# THE TYPE SEMIGROUP, COMPARISON AND ALMOST FINITENESS FOR AMPLE GROUPOIDS

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Abstract. In this article, we prove that a minimal ample groupoid has dynamical comparison if and only if its type semigroup is almost unperforated. Moreover, we investigate to what extent a not necessarily minimal almost finite groupoid has an almost unperforated type semigroup. Finally, we build a bridge between coarse geometry and topological dynamics by characterizing almost finiteness of the coarse groupoid in terms of a new coarsely invariant property for metric spaces, which might be of independent interest in coarse geometry. As a consequence, we are able to construct new examples of almost finite principal groupoids lacking other desirable properties, such as amenability or even a-T-menability. This behaviour is in stark contrast to the case of principal transformation groupoids associated to group actions.

## Introduction

The type semigroup is a new invariant for ample groupoids introduced by the second and fourth author in [8] and independently in [28]. This semigroup has recently been of significant interest for both the role it plays in the study of finitely generated conical refinement monoids [2], as well as its connection to the structure theory of the associated reduced groupoid C$^*$-algebra. In particular, the following dichotomy result was proved in [8, 28]: If the type semigroup $S(G)$ of a minimal topologically principal ample groupoid $G$ with compact unit space is almost unperforated, then its reduced groupoid $C^*$-algebra $C^*_r(G)$ is a simple $C^*$-algebra, which is either stably finite or strongly purely infinite.

Consequently, it is a natural question to ask for conditions under which the type semigroup is almost unperforated. This is indeed the situation for all the monoids described in [2]. However, one can also build groupoids whose type semigroup is not almost unperforated via the (usually non-amenable) groupoids associated to the separated graphs defined in [3].

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The first main result of this article is a dynamical analogue of a celebrated result by Rørdam in [30] on the equivalence between strict comparison and almost unperforation of the Cuntz semigroup for unital simple separable exact C*-algebras:

**Theorem A.** *Let $G$ be a second countable minimal ample groupoid. Then $G$ has dynamical comparison if and only if its type semigroup $S(G)$ is almost unperforated.*

The novelty here lies in a rather elementary approach, which allows us to drop any freeness or amenability assumptions that were crucial in previous attempts to prove such a result for transformation groups [19, 24]. A great range of examples has been constructed in [12], where the authors prove that every action of a countable group with local subexponential growth on a zero dimensional compact metric space has dynamical comparison.

We then study dynamical comparison and almost unperforation of the type semigroup in the context of other important structural properties of the groupoid. In contrast to the above result, we do not limit ourselves to the minimal case and investigate two different situations: In the infinite case, i.e. when there are no non-trivial invariant measures on the base space, we show dynamical comparison is equivalent to pure infiniteness of the groupoid, extending earlier results of Ma [24].

On the other end of the spectrum, we consider almost finite (not necessarily minimal) groupoids as introduced by Matui in [25]. In [19], David Kerr specialises to almost finite group actions and proposes that almost finiteness might play a role in topological dynamics analogous to the role $\mathcal{Z}$-stability does for simple C*-algebras. In particular, he shows that almost finiteness always implies almost unperforation of the type semigroup. Moreover, in the subsequent work [20], Kerr and Szabó prove that free actions of amenable groups on compact metrizable spaces are almost finite if and only if the action has comparison and the small boundary property.

Studying almost unperforation for the type semigroup of non-minimal almost finite groupoids leads to new complications. The main obstacle is the different behaviour of almost finiteness and almost unperforation when passing to open invariant subsets of the base space. To circumvent this problem, we call a groupoid $G$ strongly almost finite if every restriction of $G$ by a compact open subset of the base space is almost finite in the sense of Matui. In particular, when the groupoid is minimal, strong almost finiteness agrees with almost finiteness. Our second main result is:

**Theorem B.** *If $G$ is a second countable strongly almost finite ample groupoid, then its type semigroup $S(G)$ is almost unperforated.*

Finally, we establish a new link between regularity properties in topological dynamics (which are in turn inspired by their counterparts in the structure theory of nuclear C*-algebras) and coarse geometry. Inspired by recent results on the structure of amenable groups in [11], we introduce a strong version of amenability for metric spaces, which asserts that the space can be tiled by uniformly bounded Følner sets of arbitrary invariance (see Definition 4.3). To explain the promised connection to the first part of this article, recall that for every discrete metric space $X$ with *bounded geometry* (i.e., for any $R > 0$ there is a uniform upper bound on the cardinalities of all the $R$-balls in $X$), we can associate an ample groupoid $G(X)$, called the *coarse groupoid* of $X$. It is well-known that this groupoid reflects many interesting properties in coarse geometry. For instance, $X$ has Yu's property A if and only if $G(X)$ is an amenable groupoid (see [33, Theorem 5.3]).

