APPENDIX: CLASSIFYING TRACIALLY FACTORIZABLE MAPS FROM NUCLEAR CONES

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The main aim of this appendix is to prove a uniqueness theorem for certain order zero maps into sequence algebras, Theorem A.3. We begin with some definitions required to state the theorem.

Definition A.1. Let A, B be C*-algebras and let $\phi : A \to B$ be a c.p.c. order zero map. Then we say that ϕ is *tracially factorizable* if there is a contractive tracial functional $\sigma \in T_{\leq 1}(C_0((0,1]))$ and a continuous affine map $\rho : T(B) \to T(A)$ such that

$$\tau \circ \big(f(\phi)\big) = \sigma(f)\rho(\tau), \quad f \in C_0((0,1]),\ \tau \in T(B).$$

Recall that a c.p.c. order zero map $\phi : A \to B$ (between C*-algebras) induces a *-homomorphism $\pi_\phi : C_0((0,1]) \otimes A \to B$, which we call the *associated *-homomorphism*. We say that a *-homomorphism $\psi : A \to B$ (between C*-algebras) is *totally full* if $\psi(a)$ is full in B, for every nonzero $a \in A$.

Let A be a separable unital C*-algebra with $T(A) \neq \emptyset$, and let ω be a free filter on A. Set $T_\omega(A)$ equal to the set of all traces σ on A_ω of the form $\sigma((a_n)_{n=1}^\infty) = \lim_{n\to\omega} \tau_n(a_n)$, for some sequence $(\tau_n)_{n=1}^\infty$ in $T(A)$. The *trace-kernel ideal* of A is

$$J_{A_\omega} := \{a \in A_\omega : \tau(|a|^2) = 0 \text{ for all } \tau \in T_\omega(A)\},$$

a closed ideal of A_ω, and allow ourselves to view $T_\omega(A) \subseteq T(A_\omega/J_{A_\omega})$.

Definition A.2 ([**?** , Definition 2.1])**.** Let A be a separable unital C*-algebra with $T(A) \neq \emptyset$, and let ω be a free ultrafilter on A. We say that A has *complemented orthogonal partitions of unity* (CPoU) if given $a_1, \ldots, a_k \in (A_\omega/J_{A_\omega})_+$, a separable subset $S \subseteq (A_\omega/J_{A_\omega})$, and $\delta > 0$ such that

$$\sup_{\tau \in T_\omega(A)} \min\{\tau(a_1), \ldots, \tau(a_k)\} < \delta,$$

there exist $p_1, \ldots, p_k \in (A_\omega/J_{A_\omega})_+ \cap S'$ summing to $1_{A_\omega/J_{A_\omega}}$ such that

$$\tau(p_i a_i) \leq \delta\tau(p_i), \quad \tau \in T_\omega(A),\ i = 1, \ldots, k.$$

It should be noted that unital, nuclear, $\mathcal{Z}$-stable, stably finite C*-algebras have automatically CPoU ([**?** , Theorem I]).

Here is the main uniqueness theorem of this appendix.

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Theorem A.3. *Let $(B_n)_{n=1}^\infty$ be a sequence of separable, simple, unital, $\mathcal{Z}$-stable C*-algebras with $T(B_n) \neq \emptyset$ and with CPoU. Let ω be a free filter and set $B_\omega := \prod_\omega B_n$. Let A be a separable, unital C*-algebra, and let $\phi, \psi : A \to B_\omega$ be tracially factorizable c.p.c. order zero maps whose associated *-homomorphisms $\pi_\phi, \pi_\psi : C_0((0,1]) \otimes A \to B_\omega$ are totally full. Then ϕ and ψ are unitarily equivalent if and only if*

$$\tau \circ \big(f(\phi)\big) = \tau \circ \big(f(\psi)\big), \quad f \in C_0((0,1]),\ \tau \in T(B_\omega).$$

In fact, we will prove the following uniqueness result with more technical hypotheses. To establish the above theorem, we will demonstrate that its hypotheses imply the more technical hypotheses in the following.

Lemma A.4. *Let $(B_n)_{n=1}^\infty$ be a sequence of separable, simple, unital, $\mathcal{Z}$-stable C*-algebras with $QT(B_n) = T(B_n) \neq \emptyset$ and with CPoU. Let ω be a free filter and set $B_\omega := \prod_\omega B_n$. Let A be a separable, unital C*-algebra, and let $\phi_1, \phi_2 : A \to B_\omega$ be c.p.c. order zero maps such that $\phi_1(x)$ is full in B_ω, for every non-zero $x \in A$, and such that the maps $T(B_\omega) \to \mathbb{R}$ defined by $\tau \to d_\tau(\phi_i(1_A))$ are continuous. Suppose that*

$$\tau \circ f(\phi_1) = \tau \circ f(\phi_2) \quad \tau \in T(B_\omega),\ f \in C_0((0,1])_+. \tag{A.1}$$

Assume moreover that for all non-zero $f \in C_0((0,1])_+$, there exists $\gamma_f > 0$ such that

$$\tau(f(\phi_1)(x)) \geq \gamma_f \tau(\phi_1^{1/m}(x)), \quad x \in A_+,\ m \in \mathbb{N},\ \tau \in T(B_\omega). \tag{A.2}$$

Then ϕ_1 and ϕ_2 are unitarily equivalent.

