Continuous Fields of C*-algebras, their Cuntz Semigroup and the Geometry of Dimension Functions

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Figure: Sea Star

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Example

AF-algebras is the class of C-algebras built as inductive limits of finite-dimensional C*-algebras. An important subclass of them are* UHF*-algebras.*

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Examples

- If $A = \mathrm{C}(X)$ for a compact zero-dimensional space X, it follows that $\mathrm{RR}(A) = 0$, and $\mathrm{sr}(A) = 1$.
- If $A = M_n(\mathbb{C})$, then $\mathrm{RR}(A) = 0$. And, furthermore, $\mathrm{RR}(B) = 0$ for any AF-algebra B since real rank zero is preserved by inductive limits.

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  The range of the above invariant consists of the class of dimension groups (Effros-Handelman-Shen)

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where $r_A : \mathrm{T}(A) \times \mathrm{K}_0(A) \to \mathbb{R}$ is the pairing between $\mathrm{K}_0(A)$ and $\mathrm{T}(A)$ given by evaluation of a trace on a K_0-class.

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Non-stably finite and simple $\implies$ $\mathrm{T}(A) = \emptyset$, but
is non-stably finite=purely infinite? (simple case)

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Definition (V(A)-The Murray-von Neumann semigroup)

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The order in $\mathrm{V}(A)$ is algebraic. (i.e. if $[p] \leq [q] \implies \exists\, [r]$ s.t. $[p] + [r] = [q]$.)

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The order in $\mathrm{W}(A)$ is usually not the algebraic order.

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Remark

- *There is a natural map* $\mathrm{V}(A) \to \mathrm{W}(A)$ *defined by* $[p] \mapsto \langle p \rangle$*, which is injective if* A *is stably finite.*
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Definition

If we consider the Grothendieck group construction, we have the following:

$$\mathrm{G}(\mathrm{V}(A)) = \mathrm{K}_0(A) \ \text{(unital case)} \qquad \mathrm{G}(\mathrm{W}(A)) = \mathrm{K}_0^*(A).$$

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Theorem (Antoine-Dadarlat-Perera-Santiago, '13, Tikuisis, '12)

The Elliott invariant can be recovered from the Cuntz semigroup after tensoring with the circle for the same class of C-algebras as above.*

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Coward-Elliott-Ivanescu in 2008 defined $\mathrm{Cu}(A)$ for any C^*-algebra, which is a modified version of the Cuntz semigroup.
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In fact, $\mathrm{Cu}(A)$ can be identified with $\mathrm{W}(A \otimes \mathcal{K})$.

Properties

- *$\mathrm{Cu}(A)$ belongs to a category of semigroups called Cu that admits inductive limits that are not algebraic.*
- *The assignment $A \mapsto \mathrm{Cu}(A)$ is sequentially continuous.*

The category Cu

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Definition

Let a, b be elements in a partially ordered set S. Then, we will say that $a \ll b$ (way-below) if for any increasing sequence $\{y_n\}$ with supremum in S such that $b \leq \sup(y_n)$, there exists m such that $a \leq y_m$.

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Definition (Cu)

An object of Cu is a partially ordered semigroup with zero element S such that:

- The order, in S, is compatible with the addition, i.e., if $x_i \leq y_i$, $i \in \{1, 2\}$ then $x_1 + x_2 \leq y_1 + y_2$,
- every increasing sequence in S has a supremum,
- for all $x \in S$ there exists a sequence $\{x_n\}$ such that $x = \sup(x_n)$ where $x_n \ll x_{n+1}$,
- the relation $\ll$ and suprema are compatible with addition.

The maps of Cu are those morphisms which preserve the order, the zero, the suprema of increasing sequences and the relation $\ll$.

Remark

In fact, $\langle (a-\varepsilon)_+ \rangle \ll \langle a \rangle$ *in* $\mathrm{Cu}(A)$ *for all* $\varepsilon > 0$ *and for all* $a \in A_+$.

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Example

- *Let* X *be a compact metric space. Then, if* $\mathcal{O}(X)$ *is the set of open sets in* X *ordered by inclusion, it follows that* $\mathcal{O}(X) \in \mathrm{Cu}$. *In this, we have that* $U \ll V$ *for* $U, V \in \mathcal{O}(X)$, *if there exists a compact subset* $K \subseteq X$ *such that* $U \subseteq K \subseteq V$.

