We know that increasing sequences always have suprema in S, and we show below that this is also the case for weakly increasing sequences. Although the concept of a weakly increasing sequence may seem somewhat artificial, it will become key to show that the category SCu introduced in Paragraph 5.4 admits inductive limits, as we prove in [1].

Lemma 5.2. *Let S be a* Cu*-semigroup. Then, every weakly increasing sequence has a supremum in S.*

Proof. We use an argument similar to the proof of Proposition 2.13 (which in turn is similar to the proof that increasing sequences in a Cu-semigroup have suprema). We give some details as this will be used again below.

Let (x_n) be a weakly increasing sequence in S. Since S is a Cu-semigroup, we may write each x_m as $x_m = \sup_n x_n^{(m)}$, where $(x_n^{(m)})$ is a rapidly increasing sequence. We construct increasing sequences of positive integers (n_i), (m_i) such that $x_{n_i+j}^{(m_i)} \ll x_{n_k}^{(m_k)}$ whenever $i + j \leq k$.

To do this, we define the sequence inductively. Let $n_1 = 0$ and $m_1 = 1$. Since $x_{n_1+1}^{(m_1)} = x_1^{(1)} \ll x_1$, there is $m_2 > 1$ such that $x_1^{(1)} \ll x_{m_2}$, and thus there is $n_2 > 0$ with $x_1^{(1)} \ll x_{n_2}^{(m_2)}$. Now, assume that n_i, m_i have been constructed for $i \leq k$. Since for each $1 \leq j \leq k$ we have $x_{n_1+k-(j-1)}^{(m_j)} \ll x_{m_j}$, and the sequence is weakly increasing, there is $m_{k+1} > m_k$ such that $x_{n_1+k-(j-1)}^{(m_j)} \ll x_{m_{k+1}}$ for all j. Thus, there is $n_{k+1} > n_k$ such that $x_{n_1+k-(j-1)}^{(m_j)} \ll x_{n_{k+1}}^{(m_{k+1})}$ for all j. This completes the inductive step.

After reindexing the sequence (n_i), we may assume that $n_i = i$, and thus $x_{i+j}^{(m_i)} \ll x_{i+j}^{(m_{i+j})}$ whenever $i, j \geq 1$. Now, the sequence $(x_k^{(m_k)})$ is rapidly increasing, since $x_k^{(m_k)} \ll x_{k+1}^{(m_k)} \ll x_{k+1}^{(m_{k+1})}$, and one may check that its supremum is the supremum of the weakly increasing sequence. □

Remark 5.3. Although we will not be using this, it is worth mentioning that weakly increasing sequences as defined in Paragraph 5.4 are compatible with other properties in the category Cu. Namely,

(i) Cu-morphisms preserve weakly increasing sequences and their suprema.
(ii) The addition in a Cu-semigroup is compatible with suprema of weakly increasing sequences.

5.4 (The Category SCu)**.** Let S be a Cu-semigroup. We say that a subset H of S is *closed under suprema of weakly increasing sequences* if, given any weakly increasing sequence (x_n) in S whose elements are in H, we have that $\sup x_n \in H$.

We define SCu to be the abstract category whose *objects* are the pairs (S, W), where $S \in \mathrm{Cu}$ and W is a submonoid of S closed under suprema of weakly increasing sequences. The *morphisms* in SCu are $f\colon (S_1, W_1) \to (S_2, W_2)$ where $f\colon S_1 \to S_2$ is a Cu-morphism such that $f(W_1) \subseteq W_2$.

Examples 5.5. The following are natural examples in the category SCu.

(i) Any pair (S, W) with $S \in \mathrm{Cu}$ and W a sub-Cu-semigroup of S is an object in SCu.
(ii) A nonzero Cu-semigroup S is said to be *simple* if the only ideals of S are $\{0\}$ and S (see e.g. [3] for the definition of ideal in a Cu-semigroup). Let S be a simple Cu-semigroup. Then $(S, \{0, \infty\})$ is an object in SCu. Note that $\{0, \infty\}$ is not always a sub-Cu-semigroup of S.
(iii) The pair $([0, \infty], \overline{\mathbb{N}})$, where $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$, is an object in SCu. This follows since every weakly increasing sequence with elements in $\overline{\mathbb{N}} \subseteq [0, \infty]$ has an increasing cofinal subsequence. (Here, $x \ll y$ if and only if $x < y$, for $x \in [0, \infty]$ and $y \in (0, \infty]$.) However, as we have observed above, $\overline{\mathbb{N}}$ is not a sub-Cu-semigroup of $[0, \infty]$.

Proposition 5.6. *Let R be a weakly s-unital ring. Then:*

(i) *The pair* $\mathrm{SCu}(R) := (\Lambda(R), \mathrm{S}(R))$ *is an object in* SCu.

(ii) *If R' is another weakly s-unital ring and $f\colon R \to R'$ is a ring homomorphism, then f induces a morphism* $\mathrm{SCu}(f)\colon (\Lambda(R), \mathrm{S}(R)) \to (\Lambda(R'), \mathrm{S}(R'))$.

Proof. (i): By Proposition 2.13, $\Lambda(R)$ is a Cu-semigroup, and by construction $\mathrm{S}(R)$ is a subsemigroup of $\Lambda(R)$. We thus have to prove that $\mathrm{S}(R)$ is closed under suprema of weakly increasing sequences.

Let $([x_m])$ be a weakly increasing sequence in $\Lambda(R)$ with $[x_m] \in \mathrm{S}(R)$ for all m. In order to construct the supremum of $([x_m])$, we follow the argument in Lemma 5.2. Write $x_m = (x_n^{(m)})$, and find $y_n^{(m)}$ such that $y_{n+1}^{(m)} x_{n+1}^{(m)} x_n^{(m)} = x_n^{(m)}$. Since R is weakly s-unital, we have that for each m the sequence $z_{m,n} = (x_1^{(m)}, x_2^{(m)}, \dots, x_n^{(m)}, x_n^{(m)}, \dots)$ satisfies that $([z_{m,n}])$ is rapidly increasing with supremum $[x_m]$ in $\Lambda(R)$ (see the proof of Proposition 2.13).

Arguing as in the proof of Lemma 5.2, we find an increasing sequence (m_k) such that $x_{k+1}^{(m_k)} \precsim_1 x_{k+1}^{(m_{k+1})}$ and $\sup[x_m] = [(x_k^{(m_k)})]$. Now, as in Lemma 4.3, since $x_k^{(m_k)} \precsim_1 x_{k+1}^{(m_{k+1})}$ there are elements a_{k+1}, b_{k+1} such that $x_{k+1}^{(m_k)} = a_{k+1} x_{k+1}^{(m_{k+1})} b_{k+1}$. Thus

$$x_k^{(m_k)} = y_{k+1}^{(m_k)} x_{k+1}^{(m_k)} x_k^{(m_k)} = y_{k+1}^{(m_k)} a_{k+1} x_{k+1}^{(m_{k+1})} b_{k+1} x_k^{(m_k)}.$$

Therefore, $[(x_k^{m_k})] = [(x_k^{(m_k)} b_k)]$, and the latter belongs to $\mathrm{S}(R)$, as

$$x_k^{(m_k)} b_k \precsim_1 x_{k+1}^{(m_{k+1})} b_{k+1} \text{ and } x_k^{(m_k)} b_k = (y_{k+1}^{(m_k)} a_{k+1})(x_{k+1}^{(m_{k+1})} b_{k+1})(x_k^{(m_k)} b_k).$$

(ii): Let R' be another weakly s-unital ring and let $f\colon R \to R'$ be a ring homomorphism, which we can extend to a homomorphism $M_\infty(R) \to M_\infty(R')$ compatible with $\precsim_1$ and $\oplus$, also denoted by f.

Thus, we obtain a morphism of positively ordered monoids $\mathrm{W}(R) \to \mathrm{W}(R')$ defined by $\mathrm{W}(f)([x]) = [f(x)]$. By the arguments in [3, Paragraph 5.5.3 and Remark 5.5.6], the assigment $\mathrm{Cu}(f)\colon \Lambda(R) \to \Lambda(R')$ defined by $\mathrm{Cu}(f)([(x_n)]) = [(f(x_n))]$ is a Cu-morphism.