We show that our new tiling property is invariant under coarse equivalence and provide a link to the main results of this article by proving:

**Theorem C** (see Theorem 4.5)**.** *Let $X$ be a bounded geometry metric space and $G(X)$ be its coarse groupoid. Then the following are equivalent:*

*(1) $G(X)$ is almost finite,*
*(2) $X$ admits tilings of arbitrary invariance.*

*In particular, $G(X)$ is strongly almost finite if and only if every subspace of $X$ admits tilings of arbitrary invariance.*

We use this result to provide a range of new examples of almost finite groupoids. It allows us to construct groupoids that exhibit a behaviour which cannot be witnessed in the setting of transformation groupoids. In particular, we elaborate on the subtle relationship between almost finiteness and amenability and provide new examples of principal almost finite groupoids which are non-amenable, in fact not even a-T-menable. This answers a query of Yuhei Suzuki (see [34, Remark 3.7])[1]. While for the purposes of this article we only use our new tiling property for metric spaces to obtain interesting examples of groupoids, we believe that it might be of independent interest in coarse geometry.

We briefly outline the contents of this paper. In section 1, we recall the necessary definitions concerning groupoids, their type semigroups, and their connection to groupoid homology. In the second section, we study dynamical comparison, and its relation to almost unperforation of $S(G)$. The main result obtained in this section is Theorem A. In section 3 we focus our study on almost finite groupoids. In particular, in order to ease reading it, we have divided this part in two subsections. In the first, we recall the definition of almost finiteness and establish that it is invariant under stable isomorphism. And, in the second subsection, we prove Theorem B, and describe some implications on the relation between the type semigroup and the positive cone of the integral group $H_0(G)$. We finish the paper with section 4, in which we introduce our new tiling property for metric spaces and prove Theorem C. We use it to provide new examples of almost finite groupoids and in particular construct non-amenable almost finite groupoids in Corollary 4.12. Finally, we use some of the methods developed in this article to give a short and conceptual proof of a classical result by Block and Weinberger, characterizing (non-)amenability of metric spaces in terms of uniformly finite homology (see Corollary 4.18).

## 1. Preliminaries and Type semigroup

Let us start reviewing the terminology and notation related to groupoids that we will use throughout the text. Given a groupoid $G$ we will denote its unit space by $G^{(0)}$ and write $r, s : G \to G^{(0)}$ for the range and source maps, respectively. **Throughout the paper, all groupoids are always assumed to be equipped with a locally compact, Hausdorff**

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[1] Non-amenable minimal almost finite groupoids are independently constructed by Gabor Elek in [13].

**topology making all the structure maps continuous.** A groupoid $G$ is called *étale* if the range map, regarded as a map $r : G \to G$, is a local homeomorphism. It is called *ample* if it is étale and the unit space $G^{(0)}$ is totally disconnected. In that case $G$ admits a basis for its topology consisting of compact and open *bisections*, i.e. compact and open subsets $V \subseteq G$ such that the restrictions of the source and range maps to $V$ are homeomorphisms onto their respective images. For two subsets $A, B \subseteq G$ we will consider their product

$$AB = \{ab \in G \mid a \in A, b \in B, s(a) = r(b)\}.$$

If $B = \{x\}$ for a single element $x \in G^{(0)}$ we will omit the braces and just write $Ax$.

For a subset $D \subseteq G^{(0)}$ we let $G_D = \{g \in G \mid s(g), r(g) \in D\}$ denote the restriction of $G$ to $D$. Note that $G_D$ is always a subgroupoid of $G$. We say that the set $D$ is $G$-invariant if for every $g \in G$ we have $r(g) \in D \Leftrightarrow s(g) \in D$, and we say that $D$ is $G$-*full* if it satisfies that $r(GD) = G^{(0)}$.

