Continuous Fields of C*-algebras, their Cuntz Semigroup and the geometry of Dimension Functions

Joan Bosa Puigredon

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Doctorand: Joan Bosa Puigredon

Director: Francesc Perera Domènech

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la direcció del Dr. Francesc Perera Domènech

Departament de Matemàtiques
de la Universitat Autònoma de
Barcelona

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Signat: Dr. Francesc Perera Domènech

Introduction

Without a doubt the most important result in chemistry in the 19th century was the classification of the simple chemical elements into Mendeleev's periodic table. If we move forward to the 20th century, the theoretical explanation of this classification, which was made by Schrödinger's equation and Pauli's exclusion principle, is a crucial achievement of physics, and more concretely, of quantum mechanics. One could look at this theory from different angles, but all of them converge to the point of view stated by Heisenberg *"physical quantities are governed by noncommutative algebra"*.

In the 19th century, a number of experiments determined with precision the lines of the emission spectra of the atoms that make up the elements. In particular, if one has a Geissler tube filed with a gas such as hydrogen, the light emitted by the tube may be analyzed with a spectrometer and one would obtain a certain number of lines, indexed by their wavelengths. This configuration is the most direct source of information of the atomic structure and constitutes an accurate description of the element under consideration. Defining the frequency as $\eta = c/\lambda$, where λ is the wavelength and c is the phase speed of the wave, one deduces that there exists a set I of frequencies such that the spectrum ot the elements may be defined by the set of differences of frequencies of I. Further, this property shows that one can combine two frequencies to get a third, this fact is known as the Ritz-Rydberg combination principle.

We point out that these experiments could not be explained within the framework of theoretical physics of the 19th century since comparing these experiments to classical mechanics (based on Newton's mechanics and Maxwell's laws) would yield a contradiction. More concretely, the range of the set of frequencies obtained from an atom is a subgroup of $\mathbb{R}$, which contradicts experimental physics. In the 20th century Heisenberg, based on the experimental groundwork, showed that the set described by the emitted frequencies is an algebra of matrices (which is noncommutative) instead of a group as classical physics predicted. Concretely, he replaced the algebra of functions on the phase space by the algebra of matrices. This was the beginning of noncommutative topology and may be understood as the birth of quantum mechanics.

Based on the need to replace ordinary measure theory (Lebesgue measure) when dealing with noncommutative spaces and to study the lattices of projections in an algebra, Murray and von Neumann defined von Neumann algebras in the 1920's. In a certain sense, this theory is like linear algebra in infinite dimensions, i.e. over an infinite-dimensional separable Hilbert space $\mathcal{H}$. Roughly speaking, we can distinguish between a *von Neumann algebra*, which is a *-subalgebra of $\mathcal{B}(\mathcal{H})$ closed for the weak topology, where $\mathcal{B}(\mathcal{H})$ is the set of bounded linear operators on $\mathcal{H}$, and a *C*-algebra*, which may be regarded as the norm-closed subalgebras of $\mathcal{B}(\mathcal{H})$. Of course every von Neumann algebra is, in particular, a C*-algebra.

The noncommutative topological spaces we mainly speak of in this memoir are C*-algebras. This theory gave birth in 1943, when Gelfand and Naimark showed that, among all the Banach algebras, C*-algebras could be characterized by a small number of axioms. In addition, they showed that a commutative C*-algebra is isomorphic to the algebra $\mathrm{C}_0(X)$ of complex-valued continuous functions on X that vanish at infinity, for some locally compact, Hausdorff space X; more concretely, X is its spectrum.

During the last seventy years, C*-algebra theory has been an extremely active and rapidly expanding area of mathematics. In particular, some of its main achievements have had great impact in other areas such as dynamical systems and complex analysis as well as applications to theoretical physics. For instance, moving back to quantum mechanics, one may describe a physical system by a unital C*-algebra A. Indeed, the self-adjoint elements of A might be thought of as the observables, i.e. the measurable quantities of the system, and any positive normalized functional φ on A (a linear map $\varphi\colon A \to \mathbb{C}$ with $\varphi(a^*a) \geq 0$ for all $a \in A$ such that $\varphi(1) = 1$) might be understood as a state of the system. Precisely, the expected value of the observable x, if the system is in state φ, is then $\varphi(x)$. This C*-algebra approach is used in the Haag-Kastler axiomatization of local quantum field theory, where every open set of the Minkowski spacetime is associated with a C*-algebra. (See [Con94] for further details.)

From the beginning, the main topics on C*-algebra theory have been the study of the structure of different classes of C*-algebras and the quest for classifying invariants. In this work we will mainly focus on the structure of a class of C*-algebras called continuous fields of C*-algebras, and the study of one of its invariants, the Cuntz semigroup.

Classification of C*-algebras

The classification of C*-algebras is based on the search of a complete invariant, i.e. an object that completely captures on its own and functorially the nature of isomorphism of C*-algebras. Another important question is the range that this invariant has. In particular, we seek a functor $\mathrm{Inv}(\_)$ from the category of C*-algebras to a suitable category such that if $\phi : \mathrm{Inv}(A) \cong \mathrm{Inv}(B)$ for two C*-algebras A, B, then there exists $\varphi : A \cong B$ such that $\mathrm{Inv}(\varphi) = \phi$.

If one looks for a starting point on the classification of C*-algebras, this could be traced back to Glimm and his study of UHF-algebras carried out in the late 50s ([Gli60]). Specifically, he classified UHF-algebras using the set of projections under a certain equivalence relation. Later, in 1976, the above classification was generalized by Elliott ([Ell76]) who showed that $(\mathrm{K}_0(A), \mathrm{K}_0(A)^+, [1_A])$ is a complete invariant for the class of approximately finite dimensional C*-algebras (AF-algebras), i.e., inductive limits of finite dimensional C*-algebras. It is important to remark that the work of Effros, Handelman and Shen ([EHS80]) determined abstractly the K_0 groups of separable AF-algebras; these are exactly the dimension groups, that is, the countable, unperforated groups that have the Riesz interpolation property. This completely determines the range of the invariant.

In 1989, based on the existing results for some special classes of C*-algebras, Elliott conjectured that all separable, simple and nuclear C*-algebras could be classified using an invariant consisting of K-theory and traces. In the most general form, Elliott conjecture was the following:

Conjecture. (Elliott, 1989) *There is a* K*-theoretic functor F from the category of separable and nuclear*

C-algebras such that if A and B are separable and nuclear, and there is an isomorphism $\phi : F(A) \to F(B)$, then there is a *-isomorphism $\Phi : A \to B$ such that $F(\Phi) = \phi$.*

The concrete form of the invariant is:

$$\mathrm{Ell}(A) = ((\mathrm{K}_0(A), \mathrm{K}_0(A)^+, [1_A]), \mathrm{K}_1(A), \mathrm{T}(A), r_A),$$

where $r_A : \mathrm{T}(A) \times \mathrm{K}_0(A) \to \mathbb{R}$ is the pairing between $\mathrm{K}_0(A)$ and $\mathrm{T}(A)$ given by evaluation of a trace on a K_0-class.

This program has been quite sucessful and has given spectacular results. One of its main achievements was the classification of purely infinite, simple, separable and nuclear C*-algebras (known as Kirchberg algebras) obtained by Kirchberg and Phillips in [KP00b] and [KP00a] taking as invariant $((\mathrm{K}_0(A), [1_A]), \mathrm{K}_1(A))$ (as $\mathrm{T}(A) = \emptyset$ and $\mathrm{K}_0(A)^+ = \mathrm{K}_0(A)$ in this case). Recall that a simple C*-algebra A is purely infinite if for all $a \neq 0 \in A$ there exist $x, y \neq 0$ such that $xay = 1$. Moreover, it was also shown that for any triple (G_0, g_0, G_1) of countable abelian groups G_0, G_1 with distinguished element $g_0 \in G_0$, there exists a unital Kirchberg algebra A such that $(\mathrm{K}_0(A), [1]_0, \mathrm{K}_1(A)) \cong (G_0, g_0, G_1)$. Thus, also in this case the range of the invariant is determined.

If we turn our attention to the stably finite case and consider the algebras that admit an inductive limit decomposition, the first results towards classification, though not phrased in this way, were the above mentioned classification results obtained by Glimm and Elliott. After other significant results obtained in the 90′ such of the classification of AT-algebras (see [Ell97]), a crucial achievement was the classification of simple unital AH-algebras with slow dimension growth obtained by Gong ([Gon02]) and by Elliott, Gong, Li ([EGL07]). An AH-algebra is an inductive limit of a sequence (A_i, φ_i) where A_i is a direct sum of algebras which are finite matrices over $\mathrm{C}(X_{n_k})$ for some compact metric spaces. Roughly speaking, the condition of having slow dimension growth means that the dimension of the spaces compared to the sizes of the matrices tend to zero as we go along the limit decomposition.

Despite the good results obtained on the way to confirm Elliott's conjecture, in the last decade there have appeared two dramatic counterexamples built out by M. Rørdam ([Rør03]) and A. Toms ([Tom08a]). We remark that the construction of both examples is based on work by Villadsen ([Vil98]), where he exhibited examples of simple nuclear C*-algebras that fail to satisfy strict comparison of projections. Concretely, in [Tom08a] two non-isomorphic unital simple AH-algebras that agreed on their Elliott invariant and other continuous and stable isomorphism invariants were produced. However, these algebras were distinguished using their Cuntz semigroup $\mathrm{W}(A)$. This semigroup was introduced by Cuntz in 1978 ([Cun78]) modelling the construction of the Murray-von Neumann semigroup $\mathrm{V}(A)$, but taking into account positive elements in arbitrary matrices over the algebra modulo an equivalence relation. In particular, the Cuntz semigroup construction generalizes the construction of $\mathrm{V}(A)$ in the stably finite case.

The appearance of the above examples opened the door to two possible directions in the classification program of C*-algebras; one can either restrict the class of separable simple nuclear C*-algebras to be classified by $\mathrm{Ell}(A)$ or else search for finer invariants. In the latter direction, the Cuntz semigroup has been intensively studied in recent years, as can be seen in [BPT08], [ERS11], [CEI08], [Rob13] among others. Further, some classifying results have been obtained using the Cuntz semigroup as an invariant ([CE08],[Rob12] in the non-simple case). It is appropriate to note that one of the main properties of the Cuntz semigroup is that its structure contains a large amount of information coming from the Elliott invariant, and, in particular, it can be recovered from the Elliott invariant in a functorial manner in some cases (see [BPT08], [PT07]). In the converse direction, it is also posible to recover the Elliott invariant from the Cuntz semigroup for a large class of C*-algebras, as have been shown in [Tik11], [ADPS13].

The Cuntz semigroup has gained relevance in the classification program, but to work with it is usually difficult due to the complexity of its definition. On the one hand, the computation of the Cuntz semigroup of a C*-algebra is very complicated as, already in the commutative case, at least this amounts to classifying vector bundles over a topological space. On the other hand, it does not preserve inductive limits of C*-algebras. Note that the latter difficulty is a crucial drawback since, as seen before, a lot of C*-algebras are built as inductive limits. Nevertheless, the latter difficulty was solved by Coward, Elliott and Ivanescu in [CEI08] considering a modified version of the Cuntz semigroup, denoted by $\mathrm{Cu}(A)$. Particularly, they used suitable equivalence classes of countably generated Hilbert modules to obtain a semigroup strongly related to the classical Cuntz semigroup which, in fact, is isomorphic to $\mathrm{W}(A \otimes \mathcal{K})$, where $\mathcal{K}$ is the algebra of compact operators over a separable infinite-dimensional Hilbert space. One of the main advantages of their construction is that they further provide a category Cu for this new semigroup, whose objects are positively ordered abelian semigroups with some additional properties of a topological nature. Moreover, the assignment $A \to \mathrm{Cu}(A)$ defines a functor from the category of C*-algebras to Cu, which preserves inductive limits. It is opportune to note that $\mathrm{Cu}(A)$ is more tractable than $\mathrm{W}(A)$ thanks to its topological properties.

Continuous Field C*-algebras

A vector bundle is a topological construction that comes from the idea of a family of vector spaces parameterized by another space X. In particular, to every point x of the space X we associate (or "attach") a vector space V_x in such a way that these vector spaces fit together to form another space which is related with X. For instance, if $X = [0,1]$ and one attaches $\mathbb{R}$ to each $x \in X$, then one obtains the trivial vector bundle $[0,1] \times \mathbb{R}$.

The Gelfand-Naimark characterization of commutative C*-algebras explained before suggested the problem of representing noncommutative C*-algebras as continuous sections of C*-algebra bundles. In that direction, a first approach to define operator fields over a space X was made in [Fel61], where it was also shown that a separable C*-algebra A with Hausdorff primitive spectrum X is isomorphic to a C*-algebra of operator fields over X.

Continuing the studies started by Fell, in order to study deformations in the C*-algebraic framework, Dixmier introduced the notion of continuous field of C*-algebras over a locally compact space in [Dix77]. A separable $\mathrm{C}(X)$-algebra over a compact space X may be thought of as a C*-algebra A which has the structure of a $\mathrm{C}(X)$-module. These algebras may be analysed via their fibers, i.e. the quotient algebras $A_x = A/\mathrm{C}_0(X \setminus \{x\})A$, and it is known that for all $a \in A$, the map $x \mapsto \|a(x)\|$ is upper semicontinuous. If the above map is continuous for all $a \in A$, we say that A is a continuous field of C*-algebras or a C*-bundle.

Adopting the spirit of the notion of trivial bundle, we say that a continuous field A is trivial if $A \cong \mathrm{C}(X) \otimes D$ for some C*-algebra D, so all fibers are isomorphic to D. On the other hand, a point $x \in X$ is called singular for A if the restriction $A(U) = \mathrm{C}_0(U)A$ is nontrivial for any open set U that contains x (i.e. $A(U)$ is not isomorphic to $\mathrm{C}_0(U) \otimes D$ for some C*-algebra D). The above definition is relevant because only a very small fraction of continous fields of C*-algebras correspond to locally trivial bundles.

Clearly, to study continuous fields of C*-algebras, one can focus either on the structure of some specific continuous field of C*-algebras (i.e. try to determine whether a continuous field C*-algebra is locally trivial or not) or on the classification of some classes of continuous fields not bothering about locally triviality.

Along the way of studying the structure of these objects there have been a lot of relevant achievements obtained, for instance, by E. Blanchard [Bla96], M. Dadarlat [Dad09a] [Dad09b] and E. Kirchberg and S. Wasserman [KW95], among others. One of the most valuable results which should be mentioned was shown by Blanchard and Kirchberg in [Bla97]. They proved that a continuous field C*-algebra A is exact if and only if there exists a monomorphism of $\mathrm{C}(X)$-algebras $A \hookrightarrow \mathrm{C}(X) \otimes \mathcal{O}_2$, where $\mathcal{O}_2$ is the universal C*-algebra generated by two isometries with orthogonal ranges that add up to 1. Notice that the above result gives an idea of the complexity of continuous fields, as there exist nowhere trivial exact continuous field C*-algebras yet they are embedded in a trivial continuous field. Another important result regarding structural properties was given by M. Dadarlat in [Dad09a], where he proved automatic and conditional local/ global trivialization results for continuous fields of Kirchberg algebras. In fact, he showed that the only obstruction to local or global triviality for such algebras is encoded in the K-theory of the fibers.

Continuous field C*-algebras have also been analysed within the framework of the classification program of C*-algebras. We wish to emphasize important results proved by Dadarlat, Elliott in [DE07] and Dadarlat, Elliott, Niu in [DEN11]. They used the classification of Kirchberg algebras to classify continuous fields over $[0, 1]$ whose fibers are either Kirchberg algebras (with certain torsion freeness assumptions on their K-theory) or AF-algebras, by means of a K-theory sheaf. For example, in [DE07], they showed that if the fibers have torsion free K_0-group and trivial K_1-group, the K_0-sheaf is a complete invariant for separable stable continuous fields of Kirchberg algebras. Roughly speaking, the K_0-sheaf consists of the collection of the K_0-groups of the restrictions of the continuous field, together with suitable connecting morphisms. In their work, one of the key ideas is the approximation of continuous fields by the so-called elementary fields, i.e., fields that are locally trivial at all but finitely many points.

Our aims

We will be concerned with the study of the structure of continuous field C*-algebras, and the computation of their Cuntz semigroup, with classification in view. We next summarize the main topics discussed in this memoir.

(1)-Structure of Continuous Fields of C*-algebras : In the literature there are two examples which clearly give an idea about the complexity of continuous field C*-algebras. The first one was constructed by M. Dadarlat and G. A. Elliott in [DE07], and it is a continuous field C*-algebra A over the unit interval with mutually isomorphic fibers and such that it is nowhere locally trivial. We remark that, in this example, all the fibers A_x of A are isomorphic to the same Kirchberg algebra D with $\mathrm{K}_0(D) \cong \mathbb{Z}^\infty$ and $\mathrm{K}_1(D) = 0$, so all fibers have non-finitely generated K-theory, and the base space is finite-dimensional. The second example that we would like to emphasize shows that, even if the K-theory of the fibers vanish, the field can be nowhere locally trivial if the base space is infinite-dimensional. This example was constructed in [Dad09b] as a separable continuous field C*-algebra A over the Hilbert cube with the property that all fibers are isomorphic to $\mathcal{O}_2$, but nevertheless A is nowhere locally trivial.

From the above examples, it is natural to ask which is the structure of continuous fields of Kirchberg algebras over a finite-dimensional space with mutually isomorphic fibers and finitely generated K-theory. This question has been adressed in [BD13].

(2)-The Cuntz semigroup of continuous field C*-algebras : For commutative C*-algebras of lower dimension where there are no cohomological obstructions, a description of their Cuntz semigroup via point evaluation has been obtained in terms of (extended) integer valued lower semicontinuous functions on their spectrum. Next, a natural class to consider consists of those algebras that have the form $\mathrm{C}_0(X, A)$ for a locally compact Hausdorff space X. As a first instance, the case when A is a unital, simple, non type I ASH-algebra with slow dimension growth was studied by A. Tikuisis in [Tik11]. Another important case was studied in [APS11], where X is compact, metric and of dimension at most one and A has stable rank one and vanishing K_1 for every closed, two sided ideal.

In the latter situation, and for more general $\mathrm{C}(X)$-algebras, the key was to describe the Cuntz semigroup classes by the corresponding classes in the fibers, i.e., the aim was to recover global information from local data. This was done in [APS11] by analysing the map

$$\alpha\colon \mathrm{Cu}(A) \to \prod_{x\in X} \mathrm{Cu}(A_x) \quad \text{given by} \quad \alpha\langle a\rangle = (\langle a(x)\rangle)_{x\in X}.$$

Focusing on this analysis, it was shown that for the above mentioned class of C*-algebras $\mathrm{C}(X, A)$, the range of the previous map can be completely identified as a semigroup of lower semicontinuous Cu-valued functions. En route to this result, it was also shown that the Cuntz semigroup functor behaves well on some pullbacks of C*-algebras and that for any $S \in \mathrm{Cu}$ the semigroup $\mathrm{Lsc}(X, S)$ also belongs to Cu, where X is any finite-dimensional compact, metric space.

In [ABP] the map α was studied in the case when X has low dimensions and all the fibers of the $\mathrm{C}(X)$-algebra A are not necessarily mutually isomorphic. Explicitely, in [ABP] we take into account continuity properties of the objects in the category Cu to study sheaves where the target values are semigroups in Cu. This may be regarded as a version of continuous fields of semigroups in Cu.

(3)-Dimension Functions on a C*-algebra : Roughly speaking, a dimension function on a ring is a real-valued function whose values measure the size of the "support projections" of the elements. In the concrete case of operator theory, these functions appeared when Murray and von Neumann used them (defined only on projections) in their classification of factors. In particular, they are a crucial tool in the study of von Neumann algebras. The study of dimension functions on C*-algebras was developed in [Cun78]. Mainly, Cuntz proved that if A is a simple C*-algebra, then A is stably finite if and only if it has a dimension function.

This idea was further developed by B. Blackadar and D. Handelman in [BH82], who introduced a general theory for dimension functions on non necessarily simple C*-algebras. They clarified which is the relation between dimension functions and quasi-traces, which are roughly tracial maps defined on the algebra that are linear on commuting elements and that extend to matrices with the same properties. Specifically, it was shown that there is an affine bijection between the set of quasitraces and those dimension functions that are lower semicontinous in a suitable sense. Further, it was shown that the set of quasi-traces for unital C*-algebras is a simplex. It is pertinent to mention that, when A is a unital C*-algebra, dimension functions can be thought of as states on the Cuntz semigroup $\mathrm{W}(A)$. Thus, the study of dimension functions on the algebra can be translated into studying states on a semigroup.

In relation to the study of dimension functions on C*-algebras, two natural questions arised from [BH82], that are referred to as the Blackadar and Handelman conjectures:

(i) The affine space of dimension functions is a simplex.

(ii) The set of lower semicontinuous dimension functions is dense in the space of all dimension functions.

We remark that that the relevance of the latter conjecture falls on that the set of lower semicontinuous dimension functions is more tractable than the set of all dimension functions since, as mentioned, it is in correspondence with the quasitraces in A.

It is relevant at this point to mention that Blackadar and Handelman made the first progress of their conjectures confirming that (i) holds for commutative C*-algebras in [BH82]. Further progress was made in [Per97], confirming (ii) for the class of unital C*-algebras with real rank zero and stable rank one, and in [BPT08], confirming both conjectures for the class of simple unital finite C*-algebras which are exact and $\mathcal{Z}$-stable (where $\mathcal{Z}$ is the Jiang-Su algebra, [JS99]).

(4)- Dimension theory on C*-algebras : While it has been known for a long time that if X, Y are compact Hausdorff spaces the covering dimension satisfies $\dim(X \times Y) \leq \dim(X) + \dim(Y)$, there is little knowledge about the analogous situation for non-commutative versions of dimension for C*-algebras. More precisely, it is not totally clear what the behaviour of the real rank and the stable rank of tensor products of C*-algebras is. An important contribution to clarify the situation were made in [NOP01] where the real and the stable rank of some trivial continuous fields was studied. Some of their main results are the following:

(i) If X is a locally σ-compact Hausdorff and A any C*-algebra, then $\mathrm{RR}(\mathrm{C}_0(X) \otimes A) \leq \dim(X) + \mathrm{RR}(A)$.

(ii) If A is unital C*-algebra and $\mathrm{RR}(A) = 0$, $\mathrm{sr}(A) = 1$, and $\mathrm{K}_1(A) = 0$, then $\mathrm{sr}(\mathrm{C}([0,1]) \otimes A) = 1$.

Note that the first inequality is a generalization of the situation that we had for X, Y compact Hausdorff spaces and covering dimension because $\mathrm{RR}(\mathrm{C}(X) \otimes \mathrm{C}(Y)) = \mathrm{RR}(\mathrm{C}(X \times Y)) = \dim(X \times Y) \leq \dim X + \dim(Y) = \mathrm{RR}(\mathrm{C}(X)) + \mathrm{RR}(\mathrm{C}(Y))$.

In our work, we have made a remarkable contribution to the computation to the stable rank for some continuous fields. These achievements and their applications to the Blackadar and Handelman conjectures can be found in [ABPP13].

The contents of the memoir

This memoir is organized in four chapters. Chapter 1 is the place where we provide some basic background on C*-algebras, K-theory and the Cuntz semigroup. Our main aim in this first chapter is to provide some basic notation, definitions and the main results so that they may be referenced in the following chapters.

Following the line of research explained before, in the second chapter we detail how we fill the gap left by the two examples in [DE07] and [Dad09b]. In pursuance of doing this, we focus our study on continuous field C*-algebras A over X such that A_x are mutually isomorphic to a stable Kirchberg algebra D satifying the UCT with finitely generated K-theory and where X is a finite-dimensional compact metrizable space. Our main result in Chapter 2 states that under the mentioned assumptions, there exists a dense open set U of X such that $A(U)$ is locally trivial. Note that by the two examples reviewed before, the assumptions that the space X is finite dimensional and that the K-theory of the fiber is finitely generated are necessary, i.e., they are optimal.

Coming back to the study of the Cuntz semigroup, as said before, the information contained in $\mathrm{Cu}(A)$ of a continuous field C*-algebra should be obtained by the analysis of the map

$$\alpha\colon \mathrm{Cu}(A) \to \prod_{x\in X} \mathrm{Cu}(A_x) \text{ given by } \alpha\langle a\rangle = (\langle a(x)\rangle)_{x\in X}.$$

This is exactly what we do in Chapter 3 considering the continous fields A over a compact metrizable and one-dimensional space X such that A_x has stable rank one and vanishing K_1 for every closed, two sided ideal. The above analysis leads us to study the natural map $F_{\mathrm{Cu}(A)} := \sqcup_{x\in X}\mathrm{Cu}(A_x) \to X$ and its sections. This is motivated by the fact that the $\mathrm{Cu}(\_)$ functor induces a presheaf Cu_A on X given by $V \mapsto \mathrm{Cu}(A(V))$ to each closed set V of X. This is a sheaf under the above hyphoteses. Hence, we may expect to relate the Cuntz semigroup of a continuous field with the semigroup of continuous sections of an étalé bundle. We remark that this was the case of the K-theory presheaf used by Dadarlat and Elliott in [DE07] when the C*-algebra under study was a separable continuous field over $[0,1]$ of Kirchberg algebras satisfying the UCT and having finitely generated K-theory group. In our case, and in order to recover Cu_A from the sheaf of continuous sections of the map $F_{\mathrm{Cu}(A)} \to X$, we need to break away from the standard approach of étalé bundles (see [Wel73]) and consider a topological structure on $F_{\mathrm{Cu}(A)}$ taking into account continuity properties of the objects in the category Cu. We finish our deep analysis of Cu-valued sheaves with the result that allows us to recover the Cuntz semigroup of the continuous field as the semigroup of global sections on $F_{\mathrm{Cu}(A)}$. Our approach extends some of the results in [APS11].

In the second part of Chapter 3, using the description above, we are able to answer the natural question about whether the Cuntz semigroup of continuous fields C*-algebras captures, on its own, all the information of the K-theory sheaf. In fact, we prove that for continuous fields A over X such that A_x has stable rank one, vanishing K_1 and real rank zero for all $x \in X$, the Cuntz semigroup and the K-theory sheaf defined by the Murray-von Neumann semigroup carry the same information. We remark that this result allows us to rephrase the classification result explained before in [DEN11] by a single invariant, the Cuntz semigroup. The confirmation of this fact is a good contribution to the starting line of reasearch based on the use of the Cuntz semigroup as a complete invariant for some classes of C*-algebras.

Although we have briefly mentioned this in passing, it is natural to ask which is the relation between $\mathrm{W}(A)$ and $\mathrm{Cu}(A)$. Related to this, a first approach was given in [ABP11], where a new category of ordered abelian semigroups, PreCu, was built, and to which the classical Cuntz semigroup belongs for a large class of C*-algebras. We note that this category contains the category Cu as a full subcategory. Precisely, PreCu differs from Cu in that semigroups need not be closed under suprema of ascending sequences (which is one of the main features of semigroups in the category Cu). Following this, the completion of a semigroup in PreCu was defined in [ABP11] in terms of universal properties. This completion gives us a functor from the category PreCu to Cu. In particular, this construction yields that $\mathrm{Cu}(A)$ is the completion of $\mathrm{W}(A)$ in good cases (for example, if $\mathrm{sr}(A) = 1$).

We start Chapter 4 by computing the stable rank of some class of continuous fields C*-algebras, which is a result of independent interest. Specifically, we show that any continuous field A over one-dimensional compact metric space X such that A_x has stable rank one and vanishing K_1 for every closed, two sided ideal for all $x \in X$ satisfies that $\mathrm{sr}(A) = 1$. Note that this fact improves the results obtained in [NOP01] since we do not restrict just to the case of trivial continuous fields.

We move on to show that, if $\mathrm{Cu}(A)$ is the completion of $\mathrm{W}(A)$, then $\mathrm{W}(A)$ is a Riesz interpolation semigroup if and only if so is $\mathrm{Cu}(A)$. This is relevant since it shows that the structure of the Grothendieck group of $\mathrm{W}(A)$ mainly depends on the description of $\mathrm{Cu}(A)$. We use this, in the particular case of continuous fields A over one-dimensional spaces and with mild assumptions on their fibers, to conclude that the set of dimension functions is a simplex when $\mathrm{sr}(A) = 1$, thus confirming the first Blackadar and Handelman conjecture for this class of algebras. Further, we represent the Grothendieck group of $\mathrm{W}(A)$ sufficiently well into the group of affine and bounded functions on $\mathrm{T}(A)$ to give an affirmative answer to the second conjecture of Blackadar and Handelman for certain continuous fields.

Agraïments

Com que aquesta és l'única part de la tesi que espero que sigui llegida detingudament per la gent que m'ha ajudat a poder-la realitzar, espero no fer cap falta i entretenir-vos una mica!

Tal i com he fet a la introducció, m'agradaria començar explicant les sensacions que vaig tenir quan només era un petit embrió a la panxa de la meva mare. Jo, en ”aquellos entonces“, ja pensava que algun dia arribaria a ser doctor, i mira... després de molts anys sembla que estic a prop d'aconseguir-ho... no està malament, no? Podria haver sigut pitjor!

Passant per alt la majoria de les històries d'infantesa, vull recordar una boda a la qual vaig anar de petit on una dona grassoneta em va preguntar ”Què vols ser de gran?” I jo, tot i que tenia menys de deu anys, ja vaig contestar: “Matemàtic!!”. Aquella resposta em va portar força problemes, ja que, si no recordo malament, aquella dona es va passar mitja boda fent-me calcular mentalment certes operacions matemàtiques i sempre anava dient: “ Ohh!! Ets molt bo. Algun dia arribaràs a ser matemàtic!”. Tot i que els anys han anat passant, i les matemàtiques s'han tornat més complicades que sumar i multiplicar, a vegades penso que aquella dona era una mica visionària!

Però bé, el camí fins aquí, tot i que no ha estat terrible, tampoc ha sigut bufar i fer ampolles, i és per això que primer de tot vull agrair a la meva família tot el suport que m'ha donat durant aquests anys. Des de mons pares, que em van arribar a oferir un cotxe si canviava de carrera i no estudiava matemàtiques quan tenia divuit anys, però sense els quals no seria la persona que sóc, fins als meus germans, l'Andreu i l'Enric, que sempre hi han estat quan ha calgut. Evidentment, també me'n recordo del padrí jove, la Tere, el Jaume, la Carme i els meus padrins!

Dintre del món matemàtic, sempre recordaré aquests últims anys a la universitat, on finalment he trobat companys de professió amb els quals comparteixo alguna cosa més que les matemàtiques. Així, fent una petita llista, vull agrair al Dani certes discussions de vida i de física molt interessants, al Pere l'ajuda constant per fer que les idees esbojarrades, com fer un penjador al despatx, es facin realitat, al Joan per ensenyar-me que hi ha matemàtiques no teòriques que són atractives, a l'Albert per mostrar-me que el primer que s'ha de fer quan s'arriba al despatx és un cafè mentre llegeixes el diari, a l'Anna per tots aquests anys de cabòries, al Yago per ser tan útil al principi, i al David per aguantar totes les meves neures sense ni despentinar-se. No voldria descuidar-me ni de l'Antonio ni de l'Alberto ni dels altres Danis ni de la Isa!

Dins de la faràndula, em veig obligat a recordar-me dels meus companys de festa, d'excursions, de viatge, de cerveses, d'alegria. Aquests són l'Ana, la Neus, el Josep, la Montse, l'Aurora, l'Alba, el Mario, l'Albert, el Javi, el Jose i el Jordi entre altres. Continuant dintre d'aquest món, vull recordar-me del cor Desacord, el qual m'ha fet passar molt bons moments en els últims dos anys!

I de tu, companya de plors i alegries, amiga que mai em defraudes, persona que no et canses de regalar-me abraçades cada dia, també me'n recordo.

No voldria acabar sense agrair al Grup de Teoria d'Anells de la UAB l'oportunitat d'estudiar un doctorat amb ells. En especial, vull donar les gràcies al Ramon Antoine per tota l'ajuda rebuda a l'hora de resoldre certes preguntes que sense ell encara estarien obertes.

I ara sí, l'últim paràgraf és pel meu company, amic i tutor. Com ja saps, sense tu jo no hauria estat capaç d'escriure aquesta tesi, així que ... gràcies per la paciència, Francesc!

I would also like to thank Marius, Hannes and Taylor for their useful help and kind friendship.

Joan Bosa Puigredon

Bellaterra, Juliol 2013

Contents

Chapter 1

Preliminaries

In this first chapter we provide some basic background on C*-algebras, K-theory and the Cuntz semigroup. Our aim is not to repeat material easily found in textbooks on the subject, but to establish some basic notation, definitions and the main results so that they may be referenced along the following chapters. Some standard references for this material are [Mur90], [LLR00], [Dix77], [APT11], [WO93], [Bla98], [Bla06].

1.1 C*-algebras

In this section we state some basic facts about C*-algebras that the reader is assumed to be familiar with.

Definition 1.1.1. *A C*-algebra A is an algebra over $\mathbb{C}$ with a norm $a \mapsto \|a\|$ and an involution $a \mapsto a^*$, for $a \in A$, such that A is complete with respect to the norm, and such that $\|ab\| \leq \|a\|\|b\|$ and $\|aa^*\| = \|a\|^2$ for every a, b in A.*

A C*-algebra A is called *unital* if it has a multiplicative identity, denoted by 1_A. A **-homomorphism* $\varphi : A \to B$ between C*-algebras A and B is a linear and multiplicative map which satisfies $\varphi(a^*) = \varphi(a)^*$ for all a in A. If A and B are unital and $\varphi(1_A) = 1_B$, then φ is called *unital*. Recall that *-homomorphisms are automatically continuous and of norm 1.

A C*-algebra is said to be *separable* if it contains a countable dense subset. A *sub-C*-algebra* of A is a non-empty subset of A which is a C*-algebra with respect to the operations given on A. Let F be a subset of A. The sub-C*-algebra of A generated by F, denoted by $\mathrm{C}^*(F)$, is the smallest sub-C*-algebra of A that contains F. In other words, $\mathrm{C}^*(F)$ is the intersection of all sub-C*-algebras of A that contain F. We write $\mathrm{C}^*(a_1, a_2 \ldots, a_n)$ instead of $\mathrm{C}^*(\{a_1, a_2, \ldots a_n\})$, when $a_1, a_2, \ldots, a_n$ are elements in A.

We next provide a list of examples of C*-algebras.

Examples 1.1.2. (i) Let $\mathcal{H}$ be any Hilbert space and $\mathcal{B}(\mathcal{H})$ be the set of bounded linear operators on $\mathcal{H}$. Then, $\mathcal{B}(\mathcal{H})$ is a C*-algebra, where the involution is given by the adjoint operator and the norm is the operator norm, that is $\|T\| = \sup_{\|x\|\leq 1} \|Tx\|$.

(ii) $\mathbb{C}$ is a C*-algebra considering the involution as the complex conjugation, and the norm as the module of a complex number. In fact, $\mathbb{C} \cong \mathcal{B}(\mathbb{C})$.

(iii) The algebra $M_n := M_n(\mathbb{C}) \cong \mathcal{B}(\mathbb{C}^n)$ is also a C*-algebra where the involution of a matrix is its traspose conjugate on $\mathbb{C}$ and the norm is the operator norm.

(iv) Let X be a compact Hausdorff space. We define

$$\mathrm{C}(X) := \{f : X \to \mathbb{C} \mid f \text{ is continuous }\}.$$

With pointwise addition and multiplication, the involution induced by complex conjugation ($f^*(x) = \overline{f(x)}$) and the norm as the supremum norm (i.e., $\|f\| = \sup_{x \in X} |f(x)|$), $\mathrm{C}(X)$ is a unital commutative C*-algebra.

(v) Let X be a locally compact Hausdorff space. We will denote by $\mathrm{C}_0(X)$ the algebra of continuous functions over X with values in $\mathbb{C}$ that vanish at infinity, i.e., continuous functions such that, for all $n \in \mathbb{N}$, the set $\{x \in X \mid f(x) \geq 1/n\}$ is a compact subset of X. As before, with the pointwise addition and multiplication, the involution as the complex conjugation and the norm as the supremum norm, it becomes a commutative C*-algebra. Note that $\mathrm{C}_0(X)$ is unital if and only if X is compact.

(vi) Given a set I and $\{A_i\}_{i \in I}$ a family of C*-algebras,

$$\prod_{i \in I} A_i := \{(a_i)_{i \in I} \mid \sup_{i \in I} \|a_i\|_i < \infty\}$$

is a C*-algebra with the involution defined in each component and the norm as the supremum norm.

Moreover,

$$\begin{aligned} \textstyle\bigoplus_{i \in I} A_i &= \{(a_i)_{i \in I} \mid \forall \varepsilon > 0 \ \exists \ F \subseteq I \text{ finite subset for which } \|a_i\| < \varepsilon \ \ \forall i \in I \setminus F\} \\ &= \{(a_i)_{i \in I} \mid \lim_i \|a_i\| = 0\} \end{aligned}$$

is also a C*-algebra with the same involution and norm. In fact, $\oplus_{i \in I} A_i$ is an ideal of $\prod_{i \in I} A_i$ (see below).

(vii) If $I = \{1, 2, \ldots, n\}$, then $A_1 \times A_2 \times \ldots \times A_n$ is a C*-algebra. In particular, given $n_1, n_2, \ldots, n_k \in \mathbb{N}$, one has that $M_{n_1}(\mathbb{C}) \times M_{n_2}(\mathbb{C}) \times \ldots \times M_{n_k}(\mathbb{C})$ is a C*-algebra.

(viii) Any *-closed and norm-closed subalgebra of $\mathcal{B}(\mathcal{H})$ is a C*-algebra.

A C*-subalgebra B of A is called *hereditary* if $0 \leq a \leq b$, with $a \in A$ and $b \in B$, implies that $a \in B$. We will denote by $A_b = \overline{bAb} = \mathrm{Her}(b)$ the hereditary C*-subalgebra of A generated by b. Further, if A is separable, any separable hereditary C*-subalgebra B of A is of the form $\overline{aAa}$ for some element $a \in B$ ([Mur90, Theorem 3.2.5])

Theorem 1.1.3. *Let B be a C*-subalgebra of a C*-algebra A. Then B is hereditary in A if and only if $bab' \in B$ for all $b, b' \in B$ and $a \in A$.*

An important class of hereditay subalgebras of a C*-algebra A is that consisting of the ideals. By an *ideal* in a C*-algebra we shall always understand a closed, two-sided ideal. Every ideal is automatically self-adjoint, and thereby a sub-C*-algebra. Assume that I is an ideal in a C*-algebra A. The quotient of A by I is

$$A/I = \{a + I \mid a \in A\}, \quad \|a + I\| = \inf\{\|a + x\| \mid x \in I\}, \quad \pi(a) = a + I.$$

In this way A/I becomes a C*-algebra, $\pi : A \to A/I$ is a *-homomorphism, called the *quotient mapping*, and $I = \mathrm{Ker}(\pi)$.

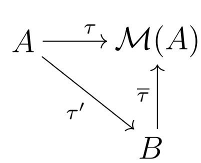
In order to associate a unital C*-algebra to any C*-algebra A there are two standard ways, the maximal and the minimal unitization of A. The first one uses the multiplier algebra, denoted by $\mathcal{M}(A)$. Let A be a C*-algebra, and let $f, g : A \to A$ be two maps (not necessarily morphisms). We shall say that (f, g) is a *double centraliser* for A if the conditions

$$\text{(i) } f(xy) = f(x)y, \;\; \text{(ii) } g(xy) = xg(y), \;\; \text{(iii) } xf(y) = g(x)y$$

are satisfied for all $x, y \in A$. We define the *multiplier algebra* of A as

$$\mathcal{M}(A) = \{(f, g) \mid (f, g) \text{ is a double centraliser for } A\}.$$

It turns out that $\mathcal{M}(A)$ is a unital C*-algebra with componentwise addition, the multiplication defined by $(f_1, g_1)(f_2, g_2) = (f_1 f_2, g_2 g_1)$ and unit as $(\mathrm{id}_A, \mathrm{id}_A)$, with the norm defined as $\|(f, g)\| = \|f\| = \|g\|$ and with the involution defined as $(f, g)^* = (g^*, f^*)$, where $f^*(a) = (f(a^*))^*$. Moreover, A is an essential ideal inside $\mathcal{M}(A)$ and there exists an injective morphism $\tau : A \to \mathcal{M}(A)$, where $\tau(a) = (L_a, R_a)$ and $(L_a(x), R_a(x)) = (ax, xa)$ for $a, x \in A$. Another important property of $\mathcal{M}(A)$ is its universality, i.e., for every unital C*-algebra B, and any injective morphism $\tau' : A \to B$ such that $\tau'(A)$ is an ideal of B there exists a unique morphism $\overline{\tau} : B \to \mathcal{M}(A)$ such that the following diagram



commutes. Note that $\mathcal{M}(A) = A$ if and only if A is unital.

Remark 1.1.4. Let $\mathcal{K}(\mathcal{H})$ be the C*-algebra of all compact operators over a Hilbert space $\mathcal{H}$. It can be checked that $\mathcal{M}(\mathcal{K}(\mathcal{H})) = \mathcal{B}(\mathcal{H})$, the set of all bounded linear operators over $\mathcal{H}$. In general the multiplier algebra of any C*-algebra A is much bigger than A. In fact, it is non-separable even when A is separable and infinite dimensional

In pursuance to define the minimal unitization of a C*-algebra A, let $\mathcal{B}(A)$ be the Banach algebra of all bounded operators on A, and consider $\iota \colon A \to \mathcal{B}(A)$ defined by $\iota(a) := [a' \mapsto aa']$. If we hence build

$$\tilde{A} := \iota(A) + \mathbb{C}I,$$

where I is the identity operator, $\tilde{A}$ is a normed algebra with the operations intherited from $\mathcal{B}(A)$ and the operator norm. We define the involution operation on $\tilde{A}$ by $(\iota(a) + \lambda I)^* := \iota(a^*) + \overline{\lambda} I$, so $\tilde{A}$ is a unital C*-algebra. Clearly A is an ideal of $\tilde{A}$, and $\tilde{A}/A$ is isomorphic to $\mathbb{C}$ when A is non-unital. When A is unital, it follows that $\mathcal{M}(A) \cong \tilde{A} \cong A$.

Definition 1.1.5. *Let A be a C*-algebra. An element a in A is called* self-adjoint *if $a = a^*$, and the set of self-adjoint elements is denoted by A_{sa}. Moreover, a is a* projection *if $a = a^* = a^2$. The set of all projections is denoted by $\mathcal{P}(A)$. If $a^*a = 1$ it is called* isometry, *and if it, furthermore, satisfies $aa^* = 1$, then a is a* unitary. *The set of all unitaries is denoted by $\mathcal{U}(A)$. We will say that $d \in A$ is a contraction if $\|d\| \leq 1$.*

Note that any element $a \in A$ can be written uniquely as a sum of two self-adjoint elements. Indeed,

$$a = \frac{1}{2}(a + a^*) + \frac{1}{2i}(a - a^*).$$

The following describes the structure of finite dimensional C*-algebras.

Theorem 1.1.6. *Let A be a finite dimensional C*-algebra. Then there exist $n_1, \ldots, n_r \in \mathbb{N}$ such that*

$$A \cong M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_r}(\mathbb{C}).$$

In the sequel we will often consider an important class of C*-algebras called AF-*algebras*. These are the C*-algebras built as inductive limits of finite dimensional C*-algebras. An important subclass of it is the class known as UHF-*algebras*. These are the inductive limit of sequences $\{A_n = M_{k_n}(\mathbb{C})\}$ with $k_n | k_{n+1}$. A specific class of UHF-algebras consists of the algebras such that $k_n = p^n$ for a prime integer p, these are usually denoted by M_{p^∞}. When $p = 2$, this is called the CAR-algebra and denoted by M_{2^∞}.

Let A be a unital C*-algebra and let a be an element in A. The *spectrum* of a is the set of complex numbers λ such that $a - \lambda 1_A$ is not invertible, and it is denoted either by $\mathrm{sp}(a)$ or by $\sigma(a)$. The spectrum $\mathrm{sp}(a)$ is a closed subset of $\mathbb{C}$, and in fact $\mathrm{sp}(a)$ is a compact subset of the complex plane. If A is non-unital, then embed A in its unitization $\tilde{A}$ and let $\mathrm{sp}(a)$ be the spectrum of a viewed as an element in $\tilde{A}$. If A is non-unital, then $0 \in \mathrm{sp}(a)$ for all a in A.

Using the above definition, an element a in A is called *positive*, denoted by $a \geq 0$, if it is self-adjoint and $\mathrm{sp}(a) \subseteq \mathbb{R}^+$ (with the convention that $0 \in \mathbb{R}^+$). We will denote the set of positive elements by A_+. It was shown by Kaplansky that $a \in A_+$ if and only if there exists $y \in A$ such that $a = yy^*$. We say that a in A is *normal* if $aa^* = a^*a$.

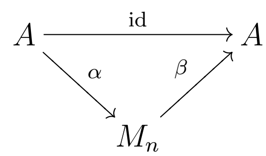
Another important class of C*-algebras we want to emphasize consists of the so-called *nuclear* C*-algebras. These class is large enough to include most C*-algebras which arise "naturally". For instance, all the finite dimensional and all commutative C*-algebras are nuclear. And, in addition, the nuclearity condition is closed under the formation of inductive limits, so AF-algebras are also nuclear.

Definition 1.1.7. *Let A and B be C*-algebras, and let $\varphi : A \to B$ be a linear function. φ is called* positive *if $a \in A_+$ implies $\varphi(a) \in B_+$. It is n-positive if $\varphi^n : M_n(A) \to M_n(B)$ is positive ($\varphi^n(a_{i,j}) = (\varphi(a_{i,j}))$), and φ is completely positive if it is n-positive for all n.*

Theorem 1.1.8. (see [Bla98, Theorem 15.8.1]) *Let A be a C*-algebra. The following are equivalent:*

(i) *For every C*-algebra B, the algebraic tensor product $A \odot B$ has a unique C*-norm.*

(ii) *The identity map A to A, as a completely positive map, approximately factors through matrix algebras, i.e., for any $x_1, \ldots x_k \in A$ and $\varepsilon > 0$, there is an n and completely positive maps α, β*



such that $\|x_j - \beta \circ \alpha(x_j)\| < \varepsilon$ for $1 \leq j \leq k$.

A C-algebra satisfying these conditions is called nuclear.*

The result below introduces what is meant by *functional calculus* for C*-algebras. It allows us to think of normal elements of a C*-algebra as "functions" over their spectra. The functional calculus is based on the following two results.

Theorem 1.1.9 (Gelfand)**.** *Let A be a commutative C*-algebra different from zero. Then, there exists a locally compact space X such that A is *-isomorphic to $\mathrm{C}_0(X)$.*

Theorem 1.1.10. *Let A be a unital C*-algebra, and let $a \in A$ be a normal element. Denote by z the inclusion of $\mathrm{sp}(a)$ to $\mathbb{C}$. Then, there exists a unique unital *-morphism $\varphi : \mathrm{C}(\mathrm{sp}(a)) \to A$ such that $\varphi(z) = a$. Moreover, φ is an isometry and $\mathrm{Im}(\varphi) = C^*(a, 1)$.*

A supplementary property of C*-algebras is their connection with Hilbert spaces, which is made via the notion of a state. We now recall the relevant concepts and make this connection explicit.

Definition 1.1.11. *Let A be a C*-algebra. A linear functional φ on A is positive, written $\varphi \geq 0$, if $\varphi(x) \geq 0$ whenever $x \geq 0$. A state on A is a positive linear functional of norm 1. Denote by $\mathrm{St}(A)$ the set of all states on A, called the state space of A. Moreover, say that a state φ is pure if it is an extreme point of $\mathrm{St}(A)$*

Note that the abundance of states is guaranteed by the following result:

Theorem 1.1.12. *Let A be a C*-algebra, $x \in A_{sa}$. Then there is a pure state φ on A which $|\varphi(x)| = \|x\|$.*

Definition 1.1.13. *Let A be a C*-algebra. We will define a representation of A as a pair $(\mathcal{H}, \pi)$, where $\mathcal{H}$ is a Hilbert space and $\pi : A \to \mathcal{B}(\mathcal{H})$ is a *-homomorphism. We say that $(\mathcal{H}, \pi)$ is faithful if π is injective. Moreover, $(\pi, \mathcal{H})$ is an* irreducible *representation of A if $\pi(A)' = \mathbb{C}$, where $\pi(A)'$ denotes the commutant of $\pi(A)$*

If $(\mathcal{H}_\lambda, \pi_\lambda)_{\lambda \in \Lambda}$ is a family of representations of A, their direct sum is the representation $(\mathcal{H}, \pi)$ obtained by setting $\mathcal{H} = \oplus_\lambda \mathcal{H}_\lambda$, and $\pi(a)((x_\lambda)_\lambda) = (\pi_\lambda(a)(x_\lambda))_\lambda$ for all $a \in A$ and all $(x_\lambda)_\lambda \in \mathcal{H}$.

Suppose now that τ is a positive linear functional on a C*-algebra A. Setting

$$N_\tau = \{a \in A \mid \tau(a^*a) = 0\},$$

it follows that it is a closed left ideal of A and that the map

$$\begin{aligned}(A/N_\tau) \times (A/N_\tau) &\to \mathbb{C}\\ (a + N_\tau, b + N_\tau) &\mapsto \tau(b^*a) = \langle a + N_\tau, b + N_\tau\rangle,\end{aligned}$$

is a well-defined inner product on A/N_τ. We denote by $\mathcal{H}_\tau$ the Hilbert completion of A/N_τ in which $\|a\|^2 = \langle a, a\rangle = \tau(a^*a)$.

If $a \in A$, define an operator $\pi(a) \in \mathcal{B}(A/N_\tau)$ by setting $\pi(a)(b + N_\tau) = ab + N_\tau$. Note that the inequality $\|\pi(a)\| \leq \|a\|$ holds. The operator $\pi(a)$ has a unique extension to a bounded operator $\pi_\tau(a)$ on $\mathcal{H}_\tau$. The map $\pi_\tau : A \to \mathcal{B}(\mathcal{H}_\tau)$ is a *-homomorphism, so $(\mathcal{H}_\tau, \pi_\tau)$ is a representation of A. The representation $(\mathcal{H}_\tau, \pi_\tau)$ of A is called the *Gelfand-Naimark-Segal representation* (or *GNS-representation*) associated to τ.