For an order zero map $\phi : A \to B$, a *supporting order zero map* for ϕ is another c.p.c. order zero map $\hat{\phi} : A \to B$ satisfying

$$\phi(ab) = \hat{\phi}(a)\phi(b) = \phi(a)\hat{\phi}(b), \quad a, b \in A. \tag{A.3}$$

By [**?** , Lemma 1.14], for any sequence $(B_n)_{n=1}^\infty$ of C*-algebras and free filter ω, every c.p.c. order zero map $\phi : A \to \prod_\omega B_n$ has a supporting order zero maps (though it need not be unique).[1] Further, note that any supporting order zero map $\hat{\phi} : A \to B_\omega$ for ϕ satisfies

$$\tau(\hat{\phi}(a)) \geq \lim_{m\to\infty} \tau(\phi^{1/m}(a)), \quad a \in A_+, \tau \in T(B_\omega). \tag{A.4}$$

If A is unital and the map $\tau \mapsto d_\tau(\phi(1_A))$ from $T(B_\omega)$ (equipped with the natural weak*-topology) to $\mathbb{R}$ is continuous, then [**?** , Lemma 1.14] says that a supporting order zero map $\hat{\phi}$ can be found such that equality holds in (A.4). Following the notational convention of [**?**], supporting order zero maps will be adorned with hats throughout.

We now fix some standing notation to be used along this appendix.

[1] Actually, the statement of [**?** , Lemma 1.14] assumes ω is an ultrafilter, but the proof only needs ω to be a free filter.

Notation A.5. The sequence $(B_n)_{n=1}^\infty$ consists of separable, simple, unital, $\mathcal{Z}$-stable C*-algebras with $QT(B_n) = T(B_n) \neq \emptyset$ and with CPoU. The symbol ω denotes a free filter, and we set $B_\omega := \prod_\omega B_n$. Moreover, A is a separable unital C*-algebra, and $\phi_1, \phi_2 : A \to B_\omega$ are c.p.c. order zero maps such that $\phi_1(x)$ is full in B_ω for every non-zero $x \in A$, and such that the maps $T(B_\omega) \to \mathbb{R}$ defined by $\tau \to d_\tau(\phi_i(1_A))$ are continuous. Assume that

$$\tau \circ f(\phi_1) = \tau \circ f(\phi_2), \quad \tau \in T(B_\omega), f \in C_0((0,1])_+. \tag{A.5}$$

Let $\hat{\phi}, \hat{\phi}_2 : A \to B_\omega$ be supporting c.p.c. order zero maps for ϕ_1, ϕ_2 respectively, satisfying

$$\tau(\hat{\phi}_i(x)) = \lim_{m\to\infty} \tau(\phi_i^{1/m}(x)), \quad x \in A,\ i = 1, 2. \tag{A.6}$$

Define a c.p.c. order zero map $\pi : A \to M_2(B_\omega)$ by

$$\pi(x) := \begin{pmatrix} \phi_1(x) & 0 \\ 0 & \phi_2(x) \end{pmatrix}, \quad x \in A. \tag{A.7}$$

and let $\hat{\pi} : A \to M_2(B_\omega)$ be given by

$$\hat{\pi}(x) := \begin{pmatrix} \hat{\phi}_1(x) & 0 \\ 0 & \hat{\phi}_2(x) \end{pmatrix}, \quad x \in A. \tag{A.8}$$

so that $\hat{\pi}$ is a supporting order zero map for π.

Finally, set

$$\begin{aligned} C :&= M_2(B_\omega) \cap \hat{\pi}(A)' \cap \{1_{M_2(B_\omega)} - \hat{\pi}(1_A)\}^\perp \\ &= \{x \in M_2(B_\omega) : \hat{\pi}(1_A)x = x,\ x\hat{\pi}(a) = \hat{\pi}(a)x,\ \text{for all } a \in A\}, \end{aligned} \tag{A.9}$$

and define positive contractions

$$a := \begin{pmatrix} \phi_1(1_A) & 0 \\ 0 & 0 \end{pmatrix},\ b := \begin{pmatrix} 0 & 0 \\ 0 & \phi_2(1_A) \end{pmatrix} \in C_+. \tag{A.10}$$

We prove Lemma A.4 via a 2×2 matrix trick. Originated in [**?**], these convert problems of classifying maps upto (approximate) unitary equivalence into problems of classifying projections or positive elements in relative commutants of ultrapowers. In particular, the version of this trick contained in [**?** , Lemma 2.3] shows that to prove ϕ_1 and ϕ_2 are unitarily equivalent, it suffices to prove that a and b are unitarily equivalent in C. This will be done using tracial information, as in particular the hypotheses ensure that C has strict comparison of positive elements by traces. We collect together this fact