A groupoid $G$ is called *minimal* if there are no proper non-trivial closed $G$-invariant subsets of $G^{(0)}$. The isotropy groupoid of $G$ is the subgroupoid $\mathrm{Iso}(G) = \{g \in G \mid s(g) = r(g)\}$, and we say that $G$ is *principal* if $\mathrm{Iso}(G) = G^{(0)}$. Finally, $G$ is called *effective* if the interior of $\mathrm{Iso}(G)$ coincides with $G^{(0)}$. This is connected with the notion of *topologically principal*, which means that the set of points of $G^{(0)}$ with trivial isotropy group is dense in $G^{(0)}$. If $G$ is second countable and effective, then $G$ is topologically principal. If $G$ is Hausdorff and topologically principal, then $G$ is effective ([29, Proposition 3.6]).

### 1.1. **Type semigroup.**

The type semigroup of an ample groupoid was introduced and studied in [8, 28]. In this section, we recall its definition and study some of its basic properties.

**Definition 1.1.** Given an ample groupoid $G$, we define an equivalence relation $\sim_G$ on $C_c(G^{(0)}, \mathbb{Z})^+$ by declaring $f_1 \sim_G f_2$ if there exist compact open bisections $W_1, \ldots, W_n$ of $G$ such that $f_1 = \sum_{i=1}^n 1_{s(W_i)}$ and $f_2 = \sum_{i=1}^n 1_{r(W_i)}$. We define the type semigroup associated to $G$ by

$$S(G) := C_c(G^{(0)}, \mathbb{Z})^+ / \sim_G .$$

We will write $[f]$ for the equivalence class of a function $f \in C_c(G^{(0)}, \mathbb{Z})^+$, and equip $S(G)$ with the addition induced by pointwise addition in $C_c(G^{(0)}, \mathbb{Z})^+$. In particular, $S(G)$ contains the class of the zero function as a neutral element and can be equipped with the algebraic preorder.

The type semigroup is clearly an isomorphism invariant for groupoids and it was shown in [28] that it is also invariant under all the various (equivalent notions) of groupoid equivalence. This observation will be important later.

Recall, that a commutative monoid $S$ is called *conical*, if for all $x, y \in S$, $x + y = 0$ only when $x = y = 0$. We say that $S$ is a *refinement monoid* if for all $a, b, c, d \in S$ such that $a + b = c + d$ there exist $w, x, y, z \in S$ such that $a = w + x$, $b = y + z$, $c = w + y$, and $d = x + z$. It is straightforward to verify, that $S(G)$ is always a conical refinement monoid.

An important part of the structure of a preordered monoid $S$ is the collection of its order units. Recall, that a non-zero element $u \in S$ is called an *order unit*, provided that for every $x \in S$ there exists $n \in \mathbb{N}$ such that $x \leq nu$. We will write $S^*$ for the collection of all order units in $S$. The monoid $S$ is called *simple*, provided that every non-zero element of $S$ is an order unit, in other words $S = S^* \cup \{0\}$. It has already been observed in [8, Lemma 5.9] that the type semigroup $S(G)$ of an ample groupoid $G$ is simple, provided that $G$ is minimal. The following Lemma extends this observation by identifying all the order units:

**Lemma 1.2.** *Let $G$ be an ample groupoid and $[f] \in S(G)$. Then $[f]$ is an order unit if and only if $supp(f)$ is $G$-full. In particular, $G$ is minimal if and only if $S(G)$ is simple.*

*Proof.* Suppose $[f] \in S(G)$ is an order unit. Let $x \in G^{(0)}$ and $K$ be a compact open set containing $x$. Then there exists some $n \in \mathbb{N}$ such that $[1_K] \leq n[f]$. Let $m \in \mathbb{Z}^+$ be the maximal value attained by $f$. Then clearly $[1_K] \leq [nm1_{supp(f)}]$ and hence there exist compact open bisections $V_1, \ldots, V_k$ such that $K = \bigsqcup_{i=1}^k r(V_i)$ and $supp(f) \supseteq \bigcup_{i=1}^k s(V_i)$. It follows that $r(Gsupp(f)) \supseteq K$ and we are done. For the converse we may proceed as in the proof of [8, Lemma 5.9]. □

Let us also identify the order ideals of the type semigroup. Recall, that an *ideal* of a monoid $S$ is a submonoid $I$ such that for all $x, y \in S$, we have $x + y \in I$ if and only if $x, y \in I$.