By definition $\mathrm{S}(R)$ is the submonoid of $\Lambda(R)$ of elements $[(x_1, x_2, \ldots)]$ such that for each n, there is y_{n+1} satisfying $y_{n+1} x_{n+1} x_n = x_n$. Thus, one has

$$f(y_{n+1}) f(x_{n+1}) f(x_n) = f(x_n)$$

and, consequently, $[(f(x_1), f(x_2), \ldots)] \in \mathrm{S}(R')$. Hence $\mathrm{Cu}(f)(\mathrm{S}(R)) \subseteq \mathrm{S}(R')$, as desired. □

As an immediate consequence, we obtain:

Corollary 5.7. *Let* Rings^{ws} *be the category of weakly s-unital rings and ring homomorphisms. The assignment*

$$\begin{array}{rccc} \mathrm{SCu}\colon & \mathrm{Rings}^{ws} & \longrightarrow & \mathrm{SCu} \\ & R & \mapsto & (\Lambda(R), \mathrm{S}(R)) \end{array}$$

is a functor.

Remark 5.8. In [1] we will see that SCu is not always sequentially continuous. However, $\Lambda(R)$ always is, and SCu ends up being continuous in some relevant situations.

6. Compact elements in $\mathrm{S}(R)$

We have shown in Section 5 that for any ring R we have that $\mathrm{S}(R)$ is a subsemigroup of $\Lambda(R)$. Both semigroups are closed under increasing sequences, and $\Lambda(R)$ is a Cu-semigroup in case R is a weakly s-unital ring. However, the question of whether $\mathrm{S}(R)$ is a Cu-semigroup remains elusive, even in the weakly s-unital case.

In this section we continue our study immersing on the *way-below* relation inherited at $\mathrm{S}(R)$ from $\Lambda(R)$. This helps on characterizing our construction in both the unit-regular and semilocal rings setting.

6.1 (Algebraic Cu-semigroups and compact elements)**.** Recall from Paragraph 2.3 that an element x in a Cu-semigroup S is termed *compact* if $x \ll x$. We also say that such a semigroup S is *algebraic* if every element is the supremum of an increasing sequence of compact elements.

For R be a weakly s-unital ring, if two elements $a, b \in \mathrm{S}(R)$ satisfy $a \ll b$ in $\Lambda(R)$, then $a \ll b$ in $\mathrm{S}(R)$. However, it is unclear when the way-below relation of $\mathrm{S}(R)$ agrees with the one in $\Lambda(R)$. For example, it is conceivable that the object $(\overline{\mathbb{N}}, \{0,\infty\})$ in SCu can be realized as $\mathrm{SCu}(R)$ for a weakly s-unital ring R, where $\infty \ll \infty$ in $\{0,\infty\}$ but $\infty \not\ll \infty$ in $\overline{\mathbb{N}}$.

Keeping this type of examples in mind, for a given weakly s-unital ring R, an element $x \in \mathrm{S}(R)$ is termed compact if $x \ll x$ in $\Lambda(R)$. Further, we will say that $\mathrm{S}(R)$ is *algebraic* if every element in $\mathrm{S}(R)$ can be expressed as the supremum of an increasing sequence of compact elements.

Lemma 6.2. *Let R be a weakly s-unital ring. If $\mathrm{S}(R)$ is algebraic then it is a* Cu*-semigroup. Moreover if $x \ll x$ in $\mathrm{S}(R)$, then $x \ll x$ in $\Lambda(R)$. Therefore, the inclusion $\mathrm{S}(R) \to \Lambda(R)$ is a* Cu*-morphism.*

Proof. The first assertion is clear. Given $x \ll x \in \mathrm{S}(R)$, write $x = \sup_n x_n$ with $x_n \ll x_n \in \Lambda(R)$. This implies that there exists m such that $x \leq x_m \ll x_m \leq x$ and hence $x \ll x$ in $\Lambda(R)$. ☐

This raises the interesting question of characterizing the elements in $\mathrm{S}(R)$ that are compact. To this end, recall that, for elements $[(x_n)], [(y_n)] \in \Lambda(R)$, we have $[(x_n)] \ll [(y_n)]$ in $\Lambda(R)$ if, and only if, there is n_0 such that $x_n \precsim_1 y_{n_0}$ for all n.

A natural source of compact elements of $\mathrm{S}(R)$ comes from the idempotent elements of $M_\infty(R)$. Indeed, if $e \in M_\infty(R)$ is idempotent, let us denote by (e) the constant sequence (in $T(R)$). We clearly have that $[(e)] \in \mathrm{S}(R)$ and, for another idempotent $f \in M_\infty(R)$, it is readily verified that $[(e)] \leq [(f)]$ if, and only if, $e \precsim_{\mathrm{MvN}} f$.

Although not all compact elements in $\mathrm{S}(R)$ come from constant sequences of idempotents, we show below that there is always a representative given by a constant sequence of an *almost* idempotent element.

Lemma 6.3. *Let R be a weakly s-unital ring and let $[(x_n)] \in \mathrm{S}(R)$ be a compact element. Then, there exists $n_0 \geq 1$ such that, for every $k \geq 1$, one can find elements $s_1, \ldots, s_k$ in $M_\infty(R)$ satisfying*

$$x_{n_0} = s_k x_{n_0} \ldots s_1 x_{n_0}.$$

In particular, an element $[(x_n)]$ *is compact if and only if there exist elements* $s, z \in M_\infty(R)$ *such that* $[(x_n)_n] = [(z)_n]$ *and* $z = sz^2$.

Proof. Since $[(x_n)] \in \mathrm{S}(R)$, there are elements y_n such that $x_n = y_{n+1}x_{n+1}x_n$ for all n. If $[(x_n)] \ll [(x_n)]$, this implies that there exists $n_0 \geq 1$ such that

$$x_{n_0+k} \precsim_1 x_{n_0} \text{ for every } k \geq 1.$$

For any given k, let r_k, t_k be such that $x_{n_0+k} = r_k x_{n_0} t_k$. Then, using that

$$x_{n_0} = (y_{n_0+1} \dots y_{n_0+k})x_{n_0+k} \dots x_{n_0+1}x_{n_0},$$

we get

$$x_{n_0} = ((y_{n_0+1} \dots y_{n_0+k})r_k)x_{n_0}(t_k r_{k-1})x_{n_0}(t_{k-1}r_{k-2}) \dots (t_1)x_{n_0}.$$

Thus, if we let $s_k = (y_{n_0+1} \dots y_{n_0+k})r_k$, $s_{k-1} = t_k r_{k-1}$, $s_{k-2} = t_{k-1}r_{k-2}, \dots, s_1 = t_1$, we obtain $x_{n_0} = s_k x_{n_0} \dots s_1 x_{n_0}$, as desired.

In particular, if $k = 2$, set $z = x_{n_0}s_1$ and $s = s_2$. Now

$$z = x_{n_0}s_1 = (s_2 x_{n_0} s_1 x_{n_0})s_1 = s_2 z^2 = sz^2,$$

and clearly the constant sequence $[(z)_n]$ belongs to $\mathrm{S}(R)$. Note that $x_{n_0} = s_2(x_{n_0}s_1)x_{n_0}$ implies $x_{n_0} \precsim_1 x_{n_0}s_1 = z$, and also that $z = sz^2 = s(x_{n_0}s_1)z$. Hence $z \precsim_1 x_{n_0}$. Therefore,

$$x_{n_0+k} \precsim_1 x_{n_0} \precsim_1 x_{n_0}s_1 = z \precsim_1 x_{n_0},$$

which implies that $[(x_n)_n] = [(z)_n]$.

Finally, if $z \in M_\infty(R)$ satisfies $z = sz^2$ for some s, then clearly $z \precsim_1 z$ and therefore $[(z)] \ll [(z)]$. □

Remark 6.4. It is reasonable to extend results such as Lemma 6.3 to general rings. To do this, one may define a transitive relation $\prec$ on $\Lambda(R)$ (or on $\mathrm{S}(R)$) as follows:

$$[(x_n)] \prec [(y_n)] \text{ if and only if there is } m \text{ such that } x_n \precsim_1 y_m \text{ for all } n.$$

Using the construction of suprema in $\Lambda(R)$ (see the proof of Proposition 2.13), it is easy to verify that $\prec$ is formally stronger than the way-below relation $\ll$ on $\Lambda(R)$, and of course it agrees with it in case R is weakly s-unital. Thus, one may term an element $x \in \mathrm{S}(R)$ $\prec$-*compact* in case $x \prec x$.