Lemma A.6. *Assume the standing conventions of Notation A.5. Then:*

(i) $\hat{\pi}$ *induces a* **-homomorphism modulo the trace-kernel ideal* $M_2(J_{B_\omega})$*;*
(ii) C *has strict comparison of positive elements with respect to traces, as defined in [**?** , Definition 1.5];*
(iii) $T(C)$ *is the closed convex hull of traces of the form* $\tau(\hat{\pi}(x)\cdot)$*, where* $\tau \in T(M_2(B_\omega))$ *and* $x \in A_+$ *satisfy* $\tau(\hat{\pi}(x)) = 1$*;*

(iv) there exists a positive contraction $e \in C$ with $e(a+b) = (a+b)$; any such e satisfies

$$(\tau_{M_2} \otimes \tau)(f(\hat{\pi}(x)e)) = (\tau_{M_2} \otimes \tau)(f(\hat{\pi}(x))) = \lim_{m\to\infty} \tau(\phi_i^{1/m}(f(x))), \quad \text{(A.11)}$$

for all $\tau \in T(B_\omega)$, $f \in C_0((0,1])$, and $i = 1, 2$.

(v) a positive element $h \in C$ is full in C if and only if there exists a constant $\gamma > 0$ such that

$$\tau(\hat{\pi}(x)h) \geq \gamma\tau(\hat{\pi}(x)), \quad \text{(A.12)}$$

for all $x \in A_+$ and all $\tau \in T(M_2(B_\omega))$.

Proof. By construction $\tau(\hat{\pi}(x)) = \lim_{m\to\infty} \tau(\pi^{1/m}(x))$ for all $x \in A$, so that by the last sentence of [**?** , Lemma 1.14], $\hat{\pi}$ is a *-homomorphism modulo the trace-kernel ideal $M_2(J_{B_\omega})$, proving (i).

Parts (ii) and (iii) are obtained from the corresponding parts of [**?** , Lemma 4.7] (applied to $\hat{\pi}$). Note that in [**?**], ω is assumed to be a free ultrafilter; however the arguments go through using a free filter instead, and letting $T_\omega(B)$ consist of all traces σ on B_ω of the form $\sigma((b_n)_{n=1}^\infty) = \lim_{n\to\omega'} \tau_n(b_n)$, where $(\tau_n)_{n=1}^\infty$ is a sequence of traces on B and $\omega' \supseteq \omega$ is an ultrafilter. The conditions in Notation A.5, together with (i), verify most hypotheses of this lemma. The only thing remaining to check is that $\hat{\pi}(x)$ is full in $M_2(B_\omega)$ for every non-zero $x \in A$. For a positive contraction $x \in A_+$,

$$\hat{\pi}(x) \geq \pi(x) \geq \begin{pmatrix} \phi_1(x) & 0 \\ 0 & 0 \end{pmatrix} \quad \text{(A.13)}$$

which is full in $M_2(B_\omega)$ by the hypothesis in Notation A.5. Thus $\hat{\pi}(x)$ is full in $M_2(B_\omega)$. Since π is positive, it follows that for every non-zero $x \in A$, $\pi(x)$ is full in $M_2(B_\omega)$. This verifies all the hypotheses of [**?** , Lemma 4.7]; hence, both (ii) and (iii) follow.

The existence of e as in (iv) follows from [**?** , Lemma 1.16]. It suffices to prove the second claim for positive $f \in C_0((0,1])$. Using part (i), and the fact that all traces on $M_2(B_\omega)$ vanish on $M_2(J_{B_\omega})$, we have

$$\begin{aligned}
(\tau_{M_2} \otimes \tau)(f(\hat{\pi}(x))) &= (\tau_{M_2} \otimes \tau)(\hat{\pi}(f(x))) \\
&= \frac{1}{2}\tau(\hat{\phi}_1(f(x)) + \hat{\phi}_2(f(x))) \\
&\overset{\text{(A.6)}}{=} \frac{1}{2}\lim_{n\to\infty} \tau(\phi_1^{1/n}(f(x)) + \phi_2^{1/n}(f(x))) \\
&\overset{\text{(A.5)}}{=} \lim_{n\to\infty} \tau(\phi_i^{1/n}(f(x)). \quad \text{(A.14)}
\end{aligned}$$

Note that $\hat{\pi}(\cdot)e$ is also a supporting order zero map for π and for all $x \in A_+$,

$$\begin{aligned}\tau(\hat{\pi}(x)e) &\leq \tau(\hat{\pi}(x)) \\ &= \lim_{m\to\infty} \tau(\pi^{1/m}(x)) \\ &\overset{(A.4)}{\leq} \tau(\hat{\pi}(x)e). \end{aligned} \tag{A.15}$$

Thus the above argument also applies to $\hat{\pi}(\cdot)e$ in place of π, and the conclusion follows.