**Lemma 1.3.** *Let $G$ be an ample groupoid. If $I$ is an order ideal in $S(G)$, then there exists an open invariant subset $U \subseteq G^{(0)}$, such that $I \cong S(G_U)$.*

*Proof.* Suppose $I$ is an order ideal in $S(G)$. Then $U := \bigcup\{supp(f) \mid [f] \in I\}$ is an open subset of $G^{(0)}$. To see that it is invariant, let $g \in G$ such that $s(g) \in U$. Then there exists $[f] \in I$ such that $s(g) \in supp(f)$. Now $f$ can be written as $f = \sum_i 1_{A_i}$ for suitable clopen sets $A_i$ and $s(g)$ must be contained in one of these. Since $I$ is an ideal, each $[1_{A_i}] \in I$. Now let $V$ be a compact open bisection containing $g$ such that $s(V) \subseteq A_i$. Upon refining the representation of $f$ if necessary, we may assume $s(V) = A_i$. Since $[1_{r(V)}] = [1_{A_i}] \in I$ we get $r(g) \in U$.

Now let $J$ denote the ideal of $S(G)$ generated by all the elements of $S(G)$ which can be represented by a function whose support is contained in $U$. Then we clearly have $I \subseteq J$. For the converse inclusion take any $[f] \in S(G)$ such that $supp(f) \subseteq U$. Since the support of $f$ is compact, we may find finitely many functions $f_1, \ldots, f_n$ such that $[f_i] \in I$ with $supp(f) \subseteq \bigcup supp(f_i)$. In particular, we have $f \leq \sum_i n_i f_i$ for suitably large $n_i \in \mathbb{N}$ and hence $[f] \leq \sum_i n_i [f_i] \in I$. Since $I$ is an order ideal, this implies $[f] \in I$ as desired. Clearly, we have $J \cong S(G_U)$. □

Once we have an order ideal $I$ in a monoid $S$ one can define a congruence on $S$ by declaring $x \sim y$ if there exist elements $a, b \in I$ such that $x + a = y + b$. Then $S/I := S/\sim$ can be equipped canonically with a monoid structure induced by $S$. To identify the quotients of the type semigroup, note that the set $G^a$ of all compact open bisections of an ample Hausdorff groupoid $G$ forms a Boolean inverse semigroup and the type semigroup $S(G)$ can be canonically identified with the type monoid $\mathrm{Typ}(G^a)$ of this inverse semigroup (see [2, Proposition 7.3]).

We shall also need the following construction: For two compact open bisections $V_1, V_2$ in $G$ let $E = s(V_1) \setminus (s(V_1) \cap s(V_2))$ and $F = r(V_2) \setminus (r(V_1) \cap r(V_2))$ and define

$$V_1 \bigtriangledown V_2 = EV_1F \cup V_2.$$

Then $V_1 \bigtriangledown V_2$ is a compact open bisection in $G$.

The proof of the following result is essentially contained in [21, Lemma 5.5]. We spell out a sketch of the proof for the readers convenience.

**Proposition 1.4.** *Let $I \subseteq S(G)$ be an order ideal. If $U$ is the corresponding open $G$-invariant subset of the unit space and $D = G^{(0)} \setminus U$ its complement, then the canonical map $S(G) \to S(G_D)$ induced by restriction of functions gives rise to an isomorphism $S(G)/I \cong S(G_D)$.*

*Proof.* Upon identifying $S(G)$ with $\mathrm{Typ}(G^a)$, the result follows from [38, Theorem 4.3.2] once we realize that the canonical semigroup homomorphism $G^a \to (G_D)^a$ is surjective. To see this proceed as follows: If $V \subseteq G_D$ is a compact open bisection, then by definition of the induced topology and using the fact that being compact does not depend on the ambient space, we can find finitely many compact open bisections $U_1, \ldots, U_n$ in $G$ such that $V = \bigcup_i U_i \cap G_D$. Then $\bigtriangledown_i U_i$ is a compact open bisection in $G$ such that $(\bigtriangledown_i U_i) \cap G_D = V$. □

1.2. **Groupoid homology and its relation with $S(G)$.** Let us now turn our attention to understand the relationship between the type semigroup of a ample groupoid and the positive cone $H_0(G)^+$ of the integral group $H_0(G)$. We refer the reader to [25, Section 3] for the relevant definitions. The relevant property here is cancellation: Recall that we say that a semigroup $S$ is *cancellative* if for $a, b, c, \in S$ satisfying $a + c = b + c$, it follows that $a = b$.