It is also worth pointing out that the relation $\prec$ is an *auxiliary relation* for the usual order in $\mathrm{S}(R)$ (and also $\Lambda(R)$). Following [19, Definition I-1.11] (see also [3, 2.1.1]), an auxiliary relation is a relation satisfying that $0 \prec x$ for any x, that $x \leq y$ whenever $x \prec y$, and whenever $x \leq y \prec z \leq u$, we have $x \prec u$.

A close inspection of the arguments in Lemma 6.3 reveals that, for any ring R, an element $[(x_n)]$ in $\mathrm{S}(R)$ is $\prec$-compact if and only if $[(x_n)] = [(z)]$, where $z = sz^2$ for some s.

As the example below shows, certain rings have very few $\prec$-compact elements.

Example 6.5. There exists a ring R such that $x \prec x$ in $\mathrm{S}(R)$ if and only if $x = 0$.

Proof. Let R be the ring in Example 4.2. That is, R is the subring of the free algebra $K[x_0, x_1, \ldots]$ on infinitely many variables subject to the non-commutative relations $x_{n+1}x_n = x_n$ consisting of all polynomials with zero constant term.

We claim that the only compact element of $\mathrm{S}(R)$ is 0. To show this, we need some easily proven facts about R. First of all, observe that the set

$$\mathcal{B} = \{x_{i_1}^{n_1} \cdots x_{i_r}^{n_r} \mid n_i \geq 1,\, i_1 < \cdots < i_r,\, r \geq 1\}$$

is a K-basis of R. This follows for instance by an immediate application of the Diamond's Lemma in Ring Theory, see [10], using the reduction system $x_j x_i \mapsto x_i$ for $j > i$.

Hence each element in $M_\infty(R)$ can be uniquely written as a linear combination $\sum_{p \in \mathcal{B}} a_p p$, where $a_p \in M_\infty(K)$. Recall from Example 4.2 the notion of the start $\mathrm{st}(p)$ of a monomial $p \in \mathcal{B}$. The numbers $\mathrm{st}(p)$ satisfy the following properties, some of which have been pointed out in Example 4.2:

(i) $\mathrm{st}(pq) = \min\{\mathrm{st}(p), \mathrm{st}(q)\}$,
(ii) If $\mathrm{st}(p) = \mathrm{st}(q)$ then $pq > p$ and $pq > q$ in the lexicographic order.
(iii) If $\mathrm{st}(p) > \mathrm{st}(q)$ then $pq = q$.

Using these properties we now show that the only compact element of $\mathrm{S}(R)$ is 0. By Lemma 6.3, it is enough to show that the equation $z = sz^2$ has no nonzero solutions in $M_\infty(R)$. Suppose that $z \in M_\infty(R)$ is a nonzero element such that $z = sz^2$. Let p be the unique monomial in $\mathcal{B}$ in the support of z such that $\mathrm{st}(p)$ is maximum amongst all monomials in the support of z, and such that p is the smallest monomial in the support of z amongst all monomials q in the support of z such that $\mathrm{st}(q) = \mathrm{st}(p)$, with respect to the lexicographic order. From the identity $z = sz^2$, it follows that there are two monomials p_1, p_2 in the support of z, and a monomial q in the support of s such that

$$p = qp_1p_2.$$

By (i) we have

$$\mathrm{st}(p) = \min\{\mathrm{st}(q), \mathrm{st}(p_1), \mathrm{st}(p_2)\}.$$

It follows that $\mathrm{st}(p) = \mathrm{st}(p_1) = \mathrm{st}(p_2) \leq \mathrm{st}(q)$. Now by (ii) we have $p_1p_2 > p_i \geq p$ in the lexicographic order, for $i = 1, 2$, and by (ii),(iii) we have, since $\mathrm{st}(q) \geq \mathrm{st}(p_1p_2)$,

$$p = q(p_1p_2) \geq p_1p_2 > p,$$

which is a contradiction.

We remark that the ring R is not weakly s-unital. Indeed, if $x_0 = rx_0s$ for some $r, s \in R$, one easily gets a contradiction expressing r and s in terms of the K-basis $\mathcal{B}$. ☐

Proposition 6.6. *Let R be a unital ring. Then, $\mathrm{S}(R)$ is an algebraic Cu-semigroup whenever every projective module of R is the direct sum of finitely generated modules.*

In particular, this is the case for weakly semi-hereditary rings, one-sided principal ideal rings and $R = C(X)$ for any strongly zero dimensional X.

Proof. Following the observations in Paragraph 6.1, we only need to show that every element in $\mathrm{S}(R)$ is the supremum of an increasing sequence of compact elements. For this, we will use the isomorphism between $\mathrm{S}(R)$ and $\mathrm{CP}(R)$ proved in Theorem 4.10.

First, note that any finitely generated projective module P has an associated sequence in $\mathcal{S}(R)$ of the form $(e, e, e, \dots)$, where $e = e^2 \in M_\infty(R)$. As we have observed before, the class of such a sequence is compact in $\mathrm{S}(R)$, and thus the class $[P]$ is compact in $\mathrm{CP}(R)$.

Now, let P be a countably generated projective module. From our assumptions on R, we may write $P = \oplus F_i$ with F_i finitely generated (and projective). We have $F_1 \precsim F_1 \oplus F_2 \precsim \dots$.

This shows that $[P] = \sup_n [\oplus_{i\leq n} F_i]$ is the supremum of an increasing sequence of compact elements in $\mathrm{CP}(R)$, as desired.

The remaining statement is a consequence of the results in [29]. ☐

Example 6.7. As an example of a ring that does not satisfy the condition in Proposition 6.6, let $R = C[0, 1]$. Then R has an indecomposable, countably projective and pure ideal (which is not finitely generated). See [27, Example 2.12].

We now do a more in-depth study of the semigroup $\mathrm{S}(R)$ when R is a unit-regular ring (Lemma 6.9) and when R is a semilocal ring (Proposition 6.13). Since $\mathrm{S}(R) \cong \mathrm{CP}(R)$, we use these two pictures interchangeably.

6.8 (Unit-regular rings)**.** Recall that a unital ring R is said to be *unit-regular* if for each $x \in R$ there is an invertible element $u \in R$ such that $x = xux$. Unit-regular ring are precisely those unital rings R such that $\mathrm{V}(R)$ is cancellative ([20, Theoremm 4.5]).

We first observe that for a unit-regular ring R, the relation $P \precsim Q$ in $\mathcal{CP}(R)$ is determined solely in terms of isomorphisms of all finitely generated submodules of P with suitable submodules of Q.

Lemma 6.9. *Let R be a unit-regular ring and let P, Q be countably generated projective modules. Then, $P \precsim Q$ if and only if every finitely generated submodule of P is isomorphic to a submodule of Q.*

Proof. Given any unital ring R, it follows from Paragraph 4.4 that whenever $P \precsim Q$, then every finitely generated submodule of P is isomorphic to a submodule of Q.

Conversely, assume that R is a unit-regular ring, and that every finitely generated submodule of P is isomorphic to a (finitely generated) submodule of Q. Then $P \precsim Q$ follows from Paragraph 4.4 and the fact that all finitely generated submodules of Q are direct summands of Q ([20, Theorem 1.11]). ☐

Next example exhibits that above characterization does not hold in general.

Example 6.10. There exist unital commutative domains R and countably generated projective modules P and Q such that all finitely generated submodules of P are isomorphic to submodules of Q but $P \precsim Q$ does not hold.

Proof. Let R be a commutative domain with an indecomposable, countably generated projective module Q, which is not free, and take $P = R_R$. Then obviously R is isomorphic to a submodule of Q. If there is a commutative diagram

$$R \xrightarrow{\phi} Q \xrightarrow{\psi} R, \quad \psi\circ\phi = \mathrm{id}_R$$

then $Q \cong R \oplus Q'$ for some projective module Q', which is impossible since Q is indecomposable and non-free. □

For a unit-regular ring, all semigroups already defined turn out to be isomorphic to either $\mathrm{V}(R)$ or $\mathrm{S}(R)$.

Proposition 6.11. *Let R be a unit-regular ring. Then we have*

(i) $\mathrm{V}(R) = \mathrm{W}(R) = \mathrm{W}_{\mathrm{M}}(R)$ *as ordered monoids, so that the orders defined in* $\mathrm{W}(R)$ *and* $\mathrm{W}_{\mathrm{M}}(R)$ *agree with the algebraic order.*

(ii) $\mathrm{V}^*(R) = \mathrm{CP}(R) = \mathrm{S}(R) = \Lambda_{\mathrm{W}}(R)$ *as semigroups. The order* $\precsim$ *in* $\mathrm{CP}(R)$ *is given by: $P \precsim Q$ if and only if P is isomorphic to a submodule of Q. We have that* $\mathrm{S}(R) = \mathrm{CP}(R) = \Lambda(R)$ *as ordered semigroups.*

Proof. (i): This follows from Lemmas 2.6 and 3.3.