Versions of (v) are repeatedly used implicitly in [**?**]. For completeness, if such a constant γ exists, let $k \in C_+$ be a positive contraction. Then given $\tau \in T(M_2(B_\omega))$, $x \in A_+$ with $\tau(\hat{\pi}(x)) = 1$ and $n \in \mathbb{N}$, we have

$$d_{\tau(\hat{\pi}(x)\cdot)}(h) \geq \tau(\hat{\pi}(x)h) \geq \gamma\tau(\hat{\pi}(x)) \geq \gamma\tau(\hat{\pi}(x)k^{1/n}), \tag{A.16}$$

so that

$$d_{\tau(\hat{\pi}(x)\cdot)}(h) \geq \gamma d_{\tau(\hat{\pi}(x)\cdot)}(k). \tag{A.17}$$

Taking $m \in \mathbb{N}$ such that $m > \gamma^{-1}$, it follows that (working in $M_n(C)$)

$$d_{\tau(\hat{\pi}(x)\cdot)}(h^{\oplus m}) > d_{\tau(\hat{\pi}(x)\cdot)}(k). \tag{A.18}$$

By (iii), this holds for all traces on C, so by strict comparison as obtained in (ii), k is in the ideal of C generated by h.

For the converse, suppose that $h \in C_+$ is full in C and let e be as in (iv). As e lies in the ideal of C generated by h, there exist $y_1, \ldots, y_m \in C$ such that

$$\|e - \sum_{i=1}^m y_i^* h y_i\| < \frac{1}{2}. \tag{A.19}$$

Then by [**?** , Proposition 2.2], there exists $z \in C$ such that

$$(e - 1/2)_+ \leq z^*(\sum_{i=1}^m y_i^* h y_i)z. \tag{A.20}$$

By parts (iii) and (iv), for any $\rho \in T(C)$, we have $\rho(e) = \rho(e^2)$, so that e is a projection modulo the ideal $\{x \in C : \rho(x^*x) = 0\}$, giving $\rho((e-1/2)_+) = \frac{1}{2}\rho(e)$. Hence,

$$\frac{1}{2}\rho(e) \leq \rho\big(h^{1/2}(\sum_{i=1}^m y_i z z^* y_i^*)h^{1/2}\big) \leq \|\sum_{i=1}^m y_i z z^* y_i^*\|\rho(h).$$

Thus (iv) enables us to take $\gamma = (2\|\sum_{i=1}^m y_i z z^* y_i^*\|)^{-1}$. □

Lemma A.7. *Assume the standing conventions of Notation A.5. If $f(a)$ is full in C for every nonzero $f \in C_0((0,1])_+$, then ϕ_1 and ϕ_2 are unitarily equivalent (by unitaries in B_ω).*

Proof. For a trace $\rho \in T(C)$ of the form $\rho = \tau(\hat{\pi}(x)\cdot)$, where $\tau \in T(M_2(B_\omega))$ and $x \in A_+$ with $\tau(\hat{\pi}(x)) = 1$, we have

$$\begin{aligned}\rho(f(a)) &= \tau\left(\begin{pmatrix}\hat{\phi}_1(x)f(\phi_1(1_A)) & 0\\ 0 & 0\end{pmatrix}\right)\\ &= \tau\left(\begin{pmatrix}f(\phi_1)(x)) & 0\\ 0 & 0\end{pmatrix}\right)\\ &\overset{(A.5)}{=} \tau\left(\begin{pmatrix}0 & 0\\ 0 & f(\phi_2)(x))\end{pmatrix}\right)\\ &= \tau\left(\begin{pmatrix}0 & 0\\ 0 & \hat{\phi}_2(x)f(\phi_2(1_A))\end{pmatrix}\right) = \rho(f(b)).\end{aligned} \tag{A.21}$$

Notice that the second and fourth equality holds since supporting order zero maps induce functional calculus (see paragraph after [**?** , Lemma 1.14] for further details). Now, via Lemma A.6 (v), it follows that b is also totally full in C, and by Lemma A.6 (iii), we have $\sigma(f(a)) = \sigma(f(b))$ for all $\sigma \in T(C)$ and $f \in C_0((0,1])$.

We now apply Theorem 5.1 of [**?**], much in the same way as in the proof of both [**?** , Theorem 5.5] and [**?** , Lemma 4.8]. The two remaining hypotheses of [**?** , Theorem 5.1] are that C has strict comparison of positive elements by traces, which is Lemma A.6 (ii) and that C is full in $M_2(B_\omega)$, which follows as for each non-zero $x \in A$, $\phi_1(x)$ is full in B_ω, so in particular a (and b) are full in $M_2(B_\omega)$. Therefore, the totally full elements a and b are unitarily equivalent by unitaries in the unitization of C.