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- *Let* X *be a finite-dimensional compact metric space, then* $\mathrm{Lsc}(X, \overline{\mathbb{N}}) \in \mathrm{Cu}$, *where* $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$.

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Remark

The main difference between the classical and the stabilized Cuntz semigroup is that $\mathrm{W}(A)$ *is not necessarily closed with respect to suprema of increasing sequences.*

C(X)-algebras

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Definition

Let X be a compact metric space.
A unital $\mathrm{C}(X)$-*algebra* is a C*-algebra A together with a unital *-homomorphism $\theta\colon \mathrm{C}(X) \to Z(A)$, where $Z(A)$ is the center of A.

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Notation

- *If $U \subset X$ is an open set, we denote $A(U) = \mathrm{C}_0(U)A$, which is a closed ideal of A. (Cohen)*
- *If $Y \subseteq X$ is a closed set, $A(Y)$, is the quotient of A by the ideal $A(X \setminus Y)$, which becomes a $\mathrm{C}(Y)$-algebra. The quotient map is denoted by π_Y.*
- *If Y reduces to a point x, we write A_x, denote by π_x the quotient map. The C^*-algebra A_x is called the fiber of A at x.*

Continuous Fields

Lemma (Blanchard)

Let A be a $\mathrm{C}(X)$-algebra and $a \in A$. Then the following conditions are satisfied:

(i) $\|a\| = \sup_{x\in X} \|a_x\|$.

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A continuous field is called trivial if there exists a C*-algebra D such that $A \cong \mathrm{C}(X, D)$.

Sheaves of semigroups

Definition (Presheaves)

A **presheaf** over X is a contravariant functor $\mathcal{S}\colon \mathcal{V}_X \to \mathcal{C}$

where $\mathcal{V}_X$ is the category of closed sets of X with non-empty interior and $\mathcal{C}$ is a subcategory of the category of sets which is closed under sequential inductive limits.

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Definition (Sheaves)

A presheaf is a **sheaf** if for all $V, V' \in \mathcal{V}_X$ such that $V \cap V' \in \mathcal{V}_X$, the map

$$\pi_V^{V\cup V'} \times \pi_{V'}^{V\cup V'} \colon \mathcal{S}(V \cup V') \to \{(f,g) \in \mathcal{S}(V) \times \mathcal{S}(V') \mid \pi^{\mathrm{V}}_{V\cap V'}(f) = \pi^{V'}_{V\cap V}(g)\}$$

is bijective.

A presheaf (respectively a sheaf) is **continuous** if for any decreasing sequence $(V_i)_{i=1}^{\infty}$ in $\mathcal{V}_X$ whose intersection $\cap_{i=1}^{\infty} V_i = V$ belongs to $\mathcal{V}_X$, the limit $\varinjlim \mathcal{S}(V_i)$ is isomorphic to $\mathcal{S}(V)$.

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$$S_x := \lim_{x \in \mathring{V}_n} \mathcal{S}(V_n),$$

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One can define $\Gamma(V, F_{\mathcal{S}(V)}) = \{f : V \to F_{\mathcal{S}(V)} \mid f \text{ is a continuous section}\}$.

Relation between a sheaf and the sheaf of sections

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Theorem (Classical Result)

Let $\mathcal{S}$ be an algebraic sheaf over X. Then,

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Example

Let $A = \mathrm{C}([0,1], \mathbb{C})$ and $\{U_m = [\frac{1}{2} - \frac{1}{m}, \frac{1}{2} + \frac{1}{m}]\}_{m \geq 2}$, which is a sequence of decreasing closed subsets of $[0,1]$ whose intersection is $\{1/2\}$.
It follows $\mathrm{Cu}(A) \cong \mathrm{Lsc}([0,1], \overline{\mathbb{N}})$, where $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$. So one has

$$\varinjlim \mathrm{Lsc}(U_n, \overline{\mathbb{N}}) = \varinjlim \mathrm{Cu}(A(U_n)) = \mathrm{Cu}(\varinjlim A(U_n)) = \mathrm{Cu}(A(1/2)) = \overline{\mathbb{N}}.$$

However, the computation of the above direct limit in **Sg** is not $\overline{\mathbb{N}}$.