(ii): By [7, Theorem 1.4], there is a monoid isomorphism

$$\Upsilon\colon \mathrm{V}^*(R) \longrightarrow \Lambda_\sigma(\mathrm{V}(R)).$$

(We warn the reader that the monoid $\mathrm{V}^*(R)$ is denoted by $\mathrm{W}(R)$ in [7].) By (i), one has that $\mathrm{V}(R) = \mathrm{W}(R)$, so $\Lambda_\sigma(\mathrm{V}(R)) = \Lambda_\sigma(\mathrm{W}(R)) = \Lambda_{\mathrm{W}}(R)$. The isomorphism Υ satifies that $\Upsilon(P)$ is the interval determined by the increasding sequence $\{[e_1R \oplus \cdots \oplus e_nR] : n \geq 1\}$ in $\mathrm{V}(R)$, where $P = \bigoplus_n e_nR$ and e_n are idempotents of R. Hence Υ factors as the composition of homomorphisms

$$\mathrm{V}^*(R) \xrightarrow{\Phi_R} \mathrm{CP}(R) = \mathrm{S}(R) \xrightarrow{\iota_R} \Lambda_{\mathrm{W}}(R),$$

where Φ_R is the canonical surjective homomorphism, and $\iota_R\colon \mathrm{S}(R) \to \Lambda_{\mathrm{W}}(R)$ is the natural inclusion. It follows that both Φ_R and ι_R are monoid isomorphisms. By [7, Proposition 1.5] the order in $\mathrm{CP}(R)$ is determined by $[P] \leq [Q]$ if and only if P is isomorphic to a submodule of Q. (Observe that this is *not* the algebraic order in $\mathrm{CP}(R)$.) □

6.12 (Semilocal rings)**.** Recall that a unital ring R is said to be semilocal if the quotient $R/J(R)$ is semisimple, i.e. if there exist divison rings $D_1, \ldots, D_r$ such that

$$R/J(R) \cong M_{n_1}(D_1) \times \ldots \times M_{n_r}(D_r).$$

Observe that we have

$$\mathrm{V}^*(R/J(R)) = \mathrm{CP}(R/J(R)) = \mathrm{S}(R/J(R)) = \Lambda_{\mathrm{W}}(R/J(R)) = \overline{\mathbb{N}}^r,$$

where the order here is the algebraic order, or equivalently, the componentwise order. The generators are the isomorphism classes of the simple $R/J(R)$-modules.

Moreover $\mathrm{V}^*(R)$ embeds in $\mathrm{V}^*(R/J(R))$ by Prihoda's Theorem [31, Theorem 2.3].

Note that we also have a surjective homomorphism of ordered monoids

$$\pi\colon \mathrm{W}(R) \to \mathrm{W}(R/J(R)) = \mathrm{V}(R/J(R)) = \mathbb{N}^r,$$

which extends to a surjective homomorphism

$$\pi\colon \Lambda_{\mathrm{W}}(R) = \Lambda_\sigma(\mathrm{W}(R)) \to \Lambda_{\mathrm{W}}(R/J(R)) = \overline{\mathbb{N}}^r.$$

Now we characterize the equivalence relation $\sim$ on $\mathcal{CP}(R)$ using the notion of dimension in the case of semilocal rings ([31]). Recall that we define $\dim(P) = (x_1, \dots, x_r) \in \overline{\mathbb{N}}^r$, where $(x_1, \dots, x_r)$ is the image of $[P]$ under the map $\mathrm{V}^*(R) \to \overline{\mathbb{N}}^r$. Further, given two countably generated projective right R-modules P, Q, we say that $\dim(P) \leq \dim(Q)$ if the corresponding tuples compare componentwise. In this case, there exists a split $R/J(R)$-monomorphism $P/PJ(R) \to Q/QJ(R)$.

Using the remarks above, we can easily characterize the order relation in $\mathrm{CP}(R)$ in the case of a semilocal ring:

Proposition 6.13. *Let R be a semilocal ring and let P, Q be two countably generated right R-modules. The following are equivalent:*
(i) $P \precsim Q$
(ii) $\dim(P) \leq \dim(Q)$ *(component-wise)*
(iii) *P is isomorphic to a pure submodule of Q.*

Proof. (i) $\implies$ (ii): Let $\mathbf{x} = (x_1, \dots, x_r) \in \mathbb{N}^r$ such that $\mathbf{x} \leq \dim(P)$. Let $\iota_R\colon \mathrm{CP}(R) \to \Lambda_{\mathrm{W}}(R)$ be the canonical inclusion. Then $\iota_R([P]) \subseteq \iota_R([Q])$ (as elements in $\Lambda_\sigma(\mathrm{W}(R))$). Take $z \in \mathrm{W}(R)$ such that $z \in \iota_R([P])$ and $\pi(z) = \mathbf{x}$. Then $z \in \iota_R([Q])$, which means that $\pi(z) \leq \pi(\iota_R[Q]) = \pi([Q]) = \dim(Q)$. It follows that $\dim(P) \leq \dim(Q)$.

(ii) $\implies$ (iii): Assume that $\dim(P) \leq \dim(Q)$. Then, one can construct a split monomorphism $\bar{r}\colon P/PJ(R) \to Q/QJ(R)$. Let $\bar{s}\colon Q/QJ(R) \to P/PJ(R)$ be such that $\bar{s}\bar{r} = \mathrm{id}_{P/PJ(R)}$.

Since P, Q are projective, both maps can be lifted to $r\colon P \to Q$ and $s\colon Q \to P$.

Moreover, since $\bar{s}\bar{r} = \mathrm{id}_{P/PJ(R)}$, we know from [31, Lemma 2.1], applied to sr, that for any finite subset $X \subseteq P$ there exists a morphism $g\colon P \to P$ such that $gsr(x) = x$ for every $x \in X$. Hence $h := gs\colon Q \to P$ satisfies that $hr(x) = x$ for all $x \in X$. It follows that r is injective and $r(P)$ is pure in Q (see [27, Exercise §4.38]).

(iii) $\implies$ (i): See [27, Exercise §4.41]. □

Corollary 6.14. *Let P, Q be two countably generated projective right modules over a unital semilocal ring R. Then, $P \sim Q$ if and only if $P \cong Q$. That is, $\mathrm{CP}(R) \cong \mathrm{V}^*(R)$ as semigroups.*

Corollary 6.15. *Let R be a unital semilocal ring. Then, $\mathrm{CP}(R)$ can be embedded into $\overline{\mathbb{N}}^r$ as a partially ordered monoid.*

7. C*-algebras

Let A be a C*-algebra. In this section we explore the relationship between the Cuntz semigroup $\mathrm{Cu}(A)$ of A and the semigroup $\mathrm{S}(A)$. We show in Theorem 7.6

that $\mathrm{Cu}(A)$ is a retract of $\mathrm{S}(A)$, in the sense that there is an ordered monoid morphism $\mathrm{Cu}(A) \to \mathrm{S}(A)$ that preserves suprema, compact containment, and has a left inverse that preserves suprema.

Remark 7.1. Given $f, g \in C([0,\infty))$ such that $\mathrm{supp}(f) = (\varepsilon, \infty)$ and $\mathrm{supp}(g) = (\varepsilon', \infty)$ with $\varepsilon' < \varepsilon$, there exists $r \in C([0,\infty))$ such that $f(t) = r(t)g(t)f(t)$ for each $t \in [0,\infty)$. In particular, we have $f \precsim_1 g$. In general, if $\mathrm{supp}(f) \subseteq \mathrm{supp}(g)$, then $f \precsim g$ (see, e.g. [8, Proposition 2.5])

7.2 (Dense subsemigroups)**.** Let S be a Cu-semigroup. We will say that a subsemigroup H of S is a *dense subsemigroup* provided that whenever $x \ll y$ in S there exists $s \in H$ such that $x \leq s \leq y$.

For example, if S is algebraic, then the subsemigroup S_c, consisting of the compact elements in S, is a dense subsemigroup.