Since each B_n is finite, simple, and $\mathcal{Z}$-stable, it has stable rank one, and consequently so does B_ω (see [**?** , Lemma 1.22 (iii)]). Our setup then matches the hypotheses of the 2×2 matrix trick in [**?** , Lemma 2.3], and so ϕ_1 and ϕ_2 are unitarily equivalent. □

To apply the above lemma, we need to give conditions in which the image of the unit under an order zero map is totally full in the relevant relative commutant. That is our next aim.

Lemma A.8. *Assume the standing conventions of Notation A.5. Let $f \in C_0((0,1])_+$. Then $f(a)$ is full in C if and only if there exists $\gamma > 0$ such that*

$$\tau(f(\phi_1)(x)) \geq \gamma \lim_{m\to\infty} \tau(\phi_1^{1/m}(x)) \tag{A.22}$$

for all $x \in A_+$ and all $\tau \in T(B_\omega)$.

Proof. (⇒): Let $e \in C_+$ be an element that acts as a unit on $a + b$. By hypothesis, e is in the ideal generated by $f(a)$, so there exist $x_1, \ldots, x_n \in C$ such that

$$(e - 1/2)_+ = \sum_{i=1}^{n} x_i^* f(a) x_i. \tag{A.23}$$

Since e is a projection modulo the trace-kernel ideal, we have for every trace $\rho \in T(C)$,

$$\frac{1}{2}\rho(e) = \rho\left((e-1/2)_+\right) = \rho\left(\sum_{i=1}^{n} x_i^* f(a) x_i\right) \leq K\rho(f(a)), \tag{A.24}$$

where $K := \sum_{i=1}^{n} \|x_i\|^2$; by linearity, this holds equally for bounded tracial functionals. In particular, for any $x \in A_+$ and $\tau \in T(B_\omega)$, we may set $\rho := (\tau_{M_2} \otimes \tau(\hat{\pi}(x)\cdot)$, to get

$$(\tau_{M_2} \otimes \tau)(\hat{\pi}(x)e) \leq 2K(\tau_{M_2} \otimes \tau)(\hat{\pi}(x)f(a)). \tag{A.25}$$

Using (A.5) and Lemma A.6 (iv), the left-hand side becomes

$$(\tau_{M_2} \otimes \tau)(\hat{\pi}(x)e) = \lim_{m\to\infty} \tau(\phi_1^{1/m}(x)). \tag{A.26}$$

Thus we have

$$\begin{aligned} \frac{1}{2}\tau(\hat{\phi}_1(x)) &\leq 2K(\tau_{M_2} \otimes \tau)(\hat{\pi}(x)f(a)) \\ &= 2K(\tau_{M_2} \otimes \tau)\left(\hat{\pi}(x)\begin{pmatrix} f(\phi_1(1)) & 0 \\ 0 & 0 \end{pmatrix}\right) \\ &= 2K(\tau_{M_2} \otimes \tau)\left(\begin{pmatrix} f(\phi_1)(x) & 0 \\ 0 & 0 \end{pmatrix}\right) \\ &= K\tau\left(f(\phi_1)(x)\right), \end{aligned} \tag{A.27}$$

where the second-last equality uses functional calculus via supporting order zero maps (as noted after Lemma 1.14 in [**?**]).

($\Leftarrow$): Consider a trace $\sigma = (\tau_{M_2} \otimes \tau)(\pi(x)\cdot)$ where $\tau \in T(B_\omega)$, $x \in A_+$, and $(\tau_{M_2} \otimes \tau)(\pi(x)) = 1$. We have

$$\begin{aligned} \sigma(f(a)) &= (\tau_{M_2} \otimes \tau)(\pi(x)f(a)) \\ &= \frac{1}{2}\tau(f(\phi_1)(x)) \quad \text{(as in (A.27))} \\ &\geq \frac{1}{2}\gamma\tau(\hat{\phi}_1(x)) \\ &\overset{\text{(A.5)}}{=} \frac{1}{2}\gamma(\tau_{M_2} \otimes \tau)(\pi(x)) \\ &= \frac{1}{2}\gamma. \end{aligned} \tag{A.28}$$

By Lemma A.6 (iii), it follows that

$$\sigma(f(a)) \geq \frac{1}{2}\gamma \tag{A.29}$$

for all $\sigma \in T(C)$. By strict comparison (Lemma A.6 (ii)), it follows that $f(a)$ is full. □

We now have the pieces in place to prove Lemma A.4.

Proof of Lemma A.4. Let $(B_n)_{n=1}^\infty$, ω, A, and ϕ_1, ϕ_2 be as in the statement of Lemma A.4. Then these satisfy the standing conventions of Notation A.5. Moreover, the last part of the hypothesis of Lemma A.4 together with Lemma A.8 shows that $f(a)$ is full in C for every nonzero $f \in C_0((0,1])_+$. Hence by Lemma A.7, it follows that ϕ_1 and ϕ_2 are unitarily equivalent. □

We now check that tracially factorizable maps satisfy the hypotheses of Lemma A.4.