The sheaves of sections on Cu

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Let $\mathcal{S}\colon \mathcal{V}_X \to \mathrm{Cu}$ be a sheaf on Cu and X be a compact metric space.

- We define a topology on F_S generated by

$$U_s^{\ll} = \{y \in F_S \mid \hat{s}(x) \gg y \text{ for some } x \in U\}.$$

The induced sections are continuous with this topology.

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Theorem

Let X be a one-dimensional compact metric space, and let $\mathcal{S}\colon \mathcal{V}_X \to \mathrm{Cu}$ be a surjective sheaf. Then $\Gamma(X, F_{\mathcal{S}})$ is a semigroup in Cu.

When do we have a sheaf on Cu?

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Theorem

For a continuous field A over a one-dimensional compact metric space X whose fibers have no K_1 obstructions, the presheaves

$$\begin{array}{rccc} \mathrm{Cu}_A(\_): & \mathcal{V}_X & \to & \mathrm{Cu} \\ & U & \mapsto & \mathrm{Cu}_A(U) = \mathrm{Cu}(A(U)) \end{array} \qquad \begin{array}{rccc} \mathbb{V}_A: & \mathcal{V}_X & \to & \mathrm{Sg} \\ & U & \mapsto & \mathbb{V}_A(U) = \mathrm{V}(A(U)) \end{array}$$

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- (Lin) If $\mathrm{sr}(A) = 1$, $\mathrm{RR}(A) = 0$ and $\mathrm{K}_1(A) = \{0\}$, then A has no K_1 obstructions.

The sheaf $\mathrm{Cu}_A(\_)$

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Theorem

Let X be a one-dimensional compact metric space, and let A be a continuous field over X whose fibers have no K_1 obstructions. Consider the functors

$$\begin{array}{rccc} \mathrm{Cu}_A(\_): & \mathcal{V}_X & \to & \mathrm{Cu} \\ & V & \mapsto & \mathrm{Cu}(A(V)) \end{array} \quad \text{and} \quad \begin{array}{rccc} \Gamma(\_, F_{\mathrm{Cu}_{A(\_)}}): & \mathcal{V}_X & \to & \mathrm{Cu} \\ & V & \mapsto & \Gamma(V, F_{\mathrm{Cu}_{A(V)}})\,. \end{array}$$

Then, $\mathrm{Cu}_A(\_)$ and $\Gamma(\_, F_{\mathrm{Cu}_{A(\_)}})$ are isomorphic sheaves.

Relation between $\mathrm{Cu}(A)$ and the sheaves $\mathrm{Cu}_A(\_)$, $\mathbb{V}_A(\_)$

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Theorem

Let X be a compact metric space, and let A and B be $\mathrm{C}(X)$-algebras such that all fibers have stable rank one. Consider the following conditions:

(i) $\mathrm{Cu}(A) \cong \mathrm{Cu}(B)$ *preserving the action of* $\mathrm{Cu}(\mathrm{C}(X))$,

(ii) $\mathrm{Cu}_A(\_) \cong \mathrm{Cu}_B(\_)$,

(iii) $\mathbb{V}_A(\_) \cong \mathbb{V}_B(\_)$.

Then (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii). *If X is one-dimensional, then also* (ii) $\Longrightarrow$ (i). *If, furthermore, A and B are continuous fields such that for all $x \in X$ the fibers A_x, B_x have real rank zero and $\mathrm{K}_1(A_x) = \mathrm{K}_1(B_x) = \{0\}$, then* (iii) $\Longrightarrow$ (ii) *and so all three conditions are equivalent.*

Classification result (Dadarlat-Elliott-Niu)

Theorem

Let A, B be separable unital continuous fields of AF-algebras over $[0,1]$. Any isomorphism $\tilde{\phi} : \mathrm{Cu}(A) \to \mathrm{Cu}(B)$ that preserves the action by $\mathrm{Cu}(\mathrm{C}(X))$ and such that $\tilde{\phi}(\langle 1_A \rangle) = \langle 1_B \rangle$ lifts to an isomorphism $\phi : A \to B$ of continuous fields of C^*-algebras.

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Question

Can the above result be extended when the fibers are simple AI-algebras?