Lemma 7.3. *Let S be a* Cu*-semigroup, T a positively ordered monoid where suprema of increasing sequences exist and are compatible with addition. Let $H \subseteq S$ be a dense submonoid. Then, for any ordered monoid morphism $\varphi\colon H \to T$ that preserves suprema of increasing sequences with supremum in H, there exists an ordered monoid morphism $\phi\colon S \to T$ that preserves suprema of increasing sequences and $\phi|_H = \varphi$.*

Proof. Given any $s \in S$, write $s = \sup s_n$ where (s_n) is a rapidly increasing sequence of elements in S. Since by assumption H is dense in S, there is for each n an element $s'_n \in H$ such that $s_n \leq s'_n \leq s_{n+1}$. Thus we may assume that $s_n \in H$ for all n. Define $\phi(s) := \sup_n \varphi(s_n)$.

If (t_n) is another rapidly increasing sequence of elements in H such that $\sup s_n \leq \sup t_n$, then for any n, there is m with $s_n \leq t_m$, whence $\sup_n \varphi(s_n) \leq \sup_n \varphi(t_n)$. This implies that ϕ is well-defined and order-perserving. To see that ϕ preserves addition, let $s, t \in S$ and write $s = \sup_n s_n$, $t = \sup_n t_n$, for rapidly increasing sequences (s_n) and (t_n) in H. Thus, using our assumption on T, we get

$$\phi(s+t) = \sup_n (\varphi(s_n) + \varphi(t_n)) = \sup_n \varphi(s_n) + \sup_n \varphi(t_n) = \phi(s) + \phi(t),$$

and therefore ϕ is an ordered monoid morphism.

Further, note that for every $h \in H$, we can write $h = \sup_n h_n$ for a rapidly increasing sequence (h_n) of elements in H. Since φ preserves the supremum of such sequences, we have $\phi(h) = \sup_n \varphi(h_n) = \varphi(h)$, and thus $\phi|_H = \varphi$.

To see that ϕ preserves suprema, fix $s \in S$ and let (t_n) be an increasing sequence in S with supremum s. Choose (s_n) to be a rapidly increasing sequence of elements in H with supremum s. Since ϕ is order-preserving, we have that $\sup_n \phi(t_n) \leq \phi(s)$. Also, for every n, there is m with $s_n \leq t_m$. Using that $\phi|_H = \varphi$ and that ϕ is order-preserving, we obtain $\varphi(s_n) = \phi(s_n) \leq \phi(t_m) \leq \sup_k \phi(t_k)$, which implies that $\phi(s) \leq \sup_k \phi(t_k)$ as required. □

7.4 (Retracts)**.** Let S be a Cu-semigroup, and let T be a positively ordered semigroup admitting suprema of increasing sequences, which are compatible with addition. Adapting the definition introduced in [33, Definition 3.14], we shall say that S is a *retract* of T if there exist ordered monoid morphisms $\varphi\colon S\to T$ and $\phi\colon T\to S$ such that

(i) φ preserves suprema and compact containment.
(ii) ϕ preserves suprema.
(iii) $\phi\varphi=\mathrm{id}_S$.

Given $\varepsilon>0$, we shall denote by $f_\varepsilon\in C([0,\infty))$ the continuous function that is 0 on $[0,\varepsilon)$, linear on $[\varepsilon,2\varepsilon]$, and 1 elsewhere. Notice that, for each positive element a in a C*-algebra A, the element $f_\varepsilon(a)$ has a unit in A, namely $f_{\varepsilon/2}(a)f_\varepsilon(a)=f_\varepsilon(a)$.

Lemma 7.5. *Let A be a* C**-algebra and let $a,b\in M_\infty(A)_+$ be such that $a\precsim b$ in $A\otimes\mathcal{K}$. Put $\varepsilon_n=1/2^n$. Then, for any n, there is m such that $f_{\varepsilon_n}(a)\precsim_1 f_{\varepsilon_m}(b)$.*

Proof. Since $a\precsim b$ in $A\otimes\mathcal{K}$, given $n\in\mathbb{N}$, there exist $\delta_n>0$ and $r_n\in A\otimes\mathcal{K}$ such that

$$(a-\varepsilon_{n+1})_+=r_n(b-\delta_n)_+r_n^*$$

(see, for instance, [8, Proposition 2.17]). As $a,(b-\delta_n)_+\in M_\infty(A)$, we can take $r_n\in M_\infty(A)$ as well. Since $\delta_n>0$, there exists m such that $f_{\varepsilon_m}(b)$ is a unit for $(b-\delta_n)_+$. Using this observation at the third step and Remark 7.1 at the first step, one gets

$$f_{\varepsilon_n}(a)\precsim_1(a-\varepsilon_{n+1})_+=r_n(b-\delta_n)_+f_{\varepsilon_m}(b)r_n^*\precsim_1 f_{\varepsilon_m}(b). \qquad\square$$

Theorem 7.6. *Let A be a* C**-algebra. Then* $\mathrm{Cu}(A)$ *is a retract of* $\mathrm{S}(A)$.

Proof. Let $H=\{x\in\mathrm{Cu}(A)\colon x=[a]\text{ for some }a\in M_\infty(A)_+\}$, which is a dense subsemigroup of $\mathrm{Cu}(A)$. We will apply Lemma 7.3 to H. To this end we need to construct a positively ordered monoid morphism $\varphi_0\colon H\to\mathrm{S}(A)$ that preserves suprema of increasing sequences.

Let $x\in H$, and let $a\in M_\infty(A)_+$ be such that $x=[a]$. For every $n\geq1$, put $\varepsilon_n=1/2^n$ and f_{ε_n} as in Lemma 7.5. The elements of the sequence $(f_{\varepsilon_n}(a))$ are pairwise commuting and, as observed before, $f_{\varepsilon_{n+1}}(a)f_{\varepsilon_n}(a)=f_{\varepsilon_n}(a)$ for all n. Thus $f_{\varepsilon_n}(a)\precsim_1 f_{\varepsilon_{n+1}}(a)$ for all n, and so $(f_{\varepsilon_n}(a))_n\in\mathcal{S}(M_\infty(A))$.

Define $\varphi_0\colon H\to\mathrm{S}(A)$ by $\varphi_0([a])=[(f_{\varepsilon_n}(a))_n]$. To see that φ_0 is well defined and order-preserving, let $b\in M_\infty(A)_+$ be such that $a\precsim b$ in $A\otimes\mathcal{K}$. By Lemma 7.5, for each n there is m with $f_{\varepsilon_n}(a)\precsim_1 f_{\varepsilon_m}(b)$, which implies that $[(f_{\varepsilon_n}(a))_n]\leq[(f_{\varepsilon_n}(b))_n]$.

It is easy to verify that φ_0 is additive, and thus it is a positively ordered monoid morphism. In order to extend φ_0 to a positively ordered monoid morphism $\varphi\colon\mathrm{Cu}(A)\to\mathrm{S}(A)$ that preserves suprema of increasing sequences, we apply Lemma 7.3. Thus, we only need to prove that the map $\varphi_0\colon H\to\mathrm{S}(A)$ preserves suprema of increasing sequences with supremum in H.

To this end, let $(a_n)_n,a\in M_\infty(A)_+$ be such that $([a_n])_n$ is increasing and $[a]=\sup_n[a_n]$ in $\mathrm{Cu}(A)$. Since φ_0 is order-preserving, we already have $\sup_n\varphi_0([a_n])\leq\varphi_0([a])$. To show the converse inequality, note that for every $n\geq1$ one has $[(a-\varepsilon_{n+1})_+]\ll[a]$ and, consequently, $(a-\varepsilon_{n+1})_+\precsim a_i$ for some i. By Lemma 7.5, there is m such that $f_{\varepsilon_{n+2}}((a-\varepsilon_{n+1})_+)\precsim_1 f_{\varepsilon_m}(a_i)$. Let $g_{\varepsilon_n}(t)=(t-\varepsilon_n)_+$. By

Remark 7.1, we have $f_{\varepsilon_n} \precsim_1 f_{\varepsilon_{n+2}} \circ g_{\varepsilon_{n+1}}$. Thus,

$$f_{\varepsilon_n}(a) \precsim_1 f_{\varepsilon_{n+2}}((a-\varepsilon_{n+1})_+) \precsim_1 f_{\varepsilon_m}(a_i).$$

This shows that $\varphi_0([a]) \leq \sup_n \varphi_0([a_n])$, as every element in $(f_{\varepsilon_n}(a))_n$ is $\precsim_1$-bounded by an element in $(f_{\varepsilon_m}(a_i))_m$ for some i. Therefore, $\varphi_0([a]) = \sup_n \varphi_0([a_n])$, as was to be shown, and we have an extension $\varphi\colon \mathrm{Cu}(A) \to \mathrm{S}(A)$ that preserves suprema of increasing sequences.