**Lemma 1.5.** *Let $G$ be an ample groupoid with compact unit space. Then the quotient map $C(G^{(0)}, \mathbb{Z}) \to H_0(G)$ induces a surjective semigroup homomorphism*

$$S(G) \to H_0(G)^+$$

*Proof.* We need to show that the map is well-defined. Suppose $f, g \in C(G^{(0)}, \mathbb{Z})^+$ such that $f \sim g$ in $S(G)$. We will show that $f - g \in im(\partial_1)$, where $\partial_1 : C_c(G, \mathbb{Z}) \to C(G^{(0)}, \mathbb{Z})$ is the differential map from the chain complex defining groupoid homology. This immediately implies $[f] = [g]$ in $H_0(G)^+$. Since $f \sim g$ in $S(G)$ we can find bisections $V_1, \ldots, V_n$ such that

$$f = \sum_{i=1}^{n} 1_{s(V_i)} \text{ and } g = \sum_{i=1}^{n} 1_{r(V_i)}.$$

Let $h := \sum_{i=1}^{n} 1_{V_i} \in C_c(G, \mathbb{Z})$. Then

$$\partial_1(h) = \sum_{i=1}^{n} s_*(1_{V_i}) - r_*(1_{V_i}) = \sum_{i=1}^{n} 1_{s(V_i)} - 1_{r(V_i)} = f - g$$

as desired. □

Before the next result, let us recall the construction of the universal cancellative abelian semigroup. Let $S$ be an abelian semigroup with $0 \in S$, and consider the equivalence relation on $S$ given by $x \sim y$ if there exists an element $z \in S$ such that $x + z = y + z$. Then $\sim$ is an equivalence relation and $C(S) := S/\sim$ is a cancellative abelian semigroup with the (universal) property, that for every homomorphism $\Phi : S \to P$ into a cancellative abelian semigroup $P$ there exists a unique homomorphism $C(\Phi) : C(S) \to P$ such that $C(\Phi)([s]) = \Phi(s)$.

**Proposition 1.6.** *Let $G$ be an ample groupoid with compact unit space. Then the canonical map $S(G) \to H_0(G)^+$ induces an isomorphism of cancellative abelian semigroups.*

$$C(S(G)) \to H_0(G)^+$$

*Proof.* By universality, one can build a well-defined surjective homomorphism $C(S(G)) \to H_0(G)^+$. Hence, it remains to check its injectivity. Let $f, g \in C(G^{(0)}, \mathbb{Z})^+$ such that $[f] = [g]$ in $H_0(G)^+$. Then $f - g \in im(\partial_1)$, i.e. there exists a function $h \in C_c(G, \mathbb{Z})$ such that

$$f - g = \partial_1(h) = s_*(h) - r_*(h)$$

This implies that $f + r_*(h) = g + s_*(h)$. Since $h$ is compactly supported and $G$ is ample, we can write $h = \sum_{i=1}^{m} 1_{V_i}$ for appropriately chosen compact open bisections $V_1, \ldots, V_m$, which implies that $r_*(h) \sim s_*(h)$. This concludes the proof since if $x := [s_*(h)] = [r_*(h)]$, then $[f] + x = [g] + x$ in $S(G)$ and hence $[f] = [g]$ in $C(S(G))$. □

## 2. Dynamical comparison

In this section we study the relation between almost unperforation of the type semigroup and dynamical comparison, an important regularity property. Since our definition is rather general and in particular not limited to minimal groupoids we need to recall some facts about (possibly infinite) Borel measures for locally compact Hausdorff spaces.