Stable rank of Continuous Fields

In the case of trivial fields:

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Theorem (Nagisa, Osaka, Phillips, 2001)

Let A be a C-algebra.*

1. *If* $\mathrm{K}_1(A) = \{0\}$, $\mathrm{sr(A)} = 1$, $\mathrm{RR}(A) = 0$, *then* $\mathrm{sr}(\mathrm{C}([0,1], A)) = 1$.
2. *If* $\mathrm{sr}(\mathrm{C}([0,1], A)) = 1$, *then* $\mathrm{K}_1(A) = \{0\}$ *and* $\mathrm{sr(A)} = 1$.

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Is no K_1 obstructions the optimal hyphotesis to obtain $\Longleftrightarrow$?

Trivial fields

Theorem

Let A be any C^*-algebra and X be a compact metric space. Then

$$\mathrm{sr}(\mathrm{C}(X, A)) = 1 \Longleftrightarrow A \text{ has no } \mathrm{K}_1 \text{ obstructions and } \dim(X) \leq 1.$$

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Let A be a simple C^*-algebra with $\mathrm{sr}(A) = 1$ and $\mathrm{K}_1(A) = \{0\}$. Then

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Is $\mathrm{sr}(A \otimes \mathcal{Z}) = 1$ when $\mathrm{sr}(A) = 1$?

M. Rørdam : A is simple.	L. Santiago : A is commutative.

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In this case, we provide an example which shows that the converse is not true.

Example

There is a continuous field A over $[0,1]$ such that $\mathrm{sr}(A) = 1$ and $\mathrm{K}_1(A_x) \neq \{0\}$ for x in a dense subset of $[0,1]$.

The Blackadar-Handelman conjectures

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What are the dimension functions?

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Definition

The set of **dimension functions** is $St(\mathrm{W}(A), \langle 1_A \rangle)$ (normalized positive linear functionals), denoted by $\mathrm{DF}(A)$.
We denote by $\mathrm{LDF}(A)$ the subset of $\mathrm{DF}(A)$ such that the dimension functions are **lower semicontinuous**.
(If $a_n \to a$ in $M_\infty(A)_+$, then $d(\langle a \rangle) \leq lim\,inf\ d(\langle a_n \rangle)$ for $d \in \mathrm{LDF}(A)$)

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It follows by the construction of the Grothendieck group that $St(\mathrm{W}(A), \langle 1_A \rangle) = St(\mathrm{K}_0^*(A), [1_A])$.

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Theorem (Blackadar, Handelman, 1982)

There is an affine bijection between the set of traces of A and $\mathrm{LDF}(A)$, when A is exact.

Blackadar-Handelman conjectures (1982)

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History

- (1982): Blackadar-Handelman proved that 2nd conjecture holds for commutative C*-algebras.
- (1997): Perera proved that 1rst conjecture holds for unital C*-algebras with stable rank one and real rank zero.
- (2008): Brown-Perera-Toms proved both conjectures hold for all unital simple exact and $\mathcal{Z}$-stable C*-algebras.

Proof 1st conjecture (Strategy)

- We study when $(\mathrm{K}_0^*(A), [1_A])$ is an interpolation group.

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Question

When $(\mathrm{K}_0^*(A), [1_A])$ is an interpolation group?

Theorem

Let X be a compact metric space, and let A be a unital continuous field over X. Then, $(\mathrm{K}_0^*(A), [1_A])$ is an interpolation group in the following cases:

(i) *If X is a one-dimensional and A is a continuous field over X such that, for all $x \in X$, A_x has stable rank one, trivial K_1, and is either of real rank zero or simple and $\mathcal{Z}$-stable.*

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$\mathrm{sr}(A) = 1$ $\qquad$ $\mathrm{Cu}(A) = \Gamma(X, \sqcup_{x \in X} \mathrm{Cu}(A_x))$

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Proof: (Sketch)

- If $\mathrm{W}(A)$ has interpolation, then $\mathrm{K}_0^*(A)$ does.
- If $\mathrm{Cu}(A)$ has interpolation and $\mathrm{W}(A) \subseteq \mathrm{Cu}(A)$ is hereditary, then $\mathrm{W}(A)$ has interpolation.