If A is non-zero, we define its *universal representation* as $\bigoplus_{\tau\in\mathrm{St}(A)}(\mathcal{H}_\tau, \pi_\tau)$.

Theorem 1.1.14 (Gelfand, Naimark)**.** *If A is a C*-algebra, then it has a faithful representation. Specifically, its universal representation is faithful.*

Theorem 1.1.15 (Gelfand, Naimark)**.** *For each C*-algebra A there exist a Hilbert space $\mathcal{H}$ and an isometric *-homomorphism φ from A into $\mathcal{B}(\mathcal{H})$, the algebra of all bounded linear operators on $\mathcal{H}$. In other words, every C*-algebra is isomorphic to a sub-C*-algebra of $\mathcal{B}(\mathcal{H})$. If A is separable, then $\mathcal{H}$ can be chosen to be a separable Hilbert space.*

We next introduce two important classes of maps for C*-algebras known as quasitraces and traces.

Definition 1.1.16. *A 1-quasitrace on a C*-algebra A is a function $\tau : A \to \mathbb{C}$ such that*

1. *$\tau(xx^*) = \tau(x^*x) \geq 0$ for any $x \in A$.*
2. *τ is linear on commutative *-subalgebras of A.*
3. *If $x = a + ib$, where a, b are self-adjoint, then $\tau(x) = \tau(a) + i\tau(b)$.*

If, in addition, τ extends to a map on $M_2(A)$ with the same properties, we shall say that τ is a 2-quasitrace. We will denote the space of normalized 2-quasitraces on A by $\mathrm{QT}(A)$.

Related to the last definition, we define a *trace* on A just as a linear quasitrace, and we denote the set of normalized traces on A as $\mathrm{T}(A)$. Recall that a trace or a quasitrace τ is normalized whenever its norm, $\|\tau\| = \sup\{\tau(a) \mid 0 \leq a \leq 1, \|a\| \leq 1\}$, equals one. In the case that A is unital, then this amounts to the requirement that $\tau(1) = 1$. We note that $\mathrm{T}(A) \subseteq \mathrm{QT}(A)$, and equality holds when A is exact [Haa91].

The theory of C*-algebras is often considered as non-commutative topology, which is justified by the natural duality between unital, commutative C*-algebras and the category of compact, Hausdorff spaces (see for instance part 1.11 of [WO93]). Given this fact, one tries to transfer concepts from commutative topology to C*-algebras. Along the following lines we will focus on some aspects of the theory of dimension for C*-algebras.

We say that a unital C*-algebra A is *finite*, if for any $x, y \in A$ with $xy = 1$ we also have $yx = 1$. A unital C*-algebra A is called *stably finite*, if $A \otimes M_k$ is finite for every $k \geq 1$. A C*-algebra is called *residually stably finite* if each of its quotients is stably finite.

Given $\mathcal{U}, \mathcal{V}$ two open covers of X, we say that $\mathcal{V}$ is an open refinement of $\mathcal{U}$ if for all $V \in \mathcal{V}$ there exists $U \in \mathcal{U}$ such that $V \subseteq U$. A cover $\mathcal{U} = \{U_\lambda\}_{\lambda \in \Lambda}$ has order k if every $x \in X$ belongs to at most k subsets in $\mathcal{U}$. Recall that, in a topological space X, the covering dimension, denoted by $\dim(X)$, is defined as the least n such that any open cover has an open refinement of order $\leq n+1$, or infinity in case this n does not exist.

The first generalization of classical dimension theory to non-commutative spaces was the stable rank as introduced by Rieffel [Rie83]. Later, Brown and Pedersen defined the real rank in a similar way in [BP91].

Let A be a unital C*-algebra and define

1. $Lg_n(A) := \{(a_1, \dots, a_n) \in A^n \mid \sum_{i=1}^n a_i^* a_i \in \mathrm{GL}(A)\}$,

2. $Lg_n(A)_{sa} := Lg_n(A) \cap (A_{sa})^n = \{(a_1, \dots, a_n) \in (A_{sa})^n \mid \sum_{i=1}^n a_i^2 \in \mathrm{GL}(A)\}$,

where $\mathrm{GL}(A)$ denotes the set of invertible elements of A. The abbreviation Lg stands for left generators, and the reason is that a tuple $(a_1, \dots, a_n) \in A^n$ lies in $Lg_n(A)$ if and only if $\{a_1, \dots, a_n\}$ generate A as a (not necessarily closed) left ideal.

Definition 1.1.17 (Rieffel [Rie83], Brown, Pedersen [BP91]). *Let A be a unital C*-algebra. The* stable rank *of A, denoted by $\mathrm{sr}(A)$, is the least integer $n \geq 1$ (or ∞) such that $Lg_n(A)$ is dense in A^n. The* real rank *of A, denoted by $\mathrm{RR}(A)$, is the least integer $n \geq 0$ (or ∞) such that $Lg_{n+1}(A)_{sa}$ is dense in A_{sa}^{n+1}.*

If A is not unital, define $\mathrm{sr}(A) := \mathrm{sr}(\tilde{A})$ and $\mathrm{RR}(A) := \mathrm{RR}(\tilde{A})$.

Remark 1.1.18. We have:

1. $\mathrm{sr}(A) \leq n \iff Lg_n(\tilde{A}) \subset (\tilde{A})^n$ is dense,

2. $\mathrm{RR}(A) \leq n \iff Lg_{n+1}(\tilde{A})_{sa} \subset (\tilde{A}_{sa})^{n+1}$ is dense.

Observe that the smallest possible value of the stable rank is one. If A is unital, then $Lg_1(A)$ consists precisely of the left invertible elements of A. Thus, A has stable rank one if and only if the left-invertible elements of A are dense in A. It was shown in Proposition 3.1 of [Rie83] that this is also equivalent to the condition that the invertible elements of A are dense in A.

Concerning the real rank, the smallest possible value of that is zero, which happens precisely if the self-adjoint invertible elements in A are dense in A_{sa}.

We next detail a collection of C*-algebras whose real rank is zero.

Examples 1.1.19. 1. Let A be a commutative C*-algebra; hence, $A = \mathrm{C}_0(X)$ for some locally compact Hausdorff space. It follows that $\mathrm{RR}(A) = \mathrm{RR}(\tilde{A}) = \dim(\alpha X)$, where αX is the one point compactification of X. So $\mathrm{RR}(A) = 0$ if and only if $\dim(X) = 0$. In this situation, as proved by Vaserstein, $\mathrm{sr}(A) = [\dim(X)/2] + 1$ (see e.g. [Bla06]).

2. The C*-algebra of bounded linear operators of a Hilbert space, $\mathcal{B}(\mathcal{H})$, has real rank zero. Concretely, when $\mathcal{H}$ has finite dimension n, one has $\mathcal{B}(\mathcal{H}) = M_n(\mathbb{C})$, so $\mathrm{RR}(M_n(\mathbb{C})) = 0$. Furthermore, $\mathrm{sr}(\mathcal{B}(\mathcal{H})) = \infty$ if $\mathcal{H}$ is an infinite dimensional Hilbert space.

3. Let $p \in A$ be a projection. We shall say that p is a *finite* projection if $p \neq 0$ and $p \sim_{M.-v.N.} q \leq p$ implies that $q = p$ (see Definition 1.3.1). A projection p is called *infinite* if it is not finite. A C*-algebra A is called *purely infinite* if every non zero hereditary C*-subalgebra of A has an infinite projection. It follows that if A is purely infinite and simple, it has real rank zero [Zha90, Theorem 1]. A standard example of purely infinite and simple C*-algebras are the *Cuntz algebras*. If $n \geq 2$, these are defined as the universal C*-algebras generated by isometries $s_1, \ldots, s_n$ such that $\sum_{i=1}^n s_i s_i^* = 1$, and they are denoted by $\mathcal{O}_n$ ([Cun81]).

4. Clearly, if $A_1, \ldots, A_n$ are C*-algebras with real rank zero, then $A_1 \oplus \ldots \oplus A_n$, with the maximum norm, is a C*-algebra with real rank zero. Therefore, the finite dimensional C*-algebras have real rank zero since their form is $M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_k}(\mathbb{C})$, for positive integers $n_1, \ldots, n_k$. In addition, AF-algebras have real rank zero since real rank zero is preserved under inductive limits ([BP91]).

5. A simple C*-algebra A has *slow dimension growth* if it can be written as $A = \varinjlim_n A_n$, where $A_n = \oplus_{1\leq k\leq L_n} M_{n_k}(\mathrm{C}(X_{n_k}))$ for some lenght L_n and for some compact Hausdorff spaces $\{X_{n_k}\}_{1\leq k\leq L_n}$, and it satisfies

$$\lim_n \max_{1\leq k\leq L_n} \left(\frac{\dim(X_{n_k})}{n_k}\right) = 0. \tag{1.1}$$

Recall that when $X_{n_k} = \{*\}$ for all n, k, then this is the definition of an AF-algebra. Further, when the condition 1.1 is not satisfied, but $X_{n_k} = [0,1]$ for all n, k, these algebras are called AI-algebras. It is proved in [BDR91] that if A is a C*-algebra with slow dimension growth, it has stable rank one. Moreover, A has real rank zero if and only if the projections set separate quasitraces (i.e., if $\tau \neq \tau'$ in $\mathrm{QT}(A)$, then there exists p projection such that $\tau(p) \neq \tau'(p)$).

6. A specific class of the algebras explained above consists of the so-called Goodearl algebras. Let X be a compact Hausdorff space. Consider $(k_n)_{n=1}^\infty$ and $(l_n)_{n=1}^\infty$ positive integers such that k_n divides k_{n+1} for each n and $l_n < k_{n+1}/k_n$, and take points $x_{n,i}$ in X for $i = 1, 2, \ldots, l_n$. Put $F_n = \{x_{n,1}, \ldots, x_{n,l_n}\}$. Associate to this the sequence:

$$M_{k_1}(\mathrm{C}(X)) \xrightarrow{\varphi_1} M_{k_2}(\mathrm{C}(X)) \xrightarrow{\varphi_2}, M_{k_3}(\mathrm{C}(X)) \xrightarrow{\varphi_3} \ldots,$$

where φ_n is the unital *-homomorphism defined by

$$\varphi_n(f)(x) = \mathrm{diag}(f(x_{n,1}), f(x_{n,2}), \ldots, f(x_{n,l_n}), f(x), f(x), \ldots, f(x)),$$

for $x \in X$ and $f \in M_{k_n}(\mathrm{C}(X))$. Let A be the inductive limit of the above sequence. One finds that A is simple if and only if $\bigcup_{n=k}^{\infty} F_k$ is dense in X for each k. A simple C*-algebra arising as the inductive limit of the above sequence is called *Goodearl algebra*. (See e.g. [Rør02, Example 3.1.7].)

Theorem 1.1.20. *Let A be a Goodearl algebra. Then:*

(i) $\mathrm{sr}(A) = 1$;

(ii) *if X is connected, then X has a unique trace if and only if*

$$\prod_{n=1}^{\infty} \frac{k_{n+1} - k_n l_n}{k_{n+1}} = 0,$$

and in this case, A has real rank zero.

1.1.1 $\mathrm{C}(X)$-algebras

Let X be a compact Hausdorff space. A $\mathrm{C}(X)$-*algebra* is a C^*-algebra A together with a unital *-homomorphism $\theta\colon \mathrm{C}(X) \to Z(\mathcal{M}(A))$, where $Z(\mathcal{M}(A))$ is the center of the multiplier algebra of A. It is pertinent to mention in this point that by a result of Cohen [Coh59] it follows that $\theta(\mathrm{C}(X))A$ is dense in A. The map θ is usually referred to as the *structure map*. We write fa instead of $\theta(f)a$ where $f \in \mathrm{C}(X)$ and $a \in A$. Note that if X is compact, then $\theta(1) = 1_{\mathcal{M}(A)}$.

If $U \subset X$ is an open set, then $A(U) = \mathrm{C}_0(U)A$ is a closed ideal of A. If $Y \subseteq X$ is a closed set, the restriction of $A = A(X)$ to Y, denoted by $A(Y)$, is the quotient of A by the ideal $A(X \setminus Y)$, which also becomes a $\mathrm{C}(X)$-algebra with the structure map $\theta\colon \mathrm{C}(Y) \to Z(\mathcal{M}(A(Y)))$. The quotient map is denoted by $\pi_Y\colon A \to A(Y)$, and if Z is a closed subset of Y we have a natural restriction map $\pi_Z^Y\colon A(Y) \to A(Z)$. Notice that $\pi_Z = \pi_Z^Y \circ \pi_Y$. If Y reduces to a point x, we write either A_x or $A(x)$ instead of $A(\{x\})$, and we denote by π_x the quotient map. The C^*-algebra A_x is called the *fiber* of A at x, and the image of $\pi_x(a) \in A_x$ will be denoted by either $a(x)$ or a_x.

A morphism of $\mathrm{C}(X)$-algebras $\eta : A \to B$ is defined as a *-morphism η such that $\eta(fa) = f\eta(a)$ for all $f \in \mathrm{C}(X)$, and it induces a morphism $\eta_Y : A(Y) \to B(Y)$ whenever $Y \subset X$ is a closed set.

Notation. *Let $a \in A$ and $\mathcal{F}, \mathcal{G} \subseteq A$, we write $a \in_\varepsilon \mathcal{F}$ for $\varepsilon > 0$ if there is $b \in \mathcal{F}$ such that $\|a - b\| < \varepsilon$. Similarly, we write $\mathcal{F} \subset_\varepsilon \mathcal{G}$ if $a \in_\varepsilon G$ for every $a \in \mathcal{F}$.*

The following Lemma collects some basic properties of $\mathrm{C}(X)$-algebras.

Lemma 1.1.21 ([Bla96], [Dad09a]). *Let A be a $\mathrm{C}(X)$-algebra and let $B \subset A$ be a $\mathrm{C}(X)$-subalgebra. Let $a \in A$ and let Y be a closed subset of X. Then the following conditions are satisfied:*

(i) $\|a\| = \sup_{x \in X} \|a_x\|$, so $\|\pi_Y(a)\| = \sup\{\|\pi_x(a)\| \mid x \in Y\}$.

(ii) *The map $x \mapsto \|a(x)\|$ is upper semicontinuous.*

(iii) *If $a(x) \in \pi_x(B)$ for all $x \in X$, then $a \in B$.*

(iv) *If* $\delta > 0$ *and* $a(x) \in_\delta \pi_x(B)$ *for all* $x \in X$*, then* $a \in_\delta B$*.*

(v) *The restriction of* $\pi_x : A \to A(x)$ *to* B *induces an isomorphism* $B_x \simeq \pi_x(B)$ *for all* $x \in X$*.*

Proof. (i): Let a be a nonzero element of A. By Theorem 1.1.12, there exists a pure state φ of A such that $\varphi(a^*a) = \|a\|^2$. If π_φ is the canonical irreducible representation of $\mathcal{M}(A)$ into $\mathcal{H}_\varphi$ associated to φ by the Gelfand-Naimark-Segal representation ([Bla06, Proposition 6.4.8]), it follows that $\pi_\varphi(\mathrm{C}(X)) \subset \mathbb{C}$. This means there exists $x \in X$ such that $\pi_\varphi(f) = f(x)$ for all $f \in \mathrm{C}(X)$. The representation π_φ factorizes through A_x, so $\varphi(a^*a) \le \|a_x\|^2$ implying the desired result.

(ii): Let $a \in A$, $x \in X$ and $\varepsilon > 0$. By the definition of the quotient norm, there exists $t = \sum_{i=1}^n f_i a_i$ where $f_i \in \mathrm{C}_0(X \setminus x)$ and $a_i \in A$ such that

$$\|\pi_x(a)\| \ge \|a - t\| - \varepsilon.$$

Let $g \in \mathrm{C}_0(X)$ with $\|g\| = 1$ such that $g = 1$ in a neighborhood $\mathcal{U}_x$ of x, but such that all gf_i are small enough so that $\|gt\| < \varepsilon$. Then

$$\|\pi_x(a)\| \ge \|a - t\| - \varepsilon \ge \|g(a - t)\| - \varepsilon \ge \|ga\| - 2\varepsilon = \|a - (1 - g)a\| - 2\varepsilon.$$

Since $1 - g$ is in $\mathrm{C}_0(X \setminus y)$ for all $y \in \mathcal{U}_x$, it follows that

$$\|\pi_x(a)\| \ge \|\pi_y(a)\| - 2\varepsilon.$$

(iii): This follows from (iv).

(iv): By assumption, for each $x \in X$, there is $b_x \in B$ such that $\|\pi_x(a - b_x)\| < \delta$. Using (i) and (ii), we find a closed neighborhood U_x of x such that $\|\pi_{U_x}(a - b_x)\| < \delta$. Since X is compact, there is a finite subcover (U_{x_i}). Let (α_i) be a partition of unity subordinated to this cover. Setting $b = \sum \alpha_i b_{x_i} \in B$, one checks immediately that $\|\pi_x(a - b)\| \le \sum \alpha_i(x)\|\pi_x(a - b_{x_i})\| < \delta$, for all $x \in X$. Thus, $\|a - b\| < \delta$ by (i).

(v): If $\iota : B \to A$ is the inclusion map, then $\pi_x(B)$ coincides with the image of

$$\iota_x : B/\mathrm{C}_0(X \setminus x)B \to A/\mathrm{C}_0(X \setminus x)A.$$

Thus, it suffices to check that ι_x is injective. If $\iota_x(b + \mathrm{C}_0(X \setminus x)B) = \pi_x(b) = 0$ for some $b \in B$, then $b = fa$ for some $f \in \mathrm{C}_0(X \setminus x)$ and some $a \in A$. If (f_λ) is an approximate unit of $\mathrm{C}_0(X \setminus x)$, then $b = \lim_\lambda f_\lambda f a = \lim_\lambda f_\lambda b$ and hence $b \in \mathrm{C}_0(X \setminus x)B$. □

A $\mathrm{C}(X)$-algebra such that the map $x \mapsto \|a(x)\|$ is continuous for all $a \in A$ is called either a *continuous* $\mathrm{C}(X)$-algebra or a C*-bundle or a continuous field C*-algebras, see [Dix77, Bla96]. Further, a $\mathrm{C}(X)$-algebra A is called trivial if there exists a C*-algebra D such that $A \cong \mathrm{C}(X) \otimes D$. We will see examples of non-trivial continuous fields in Chapter 2.

A remarkable characterization of separable exact continuous field C*-algebras is given by E. Blanchard and E. Kirchberg in [Bla97] where they proved the following.

Theorem 1.1.22. *Let* X *be a metrizable, compact Hausdorff space, and let* A *be a separable continuous field of C*-algebras. Then,* A *is exact if and only there exists a monomorphism of* $\mathrm{C}(X)$*-algebras* $A \hookrightarrow \mathrm{C}(X) \otimes \mathcal{O}_2$

Remark 1.1.23. For a continuous field A, a useful criterion to determine when an element $(a_x) \in \prod_{x \in X} A_x$ comes from an element of A is the following: given $\epsilon > 0$ and $x \in X$, if there is $b \in A$ and a neighborhood V of x such that $\|b(y) - a_y\| < \epsilon$ for $y \in V$, then there is $a \in A$ such that $a(x) = a_x$ for all x (see [Dix77, Proposition 10.2.2]).

1.2 Partially ordered semigroups

The main purpose of this section is to provide a quick review of those basic concepts from the general theory of partially ordered semigroups with neutral element that will be needed in the sequel. Throughout, all the semigroups will be abelian. We will use the additive notation, and so we will denote by 0 the neutral element.

Definition 1.2.1. *A* partial order *on a semigroup S is any reflexive, antisymmetric, transitive relation $\leq$ on S. A partially ordered semigroup is an abelian semigroup equipped with a specified translation-invariant partial order.*

Let S, R be partially ordered semigroups. We shall assume that a morphism between semigroups, $f : S \to R$, always preserves the order, i.e., if $x \leq y$ in S, then $f(x) \leq f(y)$ in R.

Definition 1.2.2. *Let S be a partially ordered semigroup. We say that an element u in S is an order-unit if $u \neq 0$ and for any $x \in S$ there exists $n \in \mathbb{N}$ such that $x \leq nu$. We denote a partially ordered semigroup with an order-unit u as (S, u).*

Definition 1.2.3. *Let (S, u) and (R, v) be partially ordered semigroups with order-units u and v, and $f : S \to R$ be a morphism of semigroups. We say f is normalized if $f(u) = v$.*

Let (S, u) be a partially ordered semigroup with order-unit. We will say that the order is *cancellative* if given $x, y, z \in S$ such that $x + z \leq y + z$, then $x \leq y$. Furthermore, S is *cancellative* if given $x, y, z \in S$ such that $x + z = y + z$, then $x = y$.

Definition 1.2.4. *Let $(S, \leq)$ be a partially ordered semigroup. We say that S is an* interpolation semigroup *provided that S satisfies the* Riesz Interpolation Property, *that is, whenever $x_1, x_2, y_1, y_2 \in S$ such that $x_i \leq y_j$ for $i, j = 1, 2$, there exists $z \in S$ such that $x_i \leq z \leq y_j$ for $i, j = 1, 2$.*

We next show a natural relation between semigroups and abelian groups. Let $(S, \leq)$ be a partially ordered semigroup. Define an equivalence relation on S by $x \sim y$ if an only if there exists $z \in S$ such that $x + z = y + z$. Set $S_c = S/\sim$ and denote the equivalence classes of the elements of S by $[x]$. Define an addition by $[x] + [y] = [x + y]$ for $x, y \in S$, and take $[x] \leq [y] \iff x + z \leq y + z$ for some $z \in S$, as an ordering. Notice that, with this structure, S_c is order-cancellative. It is called the cancellative monoid associated to S. If u is an order-unit for S, then $[u]$ will be an order-unit for S_c. Observe that as S is partially ordered, then so is S_c.

Now, by adjoining formal inverses to the elements of S_c we can define an abelian group, called the *Grothendieck group* of S and denoted by $\mathrm{G}(S)$. It is clear that $\mathrm{G}(S) = \{[x] - [y] \mid [x], [y] \in S_c\}$. Order $\mathrm{G}(S)$ by taking as positive cone:

$$\mathrm{G}(S)^+ = \{[x] - [y] \mid x, y \in S \text{ and } y \leq x\}.$$

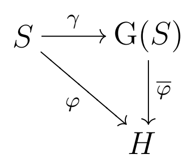
Notice that $\mathrm{G}(S)$ is partially ordered if S is, and that for $x, y, v, w \in S$,

$$[x] - [v] \leq [y] - [w] \text{ in } \mathrm{G}(S) \iff$$

$$\iff x + w + e \leq y + v + e \text{ in } S \text{ for some } e \in S.$$

One has a natural ordered map $\gamma\colon S \to \mathrm{G}(S)$ defined by $\gamma(x) = [x]$, called the Grothendieck map.

An important fact about the Grothendieck group is the universal property that it satisfies; it says that if there exists an ordered group H and an order preserving morphism $\varphi : S \to H$, then there exists a unique order-preserving morphism $\overline{\varphi} : \mathrm{G}(S) \to H$ such that the following diagram



is commutative.

We would like to remark that the notions already stated for semigroups have analogous versions for abelian groups. Therefore, we will speak of interpolation groups and states on groups without further discussion. Recall that an order-unit on a partially ordered group G is an element $u \in G \setminus \{0\}$ such that for all $x \in G$ there exists $n \in \mathbb{N}$ with $-nu \leq x \leq nu$.

Define a *convex set* as any subset of a real vector space E which is closed under convex combinations (linear combinations of the form $\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n$ such that $\alpha_i \geq 0$ for all i and $\alpha_1 + \ldots + \alpha_n = 1$). The convex hull of a subset $X \subset E$ is the smallest convex subset of E that contains X. We will denote by $\partial_e(K)$ the *extreme boundary* of any convex set K.

Definition 1.2.5. *Let (S, u) be a partially ordered semigroup with order-unit. A normalized* state *on S is an order preserving morphism $s : S \to \mathbb{R}$ such that $s(u) = 1$. We denote the set of states by $\mathrm{St}(S, u)$, which is a compact convex set* (see [Goo86, Proposition 6.2]).

In the sequel we will be interested in a specific type of convex sets, called Choquet simplices. This is motivated by the generalization of the classical notion of simplex to infinite dimension. Recall that a classical simplex in a vector space E is a convex subset of E built as the convex hull of a finite number of affinely independent points of E.

Definition 1.2.6. *Let E be a real vector space, and let $C \subseteq E$ be a convex subset of E. We say that C is a* convex cone *if $0 \in C$ and also $\alpha_1 x_1 + \alpha_2 x_2 \in C$ for any $\alpha_1, \alpha_2 \in \mathbb{R}^+$ and any $x_1, x_2 \in C$. A* strict convex cone *satisfies that the only point $x \in E$ for which $x, -x \in C$ is the point $x = 0$. In addition, we define a* base *for C as any convex subset K of C such that every nonzero point of C may be uniquely expressed in the form αx for $\alpha \in \mathbb{R}^+$ and $x \in K$.*

Definition 1.2.7. *A* lattice cone *in a real vector space E is any strict convex cone C in E such that $(C, \leq_C)$ is a lattice. Moreover, a* simplex *in E is any convex subset K of E that is affinely isomorphic to a base for a lattice cone in some real vector space.*

Definition 1.2.8. *Let E be locally convex Hausdorff vector space. If $K \subseteq E$ is a compact simplex, then we say that K is* Choquet simplex.

The result below relates some notions stated before, and it will be an important tool in Chapter four.

Theorem 1.2.9. (see [Goo96, Theorem 10.17]) *If (G, u) is an interpolation group with order-unit, then $\mathrm{St}(G, u)$ is a Choquet simplex.*

Remark 1.2.10. (i) If (S, u) is a partially ordered semigroup with the Riesz interpolation property, then $(\mathrm{G}(S), [u])$ is an interpolation group ([Per97]).

(ii) By the universal property of the Grothendieck group it follows that

$$\mathrm{St}(S, u) = \mathrm{St}(\mathrm{G}(S), [u]).$$

1.3 Invariants for C*-algebras

In this section we give a brief exposition on the more commonly used invariants for C*-algebras. This section is divided in two parts, where the first is about K-theory, and the second is about the Cuntz Semigroup. We have deeply used [LLR00], [WO93] for K-theory and [APT11] for the Cuntz semigroup.

1.3.1 The Murray-von Neuman Semigroup. K-theory.

Let A be a C*-algebra and consider $p, q \in \mathcal{P}(A)$. We say that p is Murray-von Neuman equivalent to q, denoted by $p \sim_{M-v.N} q$, if there exists $v \in A$ such that $p = v^*v$ and $q = vv^*$. An element v in A for which v^*v is a projection is called *partial isometry*. It follows that this is an equivalence relation.

Write $\mathcal{P}_n(A) = \mathcal{P}(M_n(A))$ and $\mathcal{P}_\infty(A) = \bigcup_{n=1}^{\infty} \mathcal{P}_n(A)$, where n is a positive integer. We view the sets $\mathcal{P}_n(A)$, $n \in \mathbb{N}$, as being pairwise disjoint.

In order to define the Murray-von Neumann semigroup, we consider the equivalence relation $\sim_0$ on $\mathcal{P}_\infty(A)$ given as follows. Suppose that p is a projection in $\mathcal{P}_n(A)$ and q is a projection in $\mathcal{P}_m(A)$. Then $p \sim_0 q$ if there is an element $v \in M_{m,n}(A)$ with $p = vv^*$ and $q = vv^*$.

Define a binary operation $\oplus$ on $\mathcal{P}_\infty(A)$ by $p \oplus q = \mathrm{diag}(p, q) = \left(\begin{smallmatrix} p & 0 \\ 0 & q \end{smallmatrix}\right)$, so that $p \oplus q$ belongs to $\mathcal{P}_{n+m}(A)$ when p is in $\mathcal{P}_n(A)$ and q is in $\mathcal{P}_m(A)$. The relation $\sim_0$ is an equivalence relation on $\mathcal{P}_\infty(A)$

Definition 1.3.1. *With $\mathcal{P}_\infty(A)$ and $\sim_0$ we define the* Murray-von Neumann semigroup *of A as*

$$\mathrm{V}(A) = \mathcal{P}_\infty(A)/ \sim_0 .$$

For each $p \in \mathcal{P}_\infty(A)$, let $[p]$ in $\mathrm{V}(A)$ denote the equivalence class containing p. Defining the addition as $[p] + [q] = [p \oplus q]$, it follows that $(\mathrm{V}(A), +)$ is an Abelian semigroup. We will say p is Murray-von Neumann subequivalent to q if p is equivalent to $p' \leq q$ (i.e. $p'q = q$), that is, $p = vv^*$, with $v^*v \leq q$.

Note that the above equivalence relation defines an order in $\mathrm{V}(A)$. Moreover, it is algebraic since $[p] \leq [q]$ in $\mathrm{V}(A)$ if and only if there exists $[r] \in \mathrm{V}(A)$ such that $[p]+[r]=[q]$. Indeed, if $p \sim_{M-v.N} p' \leq q$, considering $r = q - p'$, it follows that $[p]+[r]=[q]$.
We now proceed to define K_0 and K_1.

Definition 1.3.2. *Let A be a unital C*-algebra. Define $\mathrm{K}_0(A)$ to be the Grothendieck group of $\mathrm{V}(A)$, i.e.,*

$$\mathrm{K}_0(A) = \mathrm{G}(\mathrm{V}(A)).$$

Denote by $\gamma : \mathrm{V}(A) \to \mathrm{K}_0(A)$ the Grothendieck map.

Proposition 1.3.3. *Let A be a unital C*-algebra, then*

$$\mathrm{K}_0(A) = \{[p]_0 - [q]_0 \mid p, q \in \mathcal{P}_\infty(A)\} = \{[p]_0 - [q]_0 \mid p, q \in \mathcal{P}_n(A),\ n \in \mathbb{N}\}.$$

When A is not unital the definition of $\mathrm{K}_0(A)$ is a bit more elaborated. Let A be a non-unital C*-algebra, and consider the split exact sequence

$$0 \to A \xrightarrow{\iota} \tilde{A} \underset{\lambda}{\overset{\pi}{\rightleftarrows}} \mathbb{C} \to 0.$$

Applying the functor K_0, one has

$$\mathrm{K}_0(\tilde{A}) \xrightarrow{\pi_*} \mathrm{K}_0(\mathbb{C}).$$

Define $\mathrm{K}_0(A)$ to be the kernel of the homomorphism $\pi_* : \mathrm{K}_0(\tilde{A}) \to \mathrm{K}_0(\mathbb{C})$.
We remark that in ([Gli60]) was classified the UHF-algebras using the set of projections, and that this result was generalized to the AF-algebra case in [Ell76] in 1976. In fact, there was shown that $(\mathrm{K}_0(A), \mathrm{K}_0(A)^+, [1_A])$ is a complete invariant for this class of C*-algebras.

Let A be a unital C*-algebra and set $\mathcal{U}_n(A) = \mathcal{U}(M_n(A)),\ \mathcal{U}_\infty(A) = \bigcup_{n=1}^{\infty} \mathcal{U}_n(A)$. Consider a binary operation $\oplus$ on $\mathcal{U}_\infty(A)$ by

$$u \oplus v = \mathrm{diag}(u, v) = \left(\begin{smallmatrix} u & 0 \\ 0 & v \end{smallmatrix}\right) \in \mathcal{U}_{n+m}(A), \quad u \in \mathcal{U}_n(A),\ v \in \mathcal{U}_m(A).$$

Define a relation $\sim_1$ on $\mathcal{U}_\infty(A)$ as follows. For $u \in \mathcal{U}_n(A)$ and $v \in \mathcal{U}_m(A)$, write $u \sim_1 v$ if there exists a natural number $k \geq \max\{m, n\}$ such that $u \oplus 1_{k-n} \sim_h v \oplus 1_{k-m}$ in $\mathcal{U}_k(A)$, where 1_r is the unit in $M_r(A)$ and $\sim_h$ means homotopy. Let $\mathcal{U}_0(A)$ be the set of all $u \in \mathcal{U}(A)$ such that $u \sim_h 1$ in $\mathcal{U}(A)$. (Recall that $a \sim_h b$ in X if there is a continuous function $v : [0,1] \to X$ such that $v(0) = a$ and $v(1) = b$.)

Definition 1.3.4. *For each C*-algebra A define*

$$\mathrm{K}_1(A) = \mathcal{U}_\infty(\tilde{A}) / \sim_1 .$$

Let $[u]_1$ in $\mathrm{K}_1(A)$ denote the equivalence class containing $u \in \mathcal{U}_\infty(\tilde{A})$. Define a binary operation $+$ on $\mathrm{K}_1(A)$ by $[u]_1 + [v]_1 = [u \oplus v]_1$, where $u, v \in \mathcal{U}_\infty(\tilde{A})$. It follows that $+$ is well-defined, commutative, associative, has zero element $0 = [1]_1 (= [1_n]_1$ for each $n \in \mathbb{N})$, and that $0 = [1_n]_1 = [uu^*]_1 = [u]_1 + [u^*]_1$ for each $u \in \mathcal{U}_n(\tilde{A})$. This shows that $(\mathrm{K}_1(A), +)$ is an Abelian group. Concretely, $-[u]_1 = [u^*]_1$ for all $u \in \mathcal{U}_\infty(\tilde{A})$.

The following result states the universal property satisfied by $\mathrm{K}_1(A)$.

Proposition 1.3.5. *Let A be a C*-algebra, let G be an Abelian group, and let $\nu : \mathcal{U}_\infty(\tilde{A}) \to G$ be a map with the following properties:*

(i) $\nu(u \oplus v) = \nu(u) + \nu(v)$,

(ii) $\nu(1) = 0$,

(iii) *if u, v belong to $\mathcal{U}_n(\tilde{A})$ and $u \sim_h v$, then $\nu(u) = \nu(v)$.*

Then there exists a unique group homomorphism $\alpha : \mathrm{K}_1(A) \to G$ making the diagram

$$\begin{array}{ccc} \mathcal{U}_\infty(\tilde{A}) & & \\ \downarrow [\ ]_1 & \searrow \nu & \\ \mathrm{K}_1(A) & \xrightarrow{\alpha} & G \end{array}$$

commutative.

At the end of the 80s, G. A. Elliott conjectured that all simple separable nuclear C*-algebras would be classified by K-theory invariants. More precisely, the invariant that should be used is

$$\mathrm{Ell}(A) = ((\mathrm{K}_0(A), \mathrm{K}_0(A)^+, [1_A]), \mathrm{K}_1(A), \mathrm{T}(A), r_A),$$

where $r_A : \mathrm{T}(A) \times \mathrm{K}_0(A) \to \mathbb{R}$ is the pairing between $\mathrm{K}_0(A)$ and $\mathrm{T}(A)$ given by evaluation of a trace at a K_0-class. Along the 90s, Elliott's programme had a lot of good results such as the classification of the purely infinite simple C*-algebras carried out by Kirchberg and Phillips ([KP00b, KP00a]). But in the last decade some examples given by M. Rørdam and A. S. Toms ([Rør03, Tom08a]) have appeared and they show the conjecture is not true in general. As a consequence of the example described by A. S. Toms, the Cuntz semigroup defined by J. Cuntz in 1978 ([Cun78]) has obtained more relevance. Concretely, [Tom08a] provides an example of a simple, separable, unital, nuclear C*-algebra A such that A and $A \otimes U$, where U is a UHF-algebra, agree, not only on the Elliott invariant, but also on many more topological invariants, but they are differentiated by their Cuntz semigroup.

We next state some importants results on K-theory that relate K_0 and K_1. These will be used a number of times in the sequel.

Given an exact sequence

$$0 \longrightarrow J \xrightarrow{\iota} A \xrightarrow{\pi} A/J \longrightarrow 0,$$

there are induced sequences

$$\mathrm{K}_0(J) \xrightarrow{\iota_*} \mathrm{K}_0(A) \xrightarrow{\pi_*} \mathrm{K}_0(A/J)$$

$$\mathrm{K}_1(J) \xrightarrow{\iota_*} \mathrm{K}_1(A) \xrightarrow{\pi_*} \mathrm{K}_1(A/J)$$

which are exact in the middle (see e.g. [WO93]). Although these sequences cannot be made exact at the ends by adding 0's in general, there exists a connecting map $\delta : \mathrm{K}_1(A/J) \to \mathrm{K}_0(A)$ which makes the following long exact sequence:

$$\mathrm{K}_1(J) \xrightarrow{\iota_*} \mathrm{K}_1(A) \xrightarrow{\pi_*} \mathrm{K}_1(A/J) \xrightarrow{\delta} \mathrm{K}_0(J) \xrightarrow{\iota_*} \mathrm{K}_0(A) \xrightarrow{\pi_*} \mathrm{K}_0(A/J).$$

Theorem 1.3.6. *Let A be a C*-algebra, and let $SA = \mathrm{C}_0(\mathbb{R}) \otimes A$ (called the suspension of A). Then, there exists isomorphisms $\theta_1 : \mathrm{K}_1(A) \to \mathrm{K}_0(SA)$ and $\theta_0 : \mathrm{K}_0(A) \to \mathrm{K}_1(SA)$.*

Using the above Theorem and the defined connecting map δ, one has the following map

$$\beta\colon \mathrm{K}_0(A/J) \cong \mathrm{K}_1(S(A/J)) \xrightarrow{\delta} \mathrm{K}_0(SJ) \cong \mathrm{K}_1(J),$$

which allows us to obtain the cyclic 6-term exact sequence:

$$\begin{array}{ccccc} \mathrm{K}_0(J) & \xrightarrow{\iota_*} & \mathrm{K}_0(A) & \xrightarrow{\pi_*} & \mathrm{K}_0(A/J) \\ {\scriptstyle\delta}\uparrow & & & & \downarrow{\scriptstyle\beta} \\ \mathrm{K}_1(A/J) & \xleftarrow[\pi_*]{} & \mathrm{K}_1(A) & \xleftarrow[\iota_*]{} & \mathrm{K}_1(J). \end{array}$$

(see [WO93] for further details).

Considering the following diagram of C*-algebras

$$A \xrightarrow{\pi} C \xleftarrow{\gamma} B,$$

the pullback of the above diagram is the C*-algebra

$$P = \{(a, b) \in A \oplus B \mid \pi(a) = \gamma(b)\}.$$

The next result we want to provide is the computation of K-theory of a pullback of C*-algebras. This can be achieved thanks to the Mayer-Vietoris sequence in K-theory. Consider a pullback diagram of C*-algebras

$$\begin{array}{ccc} P & \xrightarrow{\pi_1} & A_1 \\ \downarrow{\scriptstyle\pi_2} & & \downarrow{\scriptstyle\alpha_1} \\ A_2 & \xrightarrow{\alpha_2} & B \end{array} .$$

Then, one obtains the following long exact sequence where $\mathrm{K}_n(A) = \mathrm{K}_0(S^n A)$.

$$\cdots \longrightarrow \mathrm{K}_2(B) \xrightarrow{\delta} \mathrm{K}_1(P) \xrightarrow{(\pi_1)_* \oplus (\pi_2)_*} \mathrm{K}_1(A_1) \oplus \mathrm{K}_1(A_2) \xrightarrow{(\alpha_2)_* - (\alpha_1)_*} \mathrm{K}_1(B) \xrightarrow{\delta}$$

$$\xrightarrow{\delta} \mathrm{K}_0(P) \xrightarrow{(\pi_1)_* \oplus (\pi_2)_*} \mathrm{K}_0(A_1) \oplus \mathrm{K}_0(A_2) \xrightarrow{(\alpha_2)_* - (\alpha_1)_*} \mathrm{K}_0(B).$$

which can be turned into a cyclic sequence by the inductive definition of $\mathrm{K}_n(\_)$

$$\begin{array}{ccccc} \mathrm{K}_1(P) & \xrightarrow{(\pi_1)_*\oplus(\pi_2)_*} & \mathrm{K}_1(A_1)\oplus \mathrm{K}_1(A_2) & \xrightarrow{(\alpha_2)_*-(\alpha_1)_*} & \mathrm{K}_1(B) \\ \delta\uparrow & & & & \downarrow\delta \\ \mathrm{K}_0(B)\cong \mathrm{K}_2(B) & \xleftarrow[(\alpha_2)_*-(\alpha_1)_*]{} & \mathrm{K}_0(A_1)\oplus \mathrm{K}_0(A_2) & \xleftarrow[(\pi_1)_*\oplus(\pi_2)_*]{} & \mathrm{K}_0(P). \end{array}$$

The last result on K-theory we want to emphasize relates the tensor product of C*-algebras and $\mathrm{K}_0, \mathrm{K}_1$. Let us denote $\mathrm{K}_*(A) = \mathrm{K}_0(A) \oplus \mathrm{K}_1(A)$ and next define the *(large) bootstrap class* of C*-algebras, denoted by $\mathcal{N}$.

Definition 1.3.7. *$\mathcal{N}$ is the smallest class of separable nuclear C*-algebras satisfying:*

(i) *$\mathcal{N}$ contains $\mathbb{C}$,*

(ii) *$\mathcal{N}$ is closed under inductive limits,*

(iii) *If $0 \to J \to A \to A/J \to 0$ is an exact sequence, and two of $J, A, A/J$ are in $\mathcal{N}$, so is the third.*

Theorem 1.3.8 (Künneth Theorem)**.** *Let A and B be C*-algebras, with $A \in \mathcal{N}$. Then there is a short exact sequence natural in each variable*

$$0 \to \mathrm{K}_*(A) \otimes \mathrm{K}_*(B) \xrightarrow{\alpha} \mathrm{K}_*(A \otimes B) \xrightarrow{\sigma} \mathrm{Tor}_1^{\mathbb{Z}}(\mathrm{K}_*(A), \mathrm{K}_*(B)) \to 0.$$

So, if $\mathrm{K}_*(A)$ or $\mathrm{K}_*(B)$ is torsion-free, α is an isomorphism. Without entering on the defintion of $\mathrm{Tor}_1^{\mathbb{Z}}(\mathrm{K}_*(A), \mathrm{K}_*(B))$, we say it measures the deviation from exactness of the tensor product functor on groups.

We end this section with the next important example of K-theory.

Example 1.3.9. Let $\mathbb{C}P^1$ be the complex projective plane. Note that writing ∞ as the point $[(0,1)] \in \mathbb{C}P^1$, it follows that $\mathbb{C}P^1 \setminus \{\infty\}$ is homeomorphic to $\mathbb{R}^2$, whereas $\mathbb{C}P^1$ is homeomorphic to S^2. On the other hand, it can be seen that there exists an isomorphism between $\mathbb{C}P^1 \setminus \{\infty\}$ and $\mathcal{P}(M_2(\mathbb{C})) \setminus \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ given by

$$[(1, z)] \mapsto 1/(1 + |z|^2) \begin{pmatrix} 1 & \overline{z} \\ z & |z|^2 \end{pmatrix}.$$

Clearly, composing the above isomorphism with the homemorphism, it follows that S^2 is isomorphic to $\mathcal{P}(M_2)$. The function $\mathrm{Bott} \in \mathrm{C}(\mathbb{R}^2, M_2)$ defined by above map is called the *Bott projection* for $\mathbb{R}^2$ and is extremely important in K-theory as it shown below.

Let $A = \mathrm{C}(\mathbb{T})$, where $\mathbb{T}$ can be thought of as the one point compatification of $\mathbb{R}$. It can be seen that $\mathrm{K}_0(\mathrm{C}(\mathbb{T})) = \mathbb{Z}$, so the equality $\mathrm{K}_0(\tilde{A}) = \mathrm{K}_0(A) \oplus \mathbb{Z}$, which comes from the natural exact sequence, tells us that $\mathrm{K}_0(\mathrm{C}_0(\mathbb{R})) = 0$. In this situation, we can see that all the projections of A come from the trivial projections arising from adjoining a unit to $\mathrm{C}_0(\mathbb{R})$.

However, when we move to two dimensions, the Bott projection plays an important role. Let $A = \mathrm{C}(S^2)$. It can be checked that $\mathrm{K}_0(A) = \mathbb{Z} \oplus \mathbb{Z}$. Thus, this K_0-group contains elements that are non-trivial, in the sense they do not come from the adjoined unit of $\mathrm{C}_0(\mathbb{R}^2)$, but rather from $\mathrm{K}_0(\mathrm{C}_0(\mathbb{R}^2)) \cong \mathbb{Z}$. The importance of the Bott projection falls on the fact that it can be used to built one of these non-zero element of $\mathrm{K}_0(\mathrm{C}_0(\mathbb{R}^2))$ (see e.g. [WO93] for further details).

The above example shows the necessity of considering projections in matrix algebras over A since unexpected projections sometimes appear, which have nothing to do with projections in A.

1.3.2 The Cuntz Semigroup

Cuntz comparison

Let A be a C*-algebra, and let $M_n(A)$ denote the $n \times n$ matrices whose entries are elements of A. Let $M_\infty(A)$ denote the algebraic limit of the directed system $(M_n(A), \phi_n)$, where $\phi_n : M_n(A) \to M_{n+1}(A)$ is given by $a \mapsto \left(\begin{smallmatrix} a & 0 \\ 0 & 0 \end{smallmatrix}\right)$.

Definition 1.3.10. *Let A be a C*-algebra, and let $a, b \in A_+$. We say that a is* Cuntz subequivalent *to b, in symbols $a \precsim b$, provided there is a sequence (x_n) in A such that $x_n b x_n^*$ converges to a in norm. We say that a and b are* Cuntz equivalent *if $a \precsim b$ and $b \precsim a$, and in this case we write $a \sim b$. Upon extending this relation to $M_\infty(A)_+$, one obtains an abelian semigroup $\mathrm{W}(A) = M_\infty(A)_+/\sim$. Denote the equivalence classes by $\langle a \rangle$. The operation and order are given by*

$$\langle a \rangle + \langle b \rangle = \langle \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \rangle = \langle a \oplus b \rangle, \quad \langle a \rangle \leq \langle b \rangle \text{ if } a \precsim b.$$

The semigroup $\mathrm{W}(A)$ is referred to as the Cuntz semigroup.

Proposition 1.3.11. *Let X be a compact Hausdorff space, and let $f, g \in \mathrm{C}(X)_+$. Then $f \precsim g$ if and only if $\mathrm{supp}(f) \subseteq \mathrm{supp}(g)$.*

Proof. Assume that $\mathrm{supp}(f) \subseteq \mathrm{supp}(g)$. Given $\varepsilon > 0$, set $K = \{x \in X \mid f(x) \geq \varepsilon\}$. Necessarily then $K \subset \mathrm{supp}(g)$. Since g is continuous on K and K is compact, there is a positive $\delta < \varepsilon$ such that $g > \delta$ on K. Put $U = \{x \in X \mid g(x) > \delta\}$, which is open and contains K. Use Urysohn's Lemma to find a funcion h such that $h_{|K} = 1$ and $h_{|X\setminus U} = 0$, and then consider the function e defined $e_{|U} = (\frac{h}{g})_{|U}$ and $e_{|X\setminus U} = 0$. Then one may check that e is continous and $\|f - egf\| < \varepsilon$. $\square$

Corollary 1.3.12. *For any C*-algebra A and any $a \in A_+$, we have $a \sim a^n$ $(n \in \mathbb{N})$. For any $a \in A$, we have $aa^* \sim a^*a$.*

Proof. Notice that $a \in C^*(a) \cong C^*(\sigma(A))$, so the first part of the statement follows from Proposition 1.3.11 since a and a^n have the same support. Next, for any a one has $aa^* \sim (aa^*)^2 = aa^*aa^* \precsim a^*a$, and by symmetry, $aa^* \sim a^*a$. $\square$

As it follows from the example below, in general, the natural order on $\mathrm{W}(A)$ does not agree with the algebraic order.

Example 1.3.13. Let $A = \mathrm{C}([0,1])$. Take $f, g \in A$ such that f is defined as linear increasing in $[1/3, 1/2]$, linear (decreasing) in $[1/2, 2/3]$, and zero elsewhere; g is linear (increasing) in $[0, 1/3]$, linear (decreasing) in $[1/3, 2/3]$, and zero elsewhere. By Proposition 1.3.11 $f \precsim g$. Seeking a contradiction, suppose there exists $y = \langle h \rangle \in \mathrm{W}(A)$, with $h \in M_n(A)_+$, such that $\langle f \rangle + y = \langle g \rangle$. Therefore,

$$\begin{pmatrix} f & 0 & \cdots & 0 \\ 0 & h_{1,1} & \cdots & h_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & h_{n,1} & \cdots & h_{n,n} \end{pmatrix} \sim \begin{pmatrix} g & 0 \\ 0 & 0 \end{pmatrix}.$$

By the Cuntz equivalence, there exist functions $\alpha_0^k, \alpha_1^k, \ldots, \alpha_n^k$ for $k \in \mathbb{N}$ with

$$\begin{pmatrix} \alpha_0^k & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_n^k & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} g & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \overline{\alpha_0^k} & \cdots & \overline{\alpha_n^k} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \longrightarrow_k \begin{pmatrix} f & 0 & \cdots & 0 \\ 0 & h_{1,1} & \cdots & h_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & h_{n,1} & \cdots & h_{n,n} \end{pmatrix}.$$

Hence, $\alpha_0^k\overline{\alpha_i^k} \to 0$, $|\alpha_0^k|^2 g \to f$, $\alpha_i^k\overline{\alpha_j^k} g \to h_{i,j}$ when $k \to \infty$. This means that

$$f = \varinjlim |\alpha_0^k|^2 g \quad \text{and that} \quad |h_{i,j}| = \varinjlim |\alpha_i^k \alpha_j^k| g.$$

It follows that $f|h_{i,j}| = 0$ for all i, j and that $\operatorname{supp}(f) \sqcup (\cup_{i,j} \operatorname{supp}(h_{i,j})) \subseteq \operatorname{supp}(g)$ since $\operatorname{supp}(h_{i,j}) = \operatorname{supp}(|h_{i,j}|)$.