Next, to prove that φ preserves the way-below relation, take first $[a],[b] \in \mathrm{Cu}(A)$ with $a,b \in M_\infty(A)_+$ and suppose that $[a] \ll [b]$ in $\mathrm{Cu}(A)$. Then there is $\varepsilon > 0$ such that $a \precsim (b-\varepsilon)_+$. Since again by Remark 7.1, we have $f_{\varepsilon_m} \circ g_\varepsilon \precsim_1 f_\varepsilon$ for each m, another usage of Lemma 7.5 implies that, for each $n \geq 1$, there is $m \geq 1$ such that

$$f_{\varepsilon_n}(a) \precsim_1 f_{\varepsilon_m}((b-\varepsilon)_+) \precsim_1 f_\varepsilon(b),$$

and therefore $\varphi_0([a]) = [(f_{\varepsilon_n}(a))_n] \ll [(f_{\varepsilon_m}(b))_m] = \varphi_0([b])$. If now $a,b \in (A\otimes\mathcal{K})_+$ satisfy $[a] \ll [b]$ in $\mathrm{Cu}(A)$, then as before there is $\varepsilon > 0$ such that $a \precsim (b-\varepsilon)_+$. Note that there is $b_\varepsilon \in M_\infty(A)_+$ such that $(b-\varepsilon)_+ \sim b_\varepsilon$, and since $[b_\varepsilon] \ll [b_{2\varepsilon}]$, we have

$$\varphi([a]) \leq \varphi([(b-\varepsilon)_+]) = \varphi_0([b_\varepsilon]) \ll \varphi_0([b_{2\varepsilon}]) \leq \varphi([b]).$$

To finish the proof, we have to construct an ordered monoid morphism $\phi\colon \mathrm{S}(A) \to \mathrm{Cu}(A)$ that preserves suprema of increasing sequences and is a left inverse for φ. To do this, let $(a_n)_n \in \mathcal{S}(M_\infty(A))$. By Lemma 2.10, the sequence $(a_n^*a_n)_n$ is $\precsim$-increasing and we can consider $[a] = \sup_n [a_n^*a_n]$ in $\mathrm{Cu}(A)$.

Define $\phi\colon \mathrm{S}(A) \to \mathrm{Cu}(A)$ by $\phi([(a_n)_n]) = \sup_n [a_n^*a_n]$. Let $[(a_n)],[(b_n)] \in \mathrm{S}(A)$ be such that $[(a_n)_n] \leq [(b_n)_n]$ in $\mathrm{S}(A)$. Then, for each n, there is m such that $a_n \precsim_1 b_m$. Again by Lemma 2.10, this implies that $a_n^*a_n \precsim b_m^*b_m$. Therefore, if we put $[a] = \sup_n [a_n^*a_n]$ and $[b] = \sup_n [b_n^*b_n]$ in $\mathrm{Cu}(A)$, we obtain that $[a] \leq [b]$. This shows that ϕ is well defined and order-preserving. It is easy to verify that ϕ is also additive, hence a positively ordered monoid morphism.

Now, given an increasing sequence $([(a_{i,n})_i])_n$ in $\mathrm{S}(A)$, there is a subsequence (n_i) of the natural numbers and elements $r_i \in A$ such that $\sup_n [(a_{i,n})_i] = [(a_{i,n_i} r_i)_i]$ (see Lemma 4.3). Let $[a_n] = \phi((a_{i,n})_i)$. We have, for each i,

$$r_i^* a_{i,n_i}^* a_{i,n_i} r_i \precsim a_{i,n_i}^* a_{i,n_i} \precsim a_{n_i},$$

and thus $[r_i^* a_{i,n_i}^* a_{i,n_i} r_i] \leq [a_{n_i}] \leq \sup_n [a_n]$. Therefore

$$\phi(\sup_n [(a_{i,n})_i]) = \phi([(a_{i,n_i} r_i)_i]) = \sup_i [r_i^* a_{i,n_i}^* a_{i,n_i} r_i] \leq \sup_n [a_n] = \sup_n \phi([(a_{i,n})_i]).$$

Since ϕ is order-preserving we always have $\sup_n \phi([(a_{i,n})_i]) \leq \phi(\sup_n [(a_{i,n})_i])$, and thus $\phi(\sup_n [(a_{i,n})_i]) = \sup_n \phi([(a_{i,n})_i]$, as required.

By construction, ϕ is a left-inverse for $\varphi_0 = \varphi|_H$. By definition of φ and since ϕ preserves suprema of increasing sequences, it follows that ϕ is a left-inverse for φ. □

7.7 (Hilbert C*-modules)**.** For a C*-algebra A, we consider the class $\mathcal{CH}(A)$ of countably generated Hilbert A-modules, see for instance [28] for definitions and background. Let H_A be the Hilbert A-module consisting of sequences (a_n) of elements in A such that $\sum_{n=1}^{\infty} a_n^* a_n$ is norm-converging in A. Note that H_A is the Hilbert A-module completion of the A-module $A^{(\mathbb{N})}$. By Kasparov's Theorem (see e.g. [28, Theorem 1.4.2]) each countably generated Hilbert A-module is isometrically isomorphic to a complemented A-submodule of H_A.

Denote by $\mathcal{K}(X)$ the C*-algebra of compact operators on a Hilbert A-module X. If $X \subseteq Y$ are Hilbert A-modules, we say that X is *compactly contained* in Y if there exists a self-adjoint compact operator $\theta \in \mathcal{K}(Y)$ such that $\theta|_X = \mathrm{id}_X$. Given Hilbert A-modules X and Y, we say that X is *Cuntz subequivalent* to Y, written $X \precsim Y$, if each Hilbert submodule X_0 of X which is compactly contained in X is isometrically isomorphic to a Hilbert module Y_0 which is compactly contained in Y. We say that X and Y are *Cuntz equivalent*, written $X \sim Y$, if $X \precsim Y$ and $Y \precsim X$. The semigroup $\mathrm{CH}(A)$ is then the semigroup of Cuntz equivalence classes of countably generated Hilbert A-modules, endowed with the operation induced by the direct sum of Hilbert A-modules.

It was shown in [14] that there is an isomorphism $\mathrm{Cu}(A) \cong \mathrm{CH}(A)$ in the category Cu. This isomorphism sends the class of a positive element a in $A \otimes \mathcal{K}$ to $\overline{a(H_A)}$, where we use the isomorphism $A \otimes \mathcal{K} \cong \mathcal{K}(H_A)$, see e.g. [8, Proposition 3.15(iii)].

Let A be a C*-algebra. Then we have, on the one hand, an isomorphism $\gamma_c\colon \mathrm{Cu}(A) \cong \mathrm{CH}(A)$, and on the other hand an isomorphism $\gamma_a\colon \mathrm{S}(A) \cong \mathrm{CP}(A)$ by Theorem 4.13, where $\mathrm{CP}(A)$ is built from the category $\mathcal{CP}(A)$ of countably generated projective unital right A^+-modules P such that $P = PA$, see Paragraph 4.11.

Hence there is a unique morphism $\tilde{\phi}\colon \mathrm{CP}(A) \to \mathrm{CH}(A)$ making commutative the following diagram:

$$\begin{array}{ccc} \mathrm{Cu}(A) & \xrightarrow{\gamma_c} & \mathrm{CH}(A) \\ {\scriptstyle\phi}\uparrow & & \uparrow{\scriptstyle\tilde{\phi}} \\ \mathrm{S}(A) & \xrightarrow{\gamma_a} & \mathrm{CP}(A) \end{array}$$

namely $\tilde{\phi} = \gamma_c \circ \phi \circ \gamma_a^{-1}$.

Proposition 7.8. *Let P be an object in $\mathrm{CP}(A)$, and let $x_n \in M_\infty(A)$ be a sequence such that $x_{n+1}x_n = x_n$ for each $n \geq 1$ and $P \cong Q := \bigcup_{i=1}^{\infty} x_n A^{(\mathbb{N})}$. We then have*

$$\tilde{\phi}([P]) = \tilde{\phi}([Q]) = [\overline{Q}],$$

where $\overline{Q}$ is the Hilbert A-module obtained by taking the closure of Q in H_A.