$\mathrm{sr}(A) = 1$ $\qquad\qquad$ $\mathrm{Cu}(A) = \Gamma(X, \sqcup_{x\in X}\mathrm{Cu}(A_x))$

either $\mathrm{sr}(A_x) = 1$ and $\mathrm{RR}(A_x) = 0$ or A_x is simple and $\mathcal{Z}$-stable.

Blackadar-Handelman conjectures

1. The set $\mathrm{DF}(A)$ of dimension functions is a simplex.
2. The set $\mathrm{LDF}(A)$ of lower semicontinuous dimension functions is dense in $\mathrm{DF}(A)$.

Theorem

Let X be a finite dimensional, compact metric space, and let A be a unital, separable infinite dimensional and exact C^*-algebra of stable rank one such that $\mathrm{T}(A)$ is a Bauer simplex. Then $\mathrm{LDF}(\mathrm{C}(X,A))$ is dense in $\mathrm{DF}(\mathrm{C}(X,A))$ in the following cases:

1. $\dim X \leq 1$, *A is simple with $\mathrm{K}_1(A) = 0$ and $\mathrm{W}(A)$ is almost unperforated.*
2. *A is a non-type I, simple, unital ASH algebra with slow dimension growth.*

Nowhere locally trivial continuous fields

Definition

A point $x \in X$ is called singular for A if $A(U)$ is nontrivial for any open set U that contains x (i.e. $A(U)$ is not isomorphic to $\mathrm{C}_0(U, D)$ for some C*-algebra D).

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Cuntz Algebras $\mathcal{O}_n$

If $n \geq 2$. The **Cuntz Algebras** are defined as the universal C*-algebras generated by isometries $s_1, \ldots, s_n$ with orthogonal ranges such that $\sum_{i=1}^{n} s_i s_i^* = 1$.

Local triviality

Example (Dadarlat, Elliott-'08)

A nowhere locally trivial continuous field over $[0, 1]$ *(finite-dimensional) such that its fibers are the same Kirchberg algebra (UCT) with infinitely generated K-theory.*

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A nowhere locally trivial continuous field over Hilbert cube (infinite-dimensional) such that its fibers are the same Kirchberg algebra (UCT) with finitely generated K-theory.

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A nowhere locally trivial continuous field over Hilbert cube (infinite-dimensional) such that its fibers are the same Kirchberg algebra (UCT) with finitely generated K-theory.

Theorem

Let X *be a finite dimensional compact metric space, and let* D *be a stable Kirchberg algebra that satisfies the* UCT *and such that* $\mathrm{K}_j(D)$ *is finitely generated for* $j = 0, 1$. *Let* A *be a separable continuous field C*-algebra over* X *such that* $A(x) \cong D$ *for all* $x \in X$. *Then there exists a dense open subset* U *of* X *such that* $A(U)$ *is locally trivial.*

Corollary

Fix $n \in \mathbb{N} \cup \{\infty\}$. Let X be a finite dimensional compact metrizable space and A be a continuous field over X such that $A(x) \cong \mathcal{O}_n \otimes \mathcal{K}$ for all $x \in X$. Then there exists a closed subset V of X with nonempty interior such that $A(V) \cong \mathrm{C}(V) \otimes \mathcal{O}_n \otimes \mathcal{K}$.

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Example

If $F \subset X$ is a closed nowhere dense set, we provide a continuous field C-algebra A with all fibers isomorphic to a fixed Cuntz algebra $\mathcal{O}_n \otimes \mathcal{K}$, $3 \leq n \leq \infty$, and such that the set of singular points of A coincides with F.*

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Our result is in a certain sense OPTIMAL!.

Bibliography

- R. Antoine, J. Bosa and F. Perera, *Completions of Monoids with Applications to the Cuntz Semigroup*, International Journal of Mathematics 22 (2011), no.6, 837-861.
- R. Antoine, J. Bosa and F. Perera, *The Cuntz semigroup of Continuous Fields*, to appear in Indiana University Mathematics Journal.
- R. Antoine, J. Bosa, F. Perera and H. Petzka, *Geometric structure of dimension functions on certain continuous fields*, to appear in Journal Functional Analysis.
- J. Bosa and M. Dadarlat, *Local triviality for continuous field C*-algebras*, to appear in International Mathematics Research Notices.

Bibliography

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