Lemma A.9. *Let B be a unital C*-algebra with $T(B) \neq \emptyset$. Let A be a unital C*-algebra and let $\phi : A \to B$ be a tracially factorizable c.p.c. order zero map whose associated *-homomorphisms $\pi_\phi : C_0((0,1]) \otimes A \to B$ is totally full. Then:*

(i) the map $\tau \mapsto d_\tau(\phi(1_A))$ from $T(B)$ to $\mathbb{R}$ is continuous; and
(ii) for every non-zero $f \in C_0((0,1])_+$, there exists $\gamma_f > 0$ such that

$$\tau(f(\phi_1)(x)) \geq \gamma_f \tau(\phi_1^{1/m}(x)), \quad x \in A_+,\ m \in \mathbb{N},\ \tau \in T(B). \tag{A.30}$$

Proof. By the definition of tracially factorizable, let $\sigma \in T_{\leq 1}(C_0((0,1]))$ and let $\rho : T(B) \to T(A)$ be a continuous affine map such that

$$\tau \circ f(\phi) = \sigma(f)\rho(\tau), \quad f \in C_0((0,1]), \tau \in T(B). \tag{A.31}$$

(i): Compute

$$\begin{aligned} d_\tau(\phi(1_A)) &= \lim_{k\to\infty} \tau(\phi(1_A)^{1/k}) \\ &= \lim_{k\to\infty} \tau(\phi^{1/k}(1_A)) \\ &\overset{(A.31)}{=} \lim_{k\to\infty} \sigma(\mathrm{id}_{(0,1]}^{1/k})\rho(\tau)(1_A) \\ &= d_\sigma(\mathrm{id}_{(0,1]})\rho(\tau)(1_A). \end{aligned} \tag{A.32}$$

This is continuous in τ, since ρ is continuous.

(ii): Given a non-zero function $f \in C_0((0,1])_+$, define $\gamma_f := \frac{\sigma(f)}{\|\sigma\|} \geq 0$. Since π_ϕ is totally full, $f(\phi)(1_A) = \pi_\phi(f \otimes 1_A)$ is full in B, so that

$$\gamma_f = \gamma_f \rho(\tau)(1_A) = \|\sigma\|^{-1}\tau(f(\phi)(1_A)) > 0 \tag{A.33}$$

(using any $\tau \in T(B)$).

Now, for $a \in A_+$, $m \in \mathbb{N}$, and $\tau \in T(B)$,

$$\begin{aligned} \tau(f(\phi)(a)) &= \sigma(f)\rho(\tau)(a)) \\ &= \gamma_f\|\sigma\|\rho(\tau)(a)) \\ &\geq \gamma_f\sigma(\mathrm{id}_{(0,1]}^{1/m})\rho(\tau)(a) \\ &= \gamma_f\tau(\phi^{1/m}(a)). \end{aligned} \tag{A.34}$$

□

Proof of Theorem A.3. Setting $\phi_1 := \phi$ and $\phi_2 := \psi$, the previous lemma shows that the hypotheses of Theorem A.3 imply those of Lemma A.4. Hence it follows from Lemma A.4 that ϕ and ψ are unitarily equivalent. □

We finish this appendix by showing by an example a situation where the assumptions on Lemma A.4 don't hold, even though the *-homomorphisms associated to the order zero maps are totally full.

Example A.10. Let A be a unital simple AF algebra, such that $\partial_e T(A) \cong [0,1]$, and let $B_n = B = \mathcal{Q}$ for all n. We will show how to use Robert's classification theorem [?] to obtain a c.p.c. map $\phi_1 : A \to B \subset B_\omega$ such that

$$\tau_{\mathcal{Q}_\omega}(f(\phi_1)(x)) = \frac{1}{10}\int_0^1 \tau_s(x)\left(s^2\int_0^1 f(t)\,dt + \int_0^s f(t)\,dt\right)\,ds, \tag{A.35}$$

where we parametrize $\partial_e T(A)$ as $(\tau_s)_{s\in[0,1]}$. We can rewrite this as

$$\tau_{\mathcal{Q}_\omega}(f(\phi_1)(x)) = \int_{[0,1]\times(0,1]} \tau_s(x) f(t)\,d\mu(s,t), \tag{A.36}$$

for a particular measure μ on $[0,1]\times(0,1]$.

We can see that $f(\phi_1)(x)$ is full in $\mathcal{Q}_\omega$, for all non-zero $f \in C((0,1]), x \in A$, and therefore the associated *-homomorphism π_{ϕ_1} of ϕ_1 is totally full. However, ϕ_1 does not satisfy the hypothesis of Lemma A.4, as we now show.