Proof. We first observe that a sequence (x_n) as in the statement always exists by Corollary 4.14. Given such a sequence (x_n), we have $x_nH_A \subseteq x_{n+1}H_A$ and in particular $\overline{x_nH_A} \subseteq \overline{x_{n+1}H_A}$, so that by [8, Proposition 4.12] we have

$$\sup_n [\overline{x_nH_A}] = [\overline{\cup_{n=1}^{\infty} \overline{x_nH_A}}]$$

in $\mathrm{CH}(A)$. On the other hand, by [8, Lemma 4.10], we have

$$\overline{x_n^* x_n H_A} \cong \overline{x_n x_n^* H_A} = \overline{x_n H_A}$$

for all $n \geq 1$. Therefore we get $\gamma_c([x_n^* x_n]) = [\overline{x_n H_A}]$. Using that γ_c preserves suprema of increasing sequences, we have

$$\begin{aligned}\gamma_c \circ \phi \circ \gamma_a^{-1}([Q]) = \gamma_c \circ \phi([(x_n)]) = \gamma_c(\sup_n [x_n^* x_n]) = \sup_n \gamma_c([x_n^* x_n]) \\ = \sup_n [\overline{x_n H_A}] = [\overline{\cup_{n=1}^\infty \overline{x_n H_A}}] = [\overline{Q}],\end{aligned}$$

as desired. □

8. Nearly simple domains

In this section we study nearly simple domains, a class of rings where one can explicitly compute the monoid $\mathrm{W}(R)$; see Paragraph 8.1.

As we will prove in Proposition 8.3, the Jacobson radical J of any nearly simple domain R is always weakly s-unital, although it is not s-unital in general. The invariants $\Lambda(J)$ and $\mathrm{S}(J)$ are computed in Theorem 8.4 and Remark 8.6 respectively.

8.1 (Uniserial domains and nearly simple domains)**.** Recall that a module over a ring R is *uniserial* if its submodules are totally ordered by inclusion, and that the ring R is said to be *right uniserial* if it is uniserial as a right module over itself.

One defines left uniserial rings analoguously, and says that R is *uniserial* if it is both right and left uniserial. Uniserial rings will be assumed to be unital throughout the section.

Note that any right uniserial domain R is a local ring. That is, R has a unique maximal left ideal. The reader is referred to [16] for a thorough exposition.

Let R be a uniserial domain. We will say that R is a *nearly simple domain* if R is not simple and the only two-sided ideals of R are $\{0\}$, $J(R)$ and R.

Given elements r, s in a unital, uniserial ring R, it is well-known that $RrR = RsR$ if and only if there exist units $u, v \in R$ such that $r = usv$; see [32, Lemma 4.2].

In particular, if R is a nearly simple domain, this implies that $J := J(R)$ is a 1-*simple ring*, that is, for each $r, s \in J \setminus \{0\}$ there exist $a, b \in J$ such that $r = asb$ (see [13], where the concept of an n-simple ring is introduced for unital rings, for every $n \geq 1$). Indeed, applying [32, Lemma 4.2] to r, s^3, we obtain units $u, v \in R$ such that

$$r = us^3v = (us)s(sv).$$

The elements $a := us$ and $b := vs$ are in J and satisfy the desired equality.

As shown in [5, Theorem 2.10], every regular square matrix over an exchange separative ring can be diagonalized by using row and column elementary transformations. Since local rings are separative exchange rings, this applies in particular to any uniserial ring. We will see in Lemma 8.2 below that *all* square matrices over a uniserial ring are equivalent to diagonal matrices. This will allow us to compute $\mathrm{W}(R)$ for a nearly simple domain R in Theorem 8.4. Note that there exist artinian local commutative rings R such that some 2×2 matrices over R are not diagonalizable (see [5, Remark 2.12]).

Let us denote by $E_n(R)$ the set of $n \times n$ elementary matrices.

Lemma 8.2. *Let R be a uniserial ring. Then, for each square matrix $A \in M_n(R)$ there exist elementary matrices $U, V \in E_n(R)$ such that UAV is diagonal.*

Proof. We proceed by induction on n and note that the case $n = 1$ is trivial.

Thus, let $n > 1$ be fixed and assume that we have proven the result for every $k \times k$ matrix with $k \leq n - 1$.

Take $A \in M_n(R)$. If A has an invertible entry, we can move such entry to the position $(1, 1)$ by means of elementary transformations. Further, since this entry is now invertible, there exist elementary matrices U, V such that the product $A' = UAV$ satisfies $A'(1, i) = A'(i, 1) = 0$ for all $i > 1$. The desired result now follows by induction.

Thus, it remains to consider the case $A \in M_n(J)$, where we will show by induction on k that there exist elementary matrices $U_k, V_k \in E_n(R)$ such that the product

$$B_k = U_k A V_k$$

satisfies $B_k(i, j) = 0$ for every pair (i, j) such that $i \leq k$ and $i \neq j$. That is, B_k is of the form

$$B_k = \left(\begin{array}{ccc|c} B_k(1,1) & & 0 & \\ & \ddots & & 0 \\ 0 & & B_k(k,k) & \\ \hline \multicolumn{4}{c}{C_k} \end{array}\right)$$

for some matrix C_k.

If $k = 1$, use that R is uniserial to find $i \geq 1$ such that $A(1, j)R \subseteq A(1, i)R$ for every j. Using elementary transformations, we may assume that $i = 1$. This shows that A can be transformed into a matrix B_1 satisfying the required conditions.

Now fix $k < n$ and assume that we have proven the result for every $k' \leq k$. In particular, we can find $U_k, V_k \in E_k(R)$ such that $U_k A V_k = B_k$.

Then, for every $i \leq k$, we either have that $RB_k(k + 1, i) \subseteq RB_k(i, i)$ or $RB_k(i, i) \subseteq RB_k(k + 1, i)$. Performing elementary row operations, we may assume that $B_k(k + 1, i) = 0$ whenever $RB_k(k + 1, i) \subseteq RB_k(i, i)$.

Let k' be such that $B_k(k + 1, i)R \subseteq B_k(k + 1, k')R$ for every i. We may assume that $B_k(k + 1, k') \neq 0$, since we are done otherwise.

If $k' \geq k + 1$, we can perform elementary column operations in order to get $k' = k + 1$. Using once again column operations, we obtain a matrix of the form

$$\left(\begin{array}{cccc|c} B_k(1,1) & & & 0 & \\ & \ddots & & & 0 \\ & & B_k(k,k) & & \\ 0 & & & B_k(k+1,k+1) & \\ \hline \multicolumn{5}{c}{C_{k+1}} \end{array}\right)$$

for some C_{k+1}, as desired.

Finally, assume that $k' \leq k$. Since $B_k(k + 1, k') \neq 0$, we have

$$RB_k(k', k') \subseteq RB_k(k + 1, k').$$

Let B' be the matrix resulting from adding a multiple of the $(k+1)$-th row to the k'-th row in such a way that $B'(k',k')=0$.

Performing elementary column operations, we obtain yet another matrix B'' with $B''(k+1,i)=0$ for every $i\neq k'$.

Swapping the k'-th row with the $(k+1)$-th row, we find a matrix B'_k with $B'_k(i,j)=0$ for every (i,j) such that $i\leq k$ and $i\neq j$. Further, B'_k has at least one more zero than B_k in the $(k+1)$-th row. Proceeding by induction, we get matrices U_{k+1}, V_{k+1} and B_{k+1} with the desired properties. This finishes the inductive argument.

Since B_n is a diagonal matrix, the matrices $U:=U_n$ and $V:=V_n$ satisfy the required conditions. This finishes the proof. ☐

Let $(M,\leq)$ be a partially ordered monoid, and let I be a submonoid of M. Recall that I is said to be an *o-ideal* of M if I is hereditary for $\leq$, that is, if whenever $x\leq y$ with $y\in I$ we have $x\in I$.

Proposition 8.3. *Let R be a nearly simple domain. Then $J(R)$ is a weakly s-unital ring, and there is an order-embedding of* $\mathrm{W}(J(R))$ *into an o-ideal of* $\mathrm{W}(R)$.