On the other hand, by our hyphoteses, there exist functions $a_{i,j}^k$ for $0 \leq i, j \leq n$ such that

$$(a_{i,j}^k) \begin{pmatrix} f & 0 \\ 0 & h \end{pmatrix} (\overline{a_{i,j}^k}) \to \begin{pmatrix} g & 0 \\ 0 & 0 \end{pmatrix}.$$

Concretely, $|a_{0,0}^k|^2 f + (|a_{0,1}^k|^2 h_{1,1} + \ldots + \overline{a_{0,1}^k} a_{0,n}^k h_{n,1}) + \ldots + (\overline{a_{0,n}^k} a_{0,1}^k h_{1,n} + \ldots + |a_{0,n}^k|^2 h_{n,n}) \to g$. So $\operatorname{supp}(g) \subseteq \operatorname{supp}(f) \sqcup (\cup_{i,j} \operatorname{supp}(h_{i,j}))$. Now, as $\operatorname{supp}(g)$ is connected and $\operatorname{supp}(f) \neq \emptyset$, we see that $\operatorname{supp}(g) = \operatorname{supp}(f)$, but this is a contradiction by construction of g and f.

Given $a \in M_\infty(A)_+$ and $\varepsilon > 0$, we denote by $(a-\varepsilon)_+$ the element of $C^*(a)$ corresponding (via functional calculus) to the function

$$f(t) = \max\{0, t - \varepsilon\}, \; t \in \sigma(a).$$

We quote below some results which will be used frequently in the sequel; however, we will not prove all of them.

Theorem 1.3.14. ([KR02, Lemma 2.2]) *Let A be a C*-algebra, and $a, b \in A_+$. Let $\varepsilon > 0$, and suppose that $\|a - b\| < \varepsilon$. Then there is a contraction d in A with $(a-\varepsilon)_+ = dbd^*$.*

Theorem 1.3.15. ([Ped87, Corollary 8]) *Let A be a C*-algebra and $a \in A$ with $\operatorname{dist}(a, \operatorname{GL}(\tilde{A})) = 0$. If a has polar decomposition $a = v|a|$, then given $f \in \mathrm{C}(\sigma(|a|))_+$ that vanishes on a neighbourhood of zero, there is a unitary $u \in U(\tilde{A})$ such that $vf(|a|) = uf(|a|)$, and therefore $uf(|a|)u^* = vf(|a|)v^* = f(|a^*|)$.*

Lemma 1.3.16. *Let A be a C*-algebra. Given $b \in A_+$, the set $\{a \in A_+ \mid a \precsim b\}$ is norm-closed.*

Proof. Suppose $a = \varinjlim a_n$ with $a_n \precsim b$ for all n. Given $n \in \mathbb{N}$, choose $a_m \in A$ with m depending on n such that $\|a - a_m\| < \frac{1}{2n}$ and $x_n \in A$ such that $\|a_m - x_n b x_n^*\| < \frac{1}{2n}$. Therefore

$$\|a - x_n b x_n^*\| \leq \|a - a_m\| + \|a_m - x_n b x_n^*\| \leq 1/n.$$

□

Making an abuse of notation, we will denote by $\mathcal{K}$ the C*-algebra of compact operators on a separable, infinite-dimensional Hilbert space. We will say that a C*-algebra A is *stable* if $A \otimes \mathcal{K} \cong A$.

The following summarizes some technical properties of Cuntz subequivalence.

Proposition 1.3.17 ([Rør92],[KR02]). *Let A be a C*-algebra, and $a, b \in A_+$. The following conditions are equivalent:*

(i) $a \precsim b$.

(ii) *For every $\epsilon > 0$, $(a - \epsilon)_+ \precsim b$.*

(iii) *For every $\epsilon > 0$, there exists $\delta > 0$ such that $(a - \epsilon)_+ \precsim (b - \delta)_+$.*

Furthermore, if A is stable, these conditions are equivalent to

(iv) *For every $\epsilon > 0$ there is a unitary $u \in U(\tilde{A})$ such that $u(a - \epsilon)_+ u^* \in \mathrm{Her}(b)$.*

Proof. (i)⇒(ii): There is by assumption a sequence (x_n) such that $a = \varinjlim_n x_n^* b x_n$. Given $\varepsilon > 0$, we find n such that $\|a - x_n^* b x_n\| < \varepsilon$. Thus, Theorem 1.3.14 implies that $(a - \varepsilon)_+ = d x_n^* b x_n d$, for some d. Therefore, $(a - \varepsilon)_+ \precsim b$.

(ii)⇒(iii): Given $\varepsilon > 0$, there is by (ii) an element x such that $\|(a - \varepsilon/2)_+ - xbx^*\| = \varepsilon_1 < \varepsilon/2$. Since $(b - \delta)_+$ is monotone increasing and converges to b as $\delta \to 0$, we may choose $\delta < \frac{\varepsilon/2 - \varepsilon_1}{\|x\|^2}$. Therefore,

$$\|(a - \varepsilon/2) - x(b - \delta)_+ x^*\| \leq \|(a - \varepsilon/2)_+ - xbx^*\| + \|xbx^* - x(b - \delta)_+ x^*\| \leq \varepsilon_1 + \|x\|^2 \delta < \varepsilon/2,$$

so by Theorem 1.3.14 $(a - \varepsilon)_+ = y(b - \delta)_+ y^* \precsim (b - \delta)_+$.

(iii)⇒(i): By assumption we have that $(a - \varepsilon)_+ \precsim b$ for all $\varepsilon > 0$, so Lemma 1.3.16 implies $a \precsim b$. The "if" direction in the last part of our statement holds without any stability conditions. Namely, assume that $\varepsilon > 0$ is given, and that we can find a unitary u such that $u^*(a - \varepsilon)_+ u \in \overline{bAb}$. This implies that $u^*(a - \varepsilon)_+ u \precsim b$, and so $(a - \varepsilon)_+ \precsim b$ and condition (ii) is verified.

For the converse, assume that A is stable and that $a \precsim b$. Given $\varepsilon > 0$, find an element x such that $(a - \varepsilon/2)_+ = xbx^* = zz^*$, where $z = xb^{1/2}$ and $z^* z = b^{1/2} x^* x b^{1/2} \in \mathrm{Her}(b)$. By [BRT+12, Lemma 4.8], we know that $A \subset \overline{\mathrm{GL}(\tilde{A})}$. Since then $\mathrm{dist}(z^*, \mathrm{GL}(\tilde{A})) = 0$, one applies Theorem 1.3.15 to find $u \in \mathcal{U}(\tilde{A})$ with

$$u(a - \varepsilon)_+ u^* = v(a - \varepsilon)_+ v^* = (z^* z - \varepsilon/2)_+,$$

where v is the partial isometry in the polar decomposition of z^*, and $(z^* z - \varepsilon/2)_+ \in \mathrm{Her}(b)$. □

For a unital C*-algebra A given $\tau \in \mathrm{T}(A)$ and $a \in M_\infty(A)_+$, we may define

$$d_\tau(a) = \lim_{n\to\infty} \tau(a^{1/n}).$$

It turns out that $d_\tau : M_\infty(A)_+ \to \mathbb{R}^+$ is lower semicontinuous, that $d_\tau(1) = 1$ and that it does not depend on the Cuntz class of a; thus, it defines a state on $\mathrm{W}(A)$. Using this, we say that A has *strict comparison* if for $a, b \in M_\infty(A)_+$ with $d_\tau(a) < d_\tau(b)$ for all $\tau \in \mathrm{QT}(A)$, then $a \precsim b$. Related to this, we say that $\mathrm{W}(A)$ is *almost unperforated* provided that whether $(n+1)x \leq ny$ for $x, y \in \mathrm{W}(A)$ and $n \in \mathbb{N}$, this implies that $x \leq y$. In particular, a simple C*-algebra A has strict comparison if and only if $\mathrm{W}(A)$ is almost unperforated ([Rør92]).

The states on $\mathrm{W}(A)$ are called *dimension functions* on A, and the set of them is denoted by $\mathrm{DF(A)} = \mathrm{St}(\mathrm{W}(A), \langle 1\rangle)$. We will denote $\mathrm{G}(\mathrm{W}(A))$ by $\mathrm{K}_0^*(A)$ ([Cun78]). Recall that, by Remark 1.2.10, $\mathrm{DF}(A) = \mathrm{St}(\mathrm{K}_0^*(A), [1]_A)$. A dimension function d is *lower semicontinuous* if $d(\langle a\rangle) \leq \liminf_{n\to\infty} s(\langle a_n\rangle)$ whenever $a_n \to a$ in norm. We shall denote the set of them as $\mathrm{LDF}(A)$.

Theorem 1.3.18 ([BH82]). *There is an affine bijection* $\mathrm{QT}(A) \to \mathrm{LDF}(A)$*, defined by* $\tau \mapsto d_\tau$*, whose inverse is continuous.*

It is pertinent at this stage to mention that Blackadar and Handelman posed in [BH82] two conjectures related with the structure of dimension functions of a C*-algebra A. As explained in the introduction, they conjectured that $\mathrm{DF}(A)$ is a Choquet simplex and that $\mathrm{LDF}(A)$ is dense in $\mathrm{DF}(A)$. We will deeply work on these conjectures in chapter 4, where we study them for some classes of $\mathrm{C}(X)$-algebras, where X is a compact Hausdorff space of dimension less or equal to one.

We remark that these questions are known to have positive answers for algebras that have a good representations of their Cuntz semigroup (see [BH82], [Per97], [APT11] for further details).

Cuntz comparison and projections

In this part we will explain the important role of the projections inside the Cuntz semigroup $\mathrm{W}(A)$. We will mainly relate $\mathrm{V}(A)$ with $\mathrm{W}(A)$ (see Definition 1.3.10). Recall that given projections p, q, we have $p \leq q$ (as positive elements) if and only if $p = pq$, and that p is Murray-von Neumann (M-v.N) subequivalent to q if p is equivalent to $p' \leq q$, that is, $p = vv^*$, with $v^*v \leq q$.

Lemma 1.3.19. *For projections p and q, we have that p is M-v.N subequivalent to q if and only if $p \precsim q$.*

Proof. It is clear that, if p is M-v.N subequivalent to q, then $p \precsim q$. For the converse, if $p \precsim q$, then given $0 < \varepsilon < 1$, we have $(p-\varepsilon)_+ = xqx^*$, and $(p-\varepsilon)_+ = \lambda p$ for some positive λ by functorial calculus. Therefore, changing notation we have $p = xqx^*$, which implies that $qx^*xq \leq q$ is a projection equivalent to p □

Let $\varepsilon > 0$. We define $f_\varepsilon(t)$ as the real function such that it is zero in $(-\infty, \varepsilon/2]$, it is linear in $[\varepsilon/2, \varepsilon]$ and it is 1 in $[\varepsilon, \infty)$.

Lemma 1.3.20. *If p is a projection, a is a positive element, and $p \precsim a$, then there is $\delta > 0$ and a projection $q \leq \lambda a$ ($\lambda \in \mathbb{R}^+$) with $p \sim_{M-v.N} q$ and $f_\delta(a)q = q$.*

Proof. Let $\varepsilon > 0$, so $(p-\varepsilon)_+ = \lambda' p$ for a positive number λ'. Then, by Proposition 1.3.17, there is a $\delta' > 0$ and an element $x \in pA$ with $\lambda' p = x(a-\delta')_+ x^*$. Making an abuse of notation, we have $p = x(a-\delta')_+ x^*$. Thus $q := (a-\delta')_+^{1/2} x^* x (a-\delta')_+^{1/2}$ is a projection equivalent to p and $q \leq \|x\|^2 a$. On the other hand, it is clear by the definition of q that we can choose $\delta < \delta'$ (e.g. $\delta = \delta'/2$) such that $f_\delta(a) q = q$. □

Lemma 1.3.21. *If A is stably finite (and, in particular, if it has stable rank one), then the natural map $\mathrm{V}(A) \to \mathrm{W}(A)$ is injective.*

Proof. Suppose that we are given projections p, q in $M_\infty(A)_+$ such that $p \sim q$, i.e., $p \precsim q$ and $q \precsim p$. Since the order in $\mathrm{V}(A)$ is algebraic, by Lemma 1.3.19 one concludes that there are projections p' and q' such that $p \oplus p' \sim_{M-v.N} q$ and $q \oplus q' \sim_{M-v.N} p$. Thus $p \oplus p' \oplus q' \sim_{M-v.N} q \oplus q' \sim_{M-v.N} p$. It follows from stable finiteness that $p' \oplus q' = 0$, i.e. $p' = q' = 0$, so that $p \sim_{M-v.N} q$. □

From now on, we identify $\mathrm{V}(A)$ with its image inside $\mathrm{W}(A)$ whenever A is stably finite without further comment. Note that $\mathrm{W}(A)$ has algebraic order if we restrict to projections since this is the Murray-von Neumann subequivalence. The next result states that projections behave well with respect to any other positive element.

Proposition 1.3.22. ([PT07, Proposition 2.2]) *Let A be a C*-algebra, and let a, p be positive elements in $M_\infty(A)$ with p a projection. If $p \precsim a$, then there is b in $M_\infty(A)_+$ such that $p \oplus b \sim a$.*

Proof. By Lemma 1.3.20, $p \sim q$ with $q \leq \lambda a$ for a positive number λ. Assuming that $p \leq a$, Lemma 2.21 of [APT11] states that $a \precsim pap \oplus (1-p)a(1-p)$ for any $a \in A_+$ and p projection, so $a \precsim p \oplus (1-p)a(1-p)$ because $pap \leq \|a\|^2 p \sim p$. For the converse, we just have to note that $p, (1-p)a(1-p) \in A_a$ □

The following result is also satisfied in the case that A is a unital C*-algebra with $\mathrm{sr}(A) = 1$ ([APT11]).

Proposition 1.3.23. ([BC09]) *Let A be a stable and finite C*-algebra. Then, for $a \in M_\infty(A)_+$ the following are equivalent:*

(i) $\langle a \rangle = \langle p \rangle$, *for a projection* p,

(ii) 0 *is an isolated point of* $\sigma(a)$, *or* $0 \notin \sigma(a)$.

Proof. (ii)⇒(i) is clear.

(i)⇒(ii) Suppose $a \sim p$, and that 0 is a non-isolated point of $\sigma(a)$. Using Lemma 1.3.20, find a projection $q \sim p$ and $\delta > 0$ with $f_\delta(a) q = q$. Since 0 is not isolated in $\sigma(a)$, we know $f_\delta(a)$ is not a projection, so in particular $f_\delta(a) \neq q$. This tells us that

$$q = f_\delta(a)^{1/2} q f_\delta(a)^{1/2} < f_\delta(a).$$

Choose $0 < \delta' < \delta/2$, so that $f_\delta(a) \leq (a-\delta')_+$. Next, use that $a \precsim q$, so there is by Proposition 1.3.17 a unitary u with $u(a-\delta')_+ u^* \in qAq$, and so $u f_\delta(a) u^* \in qAq$. In particular, we have $u f_\delta(a) u^* \leq u u^* \leq 1$, so $u f_\delta(a) u^* = q u f_\delta(a) u^* q \leq q$. But now

$$uqu^* + u(f_\delta(a) - q)u^* \leq q$$

and $u(f_\delta(a) - q)u^* > 0$, whence $uqu^* < q$. But this contradicts the fact that A is stably finite. □

The previous result motivates the definition of $\mathrm{W}(A)_+$ which denotes the subset of $\mathrm{W}(A)$ consisting of those classes which are not the classes of projections. In fact, as can be seen in the next Lemma, it also denotes the classes of the elements whose spectra are infinite. We remark that it can be seen that a C*-algebra A is infinite dimensional if and only if there exists $a \in A$ such that its spectrum is infinite. If $a \in A_+$ and $\langle a \rangle \in \mathrm{W}(A)_+$, then we say that a is *purely positive* and denote the set of such elements by A_{++}.

Lemma 1.3.24 ([APT11]). *Let A be a C*-algebra which is either stable and finite or unital with $\mathrm{sr}(A) = 1$. Then, A is infinite dimensional if and only if there exists $a \in A$ purely positive.*

Proof. If $a \in A_{++}$, it follows that $\mathrm{sp}(a)$ is infinite since otherwise it would be equivalent to some projection (in matrices over A). Conversely, if $\mathrm{sp}(a)$ is infinite, choose an accumulation point $x \in \mathrm{sp}(a)$. Let f be a continuous function on $\mathrm{sp}(a)$ such that $f(t)$ is nonzero if and only if $t \neq x$. Then, $f(a)$ is positive and has zero an an accumulation point of its spectrum, so by Proposition 1.3.23 it is purely positive. □

As with Proposition 1.3.23, the result below is also valid of in the unital, stable rank one case.

Corollary 1.3.25. *Let A be a stable and finite C*-algebra. Then $\mathrm{W}(A)_+$ is a semigroup, and it is absorbing in the sense that if one has $a \in \mathrm{W}(A)$ and $b \in \mathrm{W}(A)_+$, then $a + b \in \mathrm{W}(A)_+$.*

Proof. Take $\langle a \rangle, \langle b \rangle \in \mathrm{W}(A)_+$ and notice that the spectrum of $a \oplus b$ contains the union of the spectra of a and b. By Proposition 1.3.23 the result follows. □

We end this first chapter showing the computation of the Cuntz semigroup for some different classes of C*-algebras.

Examples 1.3.26.

1. Let A be a finite dimensional C*-algebra. Under these hypotheses it is automatically stably finite, so one has that $\mathrm{W}(A) = \mathrm{V}(A) \sqcup \mathrm{W}(A)_+$ by Lemma 1.3.21. Moreover, by the finite dimension of A (see Lemma 1.3.24), it follows that $\mathrm{W}(A)_+ = \emptyset$; therefore $\mathrm{W}(A) = \mathrm{V}(A)$.

2. We say that A is a *Kirchberg algebra* if it is a separable, nuclear, simple and purely infinite C*-algebra. In order to compute $\mathrm{W}(A)$ for Kirchberg algebras, we will use a result of Lin and Zhang [LZ91] which states that A is as above if and only if $a \precsim b$ for any non-zero $a, b \in A_+$. Clearly, it follows that $\mathrm{W}(A) = \{0, \infty\}$ with $\infty + \infty = \infty$.

3. One of the most important examples as a separable nuclear simple and infinite dimensional C*-algebra is the Jiang-Su algebra, denoted by $\mathcal{Z}$. It was discovered by X. Jiang and H. Su in [JS99] and can be described as a limit of a sequence of prime dimension drop C*-algebras. Given p, q two coprime positive integers, those are defined as

$$\mathcal{Z}_{p,q} := \{f \in \mathrm{C}([0,1], M_p \otimes M_q) \mid f(0) \in M_p \otimes I_q,\ f(1) \in I_p \otimes M_q\}.$$

   In fact, they proved the following:

   Theorem 1.3.27 ([JS99]). *Any sequential inductive limit of simple prime dimension drop C*-algebras with unital morphisms and only one trace is isomorphic to the Jiang-su algebra $\mathcal{Z}$.*

As a first approach relating the Cuntz semigroup and the Jiang-Su algebra, the computation of the Cuntz semigroup for $\mathcal{Z}$ was carried out in [PT07]. It turns out that

$$\mathrm{W}(\mathcal{Z}) \cong \mathbb{N}_0 \sqcup \mathbb{R}^{++},$$

with a natural addition defined in each component and $x + y \in \mathbb{R}^{++}$ when $x \in \mathbb{N}_0$ and $y \in \mathbb{R}^{++}$. The order is defined by $\leq_{\mathcal{Z}}$ which is the natural in each component and if $x \in \mathbb{N}_0$ and $y \in \mathbb{R}^{++}$ then $x \leq_{\mathcal{Z}} y$ iff $x < y$ with the natural order, whereas $x \geq_{\mathcal{Z}} y$ iff $x \geq y$ with the natural order.

In the same article of the definition of $\mathcal{Z}$ ([JS99]), it was also proved that $\mathcal{Z} \otimes \mathcal{Z} \cong \mathcal{Z}$, so this property induced the notion of $\mathcal{Z}$-stability. We will say that a C*-algebra A is $\mathcal{Z}$-stable if $A \otimes \mathcal{Z} \cong A$.

To the amount of computing the Cuntz semigroup of the $\mathcal{Z}$-stable C*-algebras, N. Brown, F. Perera and A. Toms described the Cuntz semigroup for some class of $\mathcal{Z}$-stable C*-algebras.

Theorem 1.3.28 ([BPT08]). *Let A be a separable simple nuclear unital and $\mathcal{Z}$-stable C*-algebra with $\mathrm{sr}(A) = 1$. Then,*

$$\begin{aligned} \alpha : \mathrm{W}(A) &\to \mathrm{V}(A) \sqcup \mathrm{LAff}_b(\mathrm{T}(A))^{++} \\ \langle p \rangle &\mapsto [p] \text{ if } p \text{ is a projection} \\ \langle a \rangle &\mapsto \hat{a} : \tau \mapsto d_\tau(a) \text{ otherwise} \end{aligned}$$

defines an order-isomorphism.

Here $\mathrm{LAff}_b(\_)$ denotes the set of lower semicontinuous and affine bounded functions. The addition of $\mathrm{W}(A)$ is defined naturally in each component and when $x \in \mathrm{V}(A)$ and $y \in \mathrm{LAff}_b(\mathrm{T}(A))^{++}$, then $x + y = \hat{x} + y \in \mathrm{LAff}_b(\mathrm{T}(A))^{++}$, where $\hat{x}(\tau) = \tau(x)$. The order, as before, is the standard in each component and if $x \in \mathrm{V}(A)$ and $y \in \mathrm{LAff}_b(\mathrm{T}(A))^{++}$, then $x \leq y$ iff $\hat{x}(\tau) < y(\tau)$ for all $\tau \in \mathrm{T}(A)$ and $y \leq x$ iff $y(\tau) \leq \hat{x}(\tau)$ for all $\tau \in \mathrm{T}(A)$.

Chapter 2

Local triviality for continuous field C*-algebras

As mentioned in the introduction and shown in [Bla97], continuous field C*-algebras play the role of bundles of C*-algebras (in the sense of topology). In particular, continuous field C*-algebras appear naturally since any separable C*-algebra A with Hausdorff primitive spectrum X has a canonical continuous field structure over X with fibers the primitive quotients of A [Dix77].

In this chapter we will work on the fact that the bundle structure that underlines a continuous field C*-algebra is typically not locally trivial. Concretely, in Theorem 2.5.9 we give the optimal assumptions on the space and the fibers of a continuous field of C*-algebras with all the fibers mutually isomorphic to the same stable Kirchberg algebra D to obtain a complete picture of when it is trivial. It is pertinent to mention that the structure of continuous field C*-algebras with Kirchberg algebras as fibers over a finite dimensional space was deeply studied by Marius Dadarlat in [Dad09a] and [Dad09b] and by Dadarlat and Elliott in [DE07]. The results in this chapter are contained in [BD13].

2.1 Non-locally trivial Continuous Fields

In this section we provide two examples, which show that the complexity of continuous fields of C*-algebras is related both with the fact that the K-theory of the fibers is not finitely generated and also to the dimension of the base space. The following definition is crucial.

Definition 2.1.1. *A point $x \in X$ is called* singular *for A if $A(U)$ is nontrivial for any open set U that contains x (i.e. $A(U)$ is not isomorphic to $C_0(U) \otimes D$ for some C*-algebra D). The singular points of A form a closed subspace of X. If all points of X are singular for A we say that A is* nowhere locally trivial.

We start showing that the complexity of the continuous field A ultimately reflects the property of the K-theory of the fiber of not being finitely generated. In order to illustrate it in the case of continuous fields of C*-algebras with all the fibers mutually isomorphic to a Kirchberg algebra, we reproduce an example constructed by M. Dadarlat and G. Elliott in [DE07, Example 8.4] which shows the existence of a unital continuous field C*-algebra A over the unit interval with mutually isomorphic fibers and such that it is nowhere locally trivial.

Notation. *If $\varphi : A \to B$ is a *-homomorphism, we will denote by φ_* the induced map on K-theory $\varphi_* : \mathrm{K}_0(A) \to \mathrm{K}_0(B)$.*

Example 2.1.2. Let D be a unital Kirchberg algebra satisfying the UCT such that $\mathrm{K}_0(D) = \mathbb{Z} \oplus \mathbb{Z}$, $[1_D] = (1, 0)$ and $\mathrm{K}_1(D) = 0$, and let $\gamma \colon D \to D$ be a unital *-monomorphism such that $\gamma_*(0, 1) = (0, 0)$. The existence of D and γ follows from classification (see e.g. [Rør02, Section 4.3],[Phi00], [Dad09a, Theorem 3.1]) Consider (x_n) a dense sequence in $[0, 1]$, and define, for each n, the continuous field over $[0, 1]$,

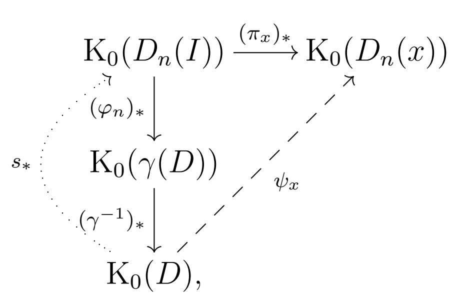
$$D_n = \{f \in \mathrm{C}([0, 1], D) \mid f(x_n) \in \gamma(D)\}.$$

Let us define inductively $A_1 = D_1$ and $A_{n+1} = A_n \otimes_{\mathrm{C}[0,1]} D_{n+1}$, which is the tensor product over $[0, 1]$ of $\mathrm{C}([0, 1])$-algebras (see [Kas88] or [Bla96] for a further discussion). Denote by $A = \varinjlim(A_n, \theta_n)$, where $\theta_n \colon A_n \to A_{n+1}$ is defined by $\theta_n(a) = a \otimes 1$. This means that A is the infinite tensor product over $\mathrm{C}([0, 1])$ of continuous fields, i.e. $A = \otimes_{n=1}^{\infty} D_n$. Note that A is a continuous field of Kirchberg algebras over $[0, 1]$ such that its fibers are isomorphic to $\otimes_1^{\infty} D$. Nevertheless, the remarkable fact about A is that it is nowhere locally trivial. This is checked by showing that for any closed nondegenerate subinterval I of $[0, 1]$ and any $x \in I$, the evaluation map $A(I) \to A(x)$ induces a non-injective map $\mathrm{K}_0(A(I)) \to \mathrm{K}_0(A(x))$ since such a situation cannot occur for trivial continuous fields. Indeed, if A is a trivial field over the interval, i.e., $A \cong \mathrm{C}([0, 1]) \otimes D$ for some C*-algebra D, then the induced K-theory exact sequence obtained from

$$0 \to \mathrm{C}_0([0, 1] \setminus \{x\}) \to A \xrightarrow{\pi_x} D \to 0$$

gives that $\mathrm{K}_j(A) \cong \mathrm{K}_j(D)$ for $j = 0, 1$ because the K-theory of a cone is zero (i.e. $\mathrm{K}_j(\mathrm{C}_0((a, b], A)) = 0$ for $a, b \in \mathbb{R}^+$ and A any C*-algebra for $j = 0, 1$). Focusing on our example, fix I as above. Observe that $D_n(I) = \mathrm{C}(I, D)$ if $x_n \notin I$ and that the map $\varphi_n \colon D_n(I) \to D$, defined by $\varphi_n(f) = f(x_n) \in \gamma(D)$, induces an isomorphism $\mathrm{K}_0(D_n(I)) \to \mathrm{K}_0(D)$ if $x_n \in I$ given by $(\gamma^{-1} \circ \varphi_n)_*$ (by the argument above).

On the other hand, if $x_n \in I$ and $x \in I \setminus \{x_n\}$, then the projection $\pi_x \colon D_n(I) \to D_n(x) \cong D$ induces a map $(\pi_x)_*$ which can be identified with γ_*; hence, it is not injective. To see this, one considers the following diagram



where $s : D \to D_n(I)$ is given by $s(d) = (\gamma(d), d)$ considering $D_n(I)$ as the pullback of

$$\begin{array}{ccc} D_n(I) & \longrightarrow & D \\ \big\downarrow & & \big\downarrow \gamma \\ C(I,D) & \xrightarrow{\pi_{x_n}} & D, \end{array}$$

and $\psi_x := (\pi_x)_* \circ s_*$. Note that $\gamma^{-1} \circ \varphi_n \circ s = \mathrm{id}$ and that s_* is bijective because $(\gamma^{-1} \circ \varphi_n)_*$ is an isomorphism. So the claim follows since $\psi_x := (\pi_x)_* \circ s_* = \gamma_*$, and this implies that $(\pi_x)_*$ is not injective because neither is γ_*.

Now, we note that identifying $A_{n+1} \cong A_n \otimes_{\mathrm{C}[0,1]} D_{n+1}$, one verifies that the inclusion $\theta_n \colon A_n \to A_{n+1} \cong A_n \otimes_{\mathrm{C}[0,1]} D_{n+1}$, given by $\theta_n(a) = a \otimes 1$, induces an injective map $\mathrm{K}_0(A_n(I)) \to \mathrm{K}_0(A_{n+1}(I))$ for any I containing x_{n+1}. Indeed, using the 6-term exact sequence of K-theory on $D_{n+1}(I)$, one has

$$\begin{array}{ccccc} \mathrm{K}_0(D_{n+1}(I \setminus \{x\})) = 0 & \longrightarrow & \mathrm{K}_0(D_{n+1}(I)) & \xrightarrow{\cong} & \mathrm{K}_0(D)) \\ \big\uparrow & & & & \big\downarrow \\ \mathrm{K}_1(D) = 0 & \longleftarrow & \mathrm{K}_1(D_{n+1}(I)) & \longleftarrow & \mathrm{K}_1(D_{n+1}(I \setminus \{x\})) = 0, \end{array}$$

so $\mathrm{K}_1(D_{n+1}(I)) = 0$. From this, we have that

$$\mathrm{K}_0(A_{n+1}(I)) \cong \mathrm{K}_0(A_n(I)) \otimes \mathrm{K}_0(D_{n+1}(I)) \cong \mathrm{K}_0(A_n(I)) \otimes (\mathbb{Z} \oplus \mathbb{Z}) \cong \mathrm{K}_0(A_n(I)) \oplus \mathrm{K}_0(A_n(I)) \quad (2.1)$$

by the Künneth Theorem. Note that the desired injectivity follows from checking how the induced maps are built. In particular, if $[p] \in \mathrm{K}_0(A_n(I))$, one has $[p] \mapsto [p \otimes 1] \mapsto [p] \otimes [1] \mapsto [p] \otimes (1, 0) \mapsto ([p], 0)$ following the isomorphisms described in (2.1).

Using this, it follows that the limit map $\eta \colon \mathrm{K}_0(A_n(I)) \to \mathrm{K}_0(A(I))$ is injective. Let $x \in I \setminus \{x_n\}$ and consider the commutative diagram induced by evaluating at x:

$$\begin{array}{ccc} \mathrm{K}_0(A_n(I)) & \xrightarrow{\eta} & \mathrm{K}_0(A(I)) \\ \big\downarrow (\pi_x^n)_* & & \big\downarrow (\pi_x^\infty)_* \\ \mathrm{K}_0(A_n(x)) & \longrightarrow & \mathrm{K}_0(A(x)). \end{array}$$

Because the sequence (x_n) is dense in $[0, 1]$, one can consider that $x_n \in I$ without loss of generality. Using the Künneth Theorem as before, one verifies that the map $(\pi_x^n)_*$ is not injective if $x \in I \setminus \{x_n\}$. Indeed, it can be identified with

$$(\pi_x^{n-1})_* \otimes (\pi_x)_* \colon \mathrm{K}_0(A_{n-1}(I)) \otimes \mathrm{K}_0(D_n(I)) \to \mathrm{K}_0(A_{n-1}(x)) \otimes \mathrm{K}_0(D_n(x))$$

and as seen before $\mathrm{K}_0(\pi_x)$ is not injective. On the other hand, since η is injective, it follows that $(\pi_x^\infty)_*$ is not injective; hence, $A(I)$ cannot be isomorphic to $\mathrm{C}(I, \otimes_1^\infty D)$.

The following example, based on an example given by Hirshberg, Rørdam and Winter in [HRW07] and built out by M. Dadarlat in [Dad09b], shows that even if the K-theory of the fiber vanishes, a field can be nowhere locally trivial if the base space is infinite-dimensional.

Example 2.1.3. Let e be the Bott projection in $M_2(\mathrm{C}(S^2))$, see Example 1.3.9. Denoting $p = \begin{pmatrix} 1 & 0 \\ 0 & e \end{pmatrix} \in M_3(\mathrm{C}(S^2))$, consider $B = \otimes_1^\infty \; pM_3(\mathrm{C}(S^2))p$.

In [HRW07, Example 4.7] it is shown that B is a continuous field over $X = \prod_1^\infty S^2$ such that $B(x) \cong M_{2^\infty}$ for all $x \in X$, but that $B \ncong \mathrm{C}(X) \otimes M_{2^\infty}$. Using this, in order to describe the above singularity type to more general spaces in the fibers, in [Dad09b] is computed $\mathrm{K}_0(B)$ which is $\mathrm{C}(K, \mathbb{Z})$, where K is the Cantor set. In particular, in [Dad09b] is noticed that for any $n \in \mathbb{N}\setminus\{1,2\}$ $A := B \otimes \mathcal{O}_n$ is a nowhere trivial continuous field. Indeed, for $n = 3$, one has

$$A(x) \cong B(x) \otimes \mathcal{O}_3 \cong M_{2^\infty} \otimes \mathcal{O}_3 \cong \mathcal{O}_2$$

since $\mathrm{K}_j(M_{2^\infty} \otimes \mathcal{O}_3) = \mathrm{K}_j(\mathcal{O}_2) = 0$ for $j = \{0,1\}$, i.e. $\mathrm{K}_j(A(x)) = 0$. However,

$$\mathrm{K}_0(A) = \mathrm{K}_0(B \otimes \mathcal{O}_3) = \mathrm{K}_0(B) \otimes \mathrm{K}_0(\mathcal{O}_3) \neq 0,$$

implying that $A \ncong \mathrm{C}(X) \otimes \mathcal{O}_2$.

2.2 KK-theory

In this section we review some basic facts about KK-theory which will be used in the sequel. We will not enter in the original definition of KK-theory made by Kasparov ([Kas88]) because it is very technical; however, we will outline the standard way of viewing the elements of $\mathrm{KK}(A,B)$ as quasihomomorphisms given by J. Cuntz. For this we follow [Rør02]. A *quasihomomorphism* from A to B is a pair $(\phi, \overline{\phi})$ of *-homomorphisms from A to $\mathcal{M}(B \otimes \mathcal{K})$ satisfying that $\phi(a) - \overline{\phi}(a) \in B \otimes \mathcal{K}$ for all $a \in A$, where $\mathcal{M}(B \otimes \mathcal{K})$ is the multiplier algebra of $B \otimes \mathcal{K}$. Two quasihomomorphisms $(\phi, \overline{\phi})$ and $(\varphi, \overline{\varphi})$ are said to be homotopic if they are connected by a path of quasihomomorphisms $(\alpha_t, \overline{\alpha}_t)$ such that for a given $a \in A$, $t \mapsto \alpha_t(a) - \overline{\alpha}_t(a)$ is norm continuous in $B \otimes \mathcal{K}$ and the map $t \mapsto (\alpha_t(a), \overline{\alpha}_t(a))$ is strictly continuous in $\mathcal{M}(B \otimes \mathcal{K})$ in each component (i.e., if $t_n \to t$, then, for a fixed $a \in A$, $\alpha_{t_n}(a)b' \to \alpha_t(a)b'$ and $b'\alpha_{t_n}(a) \to b'\alpha_t(a)'$ in norm for all $b' \in B \otimes \mathcal{K}$). We define $\mathrm{KK}(A,B)$ as the set of homotopy classes of quasihomomorphisms from A to B, and the class of $(\phi, \overline{\phi})$ is denoted by $[(\phi, \overline{\phi})]$. The set $\mathrm{KK}(A,B)$ becomes a group with addition defined as follows. Take two isometries s_1, s_2 in $\mathcal{M}(B \otimes \mathcal{K})$ satisfying $s_1 s_1^* + s_2 s_2^* = 1$ and let $(\phi, \overline{\phi}), (\varphi, \overline{\varphi})$ be two quasihomomorphisms. Then :

$$[(\phi, \overline{\phi})] + [(\varphi, \overline{\varphi})] = [(s_1\phi(a)s_1^* + s_2\varphi(a)s_2^*, \;\; s_1\overline{\phi}(a)s_1^* + s_2\overline{\varphi}(a)s_2^*)].$$

With this opperation $\mathrm{KK}(A,B)$ is an abelian group with $[(\varphi, \varphi)] = 0$ and $-[(\phi, \overline{\phi})] = [(\overline{\phi}, \phi)]$. Moreover, it follows that $\mathrm{KK}(\_,\_)$ is a homotopy-invariant bifunctor from pairs of C*-algebras to abelian groups, contravariant in the first variable and covariant in the second.

Each *-homomorphism $\varphi : A \to B \otimes \mathcal{K}$ defines a quasihomomorphism $(\varphi, 0)$, and we set $\mathrm{KK}(\varphi) = [(\varphi, 0)] \in \mathrm{KK}(A,B)$. Also, if $\varphi, \phi : A \to B \otimes \mathcal{K}$ are homotopic *-homomorphism, then $\mathrm{KK}(\varphi) = \mathrm{KK}(\phi)$. Note that any C*-algebra B, and any matrix algebra over B, can be viewed as sub-C*-algebra of $B \otimes \mathcal{K}$, so every *-homomorphism $\varphi : A \to B$, or from $A \to M_n(B)$, can be thought of as a *-homomorphism from A to $B \otimes \mathcal{K}$. Hence, $\varphi : A \to B$ represents an element $\mathrm{KK}(\varphi)$ in $\mathrm{KK}(A,B)$.

An important property of KK-theory is the Kasparov product, which, although we won't enter in to the definition because we will not use it in the sequel, we mention here some of its properties. This product associates to every triple of C*-algebras A, B, C a bi-additive map

$$\mathrm{KK}(A,B) \times \mathrm{KK}(B,C) \to \mathrm{KK}(A,C), \quad (x,y) \mapsto x \cdot y,$$

which is associative and extends composition of *-homomorphisms, i.e.

$$\mathrm{KK}(\varphi).\mathrm{KK}(\phi) = \mathrm{KK}(\phi \circ \varphi),$$

whenever $\varphi : A \to B$ and $\phi : B \to C$ are *-homomorphisms. Moreover, $\mathrm{KK}(A,A)$ is a ring with unit $\mathrm{KK}(\mathrm{id}_A)$ denoted, making an abuse of notation, by 1_A. With this, we define the additive category **KK** whose objects are separable C*-algebras and whose morphisms from A to B are the elements in $\mathrm{KK}(A,B)$ using the Kasparov product to compose two morphisms.

Definition 2.2.1. *An element $x \in \mathrm{KK}(A,B)$ is called invertible if there exists $y \in \mathrm{KK}(B,A)$ such that $xy = 1_A$ and $yx = 1_B$. Two C*-algebras A and B are said to be KK-equivalent if $\mathrm{KK}(A,B)$ contains an invertible element.*

Note that an element $x \in \mathrm{KK}(A,B)$ is invertible if it is an invertible morphism in the category **KK**, and two C*-algebras are KK-equivalent if they are isomorphic in the category **KK**.

Another important feature of KK-theory consists of its connections with K-theory. We next state some important results which show this relation (see [Bla06] for further details):

$$\mathrm{KK}(\mathbb{C}, B) \cong \mathrm{K}_0(B)$$

$$\mathrm{KK}(SA, B) \cong \mathrm{KK}(A, SB)$$

$$\mathrm{KK}(\mathrm{C}_0(\mathbb{R}), B) \cong \mathrm{K}_1(B).$$

Using the Kasparov product and the above equivalences, we may define the following group homomorphisms

$$\gamma_0 : \mathrm{KK}(A,B) \to \mathrm{Hom}(\mathrm{K}_0(A), \mathrm{K}_0(B)),$$

$$\gamma_1 : \mathrm{KK}(A,B) \to \mathrm{Hom}(\mathrm{K}_1(A), \mathrm{K}_1(B)),$$

by $\gamma_0(x)(z_0) = z_0.x$ and $\gamma_1(x)(z_1) = z_1.x$ for $z_0 \in \mathrm{KK}(\mathbb{C}, A)$ and $z_1 \in \mathrm{KK}(\mathrm{C}_0(\mathbb{R}), A)$. If $\varphi : A \to B$ is a *-homomorphism, then $\gamma_j(\mathrm{KK}(\varphi)) = \mathrm{K}_j(\varphi)$ for $j = 0, 1$. Note that the maps γ_0 and γ_1 are compatible with the Kasparov product in the sense of $\gamma_j(x.y) = \gamma_j(y) \circ \gamma_j(x)$ for $x \in \mathrm{KK}(A,B)$, $y \in \mathrm{KK}(B,C)$ and $j = 0, 1$. In particular, if x is invertible, then so are $\gamma_0(x), \gamma_1(x)$. The converse of this fact is crucial and known as the Universal Coefficient Theorem (UCT), stated in Theorem 2.2.2 below. This defines the class of C*-algebras that satisfy this Theorem, the so-called UCT class. Recall that we denote by $\mathcal{N}$ the large bootstrap class (Definition 1.3.7).

Theorem 2.2.2. (UCT)([RS87])

(i) *The homomorphism*

$$\gamma_0 \oplus \gamma_1 : \mathrm{KK}(A, B) \to \mathrm{Hom}(\mathrm{K}_0(A), \mathrm{K}_0(B)) \oplus \mathrm{Hom}(\mathrm{K}_1(A), \mathrm{K}_1(B))$$

is surjective for each $A \in \mathcal{N}$ and for each separable C-algebra B.*

(ii) *If both A and B belong to $\mathcal{N}$, then an element $x \in \mathrm{KK}(A, B)$ is invertible if and only if $\gamma_0(x) : \mathrm{K}_0(A) \to \mathrm{K}_0(B)$ and $\gamma_1(x) : \mathrm{K}_1(A) \to \mathrm{K}_1(B)$ are invertible mappings.*

(iii) *Two C*-algebras A and B in $\mathcal{N}$ are* KK*-equivalent if and only if* $\mathrm{K}_0(A) \cong \mathrm{K}_0(B)$ *and* $\mathrm{K}_1(A) \cong \mathrm{K}_1(B)$.

It is known that for any pair of countable abelian groups (G_0, G_1) there exists an abelian C*-algebra B such that $\mathrm{K}_0(B) \cong G_0$ and $\mathrm{K}_1(B) \cong G_1$. Therefore, a C*-algebra A satisfies the UCT if and only if it is KK-equivalent to an abelian C*-algebra.

Studying the KK-theory for $\mathrm{C}(X)$-algebras, Kasparov introduced parametrized KK-theory groups $\mathrm{KK}_X(A, B)$ for $\mathrm{C}(X)$-algebras A and B ([Kas88]). Concretely, this is a sort of fibration over X of KK-groups. Moreover, these groups admit a natural product structure $\mathrm{KK}_X(A, B) \times \mathrm{KK}_X(B, C) \to \mathrm{KK}_X(A, C)$, and, as before, $\mathrm{KK}_X(A, B)^{-1}$ denotes the set of invertible elements in $\mathrm{KK}_X(A, B)$. If $\mathrm{KK}_X(A, B)^{-1} \neq \emptyset$ we say that A is KK_X-equivalent to B.

One of the achievements of this theory, and the only one used in the sequel, states that if A is a trivial C(X)-algebra, i.e. $A \cong \mathrm{C}(X) \otimes D$ for some C*-algebra D, it follows that $\mathrm{KK}_X(A, B) \cong \mathrm{KK}(D, B)$ (see e.g. [Dad09b, proof of Corollary 2.8]).

2.3 Semiprojectivity

A separable C*-algebra D is *semiprojective* if, for any C*-algebra A, any increasing sequence of two-sided closed ideals (J_n) of A with $J = \overline{\bigcup_n J_n}$ and for every *-homomorphism $\varphi : D \to A/J$, there exist n and φ_n so that the diagram:

$$\begin{array}{ccc} & & A/J_n \\ & \overset{\varphi_n}{\nearrow} & \downarrow{\scriptstyle \pi_n} \\ D & \xrightarrow{\varphi} & A/J \end{array}$$

commutes.

If we weaken this condition appropriately, then D is called *weakly semiprojective*. That is, D is weakly semiprojective if for any finite subset $\mathcal{F} \subset D$, any $\varepsilon > 0$ and $A, J = \overline{\bigcup_n J_n}, \varphi$ as above, there exists n and φ_n a *-homomorphism such that $\|\pi_n \varphi_n(c) - \varphi(c)\| < \varepsilon$ for all $c \in \mathcal{F}$.

Definition 2.3.1. ([Dad09a, Definition 3.5]) *A separable C*-algebra D is* KK*-stable if there is a finite set $\mathcal{G} \subset D$ and there is $\delta > 0$ with the property that for any two *-homomorphisms $\varphi, \psi : D \to A$ such that $\|\varphi(a) - \psi(a)\| < \delta$ for all $a \in \mathcal{G}$, one has $\mathrm{KK}(\varphi) = \mathrm{KK}(\psi)$.*

The following is a generalization of a result of [EL99]; it is proved along the same general lines. We include a proof for completeness.

Proposition 2.3.2. *Let D be a separable weakly semiprojective C*-algebra. For any finite set $\mathcal{F} \subset D$ and any $\epsilon > 0$ there exists a finite set $\mathcal{G} \subset D$ and $\delta > 0$ such that for any C*-algebra $B \subset A$ and any $*$-homomorphism $\varphi : D \to A$ with $\varphi(\mathcal{G}) \subset_\delta B$, there is a $*$-homomorphism $\psi : D \to B$ such that $\|\varphi(c) - \psi(c)\| < \epsilon$ for all $c \in \mathcal{F}$. If, in addition, $\mathrm{K}_j(D)$ is finitely generated for $j = 0, 1$, then we can choose $\mathcal{G}$ and δ such that we also have $\mathrm{K}_j(\psi) = \mathrm{K}_j(\varphi)$ for $j = 0, 1$.*

Proof. Fix $\mathcal{F}$ and ε, and take $(\mathcal{G}_n)$ be an increasing sequence of finite subsets of D whose union is dense in D. Seeking a contradiction, assume that there are sequences of C*-algebras $\mathrm{C}_n \subset A_n$ and *-homomorphisms $\varphi_n : D \to A_n$ satisfying $\varphi_n(\mathcal{G}_n) \subset_{1/n} \mathrm{C}_n$ with the property that for any $n \geq 1$ there is no *-homomorphism $\psi_n : D \to \mathrm{C}_n$ such that $\|\varphi_n(c) - \psi_n(c)\| < \varepsilon$ for all $c \in \mathcal{F}$. Consider $B_i = \prod_{n \geq i} A_n$ and $E_i = \prod_{n \geq i} \mathrm{C}_n \subset B_i$. If $\nu_i : B_i \to B_{i+1}$ is the natural projection, then $\nu_i(E_i) = E_{i+1}$. We remark that if we define $\Phi_i : D \to B_i$ by $\Phi_i(d) = (\varphi_i(d), \varphi_{i+1}(d), \ldots)$, then the image of $\Phi = \varinjlim \Phi_i : D \to \varinjlim(B_i, \nu_i)$ is contained in $\varinjlim(E_i, \nu_i)$. Because D is weakly semiprojective, there is i and a *-homomorphism $\Psi_i : D \to E_i$, of the form $\Psi_i(d) = (\psi_i(d), \psi_{i+1}(d), \ldots)$ such that $\|\Phi_i(c) - \Psi_i(c)\| < \varepsilon$ for all $c \in \mathcal{F}$. Hence, $\|\varphi_i(c) - \psi_i(c)\| < \varepsilon$ for all $c \in \mathcal{F}$ which gives a contradiction. □

Remark 2.3.3. We note that if in Proposition 2.3.2 we add the extra assumption of D being KK-stable, then it also follows that $\mathcal{G}$ and δ may be choosen to obtain $\mathrm{KK}(\psi) = \mathrm{KK}(\varphi)$. This is done in [Dad09a, Proposition 3.7].

Definition 2.3.4. ([Dad09a, Definition 3.9])

(i) *A separable C*-algebra D is* KK*-semiprojective if for any separable C*-algebra A and any increasing sequence of two-sided closed ideals (J_n) of A with $J = \overline{\bigcup J_n}$, the natural map $\varinjlim \mathrm{KK}(D, A/J_n) \to \mathrm{KK}(D, A/J)$ is surjective.*

(ii) *We say that the functor* $\mathrm{KK}(D, \_)$ *is* continuous *if for any inductive system $B_1 \to B_2 \to \ldots$ of separable C*-algebras, the induced map $\varinjlim(\mathrm{KK}(D, B_n)) \to \mathrm{KK}(\mathrm{D}, \varinjlim(\mathrm{B_n}))$ is bijective.*

The following result shows the equivalence of the notions above under mild assumptions.

Theorem 2.3.5. ([Dad09a, Theorem 3.12]) *Let D be a separable C*-algebra and consider the following properties :*

(i) *D is* KK*-semiprojective.*

(ii) *The functor* $\mathrm{KK}(D, \_)$ *is continuous.*

(iii) *D is weakly semiprojective and* KK*-stable.*

Then (i) $\iff$ (ii). *Moreover, if D is a Kirchberg algebra, then* (i) $\iff$ (ii) $\iff$ (iii).