For $x \in A_+$,

$$\begin{aligned}
\tau_{\mathcal{Q}_\omega}(\hat{\phi}_1(x)) &= \lim_{m\to\infty} \tau_{\mathcal{Q}_\omega}(\phi_1^{1/m}(x)) \\
&= \lim_{m\to\infty} \frac{1}{10}\int_0^1 \tau_s(x)\left(s^2\int_0^1 t^{1/m}\,dt + \int_0^s t^{1/m}\,dt\right)\,ds \\
&= \frac{1}{10}\int_0^1 \tau_s(x)\left(s^2\int_0^1 1\,dt + \int_0^s 1\,dt\right)\,ds \\
&= \frac{1}{10}\int_0^1 \tau_s(x)(s^2+s)\,ds
\end{aligned} \tag{A.37}$$

by the Dominated Convergence Theorem. Also,

$$\begin{aligned}
\tau_{\mathcal{Q}_\omega}(\phi_1(x)) &= \frac{1}{10}\int_0^1 \tau_s(x)\left(s^2\int_0^1 t\,dt + \int_0^s t\,dt\right)\,ds \\
&= \frac{1}{10}\int_0^1 \tau_s(x)\left(s^2/2 + s^2/2\right)\,ds \\
&= \frac{1}{10}\int_0^1 \tau_s(x)s^2\,ds.
\end{aligned} \tag{A.38}$$

Now, for $k \in \mathbb{N}$, using [? , Theorem 9.3] (essentially, [? , Corollary 6.4]) we may find a positive contraction $y_k \in A_+$ such that $\tau_s(y_k) = 1$ for

$s \in [0,1/k]$ and $\tau_s(y_k) < 1/k^3$ for $s \geq 2/k$. With this y_k, we have

$$\begin{aligned}\tau_{\mathcal{Q}_\omega}(\hat{\phi}_1(y_k)) &= \frac{1}{10}\int_0^1 \tau_s(y_k)(s^2+s)\,ds \\ &\geq \frac{1}{10}\int_0^{1/k} s\,ds \\ &= \frac{1}{10}\cdot\frac{(1/k)^2}{2} \\ &= \frac{1}{20k^2}\end{aligned} \tag{A.39}$$

and

$$\begin{aligned}\tau_{\mathcal{Q}_\omega}(\phi_1(y_k)) &= \frac{1}{10}\int_0^1 \tau_s(y_k)s^2\,ds \\ &\leq \frac{1}{10}\left(\int_0^{2/k} s^2\,ds + \frac{1}{k^3}\int_0^1 s^2\,ds\right) \\ &= \frac{3}{10k^3}.\end{aligned} \tag{A.40}$$

From this, we see that there does not exist $\gamma > 0$ such that

$$\tau_{\mathcal{Q}_\omega}(\phi_1(x)) \geq \gamma\tau_{\mathcal{Q}_\omega}(\hat{\phi}_1(x)), \quad x \in A_+, \tag{A.41}$$

i.e., the hypothesis of Lemma A.4 fails for $f = \mathrm{id}_{(0,1]}$. (In other words, $\phi_1(1)$ is not full in $\mathcal{Q}_\omega \cap \hat{\phi}_1(A)' \cap \{1_{\mathcal{Q}_\omega} - \hat{\phi}_1(1_A)\}$.)

Now to explain how to get the map ϕ_1, write

$$\mathrm{Cu}(\mathcal{Q}) = V(\mathcal{Q}) \amalg \{x_t \mid t \in (0,\infty]\}. \tag{A.42}$$

Define $\alpha : \mathrm{Cu}(C_0((0,1],A))^\sim \to \mathrm{Cu}(\mathcal{Q})$ by

$$\alpha([f]) := \begin{cases} x, & \text{if } \exists\, x \in V(\mathbb{C}) \text{ s.t. } [f(t)] = x\,\forall t \in [0,1]; \\ x_{\beta(f)}, & \text{otherwise,}\end{cases} \tag{A.43}$$

where

$$\beta(f) := \int_{[0,1]\times(0,1]} d_{\tau_s}(f(t))\,d\mu(s,t), \tag{A.44}$$

with μ the measure from (A.36). One can easily see that α preserves addition in the Cuntz semigroup. If $[f] \leq [g]$ in $\mathrm{Cu}(C_0((0,1],A)^\sim)$ then $[f(t)] \leq [g(t)]$ for each $t \in [0,1]$, and from this it is not hard to see that α preserves $\leq$. To see that α preserves $\ll$, it suffices to show that if $g \in (C_0((0,1],A)^\sim \otimes \mathcal{K})_+$ and $\epsilon > 0$ then

$$\alpha([(g-\epsilon)_+]) \ll \alpha([g]). \tag{A.45}$$

If there exists $x \in V(\mathbb{C})$ such that $[g(t)] = x$ for all $t \in [0,1]$, then $\alpha([g]) = x$ and since $x \ll x$, this is automatic. Otherwise, by Lemma A.11 below, there exists a non-empty interval U and $x, y \in \mathrm{Cu}(A)$ such that $x \ll y$, $x \neq y$, and

$$[(g-\epsilon)_+(t)] < x, \quad y \leq [g(t)], \quad \text{for all } t \in U. \tag{A.46}$$

In particular, using compactness of $T(A)$, there exists $\gamma > 0$ such that $d_\tau(x) + \gamma \leq d_\tau(y)$ for all traces $\tau \in T(A)$. Therefore,

$$\begin{aligned} \beta([g]) - \beta([(g-\epsilon)_+] &= \int_{[0,1]\times(0,1]} d_{\tau_s}(g(t)) - d_{\tau_s}((g(t)-\epsilon)_+)\, d\mu(s,t) \\ &\geq \int_{[0,1]\times U} d_{\tau_s}(y) - d_{\tau_s}(x)\, d\mu(s,t) \\ &\geq \int_{[0,1]\times U} \gamma\, d\mu(s,t) \\ &> 0, \end{aligned} \tag{A.47}$$

and from this it follows that $\alpha([(g-\epsilon)_+]) \ll \alpha([g])$.