Proof. Set $J:=J(R)$ and take $A\in M_n(J)$ for some $n\geq 1$. We have to show that there exist matrices $X,Y\in M_n(J)$ such that $A=XAY$. By Lemma 8.2 there exist elementary matrices $U,V\in E_n(R)$ such that $UAV=D$, where D is diagonal matrix in $M_n(R)$. Note that, since all the entries of A belong to J, we have $D\in M_n(J)$.

Further, we know from Paragraph 8.1 that J is a 1-simple ring. Thus, we can find diagonal matrices $Z,T\in M_n(J)$ satisfying $D=ZDT$. This implies

$$A=U^{-1}DV^{-1}=U^{-1}ZDTV^{-1}=(U^{-1}ZU)A(VTV^{-1})$$

and, consequently, the matrices $X:=U^{-1}ZU$ and $Y:=VTV^{-1}$ are in $M_n(J)$ and satisfy $A=XAY$, as desired.

It follows from Paragraph 2.5 that we can form the semigroup $\mathrm{W}(J)$, which is a positively ordered monoid.

The inclusion map $J\to R$ induces a positively ordered monoid-morphism $\mathrm{W}(J)\to\mathrm{W}(R)$. To see that it is an order-embedding, let $A,B\in M_n(J)$ and assume that $A\precsim_1 B$ in $M_n(R)$. Let $P,Q\in M_n(R)$ be such that $A=PBQ$. Using the first part of the proposition, we obtain elements $X,Y\in M_n(J)$ such that $A=XAY$, and hence $A=(XP)B(QY)$.

This shows that $A\precsim_1 B$ in $M_n(J)$ and, therefore, that the map $\mathrm{W}(J)\to\mathrm{W}(R)$ is an order-embedding.

Identifying $\mathrm{W}(J)$ with its image, it is readily checked that $\mathrm{W}(J)$ is an o-ideal of $\mathrm{W}(R)$. ☐

Theorem 8.4. *Let R be a nearly simple domain, and let J be its Jacobson radical. Then,*

(i) $\mathrm{W}(J)\cong\mathbb{N}$*, with its usual order.*
(ii) $\mathrm{W}(R)\cong\mathbb{N}\times\mathbb{N}$*, with the order*

$$(r',s')\leq(r,s):\iff r'+s'\leq r+s\quad\text{and}\quad r'\leq r.$$

Proof. Take $A \in M_n(R)$. Using Lemma 8.2, we find invertible matrices U, V and a diagonal matrix $D \in M_n(R)$ such that $UAV = D$. We may assume that D is of the form

$$D = \operatorname{diag}(d_1, \dots, d_r, d_{r+1}, \dots, d_{r+s}, 0, \dots, 0)$$

for some $d_1, \dots, d_r \in R \setminus J$ and $d_{r+1}, \dots, d_{r+s} \in J \setminus \{0\}$. Let

$$\psi\colon M_\infty(R) \to \mathbb{N} \times \mathbb{N}$$

be the map defined by $\psi(A) := (r, s)$.

To see that $\psi(A)$ does not depend on the choice of U and V, set

$$C := R/d_{r+1}R \oplus \ldots \oplus R/d_{r+s}R \oplus R^{n-r-s}$$

and consider the commutative diagram

$$\begin{array}{ccccccc}
R^n & \xrightarrow{A} & R^n & \longrightarrow & R^n/AR^n & \longrightarrow & 0 \\
V \big\uparrow \cong & & U \big\downarrow \cong & & \big\downarrow \cong & & \\
R^n & \xrightarrow{D} & R^n & \longrightarrow & C & \longrightarrow & 0
\end{array}$$

Choosing $a \in J \setminus \{0\}$, we have $C \cong (R/aR)^s \oplus R^{n-r-s}$ (by [32, Lemma 4.2]). Thus, for any choice of invertible matrices U', V' and diagonal matrix D' such that $D' = U'AV'$ with ranks (r', s'), one gets

$$(R/aR)^s \oplus R^{n-r-s} \cong (R/aR)^{s'} \oplus R^{n-r'-s'}.$$

At this point we can use Puninski's Theorem [16, Theorem 9.19] asserting that every finitely presented right module M over a uniserial ring is the direct sum of cyclic uniserial modules, and any two decompositions of M as direct sums of cyclic modules are isomorphic. Using this result we immediately deduce that $s = s'$ and $n - r - s = n - r' - s'$, and thus $r = r'$.

Next we show that $A \precsim_1 B$ implies $\psi(A) \leq \psi(B)$, where recall that

$$(r', s') \leq (r, s) :\iff r' + s' \leq r + s \quad \text{and} \quad r' \leq r.$$

Thus, let A, B be such that $A \precsim_1 B$, and write $\psi(A) = (r_A, s_A)$ and $\psi(B) = (r_B, s_B)$. We may assume that $A, B \in M_n(R)$, and that there are matrices $X, Y \in M_n(R)$ such that $A = XBY$.

Let π be the quotient map $R \to R/J$. We have

$$\pi(A) = \pi(X)\pi(B)\pi(Y) \text{ in } M_n(R/J)$$

and, therefore,

$$r_A = \operatorname{rank}_{R/J}(\pi(A)) \leq \operatorname{rank}_{R/J}(\pi(B)) = r_B.$$

Let us now prove that $\psi(BY) \leq \psi(B)$ for all $B, Y \in M_n(R)$. Following the notation above, we write $\psi(B) = (r_B, s_B)$ and $\psi(BY) = (r_{BY}, s_{BY})$.

The previous argument shows that $r_{BY} \leq r_B$, so it remains to check that

$$r_{BY} + s_{BY} \leq r_B + s_B.$$

Since R^n/BR^n is a quotient of $R^n/(BY)R^n$, we obtain a surjective module homomorphism

$$(R/aR)^{s_{BY}} \oplus R^{n-r_{BY}-s_{BY}} \longrightarrow R^{n-r_B-s_B},$$

where $a \in J \setminus \{0\}$. Using that $R^{n-r_B-s_B}$ is free, we find a right R-module M such that

$$(R/aR)^{s_{BY}} \oplus R^{n-r_{BY}-s_{BY}} \cong R^{n-r_B-s_B} \oplus M.$$

By Puninski's Theorem [16, Theorem 9.19] we have that $n - r_{BY} - s_{BY} \geq n - r_B - s_B$. Thus, we get $r_{BY} + s_{BY} \leq r_B + s_B$, as desired.

Note that, by symmetry, we have $\psi(XB) \leq \psi(B)$ for all $X, B \in M_n(R)$. Consequently, one gets $\psi(A) \leq \psi(B)$ whenever $A \precsim_1 B$.

Conversely, it is also easy to see (by looking at their associated diagonal matrices) that $A \precsim_1 B$ whenever $\psi(A) \leq \psi(B)$.

Thus, ψ induces an order-isomorphism from $\mathrm{W}(R)$ to $\mathbb{N} \times \mathbb{N}$ with the stated order. This shows (ii).

To see (i), note that the image of $\mathrm{W}(J)$ through this order-isomorphism corresponds to $0 \times \mathbb{N} \cong \mathbb{N}$. The induced order in this submonoid corresponds to the usual order. □

Remark 8.5. Let R be a nearly simple domain, and let J be its Jacobson radical. Then, $V(J) = 0$ and $\mathrm{W}(J) = \mathbb{N}$ by Theorem 8.4.

Thus, any element $x \in \mathrm{W}(J)$ satisfies that, whenever $x \leq y$, there exists c with $x + c = y$. However, there are no nonzero elements in $\mathrm{V}(J)$.

In connection with Lemma 2.8, the above shows that elements $x \in \mathrm{W}(R)$ which can be complemented to each element $y \in \mathrm{W}(R)$ such that $x \leq y$ do not necessarily belong to the image of $\mathrm{V}(R)$.

Remark 8.6. It follows from Theorem 8.4 above that, if R is a nearly simple domain, the monoid $\Lambda_{\mathrm{W}}(J)$ associated to its Jacobson radical $J = J(R)$ is indistinguishable from $\Lambda_{\mathrm{W}}(D)$ with D a division ring; see Paragraph 2.16.

However, note that every sequence (x_n) defining an element in $\mathrm{S}(J)$ induces a countably generated projective module P over R such that $P = PJ(R)$. Thus, we have $P = 0$. This shows that $\mathrm{S}(J) \cong 0$.

On the other hand, $\mathrm{S}(D) \cong \overline{\mathbb{N}}$ for any division ring. Consequently, $\mathrm{S}(R)$ distinguishes these two families of rings.

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