Proposition 2.3.6. ([Dad09a, Proposition 3.10]) *Let D be a separable C*-algebra. If D is* KK*-semiprojective, then D is* KK*-stable.*

Proof. Seeking a contradiction, let D be a separable KK-semiprojective C*-algeba and $(\mathcal{G}_n)$ be an increasing sequence of finite subsets of D whose union is dense in D. If D is not KK-stable, then there exist a sequence (A_n) and *-homomorphisms $\varphi_n, \psi_n : D \to A_n$ such that $\|\varphi_n(d) - \psi(d)\| < 1/n$ for all $d \in \mathcal{G}_n$, but $\mathrm{KK}(\varphi_n) \neq \mathrm{KK}(\psi_n)$ for all $n \geq 1$. Consider $B_i = \prod_{n \geq i} A_i$ and let $\mu_i : B_i \to B_{i+1}$ be the natural projection. Define $\Phi_i, \Psi_i \colon D \to B_i$ by $\Phi_i(d) = (\varphi_i(d), \varphi_{i+1}(d), \ldots)$ and $\Psi_i(d) = (\psi_i(d), \psi_{i+1}(d), \ldots)$ for all $d \in D$. Let B_i' be the separable C*-subalgebra of B_i generated by the images of Φ_i and Ψ_i, then $\mu_i(B_i') = B_{i+1}'$ and one verifies that $\varinjlim(\Phi_i) = \varinjlim(\Psi_i) : D \to \varinjlim(B_i', \mu_i)$. Because $\mathrm{KK}(D, \_)$ is continuous by Theorem 2.3.5, we have $\mathrm{KK}(\Phi_i) = mathrmKK(\Psi_i)$ for some i; hence, $\mathrm{KK}(\varphi_n) = \mathrm{KK}(\psi_n)$ for all $n \geq i$. This is a contradiction. □

Proposition 2.3.7. ([Dad09a, Proposition 3.14]) *Let D be a separable C*-algebra satisfying the* UCT. *Then D is* KK*-semiprojective if and only if* $\mathrm{K}_j(D)$ *is finitely generated for* $j = 0, 1$.

Proof. It is proved in [RS87] that $\mathrm{K}_j(D)$ finitely generated for $j = 0, 1$ implies KK-semiprojective. We prove the converse. For the sake of simplicity, we give the proof only for K_0. Note that, since D is KK-semiprojective, $\mathrm{KK}(D, \_)$ is continuous by Theorem 2.3.5. Write $G = \mathrm{K}_0(D)$, which is a countable abelian group. Let H be a countable abelian group and find A a Kirchberg algebra with $\mathrm{K}_0(A) = H$. Define an inductive system of finitely generated abelian groups $H_1 \to H_2 \to \ldots$ such that its inductive limit is H. Then, by [Rør02, Theorem 8.4.13] one may associate an inductive sequence of Kirchberg algebras $A_1 \to A_2 \to \ldots \to A$ such that $\mathrm{K}_0(A_n) = H_n$. By the continuity of $\mathrm{KK}(D, \_)$, one has that the natural map $\varinjlim \mathrm{KK}(D, A_n) \to \mathrm{KK}(D, \varinjlim A_n)$ is bijective. Using the condition (i) of Theorem 2.2.2, one gets that there exists a surjective map from $\varinjlim \mathrm{Hom}(\mathrm{K}_0(D), H_n)$ to $\mathrm{Hom}(\mathrm{K}_0(D), H)$. Taking $H = G$, we see that the map id_G lifts to $\mathrm{Hom}(\mathrm{K}_0(D), H_n)$ for some n and finitely generated subgroup $H_n \subseteq G$. Hence, G is a quotient of H_n, implying that G is finitely generated. □

Remark 2.3.8. Notice that a Kirchberg algebra D satisfying the UCT such that $\mathrm{K}_j(D)$ is finitely generated for $j = 0, 1$ is KK-projective and KK-stable by the combination of Proposition 2.3.7 and Theorem 2.3.5.

It is pertinent to say that by previous work of Neubuser [Neu00], H.Lin [Lin07] and Spielberg [Spi07], D, under our assumptions, is weakly semiprojective. Further, in [Dad09a, Prop. 3.11] is shown that $D \otimes \mathcal{K}$ is also weakly-semiprojective.

2.4 Approximation of Continuous Fields

In this section, we state a corollary of a result on the structure of continuous fields proved in [Dad09a, Theorem 4.6]. In this, the property of weak semiprojectivity is used to approximate a continuous field A by continuous fields A_k given by n-pullbacks of trivial continuous fields. We shall use this construction several times in the sequel.

Recall that the pullback of a diagram

$$A \xrightarrow{\pi} C \xleftarrow{\gamma} B$$

is the C*-algebra

$$E = \{(a, b) \in A \oplus B \mid \pi(a) = \gamma(b)\}.$$

We are going to use pullbacks in the context of continuous field C*-algebras.

Definition 2.4.1. *Let X be a metrizable compact space, and let D be a C*-algebra. Suppose that $X = Z_0 \cup Z_1 \cup \ldots \cup Z_n$, where $\{Z_j\}_{j=0}^n$ are closed subsets, and write $Y_i = Z_0 \cup Z_1 \cup \ldots \cup Z_i$. The notion of an* n-pullback of trivial continuous fields *with fiber D over X is defined inductively by the following data. We are given continuous fields E_i over Y_i with fibers isomorphic to D and fiberwise injective morphisms of fields $\gamma_{i+1} : \mathrm{C}(Y_i \cap Z_{i+1}) \otimes D \to E_i(Y_i \cap Z_{i+1})$, $i \in \{1, \ldots, n-1\}$, with the following properties:*

(i) $E_0 = \mathrm{C}(Y_0) \otimes D = \mathrm{C}(Z_0) \otimes D$.

(ii) *E_1 is the field over $Y_1 = Y_0 \cup Z_1$ defined by the pullback of the diagram (where $\pi = \pi_{Y_0 \cap Z_1}$)*

$$E_0(Y_0) \xrightarrow{\pi} E_0(Y_0 \cap Z_1) \xleftarrow{\gamma_1 \circ \pi} \mathrm{C}(Z_1) \otimes D.$$

(iii) *In general, E_{i+1} is the field over $Y_{i+1} = Y_i \cup Z_{i+1}$ defined as the pullback of the diagram*

$$E_i(Y_i) \xrightarrow{\pi} E_i(Y_i \cap Z_{i+1}) \xleftarrow{\gamma_{i+1} \circ \pi} \mathrm{C}(Z_{i+1}) \otimes D.$$

We call the continuous field $E = E_n(Y_n) = E_n(X)$ an n-pullback (of trivial fields). Observe that all its fibers are isomorphic to D.

Remark 2.4.2. (a) If E is an n-pullback of trivial continuous fields with fiber D over X, then E_i is an i-pullback and $E_i(Z_i) \cong \mathrm{C}(Z_i) \otimes D$ for all $i = 0, 1, ..., n$.

(b) If $V \subset X$ is a closed set such that $V \cap (Z_{i+1} \cup \ldots \cup Z_n) = \emptyset$, then $E(V) \cong E_i(V)$. Moreover, if $V \subset Z_i$, then it follows that $E(V) \cong E_i(V) \cong \mathrm{C}(V) \otimes D$.

Notation. We denote by $D_n(X)$ the class of continuous fields with fibers isomorphic to D which are n-pullbacks of trivial fields in the sense of Definition 2.4.1 and which have the additional property that the spaces Z_i that appear in their representation as n-pullbacks are finite unions of closed subsets of X of the form $\overline{U(x,r)}$, where $U(x,r) = \{y \in X : \mathrm{d}(y,x) < r\}$ is the open ball of center x and radius r for a fixed metric d for the topology of X.

Recall that the condition $\mathcal{F} \subset_\epsilon B$ means that for each $a \in \mathcal{F}$ there is $b \in B$ such that $\|a-b\| < \epsilon$.

Definition 2.4.3. *Let A be a C*-algebra. We say that a sequence of C*-subalgebras $\{A_n\}$ is exhaustive if for any finite subset $\mathcal{F} \subset A$, any $\epsilon > 0$ and any n_0, there exists $n \geq n_0$ such that $\mathcal{F} \subset_\epsilon A_n$. In the case of continuous fields we will require that the algebras A_n are $\mathrm{C}(X)$-subalgebras of A.*

Note that the existence of an exhaustive sequence is ultimately related to the separability of the C*-algebra, but the important point is the structure of the C*-algebras belonging to this exhaustive sequence. It is important to remark that in [Lor97, Lemma 15.2.2] is proved that if A contains an exhaustive sequence (A_n) such that A_n is weakly semiprojective and finitely presented, then $A \cong \lim(A_{n_k}, \gamma_k)$ for some subsequence, and some connecting maps $\gamma_k : A_{n_k} \to A_{n_{k+1}}$. Furthermore, in [Rør02, Proposition 8.4.13] is proved that any Kirchberg algebra with finitely generated K-theory is isomorphic to an inductive limit of weakly semiprojective Kirchberg algebras.

The following is a result from [Dad09a] which has been rephrased checking that all conditions in the original statement are satisfied by Remark 2.3.8 and [Rør02, Proposition 8.4.13]. We shall use it to approximate a continuous field A by exhaustive sequences consisting of n-pullbacks of trivial continuous fields.

Theorem 2.4.4. ([Dad09a, Theorem 4.6]) *Let D be a stable Kirchberg algebra that satisfies the UCT. Suppose that $\mathrm{K}_j(D)$ is finitely generated for $j = 0, 1$. Let X be a finite dimensional compact metrizable space and let A be a separable continuous field over X such that all its fibers are isomorphic to D. For any finite set $\mathcal{F} \subset A$ and any $\epsilon > 0$, there exists n and $B \in D_n(X)$ with $n \leq \dim(X)$ and an injective $\mathrm{C}(X)$-linear $*$-homomorphism $\eta : B \to A$ such that $\mathcal{F} \subset_\epsilon \eta(B)$.*

We remark that Theorem 2.4.4 does not state that the sets Z_i that give the n-pullback structure of B are finite unions of closures of open balls. However, this additional condition will be easily satisfied after making a few arrangements in its proof.

Note that if A is a continuous field of C*-algebras under the assumptions of the above Theorem, by applying Theorem 2.4.4 we get an exhaustive sequence $\{\phi_k : A_k \to A\}$, where the ϕ_k are injective $\mathrm{C}(X)$-linear *-homomorphisms, such that $A_k \in D_{l_k}$ with $l_k \leq \dim(X)$.

2.5 Local triviality

We start this section by an elementary lemma which collects some useful properties of finitely generated abelian groups. It is singled out in the beginning of this section because it will be used repeatedly in this chapter, sometimes without further reference. A proof is included for completeness.

Lemma 2.5.1. *Let G be a finitely generated abelian group.*

(i) *If G is finite, then a map $\alpha : G \to G$ is bijective if and only if α is injective, if and only if α is surjective.*

(ii) *Any surjective homomorphism $\eta : G \to G$ is bijective.*

(iii) *In a commutative diagram of group homomorphisms*

$$\begin{array}{ccc} G & \xrightarrow{\gamma} & G \\ & {\scriptstyle\alpha}\searrow \quad \nearrow{\scriptstyle\beta} & \\ & G, & \end{array}$$

if α is not bijective, then γ is not bijective.

Proof. (i) This is obvious. (ii) Since G is a finitely generated abelian group, $G \cong \mathbb{Z}^k \oplus T$ where $k \geq 0$ and T is a finite torsion group. To prove the statement, consider the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T & \longrightarrow & G & \longrightarrow & \mathbb{Z}^k & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\eta} & & \downarrow{\scriptstyle\alpha} & & \\ 0 & \longrightarrow & T & \longrightarrow & G & \longrightarrow & \mathbb{Z}^k & \longrightarrow & 0. \end{array}$$

One can represent η as $\left(\begin{smallmatrix} \alpha & 0 \\ \gamma & \beta \end{smallmatrix}\right)$, where $\gamma : \mathbb{Z}^k \to T$. Note that α is surjective (and hence bijective) since η is surjective. By the Five lemma, β is surjective and so it must be bijective by (i). Applying the Five lemma again, we see that η is bijective.

(iii) If γ were bijective, β would be surjective and hence bijective by (i). Since α is not bijective, this is a contradiction. □

The next result gives necessary and sufficient K-theory conditions for triviality of continuous fields whose fibers are Kirchberg algebras. We remark that the original statement has been rephrased in K-theoretical form. This can be done using that, under our assumptions, $\mathrm{KK}(D, \_)$ is continuos and, moreover, the UCT condition is satisfied, i.e., Theorem 2.2.2.

Theorem 2.5.2. ([Dad09a, Theorem 1.2]) *Let X be a finite dimensional compact metrizable space. Let A be a separable continuous field over X whose fibers are stable Kirchberg algebras satisfying the* UCT*. Let D be a stable Kirchberg algebra that satisfies the* UCT *and such that $\mathrm{K}_j(D)$ is finitely generated for $j = 0, 1$. Then A is isomorphic to $\mathrm{C}(X) \otimes D$ if and only if there is $\sigma : \mathrm{K}_j(D) \to \mathrm{K}_j(A)$ such that $\sigma_x : \mathrm{K}_j(D) \to \mathrm{K}_j(A(x))$ is bijective for all $x \in X$ for $j = 0, 1$.*

On the way to obtain our main result of this chapter, Theorem 2.5.9, we prove an intermediate technical result, Theorem 2.5.8. To prove this, we need several lemmas.

Remark 2.5.3. Let D be a C*-algebra. We denote the set of *-homomorphisms from D to D by $\mathrm{Hom}(D, D)$ where the topology is given by $\varphi_n \to \varphi \in \mathrm{Hom}(D, D)$ if $\varphi_n(d) \to \varphi(d)$ for all $d \in D$. If A, B are two trivial continuous fields over a finite dimensional compact metric space X with A_x, B_x isomorphic to D for all $x \in X$, then any $\mathrm{C}(X)$-morphism $\phi : A \to B$ induces a continuous morphism $\tilde{\phi} : X \to \mathrm{Hom}(D, D)$. Specifically, $\tilde{\phi}(x) = (\phi_x : d \mapsto \phi(1 \otimes d)(x) = \phi_x(d))$. Moreover, any continuous morphism $\tilde{\psi} : X \to \mathrm{Hom}(D, D)$ induces a $\mathrm{C}(X)$-morphism between two trivial fields $\psi : \mathrm{C}(X, D) \to \mathrm{C}(X, D)$ defined by $\psi(f)(x) = (\tilde{\psi}(x))(f(x))$ for $f \in \mathrm{C}(X, D)$.

Lemma 2.5.4. *Let $\phi : A \to B$ be a *-homomorphism of trivial fields over a compact metric space X with all the fibers isomorphic to D. Suppose that there is $x \in X$ such that $\mathrm{K}_j(\phi_x) : \mathrm{K}_j A(x)) \to \mathrm{K}_j(B(x))$ is not bijective for $j = 0, 1$. If $\mathrm{K}_j(D)$ is finitely generated for $j = 0, 1$, then there exists a neighborhood V of x such that $\mathrm{K}_j(\phi_v) : \mathrm{K}_j(A(v)) \to \mathrm{K}_j(B(v))$ is not bijective for any $v \in V$ for $j = 0, 1$.*

Proof. For the sake of simplicity, we give the proof only for K_0. By Remark 2.5.3, one can associate to ϕ the morphism $\tilde{\phi} : X \to \mathrm{Hom}(D, D)$. Therefore, passing to K-theory, one has the map $\mathrm{Hom}(D, D) \to \mathrm{Hom}(\mathrm{K}_0(D), \mathrm{K}_0(D))$. Note that the above map is locally constant because K_0 is finitely generated. Indeed, if $\varphi, \varphi' \in \mathrm{Hom}(D, D)$ are close, i.e $\|\varphi(d) - \varphi'(d)\| < \varepsilon$ for some $\varepsilon < 1$ and for all $d \in D$, it follows that $[\varphi(e)]_0 = [\varphi'(e)]_0$ for any projection in D. Hence, $\mathrm{K}_0(\varphi) = \mathrm{K}_0(\varphi')$ since $\mathrm{K}_0(D)$ is finitely generated. Because $\mathrm{K}_j(\phi_x)$ is not bijective for $j = 0, 1$, there exists an open neighborhood U_x of x such that $\mathrm{K}_j(\phi_v)$ is not bijective for all $v \in U_x$ for $j = 0, 1$. □

Lemma 2.5.5. *Let X be a metrizable compact space. Suppose that A is a continuous field in $D_n(X)$. Then for any open set $U \subset X$ and $x \in U$, there is an open set V such that $x \in \overline{V} \subset U$ and $A(\overline{V}) \cong \mathrm{C}(\overline{V}) \otimes D$.*

Proof. We use the notation from Definition 2.4.1 with A in place of E. Let $i \in \{0, 1, ..., n\}$ be the largest number with the property that $x \in Z_i$. Set $X_i = \bigcup_{i=i+1}^{n} Z_i$ if $i < n$ and $X_n = \emptyset$. Then, $\mathrm{dist}(x, X_i) > 0$ since X_i is a closed set and $x \notin X_i$. Let W be an open ball centered at x such that $\overline{W} \subset U$ and $\overline{W} \cap X_i = \emptyset$. By the definition of $D_n(X)$, there exist $z \in Z_i$ and $r > 0$ such that $x \in \overline{U(z,r)} \subset Z_i$. Since $x \in W \cap \overline{U(z,r)}$ and W is open, there must be a sequence $z_n \in W \cap U(z,r)$ which converges to x. Setting $V = W \cap U(z,r)$, we have that $x \in \overline{V} \subset \overline{W} \subset U$ and $\overline{V} \cap X_i = \emptyset$. Because $\overline{V} \subset \overline{U(z,r)} \subset Z_i$, it follows that $A(\overline{V})$ is trivial by Remark 2.4.2 (b). □

Lemma 2.5.6. *Let D be a stable Kirchberg algebra such that $\mathrm{K}_j(D)$ is finitely generated for $j = 0, 1$. Let $\{D_n\}$ be an exhaustive sequence for D with inclusion maps $\phi_n : D_n \hookrightarrow D$. Suppose that $\mathrm{K}_j(D_n) \cong \mathrm{K}_j(D)$ for $j = 0, 1$ and for all $n \geq 1$. Then, there exists $n_1 < n_2 < \ldots < n_k < \ldots$ such that $\mathrm{K}_j(\phi_{n_k})$ is bijective for $j = 0, 1$ and for all k.*

Proof. For the sake of simplicity, we give the proof only for K_0. Let $\mathrm{K}_0(D)$ be generated by classes of projections $e_i \in D$, $i = 1, \ldots, r$. Since $\{D_n\}$ is exhaustive, there exist $n_1 < n_2 < \ldots < n_k < \ldots$ such that $\mathrm{dist}(e_i, D_{n_k}) < \varepsilon$ for $i = 1, \ldots, r$ and $k \geq 1$. Let $x \in D_{n_k}$ such that $\|e_i - x\| < \varepsilon$. Then $\|x\| < 1 + \varepsilon$ and

$$\|x^*x - e_i\| \leq \|x^*x - x^*e_i\| + \|x^*e_i - e_i\| \leq \|x^*\|\|x - e_i\| + \|(x^* - e_i)e_i\| \leq \|x\|\varepsilon + \varepsilon < 2\varepsilon + \varepsilon^2.$$

So abusing notation, we may assume that $x \geq 0$ and $\|x - e_i\| < \varepsilon$ (for sufficiently small ε). Now

$$\|x^2 - x\| \leq \|x^2 - xe_i\| + \|xe_i - x\| \leq \|x\|\varepsilon + \varepsilon + \varepsilon = 2\varepsilon + \varepsilon\|x\| < 3\varepsilon + \varepsilon^2.$$

If $\|x\| > 1$, then:

$$\Big\|e_i - \frac{x}{\|x\|}\Big\| = \Big\|\frac{e_i \cdot \|x\| - x}{\|x\|}\Big\| \leq \frac{1}{\|x\|}\Big\|e_i \cdot \|x\| - x\Big\| \leq \Big\|e_i \cdot \|x\| - x \cdot \|x\|\Big\| + \Big\|x \cdot \|x\| - x\Big\|$$
$$\leq \|x\|\|e_i - x\| + \|x\|(\|x\| - 1) < 2\|x\|\varepsilon,$$

and

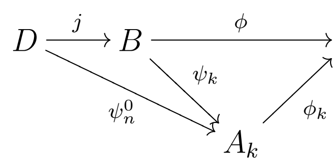
$$\Big\|\Big(\frac{x}{\|x\|}\Big)^2 - \frac{x}{\|x\|}\Big\| \leq \Big\|\frac{x^2 - \|x\|x}{\|x\|^2}\Big\| \leq \Big\|x^2 - x\Big\| + \Big\|x - \|x\| \cdot x\Big\| < 3\varepsilon + \varepsilon^2 + \|x\|\varepsilon < 4\varepsilon + 2\varepsilon^2.$$

Therefore, choosing $\varepsilon > 0$ appropriately, we may find $x \in D_{n_k}$ such that $x \geq 0$, $\|x\| \leq 1$, $\|x^2 - x\| < \varepsilon \leq 1/4$ and $\|x - e_i\| < 1/2$. Using these assumptions on x, by functional calculus it follows that x may be approximated by a projection $f_i \in D_{n_k}$ (see e.g. [WO93, Lemma 5.1.6]) such that $\|x - f_i\| < 1/2$. Thus, $\|e_i - f_i\| < 1$ and so $[e_i]_0 = [f_i]_0$ in $\mathrm{K}_0(D)$. This shows that the maps $\mathrm{K}_0(\phi_{n_k})$ are surjective. Then they must be bijective by Lemma 2.5.1. □

Lemma 2.5.7. *Let X be a finite dimensional metrizable compact space, and let D be a stable Kirchberg algebra that satisfies the UCT and such that $\mathrm{K}_j(D)$ is finitely generated for $j = 0, 1$. Let A be a separable continuous field C*-algebra over X with all fibers isomorphic to D. Let $B \in D_n(X)$ ($n < \infty$) be such that there exists a $\mathrm{C}(X)$-linear $*$-monomorphism $\phi : B \to A$. If A is nowhere locally trivial, then for any nonempty set $U \subset X$ there exists an open nonempty set W such that $\overline{W} \subset U$, $B(\overline{W})$ is trivial and for all $v \in \overline{W}$, $\mathrm{K}_j(\phi_v)$ is not bijective for $j = 0, 1$.*

Proof. In this proof, we shall denote the KK-class of homomorphism by $[\_]$. By Lemma 2.5.5 there is an open set $V \neq \emptyset$ such that $\overline{V} \subset U$ and $B(\overline{V}) \cong \mathrm{C}(\overline{V}) \otimes D$. After replacing U by V and restricting both B and A to $\overline{V}$ we may assume without any loss of generality that $B = \mathrm{C}(X) \otimes D$. By the assumption on A, using Theorem 2.4.4, we get an exhaustive sequence $\{\phi_k : A_k \to A\}$ such that $A_k \in D_{l_k}(X)$ with $l_k \leq \dim(X)$. Let us regard D as the subalgebra of constant functions of $B = \mathrm{C}(X) \otimes D$ and denote by j the corresponding inclusion map.

Let $\mathcal{F} \subset D$ be a finite set and $\varepsilon > 0$. Considering the exhaustive sequence $(A_k)_k$, and applying Proposition 2.3.2 (and Remark 2.3.3) together with Remark 2.3.8 to D and the map $\phi \circ j$, we get n and $\psi_n^0 : D \to A_n$ such that $\|\phi_k \circ \psi_n^0(d) - \phi \circ j(d)\| < \varepsilon$ for all $d \in \mathcal{F}$ and that $[\phi_k \circ \psi_n^0] = [\phi \circ j]$ in $\mathrm{KK}(D, A)$. By separability of D, after passing to a subsequence of $(A_k)_k$, if necessary, we construct a sequence of *-homomorphisms $\psi_k^0 : D \to A_k$ such that $\|\phi_k \circ \psi_k^0(d) - \phi \circ j(d)\| \to 0$ for all $d \in D$. Their canonical $\mathrm{C}(X)$-linear extension $\psi_k : B \to A_k$ form a sequence of diagrams



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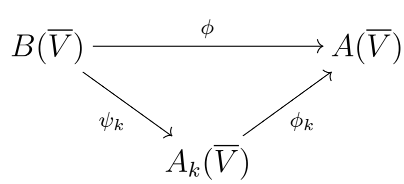
satisfying that $\|\phi_k \psi_k(b) - \phi(b)\| \to 0$ for all $b \in B$ and $[\phi_k \circ \psi_k \circ j] = [\phi \circ j] \in \mathrm{KK}(D, A)$ for all $k \geq 1$. In addition, since there is an isomorphism $\mathrm{KK}_X(\mathrm{C}(X) \otimes D, A) \cong \mathrm{KK}(D, A)$ ([Dad09b, proof of Corollary 2.8]), it follows that $[\phi_k \circ \varphi_k] = [\phi] \in \mathrm{KK}_X(\mathrm{C}(X) \otimes D, A)$, and so

$$[(\phi_k)_v \circ (\psi)_v] = [(\phi)_v] \in \mathrm{KK}(B(v), A(v)) \tag{2.2}$$

for all points $v \in X$ and $k \geq 1$ by [MN09, Proposition 3.4 (10)].

Since A is nowhere locally trivial, it follows from Theorem 2.5.2 that there exists $x \in U$ such that $\mathrm{K}_j(\phi_x)$ is not bijective for $j = 0, 1$. By applying Lemma 2.5.6 to the exhaustive sequence $\{(\phi_k)_x : A_k(x) \to A(x)\}$ we find a k, which we now fix, such that $\mathrm{K}_j((\phi_k)_x)$ is bijective for $j = 0, 1$. It follows that $\mathrm{K}_j((\psi_k)_x)$ is not bijective for this fixed k for $j = 0, 1$.

Let V be the open set given by Lemma 2.5.5 applied to A_k, U and x. Then $x \in \overline{V} \subset U$ and $A_k(\overline{V}) \cong \mathrm{C}(\overline{V}) \otimes D$. Restricting the diagram above to $\overline{V}$, we obtain a diagram



where both $B(\overline{V})$ and $A_k(\overline{V})$ are trivial and $\mathrm{K}_j((\phi_k)_v) \circ \mathrm{K}_j((\psi_k)_v) = \mathrm{K}_j(\phi_v)$ for $j = 0, 1$ for all $v \in \overline{V}$ as a consequence of (2.2). Since $\mathrm{K}_j((\psi_k)_x)$ is not bijective for $j = 0, 1$, by Lemma 2.5.4 there is $r > 0$ such that $\mathrm{K}_j((\psi_k)_v)$ is not bijective for $j = 0, 1$ and for all $v \in \overline{V} \cap U(X, r)$. Let W be an open ball whose closure is contained in $V \cap U(X, r) \subset U$. It follows by Lemma 2.5.1 (iii) that $\mathrm{K}_j(\phi_v)$ is not bijective for $j = 0, 1$ and for any $v \in W$. $\square$

Theorem 2.5.8. *Let X be a finite dimensional metrizable compact space, and let D be a stable Kirchberg algebra that satisfies the UCT and such that $\mathrm{K}_j(D)$ is finitely generated for $j = 0, 1$. Let A be a separable continuous field C*-algebra over X such that $A(x) \cong D$ for all $x \in X$. Then there exists a closed subset V of X with non-empty interior such that $A(V) \cong \mathrm{C}(V) \otimes D$.*

Proof. Arguing as in Lemma 2.5.7, that uses Theorem 2.4.4, there is an exhaustive sequence $\{A_k\}_k$, such that $A_k \in D_{l_k}(X)$, $l_k \leq \dim(X)$ and that the maps $\phi_k : A_k \to A$ are $\mathrm{C}(X)$-linear *-monomorphisms for all k. Seeking a contradiction suppose that for each open set $V \neq \emptyset$, $A(\overline{V}) \ncong \mathrm{C}(\overline{V}) \otimes D$.

Apply Lemma 2.5.7 to $\phi_1 : A_1 \to A$ to find an open set $V_1 \neq \emptyset$ such that $\mathrm{K}_j((\phi_1)_v)$ is not bijective for $j = 0, 1$ and for all $v \in \overline{V}_1$. Next, apply Lemma 2.5.7 again for $\phi_2 : A_2(\overline{V}_1) \to A(\overline{V}_1)$ and V_1 to find a nonempty open set V_2 such that $\overline{V}_2 \subset V_1$ and $\mathrm{K}_j((\phi_2)_v)$ is not bijective for $j = 0, 1$ and for all $v \in \overline{V}_2$. Using the same procedure inductively, one finds a sequence of open sets $\{V_k\}_k$ with $V_k \supset \overline{V}_{k+1}$, such that $\mathrm{K}_j((\phi_k)_v)$ is not bijective for $j = 0, 1$ and for all $v \in \overline{V}_k$ and $k \geq 1$.

Choose $x \in \bigcap_{k=1}^{\infty} \overline{V}_k$ and note that $\{A_k(x)\}_k$ is an exhaustive sequence for $A(x)$ such that none of the maps $\mathrm{K}_j((\phi_k)_x) : \mathrm{K}_j(A_k(x)) \to \mathrm{K}_j(A(x))$ are bijective for $j = 0, 1$. By Lemma 2.5.6 this implies that $\mathrm{K}_j(A(x)) \ncong \mathrm{K}_j(D)$ for $j = 0, 1$, and this is a contradiction. □

Theorem 2.5.9. *Let X be a finite dimensional compact metric space, and let D be a stable Kirchberg algebra that satisfies the UCT and such that $\mathrm{K}_j(D)$ is finitely generated for $j = 0, 1$. Let A be a separable continuous field C*-algebra over X such that $A(x) \cong D$ for all $x \in X$. Then there exists a dense open subset U of X such that $A(U)$ is locally trivial.*

Proof. Let $\mathcal{U}$ be the family of all open subsets U of X such that $A(U)$ is trivial. Since X is compact metrizable, we can find a sequence $\{U_n\}_n$ in $\mathcal{U}$ whose union is equal to the union of all elements of $\mathcal{U}$. If we set $U_\infty = \bigcup_n U_n$, then U_∞ is dense in X. Indeed, if $x \in X$ and U_x is any open neighborhood of x in X, the restriction $A(\overline{U_x})$ is a continuous field satisfying the requirements of Theorem 2.5.8; thus, there exists a closed set $W \subseteq U_x$ with $\mathring{W} \neq \emptyset$ such that $A(W)$ is trivial.

Since $A(U_\infty) = \varinjlim_n \{A(U_1 \cup \cdots \cup U_n)\} = \varinjlim_n \{A(U_1) + \cdots + A(U_n)\}$, we see immediately that $A(U_\infty)$ is locally trivial. Indeed the ideal $A(U_\infty)(U_n)$ of $A(U_\infty)$ determined by the open set U_n is equal to $A(U_n) \cong \mathrm{C}_0(U_n) \otimes D$. □

Corollary 2.5.10. *Fix $n \in \mathbb{N} \cup \{\infty\}$. Let X be a finite dimensional compact metrizable space and A be a continuous field over X such that $A(x) \cong \mathcal{O}_n \otimes \mathcal{K}$ for all $x \in X$. Then there exists a closed subset V of X with nonempty interior such that $A(V) \cong \mathrm{C}(V) \otimes \mathcal{O}_n \otimes \mathcal{K}$.*

The following example shows that the conclusion of Theorem 2.5.9 is in a certain sense optimal. Indeed, given a nowhere dense set $F \subset [0, 1]$, we construct a continuous field C*-algebra A with all fibers isomorphic to a fixed Cuntz algebra $O_n \otimes \mathcal{K}$, $3 \leq n \leq \infty$, and such that the set of singular points of A coincides with F.

Example 2.5.11. Let U be an open dense subset of the unit interval with nonempty complement F. Let D be a Kirchberg algebra with $\mathrm{K}_0(D) \neq 0$ and $\mathrm{K}_1(D) = 0$. Fix an injective *-homomorphism $\gamma : D \to D$ such that $\mathrm{K}_j(\gamma) = 0$ for $j = 0, 1$. Define a continuous field C*-algebra over $[0, 1]$ by

$$A = \{f \in C[0, 1] \otimes D \mid f(x) \in \gamma(D), \quad \forall x \in F\}.$$

It is clear that $A(U) \cong \mathrm{C}_0(U) \otimes D$. We will show that if I is any closed subinterval of $[0,1]$ such that $I \cap F \neq \emptyset$, then $A(I)$ is not trivial. This will show that F is the set of singular points of A. Let us observe that

$$A(I) = \{f \in \mathrm{C}(I) \otimes D \mid f(x) \in \gamma(D), \quad \forall x \in I \cap F\}$$

is isomorphic to the pullback of the diagram

$$\mathrm{C}(I) \otimes D \xrightarrow{\pi} \mathrm{C}(I \cap F) \otimes D \xleftarrow{id \otimes \gamma} \mathrm{C}(I \cap F) \otimes D.$$

Indeed, if we denote by P the pullback of above diagram, there is a natural *-homomorphism $\eta : P \to A(I)$ defined by $\eta(f,g) = f$. Since γ is injective, it follows that η is an isomorphism with inverse defined by $\eta^{-1}(f) = (f, \gamma^{-1} f)$.

We see that $\mathrm{K}_1(\mathrm{C}(I \cap F) \otimes D) = 0$ by the Künneth formula. Therefore, the Mayer-Vietoris exact sequence on K-theory explained in the Preliminaries (see also [Sch84, Theorem 4.5], [Bla98, Proposition 21.2.2]) gives that $\mathrm{K}_0(A(I))$ is the pullback of the following diagram of groups:

$$\mathrm{K}_0(\mathrm{C}(I) \otimes D) \xrightarrow{\pi_*} \mathrm{K}_0(\mathrm{C}(I \cap F) \otimes D) \xleftarrow{(id \otimes \gamma)_*} \mathrm{K}_0(\mathrm{C}(I \cap F) \otimes D).$$

Let $e \in D$ be a projection such that $[e] \neq 0$ in $\mathrm{K}_0(D)$. Let $\tilde{\gamma}(e)$ be the constant function on I equal to $\gamma(e)$, and let $\tilde{e}$ be the constant function on $I \cap F$ equal to e. The pair $(\tilde{\gamma}(e), \tilde{e})$ is a projection $p \in A(I)$. Since F has empty interior, there is a point $y_0 \in I \setminus F$. Choose a point $z_0 \in I \cap F$. To show that $A(I)$ is not trivial we observe that $\mathrm{K}_0(\pi_{y_0})(p) = [\gamma(e)] = 0$ in $\mathrm{K}_0(A(y_0)) = \mathrm{K}_0(D)$, whereas $\mathrm{K}_0(\pi_{z_0})(p) = [e] \neq 0$ in $\mathrm{K}_0(A(z_0)) = \mathrm{K}_0(D)$.

Chapter 3

The Cuntz Semigroup of Continuous Fields

In this chapter, we describe the Cuntz semigroup of continuous fields of C*-algebras over one-dimensional spaces whose fibers have stable rank one and trivial K_1 for each closed, two sided ideal. This is done in terms of the semigroups of global sections on a certain topological space built out of the Cuntz semigroup of the fibers of the continuous field. We use this description to show that when the fibers have furthermore real rank zero, and taking into account the action of the space, the Cuntz semigroup is a classifying invariant if and only if so is the sheaf induced by the Murray-von Neumann semigroup. The results in this chapter are contained in [ABP].

3.1 The Cuntz semigroup of a stable C*-algebra

We start this section summarizing the properties of the Cuntz semigroup of a stable C*-algebra.

Coward, Elliott and Ivanescu showed in 2008 that the Cuntz semigroup of a stable C*-algebra is richer than just being an ordered semigroup, as it belongs to a category with certain continuity properties ([CEI08]). We shall denote by $\mathrm{Cu}(A)$ the Cuntz semigroup of the stabilization of a C*-algebra, i.e. $\mathrm{Cu}(A) := \mathrm{W}(A \otimes \mathcal{K}) = (A \otimes \mathcal{K})_+/{\sim}$.

Before stating the main result of [CEI08], we recall that an element s in a semigroup S is said to be *compactly contained* in t, denoted $s \ll t$, if whenever $t \leq \sup_n z_n$ for some increasing sequence (z_n) with supremum in the semigroup, there exists m such that $s \leq z_m$. An element s in semigroup S is said to be *compact* if $s \ll s$, and a sequence (s_n) such that $s_n \ll s_{n+1}$ is termed *rapidly increasing*. For instance, if $\varepsilon > 0$, the elements $\langle (a-\varepsilon)_+ \rangle, \langle a \rangle \in \mathrm{Cu}(A)$ satisfy that $\langle (a-\varepsilon)_+ \rangle \ll \langle a \rangle$ and for any projection p, the element $\langle p \rangle$ is compact. Concretely, when A is a stably finite C*-algebra, the Murray-von Neumann semigroup can be identified with the compact elements in $\mathrm{Cu}(A)$, i.e., $\mathrm{V}(A) = \{x \in \mathrm{Cu}(A) \mid x \ll x\}$. This follows from Proposition 1.3.23.

The following summarizes some structural properties of the Cuntz semigroup of a stable C*-algebra.

Theorem 3.1.1. ([CEI08])

(i) *Every increasing sequence in* $\mathrm{Cu}(A)$ *has a supremum in* $\mathrm{Cu}(A)$.

(ii) *Every element in* $\mathrm{Cu}(A)$ *is the supremum of a rapidly increasing sequence.*

(iii) *The operation of taking suprema and* $\ll$ *are compatible with addition.*

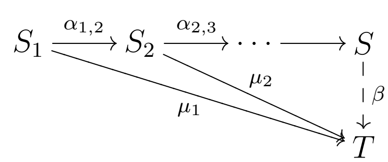
This allows us to define a category Cu whose objects are ordered semigroups whose elements satisfy conditions (i)-(iii) above and whose morphisms are those semigroup maps that preserve the zero element, the order, the compact containment relation and suprema. In [CEI08] it was also shown that the category Cu is closed under countable inductive limits. A useful description of inductive limits in Cu is available below.

Proposition 3.1.2. ([CEI08], cf. [BRT$^+$12]) *Let* $(S_i, \alpha_{i,j})_{i,j\in\mathbb{N}}$ *be an inductive system in the category* Cu. *Then* $(S, \alpha_{i,\infty})$ *is the inductive limit of this system if:*

(i) *The set* $\bigcup_i \alpha_{i,\infty}(S_i)$ *is dense in* S*, in the sense that any element* $s \in S$ *satisfies that* $s = \sup_n(s_n)$ *with* $s_n \in \bigcup_i \alpha_{i,\infty}(S_i)$ *and* $s_n \ll s_{n+1}$ *for all* n.

(ii) *For any* $x, y \in S_i$ *such that* $\alpha_{i,\infty}(x) \leq \alpha_{i,\infty}(y)$ *and* $x' \ll x$ *there is* $j \geq i$ *such that* $\alpha_{i,j}(x') \leq \alpha_{i,j}(y)$.

Proof. We will check that S satisfies the universal property of inductive limits. Suppose that $T \in \mathrm{Cu}$ and that we have a collection of morphisms $(\mu_i)_{i=1}^{\infty}$ in the category Cu such that the following diagram



commutes.

Let us prove that $\beta\colon S \to T$ exists and is unique. Since, without loss of generality, one can write any $s \in S$ as $s = \sup \alpha_{n,\infty}(s_n)$ with $s_n \in S_n$, define β as

$$\beta(s) = \sup(\mu_n(s_n)).$$

To check β is well-defined, we prove that $\mu_n(s_n)$ is an increasing sequence. Let $\alpha_{i,\infty}(s_1) \leq \alpha_{j,\infty}(s_2)$, and choose $s \ll s_1$ in S_i. By assumption (ii) $\alpha_{i,k}(s) \leq \alpha_{j,k}(s_2)$ for some k, so $\mu_i(s) = \mu_k(\alpha_{i,k}(s)) \leq \mu_k(\alpha_{j,k}(s_2)) = \mu_j(s_2)$ for all $s \ll s_1$. Taking supremum on s, we get $\mu_i(s_1) \leq \mu_j(s_2)$. Now, it is clear that β is well-defined, and further $\beta(\alpha_{i,\infty}(y)) = \mu_i(y)$ for all $y \in S_i$.

In order to check that β preserves the order, let $s \leq t$ in S, and write them as supremum of rapidly increasing sequences $(\alpha_{n,\infty}(s_n))_n, (\alpha_{n,\infty}(t_n))_n$. By the compactly containment relation, for all n there exists m such that $\alpha_{n,\infty}(s_n) \ll \alpha_{m,\infty}(t_m)$. Now, by the above fact, there exists l such that

$$\mu_n(s_n) = \mu_l(\alpha_{n,l}(s_n)) \leq \mu_l(\alpha_{m,l}(t_m)) \leq \beta(t),$$

so $\mu_n(s_n) \leq \beta(t)$ for all n. Therefore, one has $\beta(s) = \sup(\mu_n(s_n)) \leq \beta(t)$ what implies that β is well-defined and preserves the order.

To check β preserves $\ll$, consider $s \ll t$ in S and write t as a supremum of a rapidly increasing sequence, i.e., $t = \sup(\alpha_{n,\infty}(t_n))$. Then, there exists n such that $s \ll \alpha_{n,\infty}(t_n) \ll \alpha_{n+1,\infty}(t_{n+1}) \ll t$. Let $x \ll t_n$ in S_n such that $s \leq \alpha_{n,\infty}(x) \ll \alpha_{n+1,\infty}(t_{n+1}) \ll t$ and $\alpha_{n,k}(x) \ll \alpha_{n+1,k}(t_{n+1})$ for some k; therefore,

$$\beta(s) \leq \beta(\alpha_{n,\infty}(x)) = \mu_k(\alpha_{n,k}(x)) \ll \mu_k(\alpha_{n+1,k}(t_{n+1})) \leq \mu(t),$$

implying that β preserves $\ll$.

Finally, uniqueness is clear from the commutativity of the diagram. □

3.2 Sheaves of Semigroups

Here, we define what is meant by a presheaf of semigroups on a topological space and state some important results. Let X be a compact and metrizable topological space. Denote by $\mathcal{V}_X$ the category of all closed subsets of X with non-empty interior, with morphisms given by inclusion.

Definition 3.2.1. *A* presheaf *over X is a contravariant functor*

$$\mathcal{S} \colon \mathcal{V}_X \to \mathcal{C}$$

where $\mathcal{C}$ is a subcategory of the category of sets which is closed under sequential inductive limits. Thus, it consists of an assignment, for each $V \in \mathcal{V}_X$, of an object $\mathcal{S}(V)$ in $\mathcal{C}$ and a collection of maps (referred to as restriction homomorphisms*) $\pi_V^{V'} : \mathcal{S}(V') \to \mathcal{S}(V)$ whenever $V \subseteq V'$ in $\mathcal{V}_X$. We of course require that these maps satisfy $\pi_V^V = \mathrm{id}_V$ and $\pi_W^U = \pi_W^V \pi_V^U$ if $W \subseteq V \subseteq U$.*

Let $V, V' \in \mathcal{V}_X$ be such that $V \cap V' \in \mathcal{V}_X$. A presheaf is called a sheaf *if the map*

$$\pi_V^{V \cup V'} \times \pi_{V'}^{V \cup V'} : \mathcal{S}(V \cup V') \to \{(f, g) \in \mathcal{S}(V) \times \mathcal{S}(V') \mid \pi_{V \cap V'}^{V}(f) = \pi_{V' \cap V}^{V'}(g)\}$$

is bijective.

A presheaf (respectively a sheaf) is continuous *if for any decreasing sequence of closed subsets $(V_i)_{i=1}^{\infty}$ whose intersection $\cap_{i=1}^{\infty} V_i = V$ belongs to $\mathcal{V}_X$, the limit $\varinjlim \mathcal{S}(V_i)$ is isomorphic to $\mathcal{S}(V)$.*

If $\mathcal{F}$ and $\mathcal{G}$ are presheaves over X, then a *morphism* (of presheaves)

$$\alpha : \mathcal{F} \to \mathcal{G}$$

is a collection of maps

$$\alpha_U : \mathcal{F}(U) \to \mathcal{G}(U)$$

for each $U \in \mathcal{V}_X$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{F}(U) & \longrightarrow & \mathcal{G}(U) \\ {\scriptstyle \pi_V^U}\downarrow & & {\scriptstyle \pi_V^U}\downarrow \\ \mathcal{F}(V) & \longrightarrow & \mathcal{G}(V) \end{array}$$

for $V \subset U \subset X$. A presheaf $\mathcal{F}$ is said to be a *subpresheaf* of $\mathcal{G}$ if the maps α_U above are inclusions.

Consider a presheaf $\mathcal{S}$ over X. For any $x \in X$, define the *fiber* of $\mathcal{S}$ at x as

$$S_x := \lim_{x \in \mathring{V}} \mathcal{S}(V) \,,$$

with respect to the restriction maps.

We shall be exclusively concerned with continuous presheaves (or sheaves) $\mathcal{S}$ with target values in the category Sg of semigroups, in which case we will say that $\mathcal{S}$ is a (pre)sheaf of semigroups. As a general notation, we will use S to denote the semigroup $\mathcal{S}(X)$. We will also denote $\pi_x \colon S \to S_x$ the natural map from S to the fiber S_x, as well as $\pi_V \colon S \to S(V)$ rather than π_V^X.

Note that a sheaf $\mathcal{S}$ on a space X is a carrier of localized information about the space X. To obtain general information about X from $\mathcal{S}$, we have to look for another model of a sheaf.

Definition 3.2.2. (i) *An* étalé space *over a topological space X is a topological space Y together with a continuous surjective mapping $r : Y \to X$ such that r is a local homeomorphism. We will often denote an étalé space as the pair (Y, r)*

(ii) *A section of an étalé space $Y \xrightarrow{r} X$ over a closed set $V \subset X$ is a continuous map $f : V \to Y$ such that $r \circ f = 1_V$. The set of continuous sections over V is denoted by $\Gamma(V, Y)$.*

By the above definition, it is clear that the sections of an étalé space form a subsheaf of $\mathrm{C}(X, Y)$, the continuous functions from X to Y. Our next step is to associate, to any presheaf $\mathcal{S}$ over X, an étalé space (F_S, r) such that, if $\mathcal{S}$ is a sheaf, the sheaf of sections of F_S gives another model for $\mathcal{S}$.

Let

$$F_S = \bigsqcup_{x \in X} S_x$$

and $r : F_S \to X$ be the natural projection taking elements in S_x to x. Since we want to make (F_S, r) into an étalé space, all that remains to do is to give F_S a topology.

For each $s \in \mathcal{S}(V)$, define the set function $\hat{s} : V \to F_S$ by letting $\hat{s}(x) = s_x$ for each $x \in V$. Note that $r \circ \hat{s} = 1_V$. Let $\{\hat{s}(U)\}$, where U is open in V and $s \in \mathcal{S}(V)$, be a basis for the topology of F_S. Then all the functions $\hat{s}$ are continuous.

With the above techniques we have associated an étalé space to each presheaf $\mathcal{S}$ over X. We remark that as inductive limits in the category Sg are algebraic limits, the étalé space F_S inherits these properties when we compute the fibers of the sheaf. When inductive limits in Sg are given by the algebraic limit, we will say that $\mathcal{S}$ is an *algebraic* sheaf.

Suppose that $\mathcal{S}$ is a presheaf of semigroups. For a closed set $V \subset X$, the set of continuous sections of F_S over V is denoted by $\Gamma(V, F_{\mathcal{S}(V)})$, and it is a semigroup under pointwise addition. Note that, by the last construction, we have associated a presheaf $\Gamma(\_, F_{\mathcal{S}(\_)})$ to any presheaf $\mathcal{S}$.

Looking more closely at the relation between both presheaves, consider

$$\tau : \mathcal{S} \to \Gamma(\_, F_{\mathcal{S}(\_)}),$$

given by $\tau_V : \mathcal{S}(V) \to \Gamma(V, F_{\mathcal{S}(V)})$, where $\tau_{\mathrm{V}}(s) = \hat{s}$.

In the case that $\mathcal{S}$ is an algebraic sheaf, we have the following result (see section 2 of [Wel73] for further details).

Theorem 3.2.3. ([Wel73, Theorem 2.2]) *Let $\mathcal{S}$ be an algebraic sheaf over X. Then,*

$$\tau : \mathcal{S} \to \Gamma(\_, F_{\mathcal{S}(\_)})$$

is a sheaf isomorphism.

Proof. It suffices to show that τ_V is bijective for all $V \in \mathcal{V}_X$.

(i) (τ_V is injective): Let $s', s'' \in \mathcal{S}(V)$ such that $\tau_{\mathrm{V}}(s') = \tau_{\mathrm{V}}(s'')$. Then

$$\hat{s}'(x) = \hat{s}''(x) \text{ for all } x \in V;$$

i.e., $\pi_x^V(s') = \pi_x^V(s'')$ for all $x \in V$. But when $\pi_x^V(s') = \pi_x^V(s'')$ for $x \in V$, the definition of the algebraic inductive limit implies that there exists a closed neighborhood W of x such that $\pi_W^V(s') = \pi_W^V(s'')$. Since this is true for all $x \in V$, and V is compact, we can cover V with open sets $\mathring{W}_i$ such that

$$\pi_{W_i}^V(s') = \pi_{W_i}^V(s'') \text{ for all } i.$$

Because $\mathcal{S}$ is a sheaf, we have $s' = s''$.

(ii) (τ_V is surjective): Suppose $\sigma \in \Gamma(X, F_S)$. Then for $x \in V$ there is a closed neighborhood W of x and $s \in \mathcal{S}(W)$ such that $\sigma(x) = s_x = \tau_W(s)(x)$. Since sections of an étalé space are local inverses for r, any two sections which agree at a point agree in some neighborhood of that point. Hence we have a closed neighborhood W' of x such that

$$\sigma_{|W'} = \tau_W(s)_{|W'} = \tau_{W'}(\pi_{W'}^W(s)).$$

Since this is true for any $x \in V$, we can cover V with neighborhoods $\mathring{W}_i$ such that there exists $s_i \in \mathcal{S}(W_i)$ and $\tau_{W_i}(s_i) = \sigma_{|W_i}$. Moreover, we have $\tau_{W_i}(s_i) = \tau_{W_j}(s_j)$ on $W_i \cap W_j$, so $\pi_{W_i \cap W_j}^{W_i}(s_i) = \pi_{W_i \cap W_j}^{W_j}(s_j)$ by part (i). Because $\mathcal{S}$ is a sheaf and $V = \bigcup_i W_i$, there exists $s \in \mathcal{S}(V)$ such that $\pi_{W_i}^V(s) = s_i$. Thus, $\tau_{\mathrm{V}}(s)|_{W_i} = \tau_{W_i}(\pi_{W_i}^V(s)) = \tau_{W_i}(s_i) = \sigma|_{W_i}$, and finally $\tau_{\mathrm{V}}(s) = \sigma$.