All conditions have been checked for the fact that α is a Cu-map. Therefore by the existence part of [**?** , Theorems 1.0.1 and 3.2.2(i)], α comes from some *-homomorphism $\pi : C_0((0,1],A)^\sim \to B$. Define $\phi_1 : A \to B$ by

$$\phi_1(a) := \pi(\mathrm{id}_{(0,1]} \otimes a). \tag{A.48}$$

Then for positive contractions $f \in C_0((0,1])_+$ and $x \in A_+$,

$$\begin{aligned} \tau_{\mathcal{Q}_\omega}(f(\phi_1)(x)) &= \tau_{\mathcal{Q}_\omega}(\pi(f\otimes x) \\ &= \int_0^1 d_{\tau_{\mathcal{Q}_\omega}}(((f\otimes x)-\epsilon)_+)\, d\epsilon \\ &= \int_0^1 \int_{[0,1]\times(0,1]} d_{\tau_s}(((f\otimes x)-\epsilon)_+(t))\, d\mu(s,t)\, d\epsilon \\ &= \int_{[0,1]\times(0,1]} \int_0^1 d_{\tau_s}(((f\otimes x)-\epsilon)_+(t))\, d\epsilon\, d\mu(s,t) \\ &= \int_{[0,1]\times(0,1]} \tau_s((f\otimes x)(t))\, d\mu(s,t) \\ &= \int_{[0,1]\times(0,1]} \tau_s(x) f(t)\, d\mu(s,t), \end{aligned} \tag{A.49}$$

as required.

Lemma A.11. *Let A be a C^*-algebra, X be a connected compact Hausdorff space, $f \in C(X,A)$, and $\epsilon > 0$. Then either:*

- *(i) There exists $x \in \mathrm{Cu}(A)$ such that $x \ll x$ and $[f(t)] = x$ for all $t \in X$; or*
- *(ii) There exists $x, y \in \mathrm{Cu}(A)$ and a non-empty open set U of X such that $x \ll y$, $x \neq y$, and for all $t \in U$,*

$$[(f(t)-\epsilon)_+] \leq x, \quad y \leq [f(t)]. \tag{A.50}$$

Proof. We shall assume that (i) does not hold, and endeavor to prove (ii).

To this end, let us firstly claim the existence of $t_0 \in X$ such that $[(f(t_0) - \epsilon/2)_+] \neq [f(t_0)]$. For this, consider two cases :

Case 1: There exists $t_0 \in X$ such that $[f(t_0)] \not\ll [f(t_0)]$. In this case, $[(f(t_0) - \epsilon/2)_+] \neq [f(t_0)]$ follows.

Case 2: $[f(t)] \ll [f(t)]$ for all $t \in X$. In this case, there exists $y \in \mathrm{Cu}(A)$ such that

$$Y := \{t \in X : y \ll [f(t)]\} \tag{A.51}$$

is a non-empty, proper subset of X, and therefore it has a boundary point t_1. If $t_0 \in Y$ is sufficiently close to t_1, then $f(t_0) \approx_{\epsilon/2} f(t_1)$, and therefore

$$[(f(t_0) - \epsilon/2)_+] \leq [f(t_1)]. \tag{A.52}$$

Since t_1 is a boundary point of the open set Y, $y \not\ll [f(t_1)]$; however, $y \ll [f(t_0)]$ since $t_0 \in Y$. Therefore, $[f(t_0)] \neq [(f(t_0) - \epsilon/2)_+]$. This concludes the proof of the claim.

Now, since $[f(t_0)] = \sup_{\delta>0}[(f(t_0) - \delta)_+]$, there exists $\delta \in (0, \epsilon/2)$ such that $[(f(t_0) - \epsilon/2)_+] < [(f(t_0) - \delta)_+]$. By continuity of $(f - \epsilon/2)_+$ and of f, there exists a neighbourhood U of t_0 such that, for all $t \in U$,

$$[(f(t) - \epsilon)_+] \leq [(f(t_0) - \epsilon/2)_+] \quad \text{and} \quad [(f(t_0) - \delta)_+] \leq [f(t)]. \tag{A.53}$$

Setting $x := [(f(t_0) - \epsilon/2)_+]$ and $y := [(f(t_0) - \delta)_+]$, we are done. □

References

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