□

3.3 Cuntz semigroup and Pullbacks

We study in this section the behaviour of the Cuntz functor $\mathrm{Cu}(\_)$ on pullbacks of C*-algebras. This is used to define the sheaf induced by the Cuntz semigroup in the next section and to compute the Cuntz semigroup for some class of trivial $\mathrm{C}(X)$-algebras, where $\dim(X) \leq 1$. As a blanket assumption, we shall assume that X is always compact and metrizable. The main results of this section except for Theorem 3.3.6 come from [APS11].

Recall that if A, B and C are C*-algebras and $\pi : A \to C$ and $\phi : B \to C$ are *-homomorphisms, we can form the pullback

$$B \oplus_C A = \{(b, a) \in B \oplus A \mid \phi(b) = \pi(a)\}.$$

By applying the Cuntz functor to the *-homomorphisms π and ϕ we obtain Cuntz semigroup morphisms (in the category Cu)

$$\mathrm{Cu}(\pi) : \mathrm{Cu}(A) \to \mathrm{Cu}(C), \text{ and } \mathrm{Cu}(\phi) : \mathrm{Cu}(B) \to \mathrm{Cu}(C).$$

Let us consider the pullback (in the category of ordered semigroups)

$$\mathrm{Cu}(B) \oplus_{\mathrm{Cu}(C)} \mathrm{Cu}(A) = \{(\langle b\rangle, \langle a\rangle) \in \mathrm{Cu}(B) \oplus \mathrm{Cu}(A) \mid \mathrm{Cu}(\phi)\langle b\rangle = \mathrm{Cu}(\pi)\langle a\rangle\}.$$

Then, we have a natural order-preserving map

$$\beta : \mathrm{Cu}(B \oplus_C A) \to \mathrm{Cu}(B) \oplus_{\mathrm{Cu}(C)} \mathrm{Cu}(A),$$

defined by $\beta(\langle(b,a)\rangle) = (\langle b\rangle, \langle a\rangle)$. Note that the map β preserves suprema. This follows from the fact that any *-homomorphism between C*-algebras induces a morphism in Cu (see Lemma 2.1.3 [San08]) and that the order is defined in each component. Hence, if $\langle(b,a)\rangle = \sup(\langle b_n, a_n\rangle)$ in $\mathrm{Cu}(B \oplus_C A)$, one obtains $\langle b\rangle = \sup(\langle b_n\rangle)$ because $\mathrm{Cu}(\pi_B) : \mathrm{Cu}(B \oplus_C A) \to \mathrm{Cu}(B)$ preserves suprema, and likewise for $\langle a\rangle$. Therefore, $\beta(\langle(b,a)\rangle) = (\langle b\rangle, \langle a\rangle) = (\sup(\langle b_n\rangle, \langle a_n\rangle)) = \sup(\beta(\langle b_n, a_n\rangle))$.

Definition 3.3.1. *Let A be a C*-algebra. We say that A has no K_1 obstructions provided that the stable rank of A is one and $\mathrm{K}_1(I) = 0$ for any closed ideal I of A.*

The next result states the equivalence of the property of no K_1 obstructions under different hypotheses on A.

Lemma 3.3.2. *Let A be a C*-algebra.*

(i) *A has no K_1 obstructions if and only if $\mathrm{sr}(A) = 1$ and $\mathrm{K}_1(B) = 0$ for every hereditary subalgebra B of A.*

(ii) ([Lin95]) *If $\mathrm{RR}(A) = 0$, A has no K_1 obstructions if and only if $\mathrm{sr}(A) = 1$ and $\mathrm{K}_1(A) = 0$.*

Proof. If A has no K_1 obstructions and B is a hereditary subalgebra of A, then $B \otimes \mathcal{K} \cong \overline{ABA} \otimes \mathcal{K}$ by [Bro77, Theorem 2.8]. So $\mathrm{K}_1(B) \cong \mathrm{K}_1(\overline{ABA}) = 0$. The case (ii) was proved in [Lin95, Lemma 2.4]. □

The class just defined was already considered, although not quite with this terminology, in [APS11]. The following results were proved in the same article, and they help us to understand the behaviour of pullbacks in the category Cu. We just provide a sketch of their main proofs for completeness.

Recall that, if S and R are partially ordered semigroups, we say that a morphism $f : S \to R$ is an *order-embedding* if $f(x) \leq f(y)$ implies $x \leq y$ for $x, y \in S$.

Theorem 3.3.3. ([APS11, Theorem 3.1]) *Let A, B and C be C*-algebras such that C is separable and it does not have K_1 obstructions. Let $\phi : B \to C$ and $\pi : A \to C$ be *-homomorphisms such that π is surjective. Then the map*

$$\beta : \mathrm{Cu}(B \oplus_C A) \to \mathrm{Cu}(B) \oplus_{\mathrm{Cu}(C)} \mathrm{Cu}(A),$$

given by $\beta(\langle(b,a)\rangle) = (\langle b\rangle, \langle a\rangle)$ is an order-embedding.

Proof. (Sketch) By [Ped99, Theorem 3.9] applied to $Y = \mathcal{K}$ we may assume that A, B and C are stable. Let (b_1, a_1) and (b_2, a_2) be positive contractions of $B \oplus_C A$ such that $a_1 \precsim a_2$ and $b_1 \precsim b_2$, and let $\varepsilon > 0$. Then, by definition there are $x \in A$ and $y \in B$ such that

$$\|a_1 - x^*x\| < \varepsilon \quad xx^* \in \mathrm{Her}(a_2) \quad \text{and} \quad \|b_1 - y^*y\| < \varepsilon \quad yy^* \in \mathrm{Her}(b_2).$$

Since $\pi(a_1) = \phi(b_1)$ and $\pi(a_2) = \phi(b_2)$, these equations imply that

$$\|\pi(a_1) - \pi(x)^*\pi(x)\| < \varepsilon \quad \|\pi(a_1) - \phi(y)^*\phi(y)\| < \varepsilon \quad \pi(x)\pi(x)^*, \phi(y)\phi(y)^* \in \mathrm{Her}(\pi(a_2)).$$

By [APS11, Lemma 1.4], which allows to approximate x, y under the above hyphoteses, there exists a unitary $u \in \mathrm{Her}(\pi(a_2))^\sim$ such that $\|u\pi(x) - \phi(y)\| < 9\varepsilon$. It follows from [Bro77, Theorem 2.8] that $\mathrm{Her}(a_2)$ is stably isomorphic to an ideal of C. Now, by [Rie83, Theorem 2.10], $\mathcal{U}(\mathrm{Her}(\pi(a_2))^\sim) = \mathcal{U}_0(\mathrm{Her}(\pi(a_2))^\sim)$ since $\mathrm{sr}(A) = 1$. By the surjectivity of π there exists a unitary $v \in \mathrm{Her}(a_2)^\sim$ such that $\tilde{\pi}(v) = u$, where $\tilde{\pi} : \tilde{A} \to \tilde{C}$ is the natural extension of π. In addition, there exists $y' \in \mathrm{Her}(a_2)$ such that $\pi(y') = \phi(y)$.
Hence, we have $\|\pi(vx - y')\| = \|u\pi(x) - \phi(y)\| < 9\varepsilon$. Since $vx - y' \in \overline{a_2 A}$ there exists $z' \in \overline{a_2 A} \cap Ker(\pi)$ such that $\|vx - y' - z'\| < 9\varepsilon$.

Set $y' + z' = z$. Then, $\pi(z) = \pi(y') = \phi(y)$, $zz^* \in \mathrm{Her}(a_2)$, $\|vx - z\| < 9\varepsilon$. It also follows that

$$\|a_1 - z^*z\| < 118\varepsilon.$$

Since $\pi(z) = \phi(y)$, the element (y, z) belongs to $B \oplus_C A$, and $\|(b_1, a_1) - (y, z)^*(y, z)\| < 118\varepsilon$ by the above estimate. We also have

$$(y, z)(y, z)^* = \lim_{n\to\infty} (b_2, a_2)^{\frac{1}{n}}(y, z)(y, z)^*(b_2, a_2)^{\frac{1}{n}} \in \mathrm{Her}((b_2, a_2)).$$

By Theorem 1.3.14 we have $((b_1, a_1) - 118\varepsilon)_+ \precsim (b_2, a_2)$; therefore,

$$\langle (b_1, a_1)\rangle = \sup_{\varepsilon>0} \langle ((b_1, a_1) - 118\varepsilon)_+\rangle \leq \langle (b_2, a_2)\rangle.$$

□

When working with $\mathrm{C}(X)$-algebras, it useful to note that if A is a $\mathrm{C}(X)$-algebra, then this is also the case for $A \otimes \mathcal{K}$. In fact, for any closed set Y of X, there is a *-isomorphism

$$\varphi_Y \colon (A \otimes \mathcal{K})(Y) \to A(Y) \otimes \mathcal{K}$$

such that $\varphi_Y \circ \pi'_Y = \pi_Y \otimes 1_{\mathcal{K}}$, where $\pi_Y \colon A \to A(Y)$ and $\pi'_Y \colon A \otimes \mathcal{K} \to (A \otimes \mathcal{K})(Y)$. This yields, in particular, that $(A \otimes \mathcal{K})(x) \cong A_x \otimes \mathcal{K}$ for any $x \in X$, with $(a \otimes k)(x) \mapsto a(x) \otimes k$ (see [APS11, Lemma 1.5]).

Using this observation, the map induced at the level of Cuntz semigroups $\mathrm{Cu}(A) \to \mathrm{Cu}(A_x)$ can be viewed as $\langle a\rangle \mapsto \langle \pi_x(a)\rangle$. Similarly, if Y is closed in X, the map π_Y induces $\mathrm{Cu}(A) \to \mathrm{Cu}(A(Y))$, that can be thought of as $\langle a\rangle \mapsto \langle \pi_Y(a)\rangle$. Thus, when computing the Cuntz semigroup of a $\mathrm{C}(X)$-algebra A, we will assume that A, A_x and $A(Y)$ are stable.

Theorem 3.3.4. ([APS11, Theorem 3.2]) *Let X be a one-dimensional compact Hausdorff space and let Y be a closed subset of X. Let A be a C(X)-algebra and let $\pi_Y : A \to A(Y)$ be the quotient map. Let B be a C*-algebra and let $\phi : B \to A(Y)$ be a *-homomorphism. Suppose that, for every $x \in X$, the C*-algebra A_x is separable, and it has no K_1 obstructions. Then, the map*

$$\beta : \mathrm{Cu}(B \oplus_{A(Y)} A) \to \mathrm{Cu}(B) \oplus_{\mathrm{Cu}(A(Y))} \mathrm{Cu}(A),$$

is an order-embedding.

Theorem 3.3.5. ([APS11, Theorem 3.3]) *Let X be a compact Hausdorff space and let Y be a closed subset of X. Let A be a C(X)-algebra, and let B be any C*-algebra. Suppose that the map*

$$\alpha\colon \mathrm{Cu}(A) \to \prod_{x\in X} \mathrm{Cu}(A_x)$$

given by $\alpha(\langle a\rangle)(x) = \langle a(x)\rangle$ is an order-embbeding. Then

(i) *the map*
$$\beta : \mathrm{Cu}(B \oplus_{A(Y)} A) \to \mathrm{Cu}(B) \oplus_{\mathrm{Cu}(A(Y))} \mathrm{Cu}(A),$$
is surjective.

(ii) *The pullback semigroup $\mathrm{Cu}(B) \oplus_{\mathrm{Cu}(A(Y))} \mathrm{Cu}(A)$ is in the category Cu.*

Proof. (Sketch) As in the proof of Theorem 3.3.3 we may assume that $A, A(Y)$ and B are stable. Let $a \in A$ and $b \in B$ be positive elements such that $\pi_Y(a) \sim \phi(b)$. Choose $c \in A_+$ such that $\pi_Y(c) = \phi(b)$, so that $\pi_Y(a) \sim \pi_Y(c)$.

Let $\varepsilon > 0$. By the definition of Cuntz equivalence, Proposition 1.3.17 and the surjectivity of π_Y, there exists $d \in A$ and $0 < \delta < \varepsilon$ such that

$$\|\pi_Y(a-\varepsilon)_+ - \pi_Y(d)^*\pi_Y(c-\delta)_+\pi_Y(d)\| < \varepsilon.$$

We remark that the above inequality also holds in the fiber algebras A_x with $x \in Y$.

By upper semicontinuity of the norm and the normality of X since it is compact, there exists an open neighborhood U of Y such that

$$\|(a-\varepsilon)_+(x) - d(x)^*(c-\delta)_+(x)d(x)\| < \varepsilon$$

holds for all $x \in \overline{U}$. So $\|\pi_{\overline{U}}((a-\varepsilon)_+ - d^*(c-\delta)_+d)\| < \varepsilon$. By Theorem 1.3.14 and since $\pi_{\overline{U}}$ is surjective, there exists $f \in A$ such that $\pi_{\overline{U}}((a-2\varepsilon)_+) = \pi_{\overline{U}}(f^*(c-\delta)_+f)$. This implies that

$$\pi_{\overline{U}}((a-2\varepsilon)_+) \precsim \pi_{\overline{U}}((c-\delta)_+) \text{ and } \pi_{\overline{U}}((a-3\varepsilon)_+) = \pi_{\overline{U}}((f^*(c-\delta)_+f-\varepsilon)_+).$$

It follows by Proposition 1.3.17 that there exists a unitary $u \in \tilde{A}$ such that $u^*(f^*(c-\delta)_+)f-\varepsilon)_+u \in \mathrm{Her}((c-\delta)_+)$. Let us consider $a' = u^*au$. Then, passing to the fibers we have

$$(a'-3\varepsilon)_+(x) \in \mathrm{Her}((c-\delta)_+(x)) \text{ and } (a'-2\varepsilon)_+(x) \precsim (c-\delta)_+(x)$$

for all $x \in U$.

Now let us use that $\pi_Y(c) \precsim \pi_Y(a) \sim \pi_Y(a')$. Arguing as above, it can be checked that

$$(c-\delta)_+(x) \precsim (a'-3\varepsilon')_+(x) \text{ for some } \varepsilon' < \varepsilon, \text{ and for all } x \in U.$$

Let $f_1, f_2 \in C(X)$ be a partition of unity associated to the open sets U and $X \setminus Y$. Consider the element $z = f_1(c-\delta)_+ + f_2(a'-3\varepsilon)_+$. Then,

$$z(x) \begin{cases} = (c(x)-\delta)_+ & \text{if} \quad x \in Y \\ = (a'(x)-3\varepsilon)_+ & \text{if} \quad x \in X \setminus U \\ \succsim (c-\delta)_+(x) \text{ or } z(x) \succsim (a'-3\varepsilon)_+(x) & \text{if} \quad x \in U \\ \in \mathrm{Her}((c(x)-\delta)_+) & \text{if} \quad x \in U. \end{cases} \tag{3.1}$$

By the choice of c and the first equation above we have $z(x) = (c(x)-\delta)_+ = (\phi(b)-\delta)(x)$ for all $x \in Y$. Hence, $\pi_Y(z) = \phi((b-\delta)_+)$, and so $((b-\delta)_+, z)$ is an element of the pullback. By equation (3.1) we have $(a'-3\varepsilon)_+(x) \precsim z(x) \precsim (a'-3\varepsilon')_+(x)$ for all $x \in X$. Since α is order-embedding, this yields

$$(a'-3\varepsilon)_+ \precsim z \precsim (a'-3\varepsilon')_+.$$

Therefore, it is possible to choose $\delta_n < \varepsilon_n$ decreasing to zero and elements $z_n \in A_+$ such that $((b-\delta_n)_+, z_n) \in B \oplus_{A(Y)} A$, $((b-\delta_n)_+, z_n) \precsim ((b-\delta_{n+1})_+, z_{n+1})$ for all n, $\sup_n \langle (b-\delta_n)_+ \rangle = \langle b \rangle$, and $\sup_n \langle z_n \rangle = \langle a' \rangle = \langle a \rangle$. Moreover, $(\langle (b-\delta_n)_+ \rangle, \langle z_n \rangle)$ is rapidly increasing by construction and

$$\sup_n((\langle (b-\delta_n)_+ \rangle, \langle z_n \rangle)) = (\langle b \rangle, \langle a \rangle).$$

It also follows that

$$\beta(\sup_n((\langle (b-\delta_n)_+ \rangle, \langle z_n \rangle))) = \sup_n(\beta((\langle (b-\delta_n)_+ \rangle, \langle z_n \rangle))) = \sup_n((\langle (b-\delta_n)_+ \rangle, \langle z_n \rangle)) = (\langle b \rangle, \langle a \rangle),$$

which proves that β is surjective.

Conclusion (ii) follows from (i) and the fact that $\mathrm{Cu}(\pi_Y)$ and $\mathrm{Cu}(\phi)$ are morphisms in the category Cu. □

The result below provides a situation where the hipotheses of Theorem 3.3.5 are satisfied. Therefore, it shows that under mild assumptions the Cuntz functor and the pullback of C*-algebras are well-behaved. This is explicitely stated in Corollary 3.3.7.

Theorem 3.3.6. *Let X be a one-dimensional compact Hausdorff space and A be a $\mathrm{C}(X)$-algebra such that its fibers have no K_1 obstructions. Then, the map*

$$\alpha\colon \mathrm{Cu}(A) \to \prod_{x \in X} \mathrm{Cu}(A_x),$$

given by $\alpha\langle a \rangle = (\langle a(x) \rangle)_{x \in X}$ is an order-embedding.

Proof. By our assumptions on A and its fibers, we may assume that A is stable.

Let $0 < \epsilon < 1$ be fixed, and let us suppose that $a, b \in A$ are positive contractions such that $a(x) \precsim b(x)$ for all $x \in X$. Then, by the definition of the Cuntz order, since A_x is a quotient of A for each $x \in X$, there exists $d_x \in A$ such that

$$\|a(x) - d_x(x)b(x)d_x^*(x)\| < \epsilon.$$

By upper semicontinuity of the norm, the above inequality also holds in a neighborhood of x. Hence, since X is a compact set, there exists a finite cover of X, say $\{U_i\}_{i=1}^n$, and elements $(d_i)_{i=1}^n \in A$ such that $\|a(x) - d_i(x)b(x)d_i^*(x)\| < \epsilon$, for all $x \in \overline{U_i}$ and $1 \leq i \leq n$. As X is one-dimensional, we may assume that $\{U_i\}$ and $\{\overline{U_i}\}$ have order at most two (this is not restrictive, see [Pea75, Lemma 8.1.1]).

Choose, by Urysohn's Lemma, functions λ_i that are 1 in the closed sets $U_i \setminus (\bigcup_{j\neq i} U_j)$ and 0 in U_i^c. Using these functions we define $d(x) = \sum_{i=1}^n \lambda_i(x)d_i(x)$. Set $V = X \setminus (\bigcup_{i\neq j}(U_i \cap U_j))$ which is a closed set, and it is easy to check that d satisfies

$$\|a(x) - d(x)b(x)d^*(x)\| < \epsilon \tag{3.2}$$

for all $x \in V$.

Again, choose for $i < j$ functions $\alpha_{i,j}$ such that $\alpha_{i,j}$ is one on $\overline{U_i \cap U_j}$ and zero on $\overline{U_k \cap U_l}$ whenever $\{k, l\} \neq \{i, j\}$. We define $c(x) = \sum_{i<j} \alpha_{i,j}(x)d_i(x)$, put $U = \bigcup_{i\neq j}(U_i \cap U_j) = V^c$ and notice that c satisfies

$$\|a(x) - c(x)b(x)c^*(x)\| < \epsilon \tag{3.3}$$

for all $x \in \overline{U}$.

Now, by Theorem 1.3.14, equations (3.2) and (3.3), and taking into account that the norm of an element is computed fiberwise (Lemma 1.1.21), we have that

$$\pi_{\mathrm{V}}((a-\epsilon)_+) \precsim \pi_{\mathrm{V}}(b) \text{ and } \pi_{\overline{U}}((a-\epsilon)_+) \precsim \pi_{\overline{U}}(b)\,.$$

Therefore

$$(\langle \pi_{\mathrm{V}}(a-\epsilon)_+)\rangle, \langle \pi_{\overline{U}}(a-\epsilon)_+)\rangle) \leq (\langle \pi_{\mathrm{V}}(b)\rangle, \langle \pi_{\overline{U}}(b)\rangle)$$

in the pullback semigroup $\mathrm{Cu}(A(V)) \oplus_{\mathrm{Cu}(A(\overline{U}\cap V)} \mathrm{Cu}(A(\overline{U}))$. Since A can also be written as the pullback $A = A(V) \oplus_{A(\overline{U}\cap V)} A(\overline{U})$ along the natural restriction maps (see [Dad09a, Lemma 2.4], and also [Dix77, Proposition 10.1.13]), we can apply Theorem 3.3.4, to conclude that $(a-\epsilon)_+ \precsim b$. Thus $a \precsim b$, and the result follows. □

Corollary 3.3.7. *Let X be a one-dimensional compact Hausdorff space and let Y be a closed subset of X. Let A be a C(X)-algebra such that its fibers have no K_1 obstructions, and let B be any C*-algebra. Then* $\mathrm{Cu}(B) \oplus_{\mathrm{Cu}(A(Y))} \mathrm{Cu}(A)$ *belongs to the category* Cu *and it is isomorphic to* $\mathrm{Cu}(B \oplus_{A(Y)} A)$

Proof. Combine Theorems 3.3.4, 3.3.5 and 3.3.6. □

The above fact will be a basic tool in the next section, where we will define the sheaf given by the Cuntz semigroup on a $\mathrm{C}(X)$-algebra. In the particular case of trivial $\mathrm{C}(X)$-algebras, i.e. of the form $\mathrm{C}(X, A)$ for a C*-algebra A, the image of the map α in Theorem 3.3.6 can be viewed as functions from X to $\mathrm{Cu}(A)$ that are lower semicontinuous in a certain topology.

Given a semigroup $S \in \mathrm{Cu}$ and $a \in S$ the set $a^{\ll} = \{b \in S \mid b \gg a\}$ defines a basis of the so-called Scott topology on S ([GHK+03]).

Definition 3.3.8. *Let X be a topological space, S a semigroup in* Cu*, and $f : X \to S$ a function. We say that f is lower semicontinous if, for all $a \in S$, the set $f^{-1}(a^{\ll}) = \{t \in X \mid a \ll f(t)\}$ is open in X. We shall denote the set of all lower semicontinous functions from X to S by $\mathrm{Lsc}(X, S)$. If we equip S with the Scott topology, lower semicontinuous functions are the Scott continuous functions.*

Note that $\mathrm{Lsc}(X, S)$ is a partially ordered semigroup when we equip it with the pointwise order and addition.

Proposition 3.3.9. ([Tik11, Proposition 3.1]) *Let A be a C*-algebra, X a compact Hausdorff space and $f \in \mathrm{C}(X, A)$. Then, for any $b \in A$, the set $\{x \in X \mid \langle b\rangle \ll \langle f(x)\rangle\}$ is open.*

Proof. Let $x \in X$ be such that $\langle b\rangle \ll \langle f(x)\rangle$. Then, for some $\varepsilon > 0$ we have $\langle b\rangle \ll \langle (f(x) - \varepsilon)_+\rangle$. Let U be a neighborhood of x such that any $y \in U$ satisfies $\|f(y) - f(x)\| < \varepsilon$. Hence, by Theorem 1.3.14, we have that $\langle (f(x) - \varepsilon)_+\rangle \leq \langle f(y)\rangle$ for all $y \in U$. Thus, U is an open neighborhood of x contained in $\{x \in X \mid \langle b\rangle \ll \langle f(x)\rangle\}$. □

In order to compute the Cuntz Semigroup of some trivial $\mathrm{C}(X)$-algebras, it was shown in Section 5 of [APS11], that for any compact Hausdorff space X with finite covering dimension and for any countably based semigroup S in Cu, the set $\mathrm{Lsc}(X, S)$ belongs to the category Cu. We say that S is *countably based* if there is a countable subset N in S such that every element of S is the supremum of a rapidly increasing sequence of elements coming from N. If S is countably based, then S satisfies the second axiom of countability as a topological space equipped with the Scott topology (see, e.g. [GHK+03, Theorem III-4.5]).

Lemma 3.3.10. ([APS11, Lemma 1.3]) *If A is a separable C*-algebra, then* $\mathrm{Cu}(A)$ *is countably based.*

Proof. (Sketch) Let F be a countable dense subset of A_+, and consider the set

$$N = \{\langle (a - 1/n)_+\rangle \mid a \in F \ \ , n \in \mathbb{N}\}.$$

Now the result follows by the separability of A and the fact that $\mathrm{Cu}(A)$ is a semigroup in Cu. □

We give a brief explanation of the main facts of [APS11, Section 5] without proving them since the details are similar to the ones in the next section.

In [APS11] the set of so-called piecewise characteristic functions in $\mathrm{Lsc}(X, S)$ is defined. This is used to show that any $f \in \mathrm{Lsc}(X, S)$ can be written as a supremum of a rapidly increasing sequence of piecewise characteristic functions. Although this set of functions can be defined over any finite dimensional space X, we just state below the definition of the piecewise characteristic functions over $[0, 1]$ in order to clarify how they are built.

Definition 3.3.11. *Let S be a semigroup in* Cu*. Given the following data*

(i) *A partition $0 = t_0 < t_1 < \ldots < t_{n-1} < t_n = 1$ of $[0, 1]$ with $n = 2r + 1$ for some $r \geq 1$,*

(ii) *elements $x_0, \ldots, x_{n-1}$ in S, with $x_{2i}, x_{2i+2} \leq x_{2i+1}$ for $0 \leq i \leq r - 1$,* *a piecewise characteristic function is a map* $g : [0,1] \to S$ *such that*

$$g(s) = \begin{cases} x_{2i} & \text{if } s \in [t_{2i}, t_{2i+1}] \\ x_{2i+1} & \text{if } s \in (t_{2i+1}, t_{2i+2}). \end{cases}$$

If moreover $g \ll f$ *for some* $f \in \mathrm{Lsc}([0,1], S)$*, we then say that* g *is a piecewise characteristic function for* f*. We denote the set of all such functions by* $\chi(f)$*.*

Theorem 3.3.12. ([APS11, Theorem 5.15]) *Let* X *be a compact metric finite dimensional topological space, and let* S *be an object in* Cu*. If* S *is countably based, then* $\mathrm{Lsc}(X, S)$ *(with the pointwise order and addition) is also a semigroup in* Cu*.*

As a consequence of the above result (among others), in [APS11] the computation of the Cuntz semigroup for some trivial $\mathrm{C}(X)$-algebras is carried out. This result is generalized in Theorem 3.4.19

Theorem 3.3.13. ([APS11, Theorem 3.4]) *Let* X *be a one-dimensional compact metric space. Let* A *be a separable C*-algebra with no* K_1 *obstructions. Then, the map* $\alpha : \mathrm{Cu}(\mathrm{C}(X, A)) \to \mathrm{Lsc}(X, \mathrm{Cu}(A))$ *given by* $\alpha(\langle a \rangle) = \langle a(x) \rangle$*, for all* $a \in \mathrm{C}(X, A)$ *and* $x \in X$*, is an isomorphism in the category* Cu*.*

3.4 Sheaves of Cu and continuous sections

In this section, following the lines of research started in [APS11], we study the sheaf defined by the Cuntz semigroup. We use this description in order to compute the Cuntz semigroup of some $\mathrm{C}(X)$-algebras and generalize Theorem 3.3.13 to some non-trivial $\mathrm{C}(X)$-algebras. Moreover, we prove a version of Theorem 3.2.3 for non-algebraic sheaves.

Let X be a compact metric space, and let A be a $\mathrm{C}(X)$-algebra. Recall that $\mathcal{V}_X$ is the category of all closed subsets of X with non-empty interior. The assignments

$$\begin{array}{rccc} \mathrm{Cu}_A\colon & \mathcal{V}_X & \to & \mathrm{Cu} \\ & U & \mapsto & \mathrm{Cu}(A(U)) \end{array} \quad \text{and} \quad \begin{array}{rccc} \mathbb{V}_A\colon & \mathcal{V}_X & \to & \mathrm{Sg} \\ & U & \mapsto & \mathrm{V}(A(U)) \end{array}$$

define continuous presheaves of semigroups. If $U \subseteq V$ in $\mathcal{V}_X$, the restriction maps $\pi_V^U\colon A(U) \to A(V)$ and the limit maps $\pi_x\colon A \to A_x$ define, by functoriality, semigroup maps $\mathrm{Cu}(\pi_V^U)$ and $\mathrm{Cu}(\pi_x)$ in the case of the Cuntz semigroup, and likewise in the case for the semigroup of projections. For ease of notation, and unless confusion may arise, we shall still denote these maps by π_V^U and π_x.

We will say that a (pre)sheaf is *surjective* provided all the restriction maps are surjective. This is clearly the case for the presheaf Cu_A for a general $\mathrm{C}(X)$-algebra A, and also for $\mathbb{V}_A$ if A has real rank zero (which is a rather restrictive hypothesis since, as proved in [Pas05] and [Pas06], if A is a continuous field over X with real rank zero and $A_x \neq 0$ for all $x \in X$, then $\dim(X) = 0$). As we shall see in Theorem 3.5.11, still Cu_A and $\mathbb{V}_A$ determine each other under mild assumptions.

Most of the discussion in this section will consider surjective (pre)sheaves of semigroups $\mathcal{S}\colon \mathcal{V}_X \to \mathrm{Cu}$, and we will need to develop a somewhat abstract approach on how to recover the information of the sheaf from the sheaf of sections of a bundle $F_S \to X$. As seen in Theorem

3.2.3, this is classically done by endowing F_S with a topological structure that glues together the fibers (which are computed as inductive limits in the category of sets). One of the main difficulties here resides in the fact that the inductive limit in Cu is not the algebraic limit, even in the case of the fiber of a surjective presheaf. We illustrate this situation below with an easy example.

Example 3.4.1. Let $A = \mathrm{C}([0,1], M_n(\mathbb{C}))$, where $n \geq 2$. By [Rob13, Theorem 1.1] (see also Theorem 3.3.13) it follows that $\mathrm{Cu}(A) \cong \mathrm{Lsc}([0,1], \overline{\mathbb{N}})$, where $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$.

Now, let $\{U_m = [\frac{1}{2} - \frac{1}{m}, \frac{1}{2} + \frac{1}{m}]\}_{m \geq 2}$, which is a sequence of decreasing closed subsets of $[0,1]$ whose intersection is $\{1/2\}$. Since the functor $\mathrm{Cu}(\_)$ preserves limits, it follows that

$$\varinjlim \mathrm{Lsc}(U_n, \overline{\mathbb{N}}) = \varinjlim \mathrm{Cu}(A(U_n)) = \mathrm{Cu}(\varinjlim A(U_n)) = \mathrm{Cu}(A(1/2)) = \overline{\mathbb{N}}.$$

However, the computation of the direct limit above in the category of semigroups yields $\{(a,b,c) \in \mathbb{N}^3 \mid b \leq a, c\}$. Indeed, this follows from the definition of lower semicontinuous functions since, by definition, the value at any point is always smaller than the values of the points it has at each side.

As defined before, if $\mathcal{S}\colon \mathcal{V}_X \to \mathrm{Cu}$ is a sheaf, we set $F_S := \sqcup_{x \in X} S_x$, where $S_x = \varinjlim_{x \in \mathring{V}} S(V)$, and we define $\pi\colon F_S \to X$ by $\pi(s) = x$ if $s \in S_x$. We call a function $f\colon X \to F_S$ such that $f(x) \in S_x$ a *section* of F_S. We equip the set of sections with pointwise addition and order, so this set becomes an ordered semigroup. Notice also that the set of sections is closed under pointwise suprema of increasing sequences. Also, any element $s \in S$ induces a section $\hat{s}$, which is defined by $\hat{s}(x) = \pi_x(s) \in S_x$ and will be referred to as the *section induced* by s.

Lemma 3.4.2. *Let $\mathcal{S}\colon \mathcal{V}_X \to \mathrm{Cu}$ be a presheaf on X, and let $s, r \in S$.*

(i) *If $\hat{s}(x) \leq \hat{r}(x)$ for some $x \in X$ then, for each $s' \ll s$ in S there is a closed set V with $x \in \mathring{V}$ such that $\pi_V(s') \leq \pi_V(r)$. In particular, $\hat{s'}(y) \leq \hat{r}(y)$ for all $y \in V$.*

(ii) *If, further, $\mathcal{S}$ is a sheaf, U is a closed subset of X, and $\hat{s}(x) \leq \hat{r}(x)$ for all $x \in U$, then for each $s' \ll s$ there is a closed set W of X with $U \subset \mathring{W}$ and $\pi_W(s') \leq \pi_W(r)$.*

Proof. (i): Recall that $S_x = \varinjlim \mathcal{S}(V_n)$, where (V_n) is a decreasing sequence of closed sets whose intersection is x (we may take $V_1 = X$, and $x \in \mathring{V}_n$ for all n). Then, by the properties of the inductive limits in Cu described in Proposition 3.1.2, two elements s, r satisfy $\hat{s}(x) = \pi_x(s) \leq \pi_x(r) = \hat{r}(x)$ in S_x if and only if for all $s' \ll s$ there exists $j \geq 1$ such that $\pi_{V_j}(s') \leq \pi_{V_j}(r)$. In particular $\hat{s'}(y) \leq \hat{r}(y)$ for all y in V_j.

(ii): Assume now that $\mathcal{S}$ is a sheaf, and take $s' \ll s$. Apply (i) to each $x \in U$, so that we can find U_x with $x \in \mathring{U}_x$ such that $\pi_{U_x}(s') \leq \pi_{U_x}(r)$. By compactness of U, there are a finite number $U_{x_1}, \ldots, U_{x_n}$ whose interiors cover U. Put $W = \cup_i U_{x_i}$. As $\mathcal{S}$ is a sheaf and $\pi_{U_{x_i}}(s') \leq \pi_{U_{x_i}}(r)$ for all i, it follows that $\pi_W(s') \leq \pi_W(r)$. □

We will use Lemma 3.4.2 to define a topology in F_S for which the induced sections will be continuous. Instead of abstractly considering the final topology generated by the induced sections, we define a particular topology which will satisfy our needs. Given U an open set in X and $s \in S$, put

$$U_s^{\gg} = \{a_x \in F_{\mathcal{S}} \mid \hat{s'}(x) \ll a_x \text{ for some } x \in U \text{ and some } s \ll s'\},$$

and equip $F_{\mathcal{S}}$ with the topology whose sub-basis consists of these sets (i.e. the union and finite intersection of sets of the form $U_s^{\gg}$ are open for all $U \subset X$ open set and $s \in S$).

Now consider an induced section $\hat{s}$ for some $s \in S$, and an open set of the form $U_r^{\gg}$ for some $r \in S$ and $U \subseteq X$. Suppose $x \in \hat{s}^{-1}(U_r^{\gg})$. Note that $x \in U$ and that $\hat{s}(x) \gg \hat{s'}(x)$ for some $s' \gg r$. Using that $s' = \sup(s'_n)$ for a rapidly increasing sequence (s'_n), there exists n_0 such that $r \ll s'_{n_0} \ll s'$. Hence, by Lemma 3.4.2, there is a closed set V such that $x \in \mathring{V}$ and $\hat{s}'_{n_0}(y) \ll \hat{s}(y)$ for all y in V. Thus, $x \in U \cap \mathring{V} \subseteq \hat{s}^{-1}(U_r^{\gg})$, proving that $\hat{s}^{-1}(U_r^{\gg})$ is open in X, from which it easily follows that $\hat{s}$ is continuous with this topology.

Remark 3.4.3. Notice that if $\mathcal{S}$ is a surjective presheaf, then any element $a \in S_x$ can be written as $a = \sup(\hat{s}_n(x))$, where s_n is a rapidly increasing sequence in S. This is possible because the map $S \to S_x$ is surjective, i.e., $a = \pi_x(s)$ for some $s \in S$, and $s = \sup s_n$ for such a sequence.

The following result gives another characterization of continuity that will prove useful in the sequel.

Proposition 3.4.4. *Let X be a compact Hausdorff space and $\mathcal{S}$ be a continuous surjective presheaf on* Cu. *Then, for a section $f\colon X \to F_{\mathcal{S}}$, the following conditions are equivalent:*

(i) *f is continuous.*

(ii) *For all $x \in X$ and $a_x \in S_x$ such that $a_x \ll f(x)$, there exist a closed set V with $x \in \mathring{V}$ and $s \in S$ such that $\hat{s}(x) \gg a_x$ and $\hat{s}(y) \ll f(y)$ for all $y \in V$.*

Proof. Let $f\colon X \to F_{\mathcal{S}}$ be a section satisfying (ii) and consider an open set of the form $U_r^{\gg}$ for some open set $U \subseteq X$ and $r \in S$. Then

$$\begin{aligned} f^{-1}(U_r^{\gg}) &= \{y \in X \mid f(y) \gg \hat{r'}(y) \text{ for some } y \in U \text{ and for some } r' \gg r\} \\ &= \{y \in U \mid f(y) \gg \hat{r'}(y) \text{ for some } r' \gg r\}. \end{aligned}$$

For each y in the above set there exists $r' \gg r$ such that $\hat{r'}(y) \ll f(y)$. Using property (ii) there exists $s \in S$ such that $\hat{r'}(y) \ll \hat{s}(y) \ll f(y)$ and $\hat{s}(x) \ll f(x)$ for all $x \in \mathring{V}$ where V is a closed set of X. Furthermore, we can find $r'' \in S$ such that $r \ll r'' \ll r'$, and use Lemma 3.4.2 to conclude that $\hat{r''}(z) \ll \hat{s}(z) \ll f(z)$ for all z in an open set $W \subseteq X$. This proves that $f^{-1}(U_r^{\gg})$ is open, so f is continuous.

Now, assume that $f\colon X \to F_{\mathcal{S}}$ is continuous, $x \in X$ and $a_x \ll f(x)$. Using Remark 3.4.3, we can write $f(x) = \sup(\hat{s}_n(x))$ where (s_n) is a rapidly increasing sequence in S. Hence, we can find $s \ll s' \in S$ such that

$$a_x \ll \hat{s}(x) \ll \hat{s'}(x) \ll f(x),$$

where $s, s' \in S$.

Let U be any open neighborhood of x, and consider the open set $f^{-1}(U_s^{\gg})$. Note that it contains x and that for any $z \in f^{-1}(U_s^{\gg})$, we have $f(z) \gg \hat{t}(z)$ for some $t \gg s$. Thus, for any closed set V contained in $f^{-1}(U_s^{\gg})$ such that $x \in \mathring{V}$, we have $f(z) \gg \hat{s}(z)$ for all $z \in V$. So condition (ii) holds. □

Let X be a compact Hausdorff space and let $\mathcal{S}$ be a continuous presheaf on Cu. We will denote the set of *continuous sections of the space* F_S by $\Gamma(X, F_S)$, which is equipped with the pointwise order and addition. Notice that there is an order-embedding

$$\Gamma(X, F_S) \to \prod_{x \in X} S_x$$

given by $f \mapsto (f(x))_{x \in X}$.

Remark 3.4.5. By Theorem 3.3.6 it is clear that the map

$$\begin{aligned} \alpha : \mathrm{Cu}(A) &\to \Gamma(X, F_{\mathrm{Cu}(A)}) \\ \langle a \rangle &\mapsto (\langle a(x) \rangle)_{x \in X}. \end{aligned}$$

defines an order-embedding when A is a $\mathrm{C}(X)$-algebra whose fibers have no K_1 obstructions and X is a one-dimensional compact Hausdorff space.

Corollary 3.4.6. *Let X be a one-dimensional compact Hausdorff space, and let A be a $\mathrm{C}(X)$-algebra whose fibers have no K_1 obstructions. Then, $\mathrm{Cu}_A \colon \mathcal{V}_X \to \mathrm{Cu}$, $U \mapsto \mathrm{Cu}(A(U))$, is a surjective continuous sheaf.*

Proof. We know already that Cu_A is a surjective continuous presheaf. Let U and $V \in \mathcal{V}_X$ be such that $U \cap V \in \mathcal{V}_X$. Let $W = U \cup V$. We know then that $A(W)$ is isomorphic to the pullback $A(U) \oplus_{A(U \cap V)} A(V)$. Since $A(W)$ is a $\mathrm{C}(W)$-algebra whose fibers have no K_1 obstructions, we may use Remark 3.4.5 to conclude that the map $\mathrm{Cu}(A(W)) \to \prod_{x \in W} \mathrm{Cu}(A_x)$ (given by $\langle a \rangle \mapsto (\langle a(x) \rangle))$ is an order-embedding. Then Corollary 3.3.7 implies that the natural map $\mathrm{Cu}(A(W)) \to \mathrm{Cu}(A(U)) \oplus_{\mathrm{Cu}(A(U \cap V))} \mathrm{Cu}(A(V))$ is an isomorphism. □

The main result in this section is Theorem 3.4.19 which shows that, under additional assumptions, the map defined either in Theorem 3.3.6 or Remark 3.4.5 is also surjective, i.e., there exists an isomorphism in the category Cu between $\mathrm{Cu}(A)$ and $\Gamma(X, F_{\mathrm{Cu}(A)})$. We next prove some necessary Lemmas.

Recall that, if $s \ll r \in S$, then $\pi_x(s) = \hat{s}(x) \ll \hat{r}(x) = \pi_x(r)$ for all x. This comes from the fact that the induced maps belong to the category Cu, and so they preserve the compact containment relation. We shall use below $\partial(U)$ to denote the *boundary* of a set U, that is, $\partial(U) = \overline{U} \setminus \mathring{U}$.

Lemma 3.4.7. *Let $\mathcal{S} \colon \mathcal{V}_X \to \mathrm{Cu}$ be a surjective presheaf of semigroups on X.*

(i) *Let $f, g \in \Gamma(X, F_S)$, and V a closed subset of X such that $f(y) \ll g(y)$ for all $y \in V$. Put*

$$g_{V,f}(x) = \begin{cases} g(x) & \text{if } x \notin V \\ f(x) & \text{if } x \in V \end{cases}$$

Then $g_{V,f} \in \Gamma(X, F_S)$.

(ii) *If $g \in \Gamma(X, F_S)$ and $x \in X$, there exist a decreasing sequence (V_n) of closed sets (with $x \in \mathring{V}_n$ for all n) and a rapidly increasing sequence (s_n) in S such that $g = \sup_n g_{V_n,s_n}$.*

Proof. (i): Using the fact that both f and g are continuous, it is enough to check that condition (ii) in Proposition 3.4.4 is verified for $x \in \partial(V)$. Thus, let a_x be such that $a_x \ll g_{V,f}(x) = f(x) \ll g(x)$. By continuity of f, there is a closed subset U with $x \in \mathring{U}$ and $s \in S$ such that $a_x \ll \hat{s}(x)$ and $\hat{s}(y) \ll f(y)$ for all $y \in U$. As s is a supremum of a rapidly increasing sequence, we may find $s' \ll s$ with $a_x \ll \hat{s'}(x)$.

Next, as g is also continuous, there are $t \in S$ and a closed set U' with $x \in \mathring{U}'$ such that $f(x) \ll \hat{t}(x)$ and $\hat{t}(y) \ll g(y)$ for all $y \in U'$. Since $\hat{s}(x) \ll \hat{t}(x)$ and $s' \ll s$, we now use Lemma 3.4.2 to find W with $\hat{s'}(y) \ll \hat{t}(y)$ for all $y \in W$. Now condition (ii) in Proposition 3.4.4 is verified using the induced section s' and the closed set $U \cap U' \cap W$.

(ii): Write $g(x) = \sup_n \hat{s}_n(x)$, where (s_n) is a rapidly increasing sequence in S (see Remark 3.4.3).

Since $s_1 \ll s_2$ and g is continuous, condition (ii) of Proposition 3.4.4 applied to $\hat{s}_2(x) \ll g(x)$ yields $t \in S$ and a closed set U_1 whose interior contains x such that $\hat{s}_2(x) \ll \hat{t}(x)$ and $\hat{t}(y) \ll g(y)$ for all $y \in U_1$. We now apply Lemma 3.4.2, so that there is another closed set U_1' (with $x \in \mathring{U}_1'$) so that $\hat{s}_1(y) \ll \hat{t}(y)$ for any $y \in U_1'$. Let $V_1 = U_1 \cap U_1'$ and, for each $y \in V_1$, we have $\hat{s}_1(y) \ll \hat{t}(y) \ll g(y)$. Continue in this way with the rest of the s_n's, and notice that we can choose the sequence (V_n) in such a way that $\cap V_n = \{x\}$ since X is metrizable. □

Using the previous lemmas we can describe compact containment in $\Gamma(X, F_{\mathrm{Cu}(A)})$.

Proposition 3.4.8. *Let $\mathcal{S}\colon \mathcal{V}_X \to \mathrm{Cu}$ be a surjective presheaf of semigroups on X. If f, g belong to $\Gamma(X, F_{\mathrm{Cu}(A)})$, the following statements are equivalent:*

(i) $f \ll g$.

(ii) *For all $x \in X$ there exists a_x with $f(x) \ll a_x \ll g(x)$ and such that if $s \in S$ satisfies $a_x \ll \hat{s}(x)$ and $\hat{s}(y) \ll g(y)$ for y in a closed set U whose interior contains x, then there exists a closed set $V \subseteq U$ with $x \in \mathring{V}$ and $f(y) \leq \hat{s}(y) \leq g(y)$ for all $y \in V$.*

Proof. (i) $\implies$ (ii): Given $x \in X$, use Lemma 3.4.7 to write $g = \sup_n g_{V_n,s_n}$, where (s_n) is rapidly increasing in S and (V_n) is a decreasing sequence of closed sets whose interior contain x. Since $f \ll g$, there is n such that

$$f \leq g_{V_n,s_n} \leq g_{V_{n+1},s_{n+1}} \leq g\,.$$

Let $a_x = g_{V_{n+1},s_{n+1}}(x) = \hat{s}_{n+1}(x)$, which clearly satisfies

$$f(x) \leq \hat{s}_n(x) \ll \hat{s}_{n+1}(x) = a_x \ll g(x).$$

Assume now that $s \in S$ and U is a closed set with $x \in \mathring{U}$ such that $a_x \ll \hat{s}(x)$ and $\hat{s}(y) \ll g(y)$ for all $y \in U$. Since $s_n \ll s_{n+1}$ and $\hat{s}_{n+1}(x) \ll \hat{s}(x)$, there is by Lemma 3.4.2 a closed set V with $x \in \mathring{V}$ (and we may assume $V \subset V_{n+1} \cap U$) such that $\hat{s}_n(y) \leq \hat{s}(y)$ for all $y \in V$. Thus $f(y) \leq \hat{s}_n(y) \leq \hat{s}(y) \leq g(y)$ for all $y \in V$.

(ii) $\implies$ (i): Suppose now that $g \leq \sup(g_n)$, where (g_n) is an increasing sequence in $\Gamma(X, F_S)$. Let $x \in X$, and write $g = \sup g_{V_n,s_n}$ as in Lemma 3.4.7, where (s_n) is a rapidly increasing sequence in S. Our assumption provides us first with a_x such that $f(x) \ll a_x \ll g(x)$. In particular, there is m such that $a_x \ll \hat{s}_m(x) \ll \hat{s}_{m+1}(x) \ll \hat{s}_{m+2}(x) \ll g(x)$, and hence there exists k depending on x with $\hat{s}_{m+1}(x) \ll g_k(x)$.

As g_k is continuous, condition (ii) in Proposition 3.4.4 implies that we may find $s \in S$ and a closed set U with $x \in \mathring{U}$ such that $\hat{s}_{m+1}(x) \ll \hat{s}(x)$ and $\hat{s}(y) \ll g_k(y)$ for all $y \in U$. Now, as $s_m \ll s_{m+1}$, there exists a closed subset V with $x \in \mathring{V}$ and $\hat{s}_m(y) \leq \hat{s}(y)$ for all $y \in V$, whence $\hat{s}_m(y) \leq g(y)$ for all $y \in U \cap V$.

Since also $\hat{s}_m(y) \ll g(y)$ for all $y \in V_m$, there is by assumption a closed set $W \subseteq V_m \cap V$ (whose interior contains x) such that $f(y) \leq \hat{s}_m(y) \leq g_k(y)$ for all $y \in W$. Now, repeating the same argument changing x, since X is compact we find a finite cover $\{W_l\}_{l=1}^n$ of X and indices $k_1, \ldots, k_n$ such that $f(y) \leq g_{k_l}(y)$ for all $y \in W_l$. Therefore, if $K = \max\{k_1, \ldots, k_n\}$, one has $f \leq g_K$. □

Lemma 3.4.9. *Let $\mathcal{S}\colon \mathcal{V}_X \to \mathrm{Cu}$ be a surjective presheaf of semigroups. Then, the morphism*

$$\begin{array}{rccc} \alpha: & S & \to & \Gamma(X, F_S) \\ & s & \mapsto & \hat{s} \end{array}$$

preserves compact containment and suprema.

Proof. Using condition (ii) of Proposition 3.4.4, it follows that if (f_n) is an increasing sequence in $\Gamma(X, F_S)$, then its pointwise supremum is also a continuous section. Indeed, let f be the pointwise supremum of (f_n), and let $x \in X$ and $a_x \ll f(x)$. Because $f(x) \in S_x$, it can be written as a supremum of a rapidly increasing sequence, namely (f^n). Therefore, $a_x \ll f^m \leq f_k(x)$ for some m, k. Now, since f_k is continuous the claim follows.

Assume now that $s \ll r$ in S. Write $r = \sup(r_n)$, where (r_n) is a rapidly increasing sequence in S. We may find m such that

$$s \ll r_m \ll r_{m+1} \ll r\,.$$

Take $a_x = \hat{r}_{m+1}(x)$. Suppose that $t \in S$ satisfies $a_x \ll \hat{t}(x)$ and $\hat{t}(y) \ll \hat{r}(y)$ for y in a closed subset U whose interior contains x. By Lemma 3.4.2, there is a closed set V such that $x \in \mathring{V}$ and $\hat{r}_m(y) \leq \hat{t}(y)$ for $y \in V$. Thus, for any $y \in V \cap U$, we have $\hat{s}(y) \leq \hat{r}_m(y) \leq \hat{t}(y) \leq \hat{r}(y)$. This verifies condition (ii) in Proposition 3.4.8, whence $\hat{s} \ll \hat{r}$. □

Corollary 3.4.10. *Let $\mathcal{S}\colon \mathcal{V}_X \to \mathrm{Cu}$ be a surjective sheaf of semigroups on X, $f \in \Gamma(X, F_S)$, $s \in S$, and let V be a closed subset of X. If $\hat{s}(x) \leq f(x)$ for all $x \in V$ and $s' \ll s$, then there is a closed subset W of X with $V \subset \mathring{W}$ such that $\pi_W(s') \ll f_{|W}$.*

Proof. Let $s' \ll t' \ll t \ll s$ in S. For each $x \in V$, there is by Proposition 3.4.4 a closed set U_x whose interior contains x, and $r_x \in S$ such that $\hat{t}(x) \ll \hat{r}_x(x)$, and $\hat{r}_x(y) \leq f(y)$ for all $y \in U_x$. Now apply condition (i) of Lemma 3.4.2 to $t' \ll t$ to find another closed set V_x such that $x \in \mathring{V}_x$ and $\hat{t'}(y) \leq \hat{r}_x(y)$ for $y \in V_x$. Letting $W_x = U_x \cap V_x$, we have $\hat{t'}(y) \ll f(y)$ for all $y \in W_x$. Since $V \subseteq \bigcup_x \mathring{W}_x$, and V is closed, we may find a finite number of W_x's that cover V, whose union is the closed set W we are looking for. Since $\mathcal{S}$ is a sheaf, it follows that $\pi_W(t') \leq f_{|W}$, and by Lemma 3.4.9 we see that $\pi_W(s') \ll \pi_W(t') \leq f_{|W}$, as desired. □

We now proceed to define a class of continuous sections that will play an important role. This will be a version, for presheaves on spaces of dimension one, of the notion of piecewise characteristic function explained in Definition 3.3.11. We show below that, for a surjective sheaf of semigroups $\mathcal{S}\colon \mathcal{V}_X \to \mathrm{Cu}$ on a one-dimensional space X, every element in $\Gamma(X, F_S)$ can be written as the supremum of a rapidly increasing sequence of piecewise characteristic sections. From this, we can conclude that $\Gamma(X, F_S)$ is an object in Cu.

Definition 3.4.11. *(Piecewise characteristic sections) Let X be a one-dimensional compact Hausdorff space, and let $\{U_i\}_{i=1\dots n}$ be an open cover of X such that the order of $\{U_i\}$ and $\{\bar{U}_i\}$ is at most two. Assume also that $\dim(\partial(\overline{U_i})) = 0$ for all i.*

Let $\mathcal{S}\colon \mathcal{V}_X \to \mathrm{Cu}$ be a presheaf of semigroups on X, and let $s_1, \dots, s_n$ and $s_{\{i,j\}}$ be elements in S such that, whenever $i \neq j$,

$$\hat{s}_i(x) \leq \hat{s}_{\{i,j\}}(x) \text{ for all } x \text{ in } \overline{\partial(U_i \cap U_j) \cap U_i}\,.$$

We define a piecewise characteristic section *as*

$$g(x) = \begin{cases} \hat{s}_i(x) & \text{if } x \in U_i \setminus (\bigcup_{j\neq i} U_j) \\ \hat{s}_{\{i,j\}}(x) & \text{if } x \in U_i \cap U_j \end{cases}.$$

Recall that the requirement that an open cover $\{U_i\}_{i=1\dots n}$ of a one-dimensional space X satisfies that the order of $\{U_i\}$ and $\{\bar{U}_i\}$ is at most two is not restrictive (see [Pea75, Lemma 8.1.1]).

Lemma 3.4.12. *Let X be a one-dimensional compact Hausdorff space, and let $\mathcal{S}\colon \mathcal{V}_X \to \mathrm{Cu}$ be a presheaf of semigroups on X. Then the piecewise characteristic sections are continuous.*

Proof. This is basically a repetition of the arguments in Lemma 3.4.7. Using that $\hat{s}_i$ and $\hat{s}_{\{i,j\}}$ are continuous for all i, j, it is enough to check that condition (ii) in Proposition 3.4.4 is verified for $x \in \overline{\partial(U_i \cap U_j) \cap U_i}$. Thus, let $a_x \ll \hat{s}_i(x) \leq \hat{s}_{\{i,j\}}(x)$ for the corresponding i, j. By continuity of $\hat{s}_i$, there is a closed subset U with $x \in \mathring{U}$ and $s \in S$ such that $a_x \ll \hat{s}(x)$ and $\hat{s}(y) \ll \hat{s}_i(y)$ for all $y \in U$. Because s is a supremum of a rapidly increasing sequence, we may find $s' \ll s$ with $a_x \ll \hat{s}'(x)$.

Now, as $\hat{s}_{\{i,j\}}$ is also continuous, there is $t \in S$ and a closed set U' with $x \in U'$ such that $\hat{s}(x) \ll \hat{t}(x)$ and $\hat{t}(y) \ll \hat{s}_{\{i,j\}}(y)$ for all $y \in \mathring{U}'$. Because $\hat{s}(x) \ll \hat{t}(x)$ and $s' \ll s$, we may use Lemma 3.4.2 to find W with $\hat{s}'(y) \ll \hat{t}(y)$ for all $y \in W$. Now, condition (ii) of Proposition 3.4.4 is verified using the induced section s' and the closed set $U \cap U' \cap W$. □

Remark 3.4.13. In the case of zero-dimensional spaces, piecewise characteristic sections are much easier to define. Given an open cover $\{U_i\}_{i=1,\dots,n}$ consisting of pairwise disjoint clopen sets, a presheaf of semigroups $\mathcal{S}$ on Cu and elements $s_1, \dots, s_n \in S$, a piecewise characteristic section in this setting is an element $g \in \Gamma(X, F_S)$ such that $g(x) = \hat{s}_i(x)$, whenever $x \in U_i$.

As in Definition 3.3.11, if $f \in \Gamma(X, F_S)$ and g is a piecewise characteristic section such that $g \ll f$, then we say that g is a piecewise characteristic section of f and we will denote the set of these sections by $\chi(f)$.

Lemma 3.4.14. *Let X be a one-dimensional compact metric space, and $\mathcal{S}\colon \mathcal{V}_X \to \mathrm{Cu}$ be a surjective presheaf of semigroups. If $f \in \Gamma(X, F_S)$, then*

$$f = \sup\{g \mid g \in \chi(f)\}\,.$$

Proof. Let $x \in X$. By Lemma 3.4.7, we may write $f = \sup f_{V_n,s_n}$, where (V_n) is a decreasing sequence of closed sets with $x \in \mathring{V}_n$ and (s_n) is rapidly increasing in S. By construction, $f_{V_n,s_n}(y) = \hat{s}_n(y) \ll f(y)$ for all $y \in V_n$.

Now define

$$h_n(y) = \begin{cases} \hat{s}_n(y) & \text{if } y \in \mathring{V}_{n+1} \\ 0 & \text{otherwise} \end{cases}.$$

Let $x \in \mathring{V}_{n+1}$ and consider $a_x = \hat{s}_{n+1}(x)$. By definition of h_n, condition (ii) of Proposition 3.4.8 clearly holds since $s_n \ll s_{n+1}$, and so $h_n \ll f$. In addition, note that each h_n is a piecewise characteristic section for f. Using this fact for all $x \in X$, we conclude that $f = \sup\{g \mid g \in \chi(f)\}$. ☐

Proposition 3.4.15. *Let X be a compact metric space with $\dim(X) \leq 1$, and let $\mathcal{S}\colon \mathcal{V}_X \to \mathrm{Cu}$ be a surjective sheaf of semigroups. Suppose $h_1, h_2, f \in \Gamma(X, F_S)$ satisfy $h_1, h_2 \ll f$. Then, there exists $g \in \chi(f)$ such that $h_1, h_2 \ll g$. In particular, $\chi(f)$ is an upwards directed set.*

Proof. Assume first that X has dimension 0. Writing f as in condition (ii) of Lemma 3.4.7 we can find, for each $x \in X$, an open set V_x that contains x, and elements $s'_x \ll s_x \ll s''_x \in S$ such that

$$h_1(y), h_2(y) \ll \hat{s'_x}(y) \ll \hat{s}_x(y) \ll \hat{s''_x}(y) \ll f(y) \quad \text{for all } y \in \overline{V_x}. \tag{3.4}$$

Using compactness and the fact that X is zero-dimensional, there are $x_1, \dots, x_n \in X$ and (pairwise disjoint) clopen sets $\{V_i\}_{i=1,\dots,n}$ with $V_i \subseteq V_{x_i}$ and such that $X = \cup_i V_i$. Put $s_i = s_{x_i}$, $s'_i = s'_{x_i}$ and $s''_i = s''_{x_i}$. Define, using this cover, a piecewise characteristic section g as $g(x) = \hat{s}_i(x)$ if $x \in V_i$. It now follows from (3.4) that $h_1, h_2 \ll g \ll f$ (the elements s'_i, s''_i are used here to obtain compact containment).

We turn now to the case where X has dimension 1, and start as in the previous paragraph, with some additional care. Choose, for each x, a δ_x-ball V''_x (where $\delta_x > 0$) centered at x and elements $s'_x \ll s_x \ll s''_x$ such that condition (3.4) is satisfied (for all $y \in \overline{V''_x}$). Denote by $V'_x \subseteq V''_x$ the cover consisting of $\delta_x/2$-balls. By compactness we obtain a finite cover $\{V'_{x_1}, \dots, V'_{x_n}\}$. Using [Pea75, Lemma 8.1.1] together with the fact that X has dimension 1, this cover has a refinement $\{V_i\}_{i=1}^n$ such that $\{V_i\}$ and $\{\overline{V}_i\}$ have both order at most 2 and such that $\partial(V_i)$ has dimension 0 for each i. As before, set $s_i = s_{x_i}$, $s'_i = s'_{x_i}$ and $s''_i = s''_{x_i}$.

Let Y be the closed set $\cup_i \partial(V_i)$, which also has dimension 0. Put $\delta = \min\{\delta_{x_i}/3\}$. By construction, there is a δ-neighborhood V_i^δ such that $V_i^\delta \subseteq V''_i$. As in the proof of Lemma 3.4.14, we see that the sections

$$g_i(y) = \begin{cases} \hat{s''_i}(y) & \text{if } y \in V_i^\delta \\ 0 & \text{otherwise} \end{cases}$$

satisfy $g_i \ll f$. We now restrict to Y and proceed as in the argument of the zero-dimensional case above. In this way, we obtain piecewise characteristic sections g_Y, g'_Y, $g''_Y \in \Gamma(Y, F_S)$, defined by some open cover $\{W_i\}_{i=1}^m$ (of pairwise disjoint clopen sets of Y) and elements $t_i \ll t'_i \ll t''_i \in S$ in such a way that $g_Y(y) = \hat{t}_i(y)$, $g'_Y(y) = \hat{t'_i}(y)$ and $g''_Y(y) = \hat{t''_i}(y)$ whenever $y \in W_i$, and such that

$$\pi_Y(g_i) \ll g_Y \ll g'_Y \ll g''_Y \ll \pi_Y(f) \text{ for all } i = 1, \ldots, n\,. \tag{3.5}$$

Observe that we can choose the W_i of arbitrarily small size, thus in particular we may assume that each one is contained in a $\delta/6$-ball. In this way, whenever $\overline{W}_i \cap \overline{V}_j \neq \emptyset$, we have $W_i \subseteq V_j^\delta$. Therefore, if $x \in W_i$, it follows from (3.5) that

$$\hat{s''_j}(x) = g_j(x) \leq g_Y(x) = \hat{t}_i(x)\,.$$

By condition (ii) in Lemma 3.4.2, applied to the previous inequality, there is $\epsilon > 0$ such that $\hat{s}_j(x) \leq \hat{t}_i(x)$ for all $x \in W_i^\epsilon$ (where W_i^ϵ is an ϵ-neighborhood of W_i). Since the W_i are pairwise disjoint clopen sets, we can choose ϵ such that the sets W_i^ϵ are still pairwise disjoint. Further, since also $\hat{t''_i}(y) \leq f(y)$ for $y \in W_i$ and $t'_i \ll t''_i$, we may apply Corollary 3.4.10 to obtain $\pi_{\overline{W_i^\epsilon}}(t'_i) \ll \pi_{\overline{W_i^\epsilon}}(f)$ (further decreasing ϵ if necessary). As for each i, we can find U_i with $\overline{W_i^{\epsilon/2}} \subseteq U_i \subseteq W_i^\epsilon$ with zero-dimensional boundary by construction of our cover, after a slight abuse of notation we shall assume that W_i^ϵ itself has zero-dimensional boundary. Put $Y^\epsilon = \cup_{i=1}^m W_i^\epsilon$. Notice now that, for i, $k < l$, the closed sets $V_i \setminus (Y^\epsilon \cup \cup_{j\neq i} V_j)$ and $(V_k \cap V_l) \setminus (Y^\epsilon)$ are also pairwise disjoint, whence they admit pairwise disjoint ϵ'-neighborhoods (for a sufficiently small ϵ'). As before, we shall also assume that these neighborhoods have zero-dimensional boundaries.

Now consider the cover of X that consists of the sets

$$\left\{ \begin{array}{cl} W_i^\epsilon & \text{if } i = 1, \ldots, m, \\ (V_i \setminus (Y^\epsilon \cup \cup_{j\neq i} V_j))^{\epsilon'} & \text{if } i = 1, \ldots, n, \\ (V_k \cap V_l \setminus (Y^\epsilon))^{\epsilon'} & \text{if } k < l\,, \end{array} \right\}.$$

and define a piecewise characteristic section g as follows

$$g(x) = \left\{ \begin{array}{ll} \hat{t}_i(x) & \text{if } x \in W_i^\epsilon \\ \hat{s}_i(x) & \text{if } x \in (V_i \setminus (Y^\epsilon \cup \cup_{j\neq i} V_j))^{\epsilon'} \setminus Y^\epsilon \\ \hat{s}_k(x) & \text{if } x \in (V_k \cap V_l \setminus (Y^\epsilon))^{\epsilon'} \setminus Y^\epsilon \text{ for } k < l\,. \end{array} \right.$$

That h_1, $h_2 \leq g$ follows by construction of g. It remains to show that $g \ll f$. This also follows from our construction, using condition (ii) of Proposition 3.4.8. For example, given $x \in W_i^\epsilon$, we have

$$g(x) = \hat{t}_i(x) \ll \hat{t'_i}(x) \ll f(x)\,.$$

If now $s \in S$ satisfies that $\hat{t'_i}(x) \ll \hat{s}(x)$ and $\hat{s}(y) \ll f(y)$ for y in a closed set (whose interior contains x) then, since $t_i \ll t'_i$, we may find (again by Lemma 3.4.2) a smaller closed set (contained in W_i^ϵ and with interior containing x) such that $g(y) = \hat{t}_i(y) \leq \hat{s}(y) \leq f(y)$ for y in that set.

If for instance $x \in (V_k \cap V_l \setminus (Y^\epsilon))^{\epsilon'} \setminus Y^\epsilon$ for $k < l$, we have

$$g(x) = \hat{s}_k(x) \ll \hat{s}''_k(x) \ll f(x).$$

If now $s \in S$ satisfies that $\hat{s}''_k(x) \ll \hat{s}(x)$ and $\hat{s}(y) \ll f(y)$ for y in a closed set (whose interior contains x) then, since $s_k \ll s''_k$, we may find by Lemma 3.4.2 a smaller closed set contained in $(V_k \cap V_l \setminus (Y^\epsilon))^{\epsilon'} \setminus Y^\epsilon$ such that $g(y) \leq \hat{s}(y) \leq f(y)$ for y in that set. □

We remark that a second countable topological space satisfies the Lindelöf property, i.e, every open cover has a countable subcover (see e.g. [Rud74]). This will be used below.

Proposition 3.4.16. *Let X be one-dimensional compact metric space, and let $\mathcal{S}\colon \mathcal{V}_X \to \mathrm{Cu}$ be a surjective sheaf of semigroups with S countably based. If $f \in \Gamma(X, F_S)$, then f is the supremum of a rapidly increasing sequence of elements from $\chi(f)$.*

Proof. Let us define a new topology on F_S. Let $s \in S$ and U an open set in X. Consider the topology generated by the sets

$$U_s^{\ll} = \{y \in F_S \mid \hat{s}(x) \gg y \text{ for some } x \in U\}\,.$$

We claim that, under this topology, F_S is second countable. Let $\{U_n\}$ be a basis of X, and $\{s_n\}_{n\in\mathbb{N}}$ be a dense subset of S. Therefore the collections of sets $\{(U_n)_{si}^{\ll}\}_{n,i\in\mathbb{N}}$ is a countable basis for F_S. Indeed, given an open set U of X and $s \in S$, find sequences (U_{n_i}) and (s_{m_j}) such that $U = \cup U_{n_i}$ and $s = \sup s_{m_j}$. Then

$$U_s^{\ll} = \cup (U_{n_i})_{s_{m_j}}^{\ll}\,.$$

To check this, let $y \in U_s^{\ll}$, and suppose $\hat{s}(x) \ll y$ for some $x \in U$. Write $s = \sup(s^n)$ where (s^n) is a rapidly increasing sequence in S; therefore, there exists s_n in the given dense set such that $\hat{s}_n(x) \geq \hat{s}^m(x) \gg y$ for some m. Since $U = \cup U_i$, one has $x \in U_i$ for some i, so $y \in (U_i)_{s_n}^{\ll}$. The other containment is clear.

Now, for $f \in \Gamma(X, F_S)$, put $U_f = \{a_x \in F_S \mid a_x \ll f(x) \text{ for } x \in U\}$. This set is open in the topology we just have defined. To see this, let $a_x \in U_f$, and invoke Proposition 3.4.4 to find an open set V and $s \in S$ such that $a_x \ll \hat{s}(x)$ and $\hat{s}(y) \ll f(y)$ for all $y \in V$. It then follows that $a_x \in V_s^{\ll} \subseteq U_f$.

Using Lemma 3.4.14, we see that $U_f = \cup_{g\in\chi(f)} U_g$. Since F_S is second countable, it has the Lindelöf property, whence we may find a sequence (g_n) in $\chi(f)$ such that $U_f = \cup_n U_{g_n}$. This sequence may be taken to be increasing by Proposition 3.4.15. Translating this back to $\Gamma(X, F_S)$, we get $f = \sup(g_n)$. □

Assembling our observations we obtain the following:

Theorem 3.4.17. *Let X be a one-dimensional compact metric space, and let $\mathcal{S}\colon \mathcal{V}_X \to \mathrm{Cu}$ be a surjective sheaf of semigroups such that S is countably based. Then, the semigroup $\Gamma(X, F_S)$ of continuous sections belongs to the category* Cu.

The next result shows the existence of an induced section between any two compactly contained piecewise characteristic sections.

Proposition 3.4.18. *Let X be a one-dimensional compact metric space and let A be a stable continuous field over X whose fibers have no K_1 obstructions. Let $f \ll g$ be elements in $\Gamma(X, F_{\mathrm{Cu}(A)})$ such that g is a piecewise characteristic section. Then there exists an element $h \in A$ which satisfies $f(x) \leq \langle \pi_x(h) \rangle \leq g(x)$ for all $x \in X$.*

Proof. Since g is a piecewise characteristic section, there is a cover $\{U_i\}_{i=1}^n$ of X such that both $\{U_i\}$ and $\{\overline{U}_i\}$ have order at most 2, and there are elements $\langle a_i\rangle$, $\langle a_{\{i,j\}}\rangle$ in $\mathrm{Cu}(A)$ which are the values that g takes (according to Definition 3.4.11).

For $\epsilon > 0$, let g_ϵ be the section defined on the same cover as g and that takes values $\langle (a_i-\epsilon)_+\rangle$, $\langle a_{\{i,j\}}\rangle$. As $g=\sup_\epsilon g_\epsilon$ and $f \ll g$, we may choose $\epsilon>0$ such that $f\leq g_\epsilon$, and in particular

$$f(x)\leq \pi_x(\langle (a_i-\epsilon)_+\rangle) \ll \pi_x(\langle a_i\rangle) \text{ for all } x \text{ in } U_i\setminus (\cup_{j\neq i}U_j)\,.$$

Notice now that the closed sets $\overline{\partial(U_i\cap U_j)\cap U_i}$ and $\overline{\partial(U_k\cap U_l)\cap U_l}$ are pairwise disjoint whenever $(i,j)\neq (k,l)$. Indeed, since the cover $\{\overline{U}_i\}$ has order at most 2, this follows from the fact that $\overline{\partial(U_i\cap U_j)\cap U_i}\subseteq \overline{U_i}\cap\overline{U_j}$ for all i,j.

Furthermore, by definition of g we have $\pi_x(\langle a_i\rangle)\leq \pi_x(\langle a_{\{i,j\}}\rangle)$ for all $x\in \overline{\partial(U_i\cap U_j)\cap U_i}$. Therefore, there exists by Corollary 3.4.10 a neighborhood $W_{i,j}$ of $\overline{\partial(U_i\cap U_j)\cap U_i}$ for which

$$\pi_{\overline{W}_{i,j}}(\langle a_i\rangle)\leq \pi_{\overline{W}_{i,j}}(\langle a_{\{i,j\}}\rangle)\,.$$

We may assume without loss of generality that the closures $\overline{W}_{i,j}$ are pairwise disjoint sets. Since also $\overline{\partial(U_i\cap U_j)\cap U_i}\cap \overline{U}_k=\emptyset$ whenever $k\neq i,j$, we may furthermore assume that $W_{i,j}\cap\overline{U}_k=\emptyset$ for $k\neq i,j$.

By Proposition 1.3.17 there exist unitaries $u_{i,j}\in\mathcal{U}(A(\overline{W}_{i,j})^\sim)$ such that

$$u_{i,j}\pi_{\overline{W}_{i,j}}((a_i-\epsilon)_+)u_{i,j}^*\in \mathrm{Her}(\pi_{\overline{W}_{i,j}}(a_{\{i,j\}}))\,.$$

Now, as A and $A(\overline{W}_{i,j})$ are stable, the unitary groups of their multiplier algebras are connected in the norm topology (see, e.g. [WO93, Corollary 16.7]). Furthermore, since the natural map $\pi_{\overline{W}_{i,j}}\colon A\to A(\overline{W}_{i,j})$ induces a surjective morphism $\mathcal{M}(A)\to\mathcal{M}(A(\overline{W}_{i,j}))$ (by, e.g. [WO93, Theorem 2.3.9]), we can find, for each unitary $u_{i,j}$, a unitary lift $\tilde{u}_{i,j}$ in $\mathcal{M}(A)$.

We now have continuous paths of unitaries $w_{i,j}\colon[0,1]\to\mathcal{U}(\mathcal{M}(A))$ such that $w_{i,j}(0)=1$ and $w_{i,j}(1)=\tilde{u}_{i,j}$. Put $\gamma=\min\{\mathrm{dist}(\overline{W}_{i,j},\overline{W}_{k,l})\mid (i,j)\neq(k,l)\}$. Note that $\gamma>0$ as the sets $\overline{W}_{i,j}$ are pairwise disjoint. For $x\in X$, define a unitary in $\mathcal{M}(A)$ by

$$w_{i,j}^x=w_{i,j}\left(\frac{(\gamma-\mathrm{dist}(x,W_{i,j}))_+}{\gamma}\right)\,.$$

Observe that, if $x\in W_{k,l}$, then $w_{i,j}^x=\tilde{u}_{i,j}$ if $(k,l)=(i,j)$ and equals 1 otherwise. Now put

$$w_i^x=\prod_j w_{i,j}^x\,.$$

Since each π_x is norm decreasing and the w_i^x are defined by products and compositions of continuous functions, we obtain that, for each $c\in A$, the tuple $(\pi_x(w_i^xc))_{x\in X}\in\prod_{x\in X}A_x$ defines fiberwise an element in A which we denote by w_ic. Indeed, using the condition of continuous fields mentioned in Remark 1.1.23, given $x\in X$, consider the element $b=w_i^xc$. Then,

$$\|\pi_y(b)-\pi_y(w_i^yc)\|=\|\pi_y((w_i^x-w_i^y)c)\|\leq\|(w_i^x-w_i^y)c\|,$$

which is small if x and y are sufficiently close.

Now let $\{\lambda_i\}_i$ be continuous positive real-valued functions on $[0,1]$ whose respective supports are $\{(U_i\backslash(\cup_{j\neq i}U_j))\cup(\cup_j W_{i,j})\}_i$ and $\{\lambda_{\{i,j\}}\}_{i,j}$ with supports $\{U_i\cap U_j\}_{i,j}$. Define the following element in A

$$h=\sum_i \lambda_i w_i(a_i-\epsilon)_+w_i^*+\sum_{i\neq j}\lambda_{\{i,j\}}a_{\{i,j\}}.$$

We now check that $\langle\pi_x(h)\rangle=g_\epsilon(x)$, and this will yield the desired conclusion.

If $x\in U_i\backslash(\cup_{j\neq i}U_j)$, then $\pi_x(h)=\lambda_i(x)\pi_x(w_i(a_i-\epsilon)_+w_i^*)$ where $\lambda_i(x)\neq 0$, and this is equivalent to $\pi_x((a_i-\epsilon)_+)$. Hence $\langle\pi_x(h)\rangle=g_\epsilon(x)$.

On the other hand, if $x\in U_i\cap U_j$ for some i,j then $\lambda_{\{i,j\}}(x)\neq 0$, and

$$\pi_x(h)=\begin{cases}\lambda_i(x)\pi_x(\tilde u_{i,j}(a_i-\epsilon)_+\tilde u_{i,j}^*)+\lambda_{\{i,j\}}(x)\pi_x(a_{\{i,j\}}) & \text{if } x\in U_i\cap U_j\cap W_{i,j}\\ \lambda_j(x)\pi_x(\tilde u_{j,i}(a_j-\epsilon)_+\tilde u_{j,i}^*)+\lambda_{\{i,j\}}(x)\pi_x(a_{\{i,j\}}) & \text{if } x\in U_i\cap U_j\cap W_{j,i}\\ \lambda_{\{i,j\}}(x)\pi_x(a_{\{i,j\}}) & \text{if } x\in U_i\cap U_j\setminus(W_{i,j}\cup W_{j,i})\,.\end{cases}$$

If, for example, $x\in U_i\cap U_j\cap W_{i,j}$, then $\pi_x(\tilde u_{i,j}(a_i-\epsilon)_+\tilde u_{i,j}^*)\in\mathrm{Her}(\pi_x(a_{\{i,j\}}))$, and we conclude that $\langle\pi_x(h)\rangle=\langle\pi_x(a_{\{i,j\}})\rangle=g_\epsilon(x)$.

To check another case, let $x\in U_i\cap U_j\setminus(W_{i,j}\cup W_{j,i})$. Then, $\langle\pi_x(h)\rangle=\langle\lambda_{\{i,j\}}(x)\pi_x(a_{\{i,j\}})\rangle=\langle\pi_x(a_{\{i,j\}})\rangle=g_\varepsilon(x)$. The remaining case is similar.

□

This last result (together with Proposition 3.4.16) proves that, with some restrictions on X and A, the set of induced sections is a dense subset of $\Gamma(X,F_{\mathrm{Cu}(A)})$, that is, every element in $\Gamma(X,F_{\mathrm{Cu}(A)})$ is a supremum of a rapidly increasing sequence of induced sections. In fact, more is true:

Theorem 3.4.19. *Let X be a one-dimensional compact metric space and let A be a continuous field over X whose fibers have no K_1 obstructions. Then, the map*

$$\begin{array}{rccc}\alpha: & \mathrm{Cu}(A) & \to & \Gamma(X,F_{\mathrm{Cu}(A)})\\ & s & \mapsto & \hat s\end{array}$$

is an order isomorphism in Cu*.*

Proof. Let f be a continuous section in $\Gamma(X,F_{\mathrm{Cu}(A)})$ and use Propositions 3.4.16 and 3.4.18 to write f as the supremum of a rapidly increasing sequence of induced sections $f=\sup_n\hat s_n$. Since α is an order-embedding (by Theorem 3.3.6) and $\alpha(s_n)=\hat s_n$, the sequence s_n is also increasing in $\mathrm{Cu}(A)$ and thus we can define $s=\sup_n s_n\in\mathrm{Cu}(A)$. The result now follows using Lemma 3.4.9. □

3.5 The sheaf $\mathrm{Cu}_A(\_)$

For a compact metric space X, denote by $\mathcal{C}_X$ the category whose objects are the $\mathrm{C}(X)$-algebras, and the morphisms between objects are those *-homomorphisms such that commute with the (respective) structure maps.

Denote by $\mathcal{S}_{\mathrm{Cu}}$ the category which as objects has the presheaves $\mathrm{Cu}_A(\_)$ on X, where A belongs to $\mathcal{C}_X$, and the maps are presheaf homomorphisms. The following holds by definition:

Lemma 3.5.1. *The assignment*

$$\begin{array}{rccc} \mathrm{Cu}_{(\_)}\colon & \mathcal{C}_X & \to & \mathcal{S}_{\mathrm{Cu}} \\ & A & \mapsto & \mathrm{Cu}_A(\_) \end{array}$$

is a covariant functor.

Proof. Let $\varphi\colon A \to B$ be a map in $\mathcal{C}_X$. Given U a closed subset of X, we have that φ induces a map $\varphi_U\colon A(U) \to B(U)$ since $\varphi(\mathrm{C}_0(X\setminus U)A) \subseteq \mathrm{C}_0(X\setminus U)\varphi(A) \subseteq \mathrm{C}_0(X\setminus U)B$. Also, if $V \subseteq U$ is a closed subset of X, we have that the following diagram is commutative:

$$\begin{array}{ccc} A(V) & \xrightarrow{\varphi_V} & B(V) \\ \uparrow & & \uparrow \\ A(U) & \xrightarrow{\varphi_U} & B(U)\,. \end{array}$$

Because Cu is a functor, the conclusion follows. □

Theorem 3.5.2. *Let X be a one-dimensional compact metric space, and let A be a continuous field over X whose fibers have no K_1 obstructions. Consider the functors*

$$\begin{array}{rccc} \mathrm{Cu}_A(\_)\colon & \mathcal{V}_X & \to & \mathrm{Cu} \\ & V & \mapsto & \mathrm{Cu}(A(V)) \end{array} \quad \textit{and} \quad \begin{array}{rccc} \Gamma(\_, F_{\mathrm{Cu}_{A(\_)}})\colon & \mathcal{V}_X & \to & \mathrm{Cu} \\ & V & \mapsto & \Gamma(V, F_{\mathrm{Cu}_{A(V)}})\,. \end{array}$$

Then, $\mathrm{Cu}_A(\_)$ and $\Gamma(\_, F_{\mathrm{Cu}_{A(\_)}})$ are isomorphic sheaves.

Proof. That $\mathrm{Cu}_A(\_)$ is a sheaf follows from Corollary 3.4.6. Let $(h_V)_{V\in\mathcal{V}_X}$ be the collection of isomorphisms $h_V\colon \mathrm{Cu}(A(V)) \to \Gamma(V, F_{\mathrm{Cu}_{A(V)}})$ described in Theorem 3.4.19. Since, whenever $V \subset U$, the following diagram

$$\begin{array}{ccc} \mathrm{Cu}(A(V)) & \longrightarrow & \Gamma(V, F_{\mathrm{Cu}_{A(V)}}) \\ \uparrow{\scriptstyle (\mathrm{Cu}_A(\_))_V^U} & & \uparrow{\scriptstyle (\Gamma(\_, F_{\mathrm{Cu}_{A(\_)}}))_V^U} \\ \mathrm{Cu}(A(U)) & \longrightarrow & \Gamma(U, F_{\mathrm{Cu}_{A(U)}}) \end{array}$$

clearly commutes, $(h_V)_{V\in\mathcal{V}_X}$ defines an isomorphism of sheaves $h\colon \mathrm{Cu}_A(\_) \to \Gamma(\_, F_{\mathrm{Cu}_{A(\_)}})$. □

In order to relate the Cuntz semigroup $\mathrm{Cu}(A)$ and the sheaf $\mathrm{Cu}_A(\_)$, we now show that there exists an action of $\mathrm{Cu}(\mathrm{C}(X))$ on $\mathrm{Cu}(A)$ when A is a $\mathrm{C}(X)$-algebra, which is naturally induced from the $\mathrm{C}(X)$-module structure on A.

Definition 3.5.3. *Let S, T, R be semigroups in* Cu. *A* Cu*-bimorphism is a map $\varphi\colon S\times T \to R$ such that the map $\varphi(s,\_)\colon T \to R$, $s \in S$ (respectively, $\varphi(\_,t)\colon S \to R$, $t \in T$), preserves order, addition, suprema of increasing sequences, and moreover $\varphi(s',t') \ll \varphi(s,t)$ whenever $s' \ll s$ in S and $t' \ll t$ in T.*

Remark 3.5.4. We remark that if A is a C*-algebra and a, b are commuting elements, then for $\epsilon > 0$ we have $(a-\epsilon)_+(b-\epsilon)_+ \precsim (ab-\epsilon^2)_+$. Indeed, since the C*-subalgebra generated by a and b is commutative, Cuntz comparison is given by the support of the given elements, viewed as continuous functions on the spectrum of the algebra. It is then a simple matter to check that $\mathrm{supp}((a-\epsilon)_+(b-\epsilon)_+) \subseteq \mathrm{supp}((ab-\epsilon^2)_+)$ (see Proposition 1.3.11).

Proposition 3.5.5. *Let A and B be stable and nuclear C*-algebras. Then, the natural bilinear map $A \times B \to A \otimes B$ given by $(a, b) \mapsto a \otimes b$ induces a* Cu*-bimorphism*

$$\begin{array}{ccc} \mathrm{Cu}(A) \times \mathrm{Cu}(B) & \to & \mathrm{Cu}(A \otimes B) \\ (\langle a\rangle, \langle b\rangle) & \mapsto & \langle a \otimes b\rangle \end{array}$$

Proof. Since A is stable, we may think of $\mathrm{Cu}(A)$ as equivalence classes of positive elements from A. We also have an isomorphism $\Theta\colon M_2(A) \to A$ given by isometries w_1, w_2 in $\mathcal{M}(A)$ with orthogonal ranges, so that $\Theta(a_{ij}) = \sum_{i,j} w_i a_{ij} w_j^*$. Thus, in the Cuntz semigroup, $\langle a\rangle + \langle b\rangle = \langle \Theta\left(\begin{smallmatrix} a & 0 \\ 0 & b\end{smallmatrix}\right)\rangle$.

The map $\mathrm{Cu}(A) \times \mathrm{Cu}(B) \to \mathrm{Cu}(A \otimes B)$ given by $(\langle a\rangle, \langle b\rangle) \mapsto \langle a \otimes b\rangle$ is well-defined and order-preserving in each argument, by virtue of [Rør04, Lemma 4.2]. Let a, $a' \in A_+$, $b \in B_+$. As

$$\langle (w_1 a w_1^* + w_2 a' w_2^*) \otimes b\rangle = \langle w_1 a w_1^* \otimes b\rangle + \langle w_2 a' w_2^* \otimes b\rangle = \langle a \otimes b\rangle + \langle a' \otimes b\rangle\,,$$

we see that it is additive in the first entry (and analogously in the second entry).

Next, observe that if $\|a\|, \|b\| \leq 1$, $\epsilon > 0$,

$$\|a \otimes b - (a-\epsilon)_+ \otimes (b-\epsilon)_+\| \leq \|a \otimes b - (a-\epsilon)_+ \otimes b\| + \|(a-\epsilon)_+ \otimes b - (a-\epsilon)_+ \otimes (b-\epsilon)_+\| \leq$$

$$\epsilon\|b\| + \|(a-\epsilon)_+\|\epsilon \leq 2\epsilon\,,$$

and this implies $\langle a \otimes b\rangle = \sup(\langle (a-\epsilon)_+ \otimes (b-\epsilon)_+\rangle)$. If now $\langle a\rangle = \sup_n \langle a_n\rangle$ for an increasing sequence $(\langle a_n\rangle)$, then for any $\langle b\rangle$ we have $\langle a_n \otimes b\rangle \leq \langle a \otimes b\rangle$. Given $\epsilon > 0$, find n with $\langle (a-\epsilon)_+\rangle \leq \langle a_n\rangle$, hence $\langle (a-\epsilon)_+ \otimes (b-\epsilon)_+\rangle \leq \langle a_n \otimes b\rangle \leq \sup\langle a_n \otimes b\rangle$. Taking supremum as ϵ goes to zero we obtain $\langle a \otimes b\rangle = \sup\langle a_n \otimes b\rangle$.

Finally, assume that $\langle a'\rangle \ll \langle a\rangle$ in $\mathrm{Cu}(A)$, and $\langle b'\rangle \ll \langle b\rangle$ in $\mathrm{Cu}(B)$. Find $\epsilon > 0$ such that $\langle a'\rangle \leq \langle (a-\epsilon)_+\rangle$ and $\langle b'\rangle \leq \langle (b-\epsilon)_+\rangle$. Then $\langle a' \otimes b'\rangle \leq \langle (a-\epsilon)_+ \otimes (b-\epsilon)_+\rangle$.

Note that $(a-\epsilon)_+ \otimes (b-\epsilon)_+ \in A \otimes B \subseteq \mathcal{M}(A) \otimes \mathcal{M}(B)$ and, viewed in the tensor product of the multiplier algebras, we have $(a-\epsilon)_+ \otimes (b-\epsilon)_+ = ((a-\epsilon)_+ \otimes 1)(1 \otimes (b-\epsilon)_+)$. Since the map $\mathcal{M}(A) \to \mathcal{M}(A) \otimes \mathcal{M}(B)$, given by $c \mapsto c \otimes 1$ is a *-homomorphism, it induces a semigroup homomorphism $\mathrm{Cu}(\mathcal{M}(A)) \to \mathrm{Cu}(\mathcal{M}(A) \otimes \mathcal{M}(B))$ in the category Cu and, in particular, since $\langle (a-\epsilon)_+\rangle \ll \langle a\rangle$ in $\mathrm{Cu}(\mathcal{M}(A))$, it follows that $\langle (a-\epsilon)_+ \otimes 1\rangle \ll \langle a \otimes 1\rangle$ in $\mathrm{Cu}(\mathcal{M}(A) \otimes \mathcal{M}(B))$. Likewise, $\langle 1 \otimes (b-\epsilon)_+\rangle \ll \langle 1 \otimes b\rangle$, hence we may find $\epsilon' > 0$ such that $\langle (a-\epsilon)_+ \otimes 1\rangle \leq \langle (a \otimes 1 - \epsilon')_+\rangle$ and $\langle 1 \otimes (b-\epsilon)_+\rangle \leq \langle (1 \otimes b - \epsilon')_+\rangle$. Since the elements $(a-\epsilon)_+ otimes 1$, $(a \otimes 1 - \epsilon')_+$, $1 \otimes (b-\epsilon)_+$ and $(1 \otimes b - \epsilon')_+$ all commute (and using Remark 3.5.4), it follows that

$$\begin{aligned} \langle (a-\epsilon)_+ \otimes (b-\epsilon)_+\rangle &= \langle ((a-\epsilon)_+ \otimes 1)(1 \otimes (b-\epsilon)_+)\rangle \\ &\leq \langle (a \otimes 1 - \epsilon')_+ (1 \otimes b - \epsilon')_+\rangle \\ &\leq \langle (a \otimes b - \epsilon'^2)_+\rangle \ll \langle a \otimes b\rangle\,, \end{aligned}$$

whence $\langle a' \otimes b'\rangle \ll \langle a \otimes b\rangle$. ☐

Corollary 3.5.6. *Let X be a compact Hausdorff space, and let A be a stable $\mathrm{C}(X)$-algebra (with structure map θ). Then the natural map $\mathrm{C}(X) \times A \to A$, given by $(f, a) \to \theta(f)a$ induces a Cu-bimorphism*

$$\gamma_A \colon \mathrm{Cu}(\mathrm{C}(X)) \times \mathrm{Cu}(A) \to \mathrm{Cu}(A)$$

such that maps $(\langle f\rangle, \langle a\rangle)$ to $\langle \theta(f)a\rangle$, for $f \in \mathrm{C}(X)_+$ and $a \in A_+$.

Proof. Since $\mathrm{Cu}(\mathrm{C}(X)) = \mathrm{Cu}(\mathrm{C}(X) \otimes \mathcal{K})$, Proposition 3.5.5 tells us that the map

$$\begin{array}{ccc} \mathrm{Cu}(\mathrm{C}(X) \otimes \mathcal{K}) \times \mathrm{Cu}(A) & \to & \mathrm{Cu}(\mathrm{C}(X) \otimes \mathcal{K} \otimes A) \\ (\langle f\rangle, \langle a\rangle) & \mapsto & \langle f \otimes a\rangle \end{array}$$

is a Cu-bimorphism. Now the result follows after composing this map with the isomorphism $\mathrm{Cu}(\mathrm{C}(X) \otimes \mathcal{K} \otimes A) \cong \mathrm{Cu}(\mathrm{C}(X) \otimes A)$, followed by the map $\mathrm{Cu}(\theta) \colon \mathrm{Cu}(\mathrm{C}(X) \otimes A) \to \mathrm{Cu}(A)$. □

In what follows, we shall refer to the Cu-bimorphism γ_A above as the *action* of $\mathrm{Cu}(\mathrm{C}(X))$ on $\mathrm{Cu}(A)$. If A and B are $\mathrm{C}(X)$-algebras, we will say then that a morphism $\varphi\colon \mathrm{Cu}(A) \to \mathrm{Cu}(B)$ *preserves the action* provided $\varphi(\gamma_A(x, y)) = \gamma_B(x, \varphi(y))$. Notice that this is always the case if φ is induced by a *-homomorphism of $\mathrm{C}(X)$-algebras. We will write γ instead of γ_A, and we will moreover use the notation xy for $\gamma(x, y)$.

We next exhibit how the action of $\mathrm{Cu}(\mathrm{C}(X))$ on $\mathrm{Cu}(A)$ works when $\dim(X) \leq 1$.

Remark 3.5.7. Here we apply Corollary 3.5.6, in the particular case when X is a one-dimensional compact Hausdorff space. Under this assumption, it follows by [Rob13, Theorem 1.1] that the map $\alpha\colon \mathrm{Cu}(\mathrm{C}(X)) \to \mathrm{Lsc}(X, \overline{\mathbb{N}})$ given by $\alpha(\langle f\rangle)(x) = \langle f(x)\rangle$ is an order-isomorphism.

Note that any $g \in \mathrm{Lsc}(X, \overline{\mathbb{N}})$ may be written as:

$$g = \sum_{i=1}^{\infty} \mathbb{1}_{U_i} = \sup \sum_{i=1}^{n} \mathbb{1}_{U_i} \text{ where } U_i = g^{-1}((i, \infty]).$$

Given $z \in \mathrm{Cu}(\mathrm{C}(X))$, write $\alpha(z) = \sum_{i=1}^{\infty} \mathbb{1}_{U_i}$ as above, so that $z = \sup \sum_{i=1}^{n} z_i$ with $\alpha(z_i) = \mathbb{1}_{U_i}$. Thus, for $a \in A_+$, $\gamma_A(z, \langle a\rangle) = \sup \sum_{i=1}^{n} \gamma_A(z_i, \langle a\rangle)$. Therefore, to describe the action we may assume $\alpha(z) = \mathbb{1}_U$ for some open set U. Let $h \in \mathrm{C}(X)_+$ such that $\mathrm{supp}(h) = U$, and then $\alpha(\langle h\rangle) = \mathbb{1}_U$. Thus, $z = \langle h\rangle$ and

$$\gamma_A(z, \langle a\rangle) = \langle \theta(h) \cdot a\rangle.$$

As noticed above, $\mathbb{V}_A(\_)$ defines a continuous presheaf, and we show below that it becomes a sheaf when the fibers of A have no K_1 obstructions. For this we need the next result.

Lemma 3.5.8. *Let X be a one-dimensional compact metric space, and let Y, $Z \subseteq X$ be closed subsets of X. Let A be a continuous field over X whose fibers have no K_1 obstructions, and denote by $\pi_Y^Z \colon A(Y) \to A(Y \cap Z)$ and $\pi_Z^Y \colon A(Z) \to A(Y \cap Z)$ the quotient maps (given by restriction). Then, the map*

$$\beta\colon \mathrm{V}(A(Y) \oplus_{A(Y\cap Z)} A(Z)) \to \mathrm{V}(A(Y)) \oplus_{\mathrm{V}(A(Y\cap Z))} \mathrm{V}(A(Z))$$

defined by $\beta([(a, b)]) = ([a], [b])$ is an isomorphism.

Proof. We know from Corollary 3.4.6 that $\mathrm{Cu}_A(\_)$ is a sheaf in this case. Thus the map

$$\mathrm{Cu}(A(Y)\oplus_{A(Y\cap Z)}A(Z))\to \mathrm{Cu}(A(Y))\oplus_{\mathrm{Cu}(A(Y\cap Z))}\mathrm{Cu}(A(Z)),$$

given by $\langle(a,b)\rangle\mapsto(\langle a\rangle,\langle b\rangle)$, is an isomorphism in Cu, whence it maps compact elements to compact elements. Since $A(Y)\oplus_{A(Y\cap Z)}A(Z)$ is isomorphic to $A(Y\cup Z)$ and this algebra is stably finite (because all of its fibers have stable rank one), we have that the compact elements of $\mathrm{Cu}(A(Y)\oplus_{A(Y\cap Z)}A(Z))$ can be identified with $\mathrm{V}(A(Y)\oplus_{A(Y\cap Z)}A(Z))$. Using this identification, we have that $\langle(a,b)\rangle$ in $\mathrm{Cu}(A(Y)\oplus_{A(Y\cap Z)}A(Z))$ is compact if and only if $\langle a\rangle$ and $\langle b\rangle$ are compact.

On the other hand, if $\langle a\rangle$ and $\langle b\rangle$ are compact (in $\mathrm{Cu}(A(Y))$ and $\mathrm{Cu}(A(Z))$ respectively) and $\langle a\rangle=\langle b\rangle$ in $\mathrm{Cu}(A(Y\cap Z))$, then the pair $(\langle a\rangle,\langle b\rangle)$ belongs to $\mathrm{V}(A(Y))\oplus_{\mathrm{V}(A(Y\cap Z))}\mathrm{V}(A(Z))$, and every element of this pullback is obtained in this manner. The conclusion now follows easily. □

Proposition 3.5.9. *Let X be a one-dimensional compact metric space, and let A be a continuous field over X whose fibers have K_1 obstructions. Then,*

$$\begin{array}{rccc}\mathbb{V}_A(\_): & \mathcal{V}_X & \to & \mathrm{Sg}\\ & U & \mapsto & \mathrm{V}(A(U))\end{array}$$

is a sheaf and the natural transformation $\mathbb{V}_A(\_)\to\Gamma(\_,F_{\mathrm{V}(A(\_))})$ is an isomorphism of sheaves.

Proof. Note that $\mathbb{V}_A(\_)$ is a sheaf thanks to Lemma 3.5.8. On the other hand, the fact that $\mathbb{V}_A(\_)$ is isomorphic to the sheaf of continuous sections $\Gamma(\_,F_{\mathrm{V}(A(\_))})$ follows from Theorem 3.2.3. □

In the proof of Theorem 3.5.11 below, we shall use the following lemma. Recall that if X is a compact Hausdorff topological space, then the set $\mathcal{O}(X)$ consisting of open sets ordered by inclusion is a continuous lattice. Moreover, in the case X is metric, we have that $U\ll V$ if and only if there exists a compact set K such that $U\subseteq K\subseteq V$ ([GHK+03]). In particular, $\mathcal{O}(X)$ with union as addition is a semigroup in Cu.

Lemma 3.5.10. ([ADPS13, Lemma 2.4, Lemma 2.5]) *Let A be a continous field of C*-algebras over a compact Hausdorff space X and $a,b\in A_+$. Then:*

(i) *If $\pi_K(b)\precsim\pi_K(a)$ for some K such that $\mathrm{supp}(b)\subseteq K\subseteq\mathrm{supp}(a)$, then $a\precsim b$.*

(ii) *If $b\precsim a$, then $\mathrm{supp}(b)\subseteq\mathrm{supp}(a)$. Moreover, if $\langle b\rangle\ll\langle a\rangle$, then there exists a compact set K such that $\mathrm{supp}(b)\subseteq K\subseteq\mathrm{supp}(a)$.*

Proof. (i): By assumption, given $\varepsilon>0$ there exists $d\in A$ such that $\|b(x)-d(x)a(x)d^*(x)\|<\varepsilon$ for all $x\in K$, so this inequality is also satisfied in an open set $K\subseteq U$ because A is a continuous field.

Since $K\cap U^c=\emptyset$, by Urysohn's Lemma there is a continuous function $f\colon X\to[0,1]$ such that $f_K=1$ and $f_{U^c}=0$, and if $x\in\mathrm{supp}(g)\subseteq K\subseteq U$, then

$$\|b(x)-(fd)a(fd)^*(x)\|=\|b(x)-d(x)a(x)d^*(x)\|<\varepsilon.$$

Further, if $x\notin U$, then $\|b(x)-(fd)a(fd)^*(x)\|=0$. Finally, if $x\in U\setminus\mathrm{supp}(b)$, then $b(x)=0$, and

$$\|b(x)-(fd)a(fd)^*(x)\|=\|f^2b(x)-(fd)a(fd)^*(x)\|=|f^2(x)|\|b(x)-d(x)a(x)d^*(x)\|<\varepsilon.$$

Therefore, since $\|b-(fd)a(fd)^*\| = \sup_{x\in X}\|b(x)-(fd)a(fd)^*(x)\|$, we have $b \precsim a$.

(ii): Let $\langle b\rangle \ll \langle a\rangle$ for some $a, b \in A_+$. Since $\mathcal{O}(X)$ is in Cu, let us write $\mathrm{supp}(a) = \cup_{i\geq 0} U_i$ for some $U_i \ll U_{i+1}$, i.e., $U_i \subseteq \overline{U}_i \subseteq U_{i+1}$. By Urysohn's Lemma, we find a colletion of continuous functions $f_n : X \to [0,1]$ such that $f_n(\overline{U_n}) = 1$ and $f_n(U_{n+1}^c) = 0$. Because X is compact, we have $f_n a \to a$ and $f_n a \leq f_{n+1} a$. Hence, $\langle a\rangle = \sup_n \langle f_n a\rangle$. Using that $\langle b\rangle \ll \langle a\rangle$, we get $\langle b\rangle \leq \langle f_N a\rangle$ for some $N > 0$ and therefore $\mathrm{supp}(b) \subseteq \mathrm{supp}(f_N a) \subseteq U_{N+1} \subseteq \overline{U}_{N+1} \subseteq \mathrm{supp}(a)$, i.e., $\mathrm{supp}(b) \ll \mathrm{supp}(a)$. □

Theorem 3.5.11. *Let X be a compact metric space, and let A and B be $\mathrm{C}(X)$-algebras such that all fibers have stable rank one. Consider the following conditions:*

(i) $\mathrm{Cu}(A) \cong \mathrm{Cu}(B)$ *preserving the action of* $\mathrm{Cu}(\mathrm{C}(X))$,

(ii) $\mathrm{Cu}_A(\_) \cong \mathrm{Cu}_B(\_)$,

(iii) $\mathbb{V}_A(\_) \cong \mathbb{V}_B(\_)$.

Then (i) $\implies$ (ii) $\implies$ (iii). *If X is one-dimensional, then also* (ii) $\implies$ (i). *If, furthermore, A and B are continuous fields such that for all $x \in X$ the fibers A_x, B_x have real rank zero and $\mathrm{K}_1(A_x) = \mathrm{K}_1(B_x) = 0$, then* (iii) $\implies$ (ii) *and so all three conditions are equivalent.*

Proof. We may assume that both A and B are stable.

(i) $\implies$ (ii): Let $\varphi\colon \mathrm{Cu}(A) \to \mathrm{Cu}(B)$ be an isomorphism such that $\varphi(xy) = x\varphi(y)$, for any $x \in \mathrm{Cu}(\mathrm{C}(X))$ and $y \in \mathrm{Cu}(A)$. We need to verify that $\varphi(\mathrm{Cu}(\mathrm{C}_0(X\setminus V)A)) \subseteq \mathrm{Cu}(\mathrm{C}_0(X\setminus V)B)$, whenever V is a closed subset of X. Let $\langle fa\rangle \in \mathrm{Cu}(\mathrm{C}_0(X\setminus V)A)$, for $f \in \mathrm{C}_0(X\setminus V)_+$ and $a \in A_+$. Then, if $\varphi(\langle a\rangle) = \langle b\rangle$ for some $b \in B_+$, we have that $\varphi(\langle fa\rangle) = \langle f\rangle\varphi(\langle a\rangle) = \langle f\rangle\langle b\rangle = \langle fb\rangle$, and $fb \in \mathrm{C}_0(X\setminus V)B$.

The above fact entails that φ induces a semigroup map $\varphi_V\colon \mathrm{Cu}(A(V)) \to \mathrm{Cu}(B(V))$, which is an isomorphism as φ is.

(ii) $\implies$ (iii): Note that, as all fibers have stable rank one, $A(U)$ (respectively, $B(U)$) is a stably finite algebra for each closed subset U. In this case, $\mathrm{V}(A(U))$ can be identified with the subset of compact elements of $\mathrm{Cu}(A(U))$. Therefore, the given isomorphism $\mathrm{Cu}_A(U) \cong \mathrm{Cu}_B(U)$ maps $\mathbb{V}_A(U) = \mathrm{V}(A(U))$ injectively onto $\mathbb{V}_B(U) = \mathrm{V}(B(U))$.

Now assume that X is one-dimensional, and let us prove that (ii) $\implies$ (i): The isomorphism of sheaves gives, in particular, an isomorphism $\varphi\colon \mathrm{Cu}(A) \to \mathrm{Cu}(B)$. We need to verify that φ respects the action of $\mathrm{Cu}(\mathrm{C}(X))$. By Remark 3.5.7, we may reduce to the case of $\mathbb{1}_U\langle a\rangle = \langle ga\rangle$ where $g \in \mathrm{C}(X)_+$ has $\mathrm{supp}(g) = U$. Given $\langle a\rangle \in \mathrm{Cu}(A)$ we denote by $\mathrm{supp}(\langle a\rangle) = \{x \in X \mid \pi_x(\langle a\rangle) \neq 0\}$ and note that $\mathrm{supp}\varphi(\langle a\rangle) = \mathrm{supp}(\langle a\rangle)$. Also $\mathrm{supp}(\mathbb{1}_U\varphi(\langle a\rangle)) = U \cap \mathrm{supp}(\langle a\rangle) = \mathrm{supp}(\varphi(\mathbb{1}_U\langle a\rangle))$. Let $K \subseteq \mathrm{supp}(\mathbb{1}_U\varphi(\langle a\rangle)) = \mathrm{supp}(\varphi(\mathbb{1}_U\langle a\rangle))$ be a closed set. Then $\pi_K(\mathbb{1}_U\varphi(\langle a\rangle)) = \pi_K(\varphi(\mathbb{1}_U\langle a\rangle))$, where $\pi_K\colon A \to A(K)$ is the quotient map. Indeed, it follows from the commutative diagram

$$\begin{array}{ccc} \mathrm{Cu}(A) & \xrightarrow{\quad\varphi\quad} & \mathrm{Cu}(B) \\ {\scriptstyle \pi_K}\big\downarrow & & \big\downarrow{\scriptstyle \pi_K} \\ \mathrm{Cu}(A(K)) & \xrightarrow{\ \varphi_K\ } & \mathrm{Cu}(B(K)) \end{array} ,$$

that $\pi_K(\mathbb{1}_U\langle a\rangle) = \pi_K(\langle ga\rangle) = \pi_K\langle a\rangle$, since g becomes invertible in $A(K)$. Hence, $\pi_K(\varphi(\mathbb{1}_U\langle a\rangle)) = \pi_K(\varphi(\langle ga\rangle)) = \varphi_K\pi_K(\langle ga\rangle) = \varphi_K\pi_K(\langle a\rangle)$. On the other hand, $\pi_K(\mathbb{1}_U\varphi(\langle a\rangle)) = \pi_K\varphi(\langle a\rangle) = \varphi_K\pi_K(\langle a\rangle)$.

Now write $\langle a\rangle = \sup\langle a_n\rangle$, where $(\langle a_n\rangle)$ is a rapidly increasing sequence in $\mathrm{Cu}(A)$, and $\mathbb{1}_U = \sup\mathbb{1}_{V_n}$, where (V_n) is a rapidly increasing sequence of open sets. Then $(\mathbb{1}_{V_n}\langle a_n\rangle)$ is a rapidly increasing sequence with $\mathbb{1}_U\langle a\rangle = \sup\mathbb{1}_{V_n}\langle a_n\rangle$ and $\mathbb{1}_U\varphi(\langle a\rangle) = \sup\mathbb{1}_{V_n}\varphi(\langle a\rangle)$. By Lemma 3.5.10 (i) choose, for each n, a compact set K_n such that

$$\mathrm{supp}(\mathbb{1}_{V_n}\langle a_n\rangle) \subseteq \mathrm{K}_n \subseteq \mathrm{supp}(\mathbb{1}_{V_{n+1}}\langle a_{n+1}\rangle)\,.$$

Then $\mathrm{K}_n \subseteq V_{n+1}\cap\mathrm{supp}(\langle a_{n+1}\rangle) \subseteq V_{n+1}$.

By the above, $\pi_{\mathrm{K}_n}(\mathbb{1}_{V_{n+1}}\varphi(\langle a_{n+1}\rangle)) = \pi_{\mathrm{K}_n}\varphi(\mathbb{1}_{V_{n+1}}\langle a_{n+1}\rangle)$, thus:

$$\pi_{\mathrm{K}_n}(\mathbb{1}_{V_n}\varphi(\langle a_n\rangle)) \leq \pi_{\mathrm{K}_n}(\mathbb{1}_{V_{n+1}}\varphi(\langle a_{n+1}\rangle)) = \pi_{\mathrm{K}_n}(\varphi(\mathbb{1}_{V_{n+1}}\langle a_{n+1}\rangle)) \leq \pi_{\mathrm{K}_n}(\varphi(\mathbb{1}_U\langle a\rangle)),$$

and

$$\pi_{\mathrm{K}_n}(\varphi(\mathbb{1}_{V_n}\langle a_n\rangle)) \leq \pi_{\mathrm{K}_n}(\varphi(\mathbb{1}_{V_{n+1}}\langle a_{n+1}\rangle)) = \pi_{\mathrm{K}_n}(\mathbb{1}_{V_{n+1}}\varphi(\langle a_{n+1}\rangle)) \leq \pi_{\mathrm{K}_n}(\mathbb{1}_U\varphi(\langle a\rangle)).$$

Since $\mathrm{supp}(\mathbb{1}_{V_k}\langle a_k\rangle) = \mathrm{supp}(\varphi(\mathbb{1}_{V_k}\langle a_k\rangle)) = \mathrm{supp}(\mathbb{1}_{V_k}\varphi(\langle a_k\rangle))$, we may apply Lemma 3.5.10 (ii) to obtain that $\mathbb{1}_{V_n}\varphi(\langle a_n\rangle) \leq \varphi(\mathbb{1}_U\langle a\rangle))$ and $\varphi(\mathbb{1}_{V_n}\langle a_n\rangle) \leq \mathbb{1}_U\varphi(\langle a\rangle))$. Taking suprema in both inequalities we obtain $\mathbb{1}_U\varphi(\langle a\rangle) = \varphi(\mathbb{1}_U(\langle a\rangle))$.

(iii) $\Longrightarrow$ (ii): We assume now that both A and B are continuous fields such that the fibers A_x, B_x have real rank zero and trivial K_1 (see Lemma 3.3.2 (ii)). Let $\varphi\colon \mathbb{V}_A(\_) \to \mathbb{V}_B(\_)$ be a sheaf isomorphism. This induces a semigroup isomorphism $\varphi_x\colon \mathrm{V}(A_x) \to \mathrm{V}(B_x)$ for each $x \in X$. As $A(U)$ is a stably finite algebra for any closed subset U of X, we will identify $\mathrm{V}(A(U))$ with its image in $\mathrm{Cu}(A(U))$ whenever convenient.

Since A_x has real rank zero, $\mathrm{V}(A_x)$ forms a dense subset of $\mathrm{Cu}(A_x)$ so we can uniquely define an isomorphism $\mathrm{Cu}(A_x) \to \mathrm{Cu}(B_x)$ in Cu which we will still denote by φ_x. This map is defined by $\varphi_x(z) = \sup_n \varphi_x(z_n)$ where $z = \sup z_n$ and $z_n \in \mathrm{V}(A_x)$ for all $n \geq 0$ (see, e.g. [ABP11], [CEI08] for further details). Let us prove that the induced bijective map $\tilde{\varphi}\colon F_{\mathrm{Cu}(A)} \to F_{\mathrm{Cu}(B)}$ is continuous, and hence a homeomorphism. This will define an isomorphism of sheaves $\Gamma(-, F_{\mathrm{Cu}_A(-)}) \cong \Gamma(-, F_{\mathrm{Cu}_B(-)})$ from which, using Theorem 3.5.2, it follows that $\mathrm{Cu}_A(\_)$ and $\mathrm{Cu}_B(\_)$ are isomorphic. Denote by $\pi_A\colon F_{\mathrm{Cu}(A)} \to X$ and $\pi_B\colon F_{\mathrm{Cu}(B)} \to X$ the natural maps.

Let U be an open set of X and $s \in \mathrm{Cu}(B)$. We are to show that $\tilde{\varphi}^{-1}(U_s^{\gg})$ is open in $F_{\mathrm{Cu}(A)}$. Let $z \in \tilde{\varphi}^{-1}(U_s^{\gg})$, and put $x = \pi_A(z)$, so that $z \in \mathrm{Cu}(A_x)$ for some $x \in U$. Since $\tilde{\varphi}(z) = \varphi_x(z) \in U_s^{\gg}$, there exists $s'' \gg s$ such that $\hat{s''}(x) \ll \varphi_x(z)$. Choose s' such that $s \ll s' \ll s''$.

As $\hat{s''}(x) \ll \varphi_x(z)$ there exists $z' \ll z' \in \mathrm{V}(A_x)$ such that $\hat{s''}(x) \ll \varphi_x(z')$. Now we can find a closed subset W' whose interior contains x, and an element $v \in \mathrm{V}(A(W'))$ such that $\pi_x(v) = z'$. Note that $\pi_x\varphi_{W'}(v) = \varphi_x(z')$. Also, since $\hat{s'}(x) \ll \hat{s''}(x) \ll \varphi_x(z') = \widehat{\varphi_{W'}(v)}(x)$, we may use Lemma 3.4.2 to find $W \subseteq W'$ such that $x \in \mathring{W}$ and

$$\hat{s'}(y) \ll \widehat{\varphi_{W'}(v)}(y) \text{ for all } y \in W\,.$$

Let $t \in \mathrm{Cu}(A)$ be such that $\pi_W(t) = \pi_W^{W'}(v)$. We now claim that $\mathring{W}_t^{\gg} \subseteq \tilde{\varphi}^{-1}(U_s^{\gg})$. Let $w \in \mathring{W}_t^{\gg}$, and put $y = \pi_A(w) \in W$. There exists $t' \gg t$ such that $\hat{t'}(y) \ll w$, whence, applying $\tilde{\varphi}$ it follows that

$$\tilde{\varphi}(w) \gg \tilde{\varphi}(\hat{t'}(y)) \gg \tilde{\varphi}(\hat{t}(y)) = \tilde{\varphi}(\pi_W^{W'}(v)(y)) = \tilde{\varphi}(\pi_y(v)) = \pi_y(\varphi_W(v)) = \widehat{\varphi_{W'}(v)}(y) \gg \hat{s'}(y)\,,$$

and this shows that $w \in \tilde{\varphi}^{-1}(U_s^{\gg})$.

□

Remark 3.5.12. We remark that the implication (ii) $\implies$ (i) in Theorem 3.5.11 above holds whenever $\mathrm{Cu}(\mathrm{C}(X)) \cong \mathrm{Lsc}(X, \overline{\mathbb{N}})$. This is the case for spaces more general than being just one-dimensional, see [Rob13].

We can now rephrase the classification result given by M. Dadarlat, G. A. Elliott and Z. Niu in [DEN11], using the Cuntz Semigroup.

Theorem 3.5.13. ([DEN11]) *Let A, B be separable unital continuous fields of AF-algebras over $[0,1]$. Any isomorphism $\tilde{\phi} : \mathrm{Cu}(A) \to \mathrm{Cu}(B)$ that preserves the action by $\mathrm{Cu}(\mathrm{C}(X))$ and such that $\tilde{\phi}(\langle 1_A \rangle) = \langle 1_B \rangle$ lifts to an isomorphism $\phi : A \to B$ of continuous fields of C*-algebras.*

Proof. This follows from Theorem 3.5.11 together with Theorem 3.7 in [DEN11]. □

From the above result, it is natural to ask the following:

Open Problem 3.5.14. *Let $\mathcal{C}$ be the class of unital continuous fields of simple AI-algebras over $[0,1]$. Is it possible to classify the objects of $\mathcal{C}$ by using the Cuntz semigroup together with the action by $\mathrm{Cu}(\mathrm{C}([0,1]))$?*

Chapter 4

The Cuntz semigroup and Dimension functions

As mentioned in the Introduction, in this chapter we study two conjectures posed by Blackadar and Handelmann in [BH82] related with the structure of dimension functions of a C*-algebra A. Recall that they conjectured, for a unital C*-algebra, that:

(i) The set of dimension functions, $\mathrm{DF}(A)$, is a simplex.

(ii) The set of lower semicontinuous dimension functions, $\mathrm{LDF}(A)$, is dense in $\mathrm{DF}(A)$.

In particular, along the way to solve the above conjectures, we will need a categorical description of $\mathrm{W}(A)$ and $\mathrm{Cu}(A)$ ([ABP11]). We focus on the hereditariness of the Cuntz semigroup $\mathrm{W}(A)$, that is, the condition that $\mathrm{W}(A)$ can be viewed as an order-hereditary subsemigroup of $\mathrm{Cu}(A)$. As this is related to the stable rank of A, en route of determining when $\mathrm{W}(A)$ is hereditary, we obtain results of independent interest that compute the stable rank for some continuous fields. The results of this chapter come from [ABPP13].

4.1 Continuous fields of stable rank one

Let X be a compact metric space of dimension one. We will prove in this section that the algebra $\mathrm{C}(X, A)$ of continuous functions from X into a C*-algebra A, has stable rank one, if A has no K_1 obstructions. We also prove the converse direction in a setting of great generality and prove, as an application, corresponding results for continuous fields.

Recall that a C*-algebra A has *no* K_1 *obstructions* provided that A has stable rank one and $\mathrm{K}_1(I) = 0$ for every closed two-sided ideal I of A (equivalently, $\mathrm{sr}(A) = 1$ and $\mathrm{K}_1(B) = 0$ for all hereditary subalgebra B of A, see Lemma 3.3.2).

We first study the case when X is the closed unit interval. In this setting, we will use [NOP01, Proposition 5.2] to conclude that if $\mathrm{sr}(\mathrm{C}([0,1], A)) = 1$, then A has no K_1 obstructions.

Proposition 4.1.1. *Let A be any C*-algebra. If $\mathrm{sr}(\mathrm{C}([0,1], A)) = 1$, then A has no K_1 obstructions.*

Proof. In [NOP01, Proposition 5.2] it is shown that $\mathrm{sr}(A) = 1$ and $\mathrm{K}_1(A) = 0$ are necessary conditions to get $\mathrm{sr}(\mathrm{C}([0,1], A)) = 1$. We sketch the proof for completeness. One has that $\mathrm{sr}(A) = 1$ because A is a quotient of $\mathrm{C}([0,1], A)$, so it just remains to check that $\mathrm{K}_1(A) = 0$. By [Rie83, Theorem 6.1] it follows that $M_n(A)$ also satisfies the assumptions, so it suffices to show that $\mathcal{U}(\tilde{A})$ is connected. Let $u \in \tilde{A}$ be unitary and $\lambda \in \mathbb{C}$ be its image by the map $\pi : \tilde{A} \to \mathbb{C}$. Let us verify that $\lambda^{-1}u \in \mathcal{U}_0(\tilde{A})$, and this implies that $u \in \mathcal{U}_0(\tilde{A})$. We assume that $\pi(u) = 1$ to ease the proof. Define $f \in (\mathrm{C}([0,1], A))^{\sim}$ by $f(t) = t.1 + (1-t)u \in \tilde{A}$ for $t \in [0,1]$. Using $\mathrm{sr}(\mathrm{C}([0,1], A)) = 1$, choose an invertible element $g \in (\mathrm{C}([0,1], A))^{\sim}$ such that $\|g - f\| < 1/2$. Since $\|u - g(0)\| = \|f(0) - g(0)\| < 1$, there exists a continuous path in $\mathrm{GL}(\tilde{A})$ from u to $g(0)$, and, similarly, there exists a continuous path from 1 to $g(1)$. Combining both paths with the continuous path g in $\mathrm{GL}(\tilde{A})$, we see that $u \in \mathrm{GL}_0(\tilde{A})$, so $u \in \mathcal{U}_0(\tilde{A})$.

Using this, we now show that $\mathrm{K}_1(B) = 0$ for all $B \subset A$ hereditary. Let $B \subseteq A$ be a hereditary subalgebra. Let I denote the ideal generated by B. Then $\mathrm{C}([0,1], I)$ is an ideal of $\mathrm{C}([0,1], A)$ and therefore has stable rank one. Further, it follows from first part of the proof that $\mathrm{K}_1(I) = 0$. Since B is a full hereditary subalgebra of I, we conclude that $\mathrm{K}_1(B) = 0$. □

We next show that, conversely, for any C*-algebra A with no K_1 obstructions the stable rank of $\mathrm{C}([0,1], A)$ is one. It is pertinent to note that our proof follows the lines of [NOP01, Theorem 4.3], where the same result was proved under the additional assumption that $\mathrm{RR}(A) = 0$. In fact, the argument in [NOP01, Theorem 4.3] refers to Lemma 4.2 of this chapter, and our contribution is to prove the corresponding lemma in a more general setting. Our result was inspired by the next result:

Lemma 4.1.2. ([San12, Lemma 3.4]) *Let A be a C*-algebra such that $\mathcal{U}(\tilde{B})$ is connected for every hereditary subalgebra B of A. Then, for every $\varepsilon > 0$ and $M > 0$, there is $\delta > 0$ such that if $a \in \tilde{A}$ is a positive element with $a - 1 \in A$ and $\|a\| \leq M$, and $u \in A$ is a unitary with $u - 1 \in A$ satisfying*

$$\|ua - a\| < \delta,$$

there exists a path of unitaries $(u_t)_{t \in [0,1]}$ in $\tilde{A}$ with $u_t - 1 \in A$, $u_0 = u$ and $u_1 = 1$, such that

$$\|u_t a - a\| < \varepsilon$$

for all $t \in [0,1]$.

Proof. (Sketch) First, it can be proved that the universal C*-algebra

$$C^*(a, u \mid u \text{ is unitary},\ 0 \leq a \leq M.1,\ ua = a)$$

is weakly semiprojective. So, letting $0 < \varepsilon < 1$ and $\varepsilon' = \min(\frac{\varepsilon}{2}, (\frac{\varepsilon}{4M+1})^2)$, there exists $0 < \delta < \varepsilon'$ such that if $a \in \tilde{A}$ is a positive element with $a - 1 \in A$ and $u \in \mathcal{U}_0(\tilde{A})$ with $u - 1 \in A$ satisfy $\|ua - a\| < \delta$, then there are $a' \in \tilde{A}$, with $0 \leq a' \leq M.1$, and $u' \in \mathcal{U}_0(\tilde{A})$ such that

$$u'a' = a',\ \ \|u - u'\| < \varepsilon',\ \ \|a - a'\| < \varepsilon' (\text{see [Lor97, Chapter 19] for further details}).$$

If $\pi : \tilde{A} \to \mathbb{C}$ is the quotient map, one obtains

$$\pi(u')\pi(a') = \pi(a'),\ \ |\pi(u') - 1| < \varepsilon' < 1/2,\ \ |\pi(a') - 1| < \varepsilon' < 1/2.$$

Thus $\pi(a') \neq 0$, and then it follows that $\pi(u') = 1$ (i.e. $u' - 1 \in A$), and also that $|\pi(a')| > 1 - 1/2 = 1/2$. Further,

$$\|a - \frac{a'}{\pi(a')}\| \leq \|a - a'\| + \|\frac{a'}{\pi(a')}(\pi(a') - 1)\| < \varepsilon' + \frac{\|a'\|}{|\pi(a')|}|\pi(a') - 1| < \varepsilon' + 2M\varepsilon' = (2M+1)\varepsilon'.$$

Now, making an abuse of notation and denoting a' by $a'/\pi(a')$, we have $a' \in \tilde{A}$ and a unitary $u' \in \mathcal{U}_0(\tilde{A})$ such that

$$a' - 1, u' - 1 \in A,\ u'a' = a',\ \|u - u'\| < \varepsilon',\ \|a - a'\| < (2M+1)\varepsilon'.$$

Write $u' = x + 1$, with $x \in A$ a normal element. Then $xa' = 0$ since $u'a = a$. Also, $x \in \overline{x^*Ax}$, and, in particular, u' is a unitary of $(\overline{x^*Ax})^{\sim}$. Therefore, by hypothesis, we can find a path of unitaries $(v_t)_{t\in[0,1]}$ such that $v_0 = 1$ and $v_1 = u'$, and, moreover, it may be taken such that $v_t - 1 \in \overline{x^*Ax}$ for all t. Notice that $v_t a' - a' = 0$ for all t. This yields

$$\|v_t a - a\| \leq \|v_t(a - a')\| + \|a' - a\| < 2\varepsilon' \leq \varepsilon.$$

Let $z_t = tu + (1-t)u'$ with $t \in [0,1]$, and note that $\|u - z_t\| = \|(1-t)(u - u')\| < \varepsilon' < 1/2$, so z_t is invertible for all t satisfying $\|z_t\| \leq \varepsilon' + 1$. Hence, it has polar decomposition $z_t = w_t|z_t|$, where $w_t \in \mathcal{U}(\tilde{A})$, and, moreover, one has that

$$\|u^*u - z_t^*z_t\| \leq \|u^*u - u^*z_t\| + \|u^*z_t - z_t^*z_t\| \leq$$

$$\leq \|u - z_t\| + \|z_t\|\|u^* - z_t^*\| < \varepsilon' + \varepsilon' + (\varepsilon')^2 < 3\varepsilon'$$

Now, by [CES11, Lemma 2.3], one has $\|1 - |z_t|\| \leq \sqrt{3\varepsilon'} < 3\sqrt{\varepsilon'}$. Note that the path of unitaries $(w_t)_{t\in[0,1]}$ satisfies $w_0 = u'$, $w_1 = u$ and $w_t - 1 \in A$. Also, by the inequalites above, it follows that

$$\|u - w_t\| \leq \|u - w_t|z_t|\| + \|w_t(|z_t| - 1)\| < 4\sqrt{\varepsilon'}$$

for all t. This implies that

$$\|w_t a - a\| \leq \|(w_t - u)a\| + \|ua - a\| < 4M\sqrt{\varepsilon'} + \delta \leq (4M+1)\sqrt{\varepsilon'} < \varepsilon.$$

Consider the path

$$u_t := \begin{cases} v_{2t}, & t \in [0, \frac{1}{2}] \\ w_{2t-1}, & t \in [\frac{1}{2}, 1] \end{cases}$$

to conclude the proof. □

Lemma 4.1.3. *Let A be a unital C*-algebra with no K_1 obstructions. For any given $\epsilon > 0$ there is some $\delta > 0$ such that whenever a and b are two invertible contractions in A with $\|a - b\| < \delta$ then there is a continuous path $(c_t)_{t\in[0,1]}$ in the invertible elements of A such that $c_0 = a$, $c_1 = b$, and $\|c_t - a\| < \epsilon$ for all $t \in [0,1]$.*

Proof. For given $\epsilon > 0$ we choose $\delta_0 > 0$ satisfying the conclusion of Lemma 4.1.2 for $\frac{\epsilon}{2}$ and $M = 1$, i.e., for any positive contraction a and any unitary u with $\|ua - a\| < \delta_0$ there is a path of unitaries $(u_t)_{t\in[0,1]}$ in A such that $u_0 = u$, $u_1 = 1_A$, and $\|u_t a - a\| < \frac{\epsilon}{2}$ for all $t \in [0, 1]$. (It follows from our assumptions and [Rie83, Theorem 2.10] that $U(\tilde{B})$ is connected for each hereditary subalgebra B of A, which is needed for the application of Lemma 4.1.2.) Find $0 < \delta \leq \frac{\delta_0}{2}$ such that whenever $\|a - b\| < \delta$, then $\||a| - |b|\| < \frac{\delta_0}{2}$. (This is possible by [Lin96, Lemma 2.8].)

Take two invertible contractions a, b in A with $\|a - b\| < \delta$ and write $a = u|a|$ and $b = v|b|$ with unitaries $u, v \in A$. We first connect a and $u|b|$ by a path of invertible elements. To do this, define a continuous path $(w_t)_{t\in[0,1]}$ by

$$w_t := u(t|b| + (1-t)|a|),\ t \in [0, 1].$$

Then $w_0 = a$, $w_1 = u|b|$ and, for any $t \in [0, 1]$, w_t is invertible. Indeed, since $|b|$, $|a|$ are invertible, there are $\lambda_b, \lambda_a \in (0, 1]$ for which $|b| \geq \lambda_b 1_A$ and $|a| \geq \lambda_a 1_A$, so $t|b| + (1-t)|a| \geq \lambda 1_A$ for some $\lambda \neq 0$, and this implies that $t|b| + (1-t)|a|$ is invertible. Hence, w_t is also invertible. Further,

$$\|w_t - a\| = \|ut|b| - ut|a|\| = t\||b| - |a|\| < \frac{\delta_0}{2} < \epsilon.$$

Next, we connect $u|b|$ and b by a path of invertible elements. Since

$$\|v^*u|b| - |b|\| = \|u|b| - v|b|\| \leq \|u|b| - u|a|\| + \|u|a| - v|b|\| = \||a| - |b|\| + \|a - b\| < \delta_0,$$

an application of Lemma 4.1.2 provides us with a path of unitaries $(u_t)_{t\in[0,1]}$ in A such that $u_0 = v^*u$, $u_1 = 1$, and $\|u_t|b| - |b|\| < \frac{\epsilon}{2}$ for all $t \in [0, 1]$.

Define a continuous path $(z_t)_{t\in[0,1]}$ by

$$z_t := vu_t|b|,\ t \in [0, 1].$$

Then $z_0 = u|b|$, $z_1 = b$ and, for each $t \in [0, 1]$, z_t is invertible and

$$\|z_t - a\| \leq \|vu_t|b| - v|b|\| + \|v|b| - a\| = \|u_t|b| - |b|\| + \|b - a\| < \epsilon.$$

Hence

$$c_t := \begin{cases} w_{2t}, & t \in [0, \frac{1}{2}] \\ z_{2t-1}, & t \in [\frac{1}{2}, 1] \end{cases}$$

is the continuous path with the desired properties. □

Theorem 4.1.4. *Let A be any C^*-algebra with* $\mathrm{sr}(A) = 1$. *Then*

$$\mathrm{sr}(\mathrm{C}([0,1], A)) = \begin{cases} 1, & \text{if } A \text{ has no } \mathrm{K}_1 \text{ obstructions} \\ 2, & \text{else.} \end{cases}$$

Proof. It is known that $\mathrm{sr}(\mathrm{C}([0, 1], A)) \leq 1 + \mathrm{sr}(A) \leq 2$ ([Sud02]), and from Proposition 4.1.1 we know that for $\mathrm{sr}(\mathrm{C}([0, 1], A)) = 1$ it is a necessary condition that $\mathrm{K}_1(B) = 0$ for all hereditary subalgebras B of A. To show that this condition is also sufficient we follow the lines of the proof of [NOP01, Theorem 4.3], applying Lemma 4.1.3 instead of [NOP01, Lemma 4.2]. We give the argument for completeness.

Let $a \in \mathrm{C}([0,1], A)$, and let $\varepsilon > 0$. Without loss of generality we may suppose $\|a\| \leq 1$, and let us approximate a by an invertible element of $\mathrm{C}([0,1], A)$.

Use $\varepsilon/3$ in place of ε in Lemma 4.1.3 and choose δ accordingly. Choose $0 = t_0 < t_1 < \ldots < t_n = 1$ such that

$$\|a(t_j) - a(t_{j-1})\| < \delta/3 \quad \text{and} \quad \|a(t) - a(t_{j-1})\| < \varepsilon/3$$

for $1 \leq j \leq n$ and $t \in [t_{j-1}, t_j]$. Using the fact that $\mathrm{sr}(A) = 1$, there are $c_0, c_1, \ldots, c_n \in \mathrm{GL}(A)$ (invertibles of A) such that

$$\|c_j - a(t_j)\| < \min\{\varepsilon/3, \delta/3\}.$$

Then $\|c_j - c_{j-1}\| < \delta$ because $\|a(t_j) - a(t_{j-1})\| < \delta/3$. For each j, use Lemma 4.1.3 to find a continuous path $t \mapsto b(t) \in \mathrm{GL}(A)$, defined for $t \in [t_{j-1}, t_j]$, such that

$$b(t_{j-1}) = c_{j-1}, \;\; b(t_j) = c_j \quad \text{and} \quad \|b(t) - c_{j-1}\| < \varepsilon/3.$$

The two definitions at t_j agree, so $t \mapsto b(t)$ is a continuous invertible path defined for $t \in [0,1]$. Moreover, for $t \in [t_{j-1}, t_j]$ we have

$$\|b(t) - a(t)\| \leq \|b(t) - c_{j-1}\| + \|c_{j-1} - a(t_{j-1})\| + \|a(t_{j-1}) - a(t)\| < \varepsilon.$$

□

Corollary 4.1.5. *Let A be a simple C*-algebra with $\mathrm{sr}(A) = 1$ and $\mathrm{K}_1(A) = 0$. Then*

$$\mathrm{sr}(\mathrm{C}([0,1], A)) = 1\,.$$

The previous corollary answers positively a question raised in [NOP01, Question 5.9], which asks whether a simple direct limit of direct sums of homogeneous C*-algebras A with real rank one and $\mathrm{K}_1(A) = 0$ satisfies that $\mathrm{sr}(\mathrm{C}([0,1], A)) = 1$.

Corollary 4.1.6. *Let A be a Goodearl algebra with $\mathrm{K}_1(A) = 0$. Then $\mathrm{sr}(\mathrm{C}([0,1], A)) = 1$.*

Proof. It is a direct application of Corollary 4.1.5 since A is simple and $\mathrm{sr}(A) = 1$ by Theorem 1.1.20. □

Before giving more applications of Theorem 4.1.4 we state a result of L. G. Brown and G. K. Pedersen, which shows the behaviour of the stable rank and real rank of pullbacks, that will be used several times in this chapter.

Theorem 4.1.7. ([BP09, Theorem 4.1]) *Consider a pullback diagram of C*-algebras*

$$\begin{array}{ccc} A & \xrightarrow{\eta} & B \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \tau} \\ C & \xrightarrow{\pi} & D \end{array}$$

in which π (hence also η) is surjective. Then:

(i) $\mathrm{sr}(A) \leq \max\{\mathrm{sr}(B), \mathrm{sr}(C)\}$.

(ii) *If B and C have real rank zero, then A has real rank zero.*

Another application of Theorem 4.1.4 is the computation of the stable rank of a tensor product of the form $A \otimes \mathcal{Z}$ where A has no K_1 obstructions and $\mathcal{Z}$ is the Jiang-Su algebra. This was proved by Sudo in [Sud09, Theorem 1.1] assuming that A has real rank zero, stable rank one, and trivial K_1.

Corollary 4.1.8. *Let A be a C*-algebra with no K_1 obstructions. Then the stable rank of $A \otimes \mathcal{Z}$ is one.*

Proof. As mentioned in the Chapter one, we can write $A \otimes \mathcal{Z}$ as an inductive limit

$$A \otimes \mathcal{Z} = \lim_{i\to\infty} A \otimes Z_{p_i,q_i},$$

with pairs of co-prime numbers (p_i, q_i) and prime dimension drop algebras

$$Z_{p_i,q_i} = \{f \in \mathrm{C}([0,1], M_{p_i} \otimes M_{q_i}) |\, f(0) \in I_{p_i} \otimes M_{q_i},\ f(1) \in M_{p_i} \otimes I_{q_i}\}.$$

Since the stable rank of inductive limit algebras satisfies that $\mathrm{sr}(\lim_{i\to\infty}(A_i)) \leq \liminf \mathrm{sr}(A_i)$ ([Rie83]) it suffices to show that the stable rank of each $Z_{p_i,q_i} \otimes A$ is one.

Fix two co-prime numbers p and q and write $Z_{p,q} \otimes A$ as a pullback

$$\begin{array}{ccc} Z_{p,q} \otimes A & \dashrightarrow & M_p(A) \oplus M_q(A) \\ \downarrow & & \downarrow \phi \\ \mathrm{C}([0,1], M_{pq}(A)) & \xrightarrow{(\lambda_0,\lambda_1)} & M_{pq}(A) \oplus M_{pq}(A) \end{array}$$

with maps $\lambda_i(f) = f(i)$ and $\phi(A, B) = (A \otimes I_q, I_p \otimes B)$.

Our assumptions together with Theorem 4.1.4 imply that $\mathrm{sr}(\mathrm{C}([0,1], M_{pq}(A))) = 1$. Further, $\mathrm{sr}(M_m(A)) = 1$ for all $m \in \mathbb{N}$, and the map from left to right in the pullback diagram is surjective. An application of Theorem 4.1.7 implies that $\mathrm{sr}(Z_{p,q} \otimes A) = 1$. □

By the above result, it is natural to ask the following:

Open Problem 4.1.9. *Is it always true that $\mathrm{sr}(A \otimes \mathcal{Z}) = 1$ if A has stable rank one?*

We now turn our attention to $\mathrm{C}(X, A)$ for compact metric spaces X with $\dim(X) = 1$. In particular, we first study the algebras $\mathrm{C}(X, A)$, where the space X is a finite graph. As a directed graph, write $X = (V, E, r, s)$, where $V = \{v_1, \ldots, v_n\}$ is the set of vertices, $E = \{e_1, \ldots, e_m\}$ is the set of edges, and $r, s : E \to V$ are the range and source maps. For $1 \leq k < m$, denote by $\iota_k : A \to A^m \oplus A^m$ the inclusion in the kth component of the first summand. Likewise, we may define $j_k : A \to A^m \oplus A^m$ for the second summand. Next, define

$$\varphi : \mathrm{C}(V, A) \to A^m \oplus A^m$$

by

$$\varphi(g) = \sum_{l=1}^{n} \Big(\sum_{k \in s^{-1}(v_l)} \iota_k(g(v_l)) + \sum_{k \in r^{-1}(v_l)} j_l(g(v_l)) \Big).$$

Finally, let $\pi_{\{0,1\}} : \mathrm{C}([0,1], A) \to \mathrm{C}(\{0,1\}, A)$ denote the quotient map. Then

$$\mathrm{C}(X, A) \cong \mathrm{C}([0,1], A^m) \oplus_{A^m \oplus A^m} \mathrm{C}(V, A),$$

where $A^m \oplus A^m$ is identified with $\mathrm{C}(\{0,1\}, A)$ in the obvious manner. This means that we have the following diagram:

$$\begin{array}{ccc} \mathrm{C}(X, A) & \dashrightarrow & \mathrm{C}([0,1], A^m) \\ \downarrow & & \downarrow \\ A^n \cong \mathrm{C}(V, A) & \longrightarrow & A^m \oplus A^m \end{array}$$

On the way of computing $\mathrm{sr}(\mathrm{C}(X, A)$ for any compact metric space X with $\dim(X) = 1$, we will use the result that any compact Hausdorff space X that is second countable and one-dimensional can be written as a projective limit $X = \varprojlim(X_i, \mu_{i,j})_{i,j \in \mathbb{N}}$, where X_i are finite graphs and $\mu_{i,j} : X_j \to X_i$, with $i \leq j$, are surjective maps (see [Eng78] pp 153).

Corollary 4.1.10. *Let A be a separable C*-algebra A with no K_1 obstructions, and let X be a compact metric space of dimension one. Then $\mathrm{sr}(\mathrm{C}(X, A)) = 1$.*

Proof. Since X is one-dimensional, we may write X as a (countable) inverse limit of finite graphs, and so, if $\mathrm{sr}(\mathrm{C}(Y, A)) = 1$ when Y is a finite graph, we will have that the stable rank of $\mathrm{C}(X, A)$ is also one.

Let us compute the stable rank $\mathrm{sr}(\mathrm{C}(Y, A))$ when Y is a finite graph. If Y is a finite graph with m edges and V is its set of vertices, by the above constrution if follows that

$$\begin{array}{ccc} \mathrm{C}(Y, A) & \dashrightarrow & \mathrm{C}([0,1], A^m) \, . \\ \downarrow & & \downarrow \\ A^n \cong \mathrm{C}(V, A) & \longrightarrow & A^m \oplus A^m \end{array}$$

It is clear that the two algebras at the bottom of the diagram have stable rank one, and also by Theorem 4.1.4 the algebra in the upper right corner has stable rank one. It then follows from Theorem 4.1.7 that $\mathrm{sr}(\mathrm{C}(Y, A)) = 1$. □

To prove the corresponding result to Theorem 4.1.4 for more general spaces of dimension one, we need to generalize Proposition 4.1.1. This is done in the next Proposition, and its proof is inspired by [NOP01, Proposition 5.2]. (See Proposition 4.1.1.)

Recall that, given a compact metric space X, a continuous map $f\colon X \to [0,1]$ is *essential* if whenever a continuous map $g\colon X \to [0,1]$ agrees with f on $f^{-1}(\{0,1\})$, then g must be surjective. A classical result of Alexandroff shows that if X is one-dimensional space, then there is an essential map from X to $[0,1]$. (A suitable generalization of the above definition can be used to characterize when a space has dimension $\geq n$, see [Eng78].)

Proposition 4.1.11. *Let A be any C*-algebra and X be a compact metric space with $\dim(X) = 1$. If $\mathrm{sr}(\mathrm{C}(X, A)) = 1$, then A has no K_1 obstructions.*

Proof. Since A is a quotient of $\mathrm{C}(X, A)$ it is clear that the stable rank of A must be one. Further, arguing as in the proof of Proposition 4.1.1, it suffices to show that $\mathrm{K}_1(A) = 0$.

Suppose, to reach a contradiction, that there is a unitary u in A not connected to 1. Let d be the metric that induces the topology on X. Since X is one-dimensional, there is an essential map $f\colon X \to [0, 1]$. Let $S = f^{-1}(\{0\})$ and $T = f^{-1}(\{1\})$, which are disjoint closed sets and hence $\mathrm{d}(S, T) > 0$. We may assume that $\mathrm{d}(S, T) = 1$. Now define a continuous function $v\colon X \to A$ as follows:

$$v(x) = (1 - \mathrm{d}(x, T))_+ \cdot u + (1 - \mathrm{d}(x, S))_+ \cdot 1\,.$$

Notice that, by definition, $v_{|S} = 1$ and $v_{|T} = u$. As $\mathrm{C}(X, A)$ has stable rank one, there is a map $w\colon X \to \mathrm{GL}(A)$ such that $\|v - w\| < 1$. Denote by $\mathrm{GL}_0(A)$ the connected component of $\mathrm{GL}(A)$ containing the identity. We have that $S \subseteq w^{-1}(\mathrm{GL}(A))$ and $T \subseteq w^{-1}(\mathrm{GL}(A) \setminus \mathrm{GL}_0(A))$ as $u \notin \mathrm{GL}_0(A)$ by assumption. Note that $\mathrm{GL}_0(A)$ is both open and closed in $\mathrm{GL}(A)$, so by continuity of w we obtain that $S' := w^{-1}(\mathrm{GL}_0(A))$ and $T' := w^{-1}(\mathrm{GL}(A) \setminus \mathrm{GL}_0(A))$ form a partition of X consisting of clopen sets. Thus we can define a (non-surjective) continuous function $h\colon X \to [0, 1]$ such that $h(S') = 0$ and $h(T') = 1$, and this contradicts the essentiality of f.

□

We collect everything for a repetition of the arguments of the proof of Theorem 4.1.4 in a more general setting.

Theorem 4.1.12. *Let A be any C*-algebra with $\mathrm{sr}(A) = 1$ and X be a compact metric space of dimension one. Then*

$$\mathrm{sr}(\mathrm{C}(X, A)) = \begin{cases} 1, & \text{if } A \text{ has no } \mathrm{K}_1 \text{ obstructions} \\ 2, & \text{else.} \end{cases}$$

Although we won't need it in the following, we would like to point out that Theorem 4.1.12 determines the real rank of certain algebras by an application of the well-known inequality stating that $\mathrm{RR}(A) \leq 2\mathrm{sr}(A) - 1$ and [NOP01, Proposition 5.1], which states that $\mathrm{RR}(\mathrm{C}([0, 1]) \otimes A) \geq 1$ for any C*-algebra A.

Corollary 4.1.13. *Let A be C*-algebra with no K_1 obstructions, and let X be a compact metric space of dimension one. Then $\mathrm{RR}(\mathrm{C}(X, A)) = 1$.*

In view of Theorem 4.1.12, it is natural to ask if given a continuous field of C*-algebras A over a one-dimensional space X, all of whose fibers have no K_1 obstructions, is necessarily of stable rank one. And, conversely, if $\mathrm{sr}(A) = 1$ for a continuous field A over a one-dimensional space X implies $\mathrm{K}_1(A_x) = 0$ for all $x \in X$. We answer positively the first named question, but the second question can be answered in the negative, even for $X = [0, 1]$ as we show in Proposition 4.1.18.

Theorem 4.1.14. *Let X be a one-dimensional, compact metric space, and let A be a continuous field over X such that each fiber A_x has no K_1 obstructions. Then $\mathrm{sr}(A) = 1$.*

Proof. As X is metrizable and one-dimensional, we can apply [NS05, Theorem 1.2] to obtain that $\mathrm{sr}(A) \leq \sup_{x \in X} \mathrm{sr}(\mathrm{C}([0, 1], A_x))$. Now the result follows immediately from Theorem 4.1.12. □

Corollary 4.1.15. (cf. [DEN11, Lemma 3.3]) *Let X be a one-dimensional, compact metric space, and let A be a continuous field of* AF *algebras. Then* $\mathrm{sr}(A) = 1$.

Corollary 4.1.16. *Let X be a one-dimensional, compact metric space, and let A be a continuous field of simple* AI *algebras. Then* $\mathrm{sr}(A) = 1$.

We remark that if A is a locally trivial field of C*-algebras with base space the unit interval then it is clear by the methods above that $\mathrm{sr}(A) = 1$ implies that $\mathrm{K}_1(A_x)$ must be trivial for all x. For general continuous fields, this implication is false.

Proposition 4.1.17. *Let $B \subset C$ be C*-algebras with stable rank one such that C has no K_1 obstructions. Let*

$$A = \{ f \in \mathrm{C}([0,1], C) \mid f(0) \in B \} .$$

Then A is a continuous field over $[0,1]$ with stable rank one.

Proof. It is clear that A is a $\mathrm{C}([0,1])$-algebra, which is moreover a continuous field.

Observe that A can be obtained as the pullback of the diagram

$$\begin{array}{ccc} A & \dashrightarrow & B \\ \downarrow & & \big\downarrow i \\ \mathrm{C}([0,1], C) & \xrightarrow{\mathrm{ev}_0} & C \end{array}$$

where ev_0 is the map given by evaluation at 0. Since the rows are surjective, we have by Theorem 4.1.7 that $\mathrm{sr}(A) \leq \max\{\mathrm{sr}(B), \mathrm{sr}(\mathrm{C}([0,1], C))\}$. Because C has no K_1 obstructions, we have $\mathrm{sr}(\mathrm{C}([0,1]), C) = 1$. Since also $\mathrm{sr}(B) = 1$, we obtain that $\mathrm{sr}(A) = 1$ by Theorem 4.1.4. □

Proposition 4.1.18. *There exists a nowhere locally trivial continuous field A over $[0,1]$ such that $\mathrm{sr}(A) = 1$ and $\mathrm{K}_1(A_x) \neq 0$ for a dense subset of $[0,1]$.*

Proof. Let $C = \mathrm{C}(K)$ and $B = \mathrm{C}(\mathbb{T})$, where K denotes the cantor set and $\mathbb{T}$ the unit circle. There exists a continuous surjective map $\pi \colon K \to \mathbb{T}$ ([Hau57, p.226]), so there is an embedding $i \colon B \to C$. Choose a dense sequence $\{x_n\}_n \subset [0,1]$ and define

$$C_n := \{ f \in \mathrm{C}([0,1], C) \mid f(x_n) \in i(B) \} .$$

Since K is zero dimensional, C is an AF-algebra and hence it has no K_1 obstructions. Therefore, C_n is a continuous field over $[0,1]$ of stable rank one by Proposition 4.1.17. Note that $C_n(x_n) \cong B$ which satisfies that $K_1(B) = \mathbb{Z}$. We now proceed as in Example 2.1.2 to obtain a dense subset of such singularities.

Let $A_1 = C_1$, $A_{n+1} = A_n \otimes_{\mathrm{C}[0,1]} C_{n+1}$ and $A = \varinjlim(A_n, \theta_n)$ where $\theta_n(a) = a \otimes 1$ (see [Bla95]). Note that A_n can be described as

$$A_n = \{ f \in \mathrm{C}([0,1], C^{\otimes n}) \mid f(x_i) \in C^{\otimes i-1} \otimes B \otimes C^{\otimes n-i}, i = 1, \ldots, n \} ,$$

and now $\theta_n(f)(x) = f(x) \otimes 1$. Thus, A_n is clearly a continuous field which can moreover be described by the following pullback diagram

$$\begin{array}{ccc} A_n & \dashrightarrow & (B \otimes C^{\otimes n-1}) \oplus \cdots \oplus (C^{\otimes n-1} \otimes B) \\ \downarrow & & \downarrow \\ \mathrm{C}([0,1], C^{\otimes n}) & \xrightarrow{\mathrm{ev}_{x_1,\ldots,x_n}} & C^{\otimes n} \oplus \overset{n}{\cdots} \oplus C^{\otimes n} \end{array}$$

Again, since $C^{\otimes n}$ is an AF algebra it has no K_1 obstructions. Then, a similar argument as that in the proof of Proposition 4.1.17 applies to conclude that A_n has stable rank one. Moreover, A has stable rank one since it is an inductive limit of stable rank one algebras.

Now, for any $x \in [0,1]$, the fiber $A(x)$ can be computed as $\varinjlim A_n(x)$. Hence, if $x \notin \{x_n\}_n$, $A(x) \cong \varinjlim C^{\otimes n} \cong \varinjlim \mathrm{C}(K^n) \cong \mathrm{C}(\varprojlim K^n)$. Since $\varprojlim K^n$ is also zero dimensional, $A(x)$ is an AF-algebra and thus has trivial K_1.

On the other hand, assume $x = x_k \in \{x_n\}_n$. Now for any $n \geq k$, $A_n(x_k) \cong C^{\otimes k-1} \otimes B \otimes C^{\otimes n-k} \cong \mathrm{C}(K^{k-1} \times \mathbb{T} \times K^{n-k})$. An application of the Künneth formula shows that $\mathrm{K}_1(A_n(x_k)) \cong \mathrm{K}_1(C^{\otimes n-1}) \otimes \mathrm{K}_0(B) \oplus \mathrm{K}_0(C^{\otimes n-1}) \otimes \mathrm{K}_1(B)$; therefore, as $C^{\otimes n-1}$ is an AF-algebra and K^{n-1} is a totally disconnected space, it follows from [Rør03, Exercise 3.4] that $\mathrm{K}_0(C^{\otimes n-1}) \cong \mathrm{C}(K^{n-1}, \mathbb{Z})$, so $\mathrm{K}_1(A_n(x_k)) \cong \mathrm{C}(K^{n-1}, \mathbb{Z})$. Furthermore, $\mathrm{K}_1(A(x_k)) \cong \varinjlim \mathrm{C}(K^{n-1}, \mathbb{Z}) \cong \mathrm{C}(\prod_{i=1}^{\infty} K, \mathbb{Z}) \neq 0$. $\square$

4.2 Hereditariness

As mentioned in the beginning of this chapter, here we will provide the study of the classical Cuntz semigroup in categorical terms, and we will show that the classical and stabilized Cuntz semigroup carry the same information under mild assumptions.

In order to describe $\mathrm{W}(A)$, keeping in mind the description of the category Cu given in Chapter 3, a new category called PreCu was introduced in [ABP11], where $\mathrm{W}(A)$ often belongs. Moreover, it was shown in [ABP11, Proposition 4.1] that there exists a functor from PreCu to Cu which is left-adjoint to the identity functor. This functor is basically a completion of semigroups, and, for a wide class of C*-algebras, it sends $\mathrm{W}(A)$ to $\mathrm{Cu}(A) \cong \mathrm{W}(A \otimes \mathcal{K})$. We recall some of the main facts below.

Definition 4.2.1 ([ABP11]). *Let* PreCu *be the category defined as follows. Objects of* PreCu *will be partially ordered abelian semigroups* S *satisfying the properties below:*

(i) *Every element in* S *is supremum of a rapidly increasing sequence.*

(ii) *The relation* $\ll$ *and suprema are compatible with addition.*

Maps of PreCu are semigroup maps preserving

(i) *suprema of increasing sequences (when they exist), and*

(ii) *the relation* $\ll$.

From this point of view Cu is the full subcategory of PreCu whose objects are those partially ordered abelian semigroups (in PreCu) for which every increasing sequence has a supremum.

Definition 4.2.2. *Let S be an object of* PreCu. *We say that a pair (R, ι) is a completion of S if*

(i) *R is an object of* Cu,

(ii) *$\iota : S \to R$ is an order-embedding in* PreCu, *and*

(iii) *for any $x \in R$, there is a rapidly increasing sequence (x_n) in S such that $x = \sup \iota(x_n)$.*

Theorem 4.2.3. ([ABP11]) *Let S be an object of* PreCu. *Then there exists an object $\overline{S}$ of* Cu *and $\iota : S \to \overline{S}$, an order-embedding map in* PreCu, *satisfying that $(\overline{S}, \iota)$ is the completion of S.*

In [ABP11] it was shown that the category PreCu does not always admit sequential inductive limits. To remedy this fact, we define $\mathcal{C}$ as the full subcategory of PreCu whose objects admit suprema of bounded increasing sequences. It is shown in [ABP11] that $\mathcal{C}$ has sequential inductive limits and that the functor from $\mathcal{C}$ to Cu, defined by $S \mapsto \overline{S}$, preserves them.

Examples 4.2.4. *As illustrating examples, observe that $\mathbb{Q}^+$ is an object of* PreCu *but not of $\mathcal{C}$, $\mathbb{R}^+$ is an object of $\mathcal{C}$ but not of* Cu, *and finally $\mathbb{R}^+ \cup \{\infty\}$ is an object of* Cu.

Definition 4.2.5. *Let S and R be partially ordered semigroups. An order-embedding $f : S \to R$ will be called hereditary if, whenever $x \in R$ and $y \in f(S)$ satisfy $x \leq y$, it follows that $x \in f(S)$.*

Proposition 4.2.6. ([ABP11]) *Let S be in* PreCu. *Then the embedding $\iota : S \to \overline{S}$ is hereditary if and only if S is an object of $\mathcal{C}$.*

For a C*-algebra A, we say that $\mathrm{W}(A)$ is hereditary if the embedding $\mathrm{W}(A) \to \mathrm{W}(A \otimes \mathcal{K})$ is hereditary. Note that in this case $\mathrm{W}(A)$ is an object of $\mathcal{C}$ and $\mathrm{Cu}(A) = \mathrm{W}(A \otimes \mathcal{K})$ is order-isomorphic to $\overline{\mathrm{W}(A)}$. We remark that there are no examples known of C*-algebras A for which $\mathrm{W}(A)$ is not hereditary.

We next focus on the hereditariness of the classical Cuntz semigroup. We first state some known results about the hereditary character of certain C*-algebras so that study more concretely the hereditariness of certain continuous fields afterwards.

Lemma 4.2.7. ([ABP11]) *Let A be a C*-algebra with $\mathrm{sr}(A) = 1$. Then $\mathrm{W}(A)$ is hereditary.*

Proof. Let $a \in A \otimes \mathcal{K}_+$ and $b \in M_\infty(A)_+$, and assume that $a \precsim b$. We are to show that there exists $c \in M_\infty(A)_+$ such that $c \sim a$.

Since a can be approximated in norm by a sequence (a_n) of elements from $M_\infty(A)_+$, there is a sequence $\epsilon_n > 0$ decreasing to zero such that $(a - \epsilon_n)_+ \sim b_n$, where $b_n \in M_\infty(A)_+$. As $(a - \epsilon_n)_+$ is increasing and converges in norm to a, it follows that $\langle a \rangle = \sup_{\mathrm{W}(A \otimes \mathcal{K})} \langle b_n \rangle$.

Notice that the sequence $(\langle b_n \rangle)$ is bounded above in $\mathrm{W}(A)$ by $\langle b \rangle$. Therefore, it also has a supremum $\langle c \rangle$ in $\mathrm{W}(A)$, by [BPT08, Lemma 4.3]. The arguments in [BPT08] show that there exists $\delta_n > 0$ with $\delta_n \to 0$ such that $(c - 1/n)_+ \precsim (b_n - \delta_n)_+$. This implies then that

$$(c - 1/n)_+ \precsim (b_n - \delta_n)_+ \precsim b_n \precsim a$$

in $A \otimes \mathcal{K}$, whence $c \precsim a$.

On the other hand, since clearly $b_n \precsim c$ for all n, and $\langle a \rangle$ is the supremum in $\mathrm{W}(A \otimes \mathcal{K})$ of $\langle b_n \rangle$, we see that $a \precsim c$. Thus $c \sim a$. □

Definition 4.2.8. (see e.g. [Tom08b]) *A unital C*-algebra A has r-comparison if whenever $a, b \in M_\infty(A)_+$ satisfy*

$$s(\langle a \rangle) + r < s(\langle b \rangle) \text{ for every } s \in \mathrm{LDF}(A),$$

then $\langle a \rangle \leq \langle b \rangle$. The radius of comparison *of A is $\mathrm{rc}(A) = \inf\{r \geq 0 \mid A$ has r-comparison$\}$, which is understood to be ∞ if the infimum does not exist.*

Theorem 4.2.9. ([BRT+12, Theorem 4.4.1]) *Let A be a unital C*-algebra with finite radius of comparison. Then $\mathrm{W}(A)$ is hereditary.*

Moreover, if A is a unital C*-algebra, the *radius of comparison* of $(\mathrm{Cu}(A), [1_A])$, denoted by r_A, is defined as the infimum of $r \geq 0$ satisfying that if $x, y \in \mathrm{Cu}(A)$ are such that $(n+1)x + m[1_A] \leq ny$ for some n, m with $\frac{m}{n} > r$, then $x \leq y$ ([BRT+12]). We remark that, when $\mathrm{Cu}(A)$ is almost unperforated, the radius of comparison is zero, and the converse holds if A is simple ([BRT+12]). In general, $r_A \leq \mathrm{rc}(A)$, where $\mathrm{rc}(A)$ is the radius of comparison of the algebra, and equality holds if A is residually stably finite (see Proposition 3.2.3 in [BRT+12]).

Note that, in particular, if A is a continuous field over a one-dimensional, compact metric space such that each fiber has no K_1 obstructions, it follows by Theorem 4.1.14 and Lemma 4.2.7 that $\mathrm{W}(A)$ is hereditary.

The following is probably well known. We include a proof for completeness.

Lemma 4.2.10. *Let X be a compact Hausdorff space, and let A be a $\mathrm{C}(X)$-algebra such that A_x has stable rank one for all x. Then A is residually stably finite.*

Proof. Let I be an ideal of A, which is also a $\mathrm{C}(X)$-algebra, as well as is the quotient A/I, with fibers $(A/I)_x \cong A/(\mathrm{C}_0(X \setminus \{x\})A + I)$. As these are quotients of A_x, they have stable rank one, so in particular they are stably finite, and this clearly implies A/I is stably finite. □

Proposition 4.2.11. *Let X be a finite dimensional compact Hausdorff space, and let A be a continuous field over X whose fibers are simple, finite, and $\mathcal{Z}$-stable. Then $\mathrm{W}(A)$ is hereditary.*

Proof. We know from Lemma 4.2.10 that A is residually stably finite, so $\mathrm{rc}(A) = r_A$. We also know from [HRW07, Theorem 4.6] that A itself is $\mathcal{Z}$-stable, whence $\mathrm{Cu}(A)$ is almost unperforated ([Rør04, Theorem 4.5]). Thus $r_A = 0$. This implies that A has radius of comparison zero and then Theorem 4.2.9 applies to conclude that $\mathrm{W}(A)$ is hereditary. □

Remark 4.2.12. In the previous proposition, finite dimensionality is needed to ensure $\mathcal{Z}$-stability of the continuous field. Notice that in the case of a trivial continuous field $A = \mathrm{C}(X, D)$ where D is simple, finite, and $\mathcal{Z}$-stable, the same argument can be applied for arbitrary (infinite dimensional) compact Hausdorff spaces.

Definition 4.2.13. *Let X be a topological space, let S be a semigroup in PreCu. We shall denote the set of bounded lower semicontinuous functions by $\mathrm{Lsc}_{\mathrm{b}}(X, S)$. Note that, if $S \in \mathrm{Cu}$, then $\mathrm{Lsc}(X, S) = \mathrm{Lsc}_{\mathrm{b}}(X, S)$. Furthermore, the set just defined becomes an ordered semigroup when equipped with pointwise order and addition.*

Recall that in Theorem 3.3.12 it is proved that, if X is finite dimensional, compact, metric space and $S \in \mathrm{Cu}$ is countably based, then $\mathrm{Lsc}(X, S)$ is also in Cu. As it turns out from the proof of this fact ([APS11]), every function in $\mathrm{Lsc}(X, S)$ is a supremum of a rapidly increasing sequence of functions, each of which takes finitely many values.

Recall that a compact metric space X is termed *arc-like* provided X can be written as the inverse limit of intervals. Examples of such spaces include the pseudo-arc which is a one-dimensional space that does not contain an arc (see e.g. [Nad92]).

Proposition 4.2.14. *Let X be an arc-like compact metric space, and let A be a unital, simple C*-algebra with stable rank one, and finite radius of comparison. Then $\mathrm{W}(\mathrm{C}(X, A))$ is hereditary.*

Proof. We will prove that $\mathrm{C}(X, A)$ has finite radius of comparison, and then appeal to Theorem 4.2.9. Since Cu is a continuous functor and X is an inverse limit of intervals (I_n), by [APS11, Proposition 5.18] we obtain that $\mathrm{Cu}(\mathrm{C}(X, A)) = \varinjlim \mathrm{Cu}(\mathrm{C}(I_n, A))$. Further, by [ADPS13, Theorem 2.6], we get that $\mathrm{Cu}(\mathrm{C}(I_n, A)) \cong \mathrm{Lsc}(I_n, A)$, so it follows that

$$\mathrm{Cu}(\mathrm{C}(X, A)) \cong \varinjlim \mathrm{Cu}(\mathrm{C}(I_n, A)) \cong \varinjlim \mathrm{Lsc}(I_n, A) \cong \mathrm{Lsc}(X, \mathrm{Cu}(A)).$$

Now, by Lemma 4.2.10, $\mathrm{C}(X, A)$ is residually stably finite, and hence by [BRT+12, Proposition 3.3] $\mathrm{rc}(\mathrm{C}(X, A)) = r_{\mathrm{C}(X,A)}$. Since the order in $\mathrm{Lsc}(X, \mathrm{Cu}(A))$ is the pointwise order, it is easy to verify that $r_{\mathrm{C}(X,A)} \leq r_A$. Note that this is in fact an equality as A is a quotient of $\mathrm{C}(X, A)$ (see condition (i) in [BRT+12, Proposition 3.2.4]). □

4.3 Lower semicontinuous functions and Riesz interpolation

In this section we prove that the Grothendiek group of the Cuntz semigroup of certain continuous fields has Riesz interpolation. In some cases we apply the results on hereditariness from Section 4.2 together with results in either Chapter 2 or [ADPS13], and then if A is simple, unital, ASH, with slow dimension growth, we apply the description of $\mathrm{W}(\mathrm{C}(X, A))$ given in [Tik11].

Recall that we denote the Grothendieck group of $\mathrm{W}(A)$ by $\mathrm{K}_0^*(A)$, and that we say that a partially ordered semigroup $(S, \leq)$ is an *interpolation semigroup* if it satisfies the *Riesz interpolation property*, that is, whenever $a_1, a_2, b_1, b_2 \in S$ are such that $a_i \leq b_j$ for $i, j = 1, 2$, there exists $c \in S$ such that $a_i \leq c \leq b_j$ for $i, j = 1, 2$.

Lemma 4.3.1. *Let S be a semigroup in $\mathcal{C}$. Then S is an interpolation semigroup if and only if its completion $\overline{S}$ is.*

Proof. Assume that S satisfies the Riesz interpolation property and let $a_i \leq b_j$ be elements in $\overline{S}$ for $i, j \in \{1, 2\}$. Denote by $\iota\colon S \to \overline{S}$ the corresponding order-embedding completion map. We may write $a_i = \sup(\iota(a_i^n))$ and $b_j = \sup(\iota(b_j^n))$ for $i, j \in \{1, 2\}$, where (a_i^n) and (b_j^n) are rapidly increasing sequences in S. Find $m_1 \geq 1$ such that $\iota(a_i^1) \leq \iota(b_j^{m_1})$. Then $a_i^1 \leq b_j^{m_1}$ and by the Riesz interpolation property there is $c_1 \in S$ such that $a_i^1 \leq c_1 \leq b_j^{m_1}$. Suppose we have constructed $c_1 \leq \cdots \leq c_n$ in S and $m_1 < \cdots < m_n$ such that $a_i^k \leq c_k \leq b_j^{m_k}$ for each k. Find $m_{n+1} > m_n$ such that $a_i^{n+1}, c_n \leq b_j^{m_{n+1}}$, and by the interpolation property there exists $c_{n+1} \in S$ with $a_i^{n+1}, c_n \leq c_{n+1} \leq b_j^{m_{n+1}}$. Now let $\bar{c} = \sup \iota(c_n) \in \overline{S}$, and it is clear that $a_i \leq \bar{c} \leq b_j$ for all i, j.

Since ι is a hereditary order-embedding, the converse implication is immediate. □

Proposition 4.3.2. *Let $S \in \mathcal{C}$ be countably based, let $(\overline{S}, \iota)$ be its completion, and let X be finite dimensional, compact metric space. Then $\mathrm{Lsc_b}(X, S)$ is an object of $\mathcal{C}$ and $(\mathrm{Lsc}(X, \overline{S}), i)$ is its completion, where i is induced by ι.*

Proof. Notice that $\mathrm{Lsc}(X, \overline{S}) \in \mathrm{Cu}$ and that $i(f) = \iota \circ f$ defines an order-embedding.

Given $f \in \mathrm{Lsc}(X, \overline{S})$, write $f = \sup f_n$, where (f_n) is a rapidly increasing sequence of functions taking finitely many values. Since $f_n \ll f$ and thus $f_n(x) \ll f(x)$ for every $x \in X$, the range of f_n is a (finite) subset of $\iota(S)$. Therefore each f_n belongs to $\mathrm{Lsc_b}(X, S)$. □

Lemma 4.3.3. *Let $S \in \mathrm{Cu}$ satisfying the property that, whenever a_i, b_j $(i, j = 1, 2)$ are elements in S and $a_i \ll b_j$ for all i and j, then, for every $a'_i \ll a_i$, there is $c \in S$ such that $a'_i \ll c \ll b_j$. Then S is an interpolation semigroup.*

Proof. Suppose that $a_i \leq b_j$ in S (for $i, j = 1, 2$). Write $a_i = \sup a_i^n$ and $b_j = \sup b_j^m$, where (a_i^n) and (b_j^m) are rapidly increasing sequences in S. Since $a_i^1 \ll a_i^2 \ll b_j$, there is $m_1 \geq 1$ such that $a_i^2 \ll b_j^{m_1}$. By assumption, there are elements $c_1 \ll c'_1$ in S such that $a_i^1 \ll c_1 \ll c'_1 \ll b_j^{m_1}$. Now, there is $m_2 > m_1$ such that $c'_1, a_i^3 \ll b_j^{m_2}$, so a second application of the hypothesis yields elements $c_2 \ll c'_2$ with $c_1, a_i^2 \ll c_2 \ll c'_2 \ll b_j^{m_2}$. Continuing in this way we find an increasing sequence (c_n) in S whose supremum c satisfies $a_i \leq c \leq b_j$. □

Proposition 4.3.4. *Let S be a countably based, interpolation semigroup in Cu, and let X be finite dimensional, compact metric space. Then $\mathrm{Lsc}(X, S)$ is an interpolation semigroup.*

Proof. We apply Lemma 4.3.3, so assume $f_i \ll g_j$ and $f'_i \ll f_i$ for $i, j = 1, 2$. Now, given $x \in X$ and applying [APS11, Proposition 5.5] (which is a version of Proposition 3.4.8 for the case of $\mathrm{Lsc}(X, S)$), there exists a neighborhood U'_x of x and $c_{i,x} \in S$ such that $f'_i(y) \ll c_{i,x} \ll f_i(y) \ll g_j(y)$ for all $y \in U'_x$. In particular, this will hold for x. Since S is an interpolation semigroup, there is $d_x \in S$ such that $c_{i,x} \ll d_x \ll g_j(x)$ and, by lower semicontinuity of g_j, there is a neighborhood U''_x such that $d_x \ll g_j(y)$ for every $y \in U''_x$. Thus, if $U_x = U'_x \cap U''_x$, we have $f_i(y) \ll d_x \ll g_j(y)$ for all $y \in U_x$.

We may now run the argument in [APS11, Proposition 5.13] to patch the values d_x into a function $h \in \mathrm{Lsc}(X, S)$ that takes finitely many values and $f_i \ll h \ll g_j$, as desired. □

Corollary 4.3.5. *Let S be a countably based semigroup in $\mathcal{C}$ with Riesz interpolation, and let X be finite dimensional, compact metric space. Then $\mathrm{Lsc_b}(X, S)$ has Riesz interpolation.*

Proof. By Lemma 4.3.1 followed by Proposition 4.3.4, the semigroup $\mathrm{Lsc}(X, \overline{S})$ satisfies the Riesz interpolation property, where $\overline{S}$ is the completion of S. On the other hand, by Proposition 4.3.2, $\mathrm{Lsc}(X, \overline{S})$ is the completion of $\mathrm{Lsc_b}(X, S) \in \mathcal{C}$, whence another application of Lemma 4.3.1 yields the conclusion. □

We now apply our results to C*-algebras of the form $\mathrm{C}(X, A)$.

Theorem 4.3.6. *Let X be a compact metric space, and let A be a separable, C*-algebra of stable rank one. Then $\mathrm{K}_0^*(\mathrm{C}(X, A))$ is an interpolation group in the following cases:*

(i) $\dim X \leq 1$, $\mathrm{K}_1(A) = 0$ *and has either real rank zero or is simple and $\mathcal{Z}$-stable.*

(ii) *X is arc-like, A is simple and either has real rank zero and finite radius of comparison, or else is $\mathcal{Z}$-stable.*

(iii) $\dim X \leq 2$ *with vanishing second Čech cohomology group* $\check{\mathrm{H}}^2(X,\mathbb{Z})$, *and A is an infinite dimensional AF-algebra.*

Proof. (i): If A has real rank zero, it was proved in [Per97, Theorem 2.13] that $\mathrm{W}(A)$ satisfies the Riesz interpolation property, and then so does $\mathrm{Cu}(A)$ by Lemma 4.3.1. In the case that A is simple and $\mathcal{Z}$-stable, $\mathrm{Cu}(A)$ is an interpolation semigroup by [Tik11, Proposition 5.4]. Since by Theorem 3.3.13 $\mathrm{Cu}(\mathrm{C}(X,A)$ is order-isomorphic to $\mathrm{Lsc}(X,\mathrm{Cu}(A))$, we obtain, using Proposition 4.3.4, that $\mathrm{Cu}(\mathrm{C}(X,A))$ is an interpolation semigroup in both cases. By Corollary 4.1.10, $\mathrm{C}(X,A)$ has stable rank one, and so $\mathrm{W}(\mathrm{C}(X,A))$ is hereditary, hence also an interpolation semigroup by Lemma 4.3.1. Thus $\mathrm{K}_0^*(\mathrm{C}(X,A))$ is an interpolation group (using Remark 1.2.10).

(ii): By Proposition 4.2.14 and its proof, it follows that $\mathrm{W}(\mathrm{C}(X,A))$ is hereditary and that $\mathrm{Cu}(\mathrm{C}(X,A))$ is order-isomorphic to $\mathrm{Lsc}(X,\mathrm{Cu}(A))$. Now the proof follows the lines of the previous case.

(iii): This follows as above, using [APS11, Corollary 3.6], so that $\mathrm{Cu}(\mathrm{C}(X,A))$ is order-isomorphic to $\mathrm{Lsc}(X,\mathrm{Cu}(A))$, and the proof of Proposition 4.2.14, so that $\mathrm{W}(\mathrm{C}(X,A))$ is hereditary. □

We now turn our consideration to algebras of the form $\mathrm{C}(X,A)$ where A is a unital, simple, non-type I ASH-algebra with slow dimension growth. We say that a C*-algebra A is *subhomogeneous* if there is a finite bound on the dimension of its irreducible representations, and it is approximately subhomogeneous (ASH) if it can be written as a direct limit of subhomogeneous algebras. It is noted below that the condition that a simple, unital ASH algebra A is $\mathcal{Z}$-stable is equivalent to having slow dimension growth and being non-type I. Therefore, its Cuntz semigroup has Riesz interpolation (see below).

Proposition 4.3.7. ([Tik11, Proposition 2.9]) *Let A be a simple, unital, non-type I ASH algebra. Then A has slow dimension growth if and only if A is $\mathcal{Z}$-stable.*

Proposition 4.3.8. ([Tik11, Proposition 5.4]) *Let A be a simple $\mathcal{Z}$-stable C*-algebra. Then* $\mathrm{Cu}(A)$ *has Riesz interpolation.*

In this setting we are able to obtain the same conclusion as above without the necessity to go over proving interpolation of $\mathrm{Cu}(\mathrm{C}(X,A))$. We first need a preliminary result.

Proposition 4.3.9. *Let N be a partially ordered abelian semigroup and let S be an ordered subsemigroup of N such that $S+N\subseteq S$. Then $\mathrm{G}(S)$ and $\mathrm{G}(N)$ are isomorphic as partially ordered abelian groups.*

Proof. Let us denote by $\gamma\colon S\to \mathrm{G}(S)$ and $\eta\colon N\to\mathrm{G}(N)$ the natural Grothendieck maps. Fix $c\in S$, and define $\alpha\colon N\to\mathrm{G}(S)$ by $\alpha(a):=\gamma(a+c)-\gamma(c)$. Using that $S+N\subseteq S$, it is easy to verify that the definition of α does not depend on c. Now, if a, $b\in N$, we have

$$\begin{aligned}\alpha(a+b) &= \gamma(a+b+c)-\gamma(c)=\gamma(a+b+c)+\gamma(c)-2\gamma(c)\\ &=\gamma(a+c+b+c)-2\gamma(c)=(\gamma(a+c)-\gamma(c))+(\gamma(b+c)-\gamma(c))\\ &=\alpha(a)+\alpha(b)\,,\end{aligned}$$

so that α is a homomorphism. It is clear that $\alpha(N) \subseteq \mathrm{G}(S)^+$.

By the universal property of the Grothendieck group, there exists a group homomorphism $\alpha' \colon \mathrm{G}(N) \to \mathrm{G}(S)$ such that $\alpha'(\eta(a) - \eta(b)) = \alpha(a) - \alpha(b)$. Note that α' is injective. Indeed, if $\alpha(a) - \alpha(b) = 0$, then $\gamma(a+c) = \gamma(b+c)$ and so $a + c + c' = b + c + c'$ for some $c' \in S$, and thus $\eta(a) = \eta(b)$.

If $\eta(a) - \eta(b) \in \mathrm{G}(N)^+$ with $b \leq a$ in N, then $b+c \leq a+c$ in S and so $\gamma(a+c) - \gamma(b+c) \in \mathrm{G}(S)^+$. Therefore

$$\begin{aligned}\alpha'(\eta(a) - \eta(b)) &= \alpha(a) - \alpha(b) \\ &= \gamma(a+c) - \gamma(c) - (\gamma(b+c) - \gamma(c)) = \gamma(a+c) - \gamma(b+c)\,,\end{aligned}$$

which shows that $\alpha'(\mathrm{G}(N)^+) \subseteq \mathrm{G}(S)^+$.

Observe that, if $a \in S \subseteq N$, then $\alpha(a) = \gamma(a+c) - \gamma(c) = \gamma(a)$. This implies that any element in $\mathrm{G}(S)$ has the form

$$\gamma(a) - \gamma(b) = \gamma(a+c) - \gamma(b+c) = \alpha'(\eta(a+c) - \eta(b+c))$$

and so α' is surjective and $\alpha'(\mathrm{G}(N)^+) = \alpha(\mathrm{G}(S)^+)$.

□

Given semigroups S and N as above, we will say that S *absorbs* N.

Recall that given a C*-algebra A, we denote by $\mathrm{W}(A)_+$ the classes of those elements in $M_\infty(A)_+$ which are not Cuntz equivalent to a projection. Moreover, if A is unital and has $\mathrm{sr}(A) = 1$, then $\mathrm{W}(A)_+$ absorbs $\mathrm{W}(A)$ (see Corollary 1.3.25). If now X is finite dimensional, compact metric space, define

$$\mathrm{Lsc_b}(X, \mathrm{W}(A)_+) = \{f \in \mathrm{Lsc_b}(X, \mathrm{W}(A)) \mid f(X) \subseteq \mathrm{W}(A)_+\}\,.$$

We remark that it also becomes clear from Corollary 1.3.25 that the set $\mathrm{Lsc_b}(X, \mathrm{W}(A)_+)$ absorbs $\mathrm{Lsc_b}(X, \mathrm{W}(A))$.

Theorem 4.3.10. *Let X be finite dimensional, compact metric space, and let A be a unital, simple, non-type I,* ASH *algebra with slow dimension growth. Then $\mathrm{K}_0^*(\mathrm{C}(X, A))$ is an interpolation group.*

Proof. A description of $\mathrm{W}(\mathrm{C}(X, A))$ for the algebras satisfying the hypothesis is given in [Tik11, Corollary 7.1] by means of pairs (f, P), which consists of a lower semicontinuous function $f \in \mathrm{Lsc_b}(X, \mathrm{W}(A))$, and a collection P, indexed over $[p] \in V(A)$, of projection valued functions in $\mathrm{C}(f^{-1}([p]), A \otimes \mathcal{K})$ modulo a certain equivalence relation. If $f \in \mathrm{Lsc_b}(X, \mathrm{W}(A)_+)$, then clearly $f^{-1}([p]) = \emptyset$ for all $[p] \in V(A)$ thus notably simplifying the description of these elements. Namely, there is only one pair of the form (f, P_0), where P_0 does not depend on $f \in \mathrm{Lsc_b}(X, \mathrm{W}(A)_+)$. In particular, the assignment $f \mapsto (f, P_0)$ defines an order-embedding $\mathrm{Lsc_b}(X, \mathrm{W}(A)_+) \to \mathrm{W}(\mathrm{C}(X, A))$ whose image absorbs $\mathrm{W}(\mathrm{C}(X, A))$. As we also have that the set $\mathrm{Lsc_b}(X, \mathrm{W}(A)_+)$ absorbs $\mathrm{Lsc_b}(X, \mathrm{W}(A))$, we have by Proposition 4.3.9 that

$$\mathrm{K}_0^*(\mathrm{C}(X, A)) \cong \mathrm{G}(\mathrm{Lsc_b}(X, \mathrm{W}(A)_+)) \cong \mathrm{G}(\mathrm{Lsc_b}(X, \mathrm{W}(A)))\,,$$

as partially ordered abelian groups. Since $\mathrm{W}(A)$ is an interpolation semigroup (Proposition 4.3.8), we conclude, using Corollary 4.3.5 and Remark 1.2.10, that $\mathrm{K}_0^*(\mathrm{C}(X, A))$ is an interpolation group. □

We now turn our attention to continuous fields over one-dimensional spaces. For such algebras we use the representation of the Cuntz semigroup computed in terms of continuous sections over a topological space in Chapter 3.

Proposition 4.3.11. *Let X be a one-dimensional, compact metric space, and let $\mathcal{S}\colon \mathcal{V}_X \to \mathrm{Cu}$ be a surjective sheaf such that S_x is an interpolation semigroup for each $x \in X$. Then $\Gamma(X, F_S)$ is also an interpolation semigroup.*

Proof. We apply Lemma 4.3.3, and so suppose that $f_i' \ll f_i \ll g_j$, for $i, j = 1, 2$. Given $x \in X$, there are elements $a_{i,x} \in S_x$ such that $f_i'(x) \ll a_{i,x} \ll f_i(x)$ for each i, and that satisfy condition (ii) in Proposition 3.4.8 . As $a_{i,x} \ll f_i(x)$, there are by continuity a closed neighborhood V_x of x with $x \in \mathring{V}_x$ and $s_i \ll s_i' \in S$ such that $a_{i,x} \ll \hat{s}_i(x)$ and $\hat{s}_i'(y) \ll f_i(y)$ for all $y \in V_x$. Now, by Proposition 3.4.8, there is a closed neighborhood $W_x \subseteq V_x$ (whose interior contains x) such that $f_i'(y) \leq \hat{s}_i(y) \ll \hat{s}_i'(y) \ll f_i(y)$ for all $y \in W_x$.

At x, we have that $f_i(x) \ll g_j(x)$, so by the interpolation property assumed on S_x and Lemma 3.4.9 there are elements $c \ll c'$ in S depending on x such that

$$f_i'(x) \leq \hat{s}_i(x) \ll \hat{s}_i'(x) \ll \hat{c}(x) \ll \hat{c}'(x) \ll g_j(x).$$

Since $c \ll c'$, we may apply Corollary 3.4.10 to find a closed subset $W_x' \subseteq \mathring{W}_x$ such that $\pi_{W_x'}(c) \ll g_{j|W_x'}$ for all j. Since $s_i \ll s_i'$ for each i, another application of Corollary 3.4.10 yields a closed subset $W_x'' \subseteq \mathring{W}_x$ such that $\pi_{W_x''}(s_i') \ll \pi_{|W_x''}(c_x)$ for all i. We therefore conclude that $f_i'(y) \ll \hat{c}_x(y) \ll g_j(y)$ for all $y \in W_x' \cap W_x''$ and for all $i, j \in \{1, 2\}$. By compactness we obtain a finite cover $W_1, \dots, W_n$ of X and elements $c_1, \dots, c_n \in S$ such that $f_i'(y) \ll \hat{c}_i(y) \ll g_j(y)$ for all $y \in W_i$. We now run the argument in Proposition 3.4.15 to patch the sections $\hat{c}_i$ into a continuous section $h \in \Gamma(X, F_S)$ such that $f_i' \ll h \ll g_j$. □

Theorem 4.3.12. *Let X be a one-dimensional, compact metric space. Let A be a continuous field over X such that, for all $x \in X$, A_x has stable rank one, trivial K_1, and is either of real rank zero, or simple and $\mathcal{Z}$-stable. Then $\mathrm{K}_0^*(A)$ is an interpolation group.*

Proof. By Theorem 3.4.19 we have an order-isomorphism between $\mathrm{Cu}(A)$ and $\Gamma(X, F_{\mathrm{Cu}(A)})$, and the latter is an interpolation semigroup by Proposition 4.3.11. Furthermore, A has stable rank one by Theorem 4.1.14, and so $\mathrm{W}(A)$ is hereditary by Lemma 4.2.7. Hence, $\mathrm{W}(A)$ will also be an interpolation semigroup (Lemma 4.3.1) and $\mathrm{K}_0^*(A)$ is an interpolation group by Remark 1.2.10. □

4.4 Structure of dimension functions

In this section we apply the above results to confirm the conjectures of Blackadar and Handelman for certain continuous fields of C*-algebras.

Recall that if A is unital the set of dimension functions is

$$\mathrm{DF}(A) = \mathrm{St}(\mathrm{W}(A), \langle 1_A \rangle) = \mathrm{St}(\mathrm{K}_0^*(A), \mathrm{K}_0^*(A)^+, [1_A]).$$

Theorem 4.4.1. *Let X be a finite dimensional, compact metric space, and let A be a separable, unital C^*-algebra. Then $\mathrm{DF}(A)$ is a Choquet simplex in the following cases:*

(i) $\dim X \leq 1$ *and A is a continuous field such that, for all $x \in X$, A_x has stable rank one, trivial K_1 and is either of real rank zero or else simple and $\mathcal{Z}$-stable.*

(ii) *X is an arc-like space and $A = \mathrm{C}(X, B)$ where B is simple, with real rank zero, and has finite radius of comparison, or else B is simple and $\mathcal{Z}$-stable.*

(iii) $\dim X \leq 2$, $\check{\mathrm{H}}^2(X, \mathbb{Z}) = 0$, *and $A = \mathrm{C}(X, B)$ with B an infinite dimensional* AF-*algebra.*

(iv) *$A = \mathrm{C}(X, B)$, where B is a non-type I, simple,* ASH *algebra with slow dimension growth.*

Proof. By the results of Section 4.3, $\mathrm{K}_0^*(A)$ is an interpolation group in all the cases. Then, by Theorem 1.2.9, $\mathrm{DF}(A)$ is a Choquet simplex. □

Proposition 4.4.2. *Let X and Y be compact Hausdorff spaces. Put*

$$\mathrm{G_b}(X, Y) = \{f \colon X \times Y \to \mathbb{R} \mid f = g - h \text{ with } g, h \in \mathrm{Lsc_b}(X \times Y)^{++}\}\,.$$

Then

(i) *$\mathrm{G_b}(X, Y)$, equipped with the pointwise order, is a partially ordered abelian group.*

(ii) *For any $f \in \mathrm{Lsc_b}(X, \mathrm{Lsc_b}(Y)^{++})$, the map $\tilde{f} \colon X \times Y \to \mathbb{R}^+$, defined by $\tilde{f}(x, y) = f(x)(y)$, is lower semicontinuous.*

(iii) *The map $\beta \colon \mathrm{G}(\mathrm{Lsc_b}(X, \mathrm{Lsc_b}(Y)^{++})) \to \mathrm{G_b}(X, Y)$ defined by*

$$\beta([f] - [g]) = \tilde{f} - \tilde{g}$$

is an order-embedding.

Proof. (i): This is trivial.

(ii): We have to show that the set $U_\alpha = \{(x, y) \mid f(x)(y) > \alpha\}$ is open for all $\alpha > 0$.

Fix $(x_0, y_0) \in U_\alpha$. Since $f(x_0)(y_0) > \alpha$, we may consider $f(x_0)(y_0) > \alpha + \epsilon' > \alpha + \epsilon > \alpha$ for some $\epsilon, \epsilon' > 0$. Since $f(x_0)$ is lower semicontinuous, there exists an open set $V'_{y_0} \subseteq Y$ containing y_0 such that $f(x_0)(y) > \alpha + \epsilon$ for all $y \in V'_{y_0}$. Now, as Y is compact, $f(x_0)$ is bounded away from zero and there is $0 < \epsilon_0 < \alpha$ such that $f(x_0)(y) > \epsilon_0$ for all $y \in Y$.

Let V_{y_0} be an open neighboorhood of y_0 such that $V_{y_0} \subseteq \overline{V}_{y_0} \subseteq V'_{y_0}$. Define $g \in \mathrm{Lsc_b}(Y)^{++}$ by $g(y) = \alpha + \epsilon$ when $y \in V_{y_0}$ and $g(y) = \epsilon_0 < \alpha$ otherwise. Observe that, by the way we have chosen V_{y_0} and the construction of g, for every $y \in Y$, there exists U_y containing y and $\lambda_y \in \mathbb{R}^+$ such that $g(y') \leq \lambda_y < f(x_0)(y')$ whenever $y' \in U_y$. This implies that $g \ll f(x_0)$ in $\mathrm{Lsc_b}(Y)$.

Since f is lower semicontinuous, $\{x \in X \mid f(x) \gg g\}$ is an open set containing x_0. Thus, we may find an open set U_{x_0} such that $x_0 \in U_{x_0}$ and $f(x) \gg g$ for all $x \in U_{x_0}$. Now, for (x, y) in the open set $U_{x_0} \times V_{y_0} \subseteq X \times Y$ we have $\tilde{f}(x, y) = f(x)(y) > g(y) = \alpha + \epsilon > \alpha$.

(iii): We first need to check that β is well-defined. Suppose that $[f] - [g] = [f'] - [g']$ in $\mathrm{G}(\mathrm{Lsc_b}(X, \mathrm{Lsc_b}(Y)^{++}))$. Then there is h such that $f + g' + h = f' + g + h$. Since $h(x)$ is bounded for every x, we obtain $f(x)(y) + g'(x)(y) = f'(x)(y) + g(x)(y)$ for all x and y, and so $f(x)(y) - g(x)(y) = f'(x)(y) - g'(x)(y)$. By (ii), it is clear that $\beta([f] - [g]) \in G_{\mathrm{b}}(X, Y)$, and that it is a group homomorphism. If $[f] - [g] \in \mathrm{G}(\mathrm{Lsc}_{\mathrm{b}}(X, \mathrm{Lsc}_{\mathrm{b}}(Y)^{++}))$, then $g \leq f$, if and only if $g(x)(y) \leq f(x)(y)$ for each x and y, proving that β is an order-embedding. □

The result below provides a good characterization of the set of extreme points of a convex set K. As usual, we denote this set by $\partial_e K$.

Lemma 4.4.3. *Let X be a compact Hausdorff space and let A be a unital C*-algebra. Then there exists a homeomorphism between $\partial_e \mathrm{T}(\mathrm{C}(X, A))$ and $X \times \partial_e \mathrm{T}(A)$. Moreover, if $\tau \in \partial_e \mathrm{T}(\mathrm{C}(X, A))$ corresponds to (x, τ_A), then $d_\tau(b) = d_{\tau_A}(b(x))$ for any $b \in M_\infty(\mathrm{C}(X, A))_+$.*

Proof. Recall that a normalized trace on a unital C*-algebra is extremal if, and only if, the weak closure of its corresponding GNS-representation is a factor (i.e. it has trivial center), see, e.g. [Dix77, Theorem 6.7.3]. Now identify $\mathrm{C}(X, A)$ with $B := \mathrm{C}(X) \otimes A$. Let $\tau \in \partial_e \mathrm{T}(B)$ and let $(\pi_\tau, \mathcal{H}_\tau, v)$ be the GNS-triple associated to τ, and we know that $\pi_\tau(B)''$ is a factor.

Since $\mathrm{C}(X) \otimes 1_A$ is in the center of B, we have that $\pi_\tau(\mathrm{C}(X) \otimes 1_A)$ is in the center of $\pi_\tau(B)''$, whence $\pi_\tau(\mathrm{C}(X) \otimes 1_A) = \mathbb{C}$. Thus, the restriction of π_τ to $\mathrm{C}(X) \otimes 1_A$ corresponds to a point evaluation ev_{x_0} for some $x_0 \in X$.

Next,

$$\tau(f \otimes a) = \langle \pi_\tau(f \otimes a)v, v\rangle = \langle \mathrm{ev}_{x_0}(f)\pi_\tau(1 \otimes a)v, v\rangle = f(x_0)\langle \pi_\tau(1 \otimes a)v, v\rangle = f(x_0)\tau(1 \otimes a)\,,$$

for all $f \in \mathrm{C}(X)$ and $a \in A$. Therefore $\tau = ev_{x_0} \otimes \tau_A$ where τ_A is the restriction of τ to $1 \otimes A$. Note that τ_A is extremal as τ is.

We thus have a map $\psi\colon \partial_e \mathrm{T}(B) \to \partial_e \mathrm{T}(\mathrm{C}(X)) \times \partial_e \mathrm{T}(A)$ defined by $\psi(\tau) = (ev_{x_0}, \tau_A)$, which is easily seen to be a homeomorphism.

Now identify $M_n(\mathrm{C}(X, A))$ with $\mathrm{C}(X, M_n(A))$ and let $b \in \mathrm{C}(X, M_n(A))_+$. Let $\tau \in \partial_e \mathrm{T}(B)$ and $\psi(\tau) = (x, \tau_A)$. Then

$$d_\tau(b) = \lim_{k\to\infty} \tau(b^{1/k}) = \lim_{k\to\infty} \tau_A(b^{1/k}(x)) = d_{\tau_A}(b(x))\,.$$

□

Recall that if K is a compact convex set, we denote by $\mathrm{LAff}_{\mathrm{b}}(K)^{++}$ the semigroup of (real-valued) bounded, strictly positive, lower semicontinuous, and affine functions on K. In particular, this is a subsemigroup of the group $\mathrm{Aff}_{\mathrm{b}}(K)$ of all real-valued, bounded affine functions defined on K. Now, given an exact C*-algebra A, we may define a semigroup homomorphism

$$\varphi\colon \mathrm{W}(A)_+ \to \mathrm{LAff}_{\mathrm{b}}(\mathrm{T}(A))^+\,,$$

where $\mathrm{T}(A)$ is the trace simplex of A, by $\varphi(\langle a\rangle)(\tau) = d_\tau(a)$ (see, e.g. Chapter 1, [APT11], [PT07]). For ease of notation, we shall denote $\varphi(\langle a\rangle) = \hat{a}$. Notice that, if A is simple, then $\hat{a} \in \mathrm{LAff}_{\mathrm{b}}(\mathrm{T}(A))^{++}$ if a is non-zero.

Observe also that there is an ordered morphism $\alpha\colon \mathrm{W}(\mathrm{C}(X, A)) \to \mathrm{Lsc}_{\mathrm{b}}(X, \mathrm{W}(A))$, given by $\alpha(\langle b\rangle)(x) = \langle b(x)\rangle$.

We next use the notion of Bauer simplex. A Choquet simplex is called a *Bauer simplex* if the set of its extreme points is closed. Bauer simplices are also characterized as those simplices K such that every real-valued continuous function on $\partial_e(K)$ can be extended to a (unique) continuous affine function on K ([Bau61]).

Proposition 4.4.4. *Let X be a compact Hausdorff space, and let A be a separable, infinite dimensional, simple, unital and exact C*-algebra with strict comparison and such that $\mathrm{T}(A)$ is a Bauer simplex. Then there is an order-embedding*

$$\mathrm{G}(\mathrm{Lsc}_{\mathrm{b}}(X, \mathrm{W}(A))) \to \mathrm{Aff}_{\mathrm{b}}(\mathrm{T}(\mathrm{C}(X, A)))\,.$$

Moreover, given $b \in \mathrm{C}(X, M_n(A))_+$, this map sends the class of the function $\alpha(\langle b\rangle)$ to $\hat{b}$.

Proof. Since $\mathrm{Lsc}_{\mathrm{b}}(X, \mathrm{W}(A)_+)$ absorbs $\mathrm{Lsc}_{\mathrm{b}}(X, \mathrm{W}(A))$, there is by Lemma 4.3.9 an order-isomorphism between $\mathrm{G}(\mathrm{Lsc}_{\mathrm{b}}(X, \mathrm{W}(A)))$ and $\mathrm{G}(\mathrm{Lsc}_{\mathrm{b}}(X, \mathrm{W}(A)_+))$. In fact, if we take $\langle a\rangle \in \mathrm{W}(A)_+$, and let $v\colon X \to \mathrm{W}(A)_+$ be the function defined as $v(x) = \langle a\rangle$, the previous isomorphism takes $[\alpha(\langle b\rangle)]$ to $[\alpha(\langle b\rangle) + v] - [v]$. Next, as A has strict comparison, the semigroup homomorphism φ defined previous to this proposition is an order-embedding (see [PT07, Theorem 4.4]) and thus induces

$$\mathrm{G}(\mathrm{Lsc}_{\mathrm{b}}(X, \mathrm{W}(A)_+)) \to \mathrm{G}(\mathrm{Lsc}_{\mathrm{b}}(X, \mathrm{LAff}_{\mathrm{b}}(\mathrm{T}(A))^{++}))\,,$$

which is also an order-embedding, and takes $[\alpha(\langle b\rangle) + v] - [v]$ to $[\tilde{\varphi}(\alpha(\langle b\rangle)) + \hat{a}] - [\hat{a}]$, where we identify $\hat{a}$ with a constant function and $\tilde{\varphi}(\alpha(\langle b\rangle))(x) = \widehat{b(x)}$. Now, since $\mathrm{T}(A)$ is a Bauer simplex, the restriction to the extreme boundary yields a semigroup isomorphism $r\colon \mathrm{LAff}_{\mathrm{b}}(\mathrm{T}(A))^{++} \cong \mathrm{Lsc}_{\mathrm{b}}(\partial_e\mathrm{T}(A))^{++}$ (see, e.g. [Goo96, Lemma 7.2]). Combining these observations with condition (iii) in Proposition 4.4.2, we obtain an order-embedding

$$\mathrm{G}(\mathrm{Lsc}_{\mathrm{b}}(X, \mathrm{Lsc}_{\mathrm{b}}(\partial_e\mathrm{T}(A))^{++})) \to \mathrm{G}_{\mathrm{b}}(X, \partial_e\mathrm{T}(A))\,,$$

that sends $[r(\tilde{\varphi}(\alpha(\langle b\rangle)) + \hat{a})] - [r(\hat{a})]$ to $r(\tilde{\varphi}(\alpha(\langle b\rangle)) + \hat{a})^{\sim} - r(\hat{a})^{\sim}$, which equals the function $(x, \tau_A) \mapsto d_{\tau_A}(b(x))$. Finally, upon identifying the compact space $X \times \partial_e\mathrm{T}(A)$ with $\partial_e\mathrm{T}(\mathrm{C}(X, A))$ (by Lemma 4.4.3), a second use of [Goo96, Lemma 7.2] allows us to order-embed $\mathrm{G}_{\mathrm{b}}(X, \partial_e\mathrm{T}(A))$ into $\mathrm{Aff}_{\mathrm{b}}(\mathrm{T}(\mathrm{C}(X, A)))$, and the map $(x, \tau_A) \mapsto d_{\tau_A}(b(x))$ is sent to $\hat{b}$, as desired. □

Theorem 4.4.5. *Let X be a finite dimensional, compact metric space, and let A be a unital, separable infinite dimensional and exact C*-algebra of stable rank one such that $\mathrm{T}(A)$ is a Bauer simplex. Then $\mathrm{LDF}(\mathrm{C}(X, A))$ is dense in $\mathrm{DF}(\mathrm{C}(X, A))$ in the following cases:*

(i) $\dim X \leq 1$ *and A is simple, $\mathrm{K}_1(A) = 0$ and A has strict comparison.*

(ii) *X is arc-like, A is simple, has real rank zero, and strict comparison.*

(iii) $\dim X \leq 2$ *and $\check{\mathrm{H}}^2(X, \mathbb{Z}) = 0$, with A an infinite dimensional* AF*-algebra.*

(iv) *A is a non-type I, simple, unital* ASH *algebra with slow dimension growth.*

Proof. (i): By Theorem 3.3.13, one has that $\mathrm{Cu}(\mathrm{C}(X,A))$ and $\mathrm{Lsc}(X,\mathrm{Cu}(A))$ are order-isomorphic. Moreover, $\mathrm{W}(\mathrm{C}(X,A))$ is hereditary by Theorem 4.1.12 and Lemma 4.2.7. By the hereditariness of $\mathrm{W}(\mathrm{C}(X,A))$ and Proposition 4.3.2, it follows that $\mathrm{W}(\mathrm{C}(X,A))$ is order-isomorphic to $\mathrm{Lsc}_{\mathrm{b}}(X,\mathrm{W}(A))$.

Now, by Proposition 4.4.4 we obtain that $\mathrm{K}_0^*(\mathrm{C}(X,A))$ is order-isomorphic to a (pointwise ordered) subgroup G of $\mathrm{Aff}_{\mathrm{b}}(\mathrm{T}(\mathrm{C}(X,A)))$ in such a way that $[b]$ is mapped to $\hat{b}$, and in particular $[1]$ is sent to the constant function 1.

Applying the same argument as in [BPT08, Theorem 6.4], which we next sketch, we get the desired result. If $d \in \mathrm{DF}(\mathrm{C}(X,A))$, then it can be identified with a normalized state (at 1) on G. By [BPT08, Lemma 6.1], there is a net of traces (τ_i) in $\mathrm{T}(\mathrm{C}(X,A))$ such that $d(s) = \lim_i s(\tau_i)$ for any $s \in G$. In particular, $d([b]) = \lim_i \hat{b}(\tau_i) = \lim_i d_{\tau_i}(b)$ for $b \in M_\infty(\mathrm{C}(X,A))_+$.

(ii): This case uses the same arguments as (i), replacing Theorem 3.3.13 by Proposition 4.2.14 and its proof.

(iii): Proceed as in case (i), using [APS11, Corollary 3.6] instead of Theorem 3.3.13 and Remark 4.2.12.

(iv): As in the proof of Theorem 4.3.10, we see that $\mathrm{K}_0^*(\mathrm{C}(X,A)) \cong \mathrm{G}(\mathrm{Lsc}_{\mathrm{b}}(X,\mathrm{W}(A)))$ as ordered groups, and then we may use the same argument as in case (i). □

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