For a topological space $X$, we denote by $UM(X)$ the cone of positive Borel measures on $X$. For a given Borel subset $B$ of $X$, the convex subset $UM(X, B) \subseteq UM(X)$ consists of those $\mu \in UM(X)$ such that $\mu(B) = 1$. If $X$ is further locally compact and Hausdorff, we denote by $UM_c(X)$ the cone of all the positive regular Borel measures $\mu$ on $X$ such that $\mu(K) < \infty$ for all compact sets $K$ of $X$. By [32], if $X$ is in addition $\sigma$-finite, then $UM_c(X)$ can be identified with the positive part of the dual space of the space $C_c(X)$. Finally, if $X$ is compact, we will denote by $M(X)$ the compact convex set of all the positive regular Borel probability measures on $X$, which is isomorphic to the positive part of the unit ball of the dual of $C(X)$.

Now let $G$ be an étale groupoid (so that $G^{(0)}$ is a locally compact Hausdorff space), and recall that a Borel measure $\mu$ on $G^{(0)}$ is called *$G$-invariant* if $\mu(s(V)) = \mu(r(V))$ for every open bisection $V \subseteq G$. Slightly abusing notation, we write $UM(G)$ for the subcone of $UM(G^{(0)})$ of all the invariant positive Borel measures on $G^{(0)}$. Similarly, we will write $UM_c(G)$ for the subcone of $UM_c(G^{(0)})$ consisting of all the invariant positive regular Borel measures $\mu$ on $X$ such that $\mu(K) < \infty$ for all compact subsets $K$ of $G^{(0)}$. If in addition $G^{(0)}$ is compact, we denote by $M(G)$ the compact convex set of invariant positive regular Borel probability measures on $G^{(0)}$.

We now introduce a version of dynamical comparison which also works in the non-minimal case.

**Definition 2.1.** Let $G$ be an ample groupoid. For two compact open subsets $A, B \subseteq G^{(0)}$ we say that $A$ *is subequivalent to* $B$ and write $A \precsim B$, if there exist finitely many compact open bisections $V_1, \ldots, V_n$ of $G$ such that $A = \bigsqcup_{i=1}^{n} s(V_i)$ and the sets $\{r(V_i)\}_{i=1}^{n}$ are pairwise disjoint subsets of $B$.

We say that $G$ has *dynamical comparison* if for all nonempty compact open subsets $A, B \subseteq G^{(0)}$ such that $A \subset r(GB)$ and satisfying $\mu(A) < \mu(B)$ for every $\mu \in UM(G)$ such that $0 < \mu(B) < \infty$, we have $A \precsim B$. We say that $G$ has *stable dynamical comparison* if $G^m$ has dynamical comparison for all $m \geq 1$, where $G^m$ denotes $G \times \mathcal{R}_m$, where $\mathcal{R}_m = \{1, \dots, m\}^2$.

**Remark 2.2.** If $G$ is a *minimal* ample groupoid *with compact unit space* $G^{(0)}$, then $G$ has dynamical comparison if and only if for all nonempty clopen subsets $A, B \subseteq G^{(0)}$ satisfying $\mu(A) < \mu(B)$ for every $\mu \in M(G)$, then $A \precsim B$. Hence, it follows from [19, Proposition 3.6] that our notion of dynamical comparison generalizes Kerr's dynamical comparison at least in the ample case.

We now come back to the type semigroup $S(G)$, which is a useful tool to study dynamical comparison. This is due to the fact that dynamical subequivalence $A \precsim B$ translates to the inequality $[1_A] \leq [1_B]$ in the type semigroup.

Moreover, the invariant Borel measures on the base space can be canonically identified with certain functionals on the type semigroup:

For a preordered monoid $(S, +, \leq)$ we denote by $F(S)$ the set of all unnormalized states on $S$, that is the set of all the monoid homomorphisms $S \to [0, \infty]$. Note that $F(S)$ is a cone, i.e., we can sum and multiply by positive real numbers. If $x \in S$, we define the set of states on $S$ which are normalized at $x$ as $F(S, x) = \{f \in F(S) : f(x) = 1\}$. This set might be empty, but in any case it is a convex subset of $F(S)$.

In our setting, letting $G$ be an ample groupoid and $\mathbb{K}$ be the ring of open compact subsets of $G^{(0)}$, the type semigroup $S(G)$ can also be defined (see [2, Proposition 7.3]) as the commutative monoid with generators $[U]$ for each $U \in \mathbb{K}$ subject to the relations:

(1) $[\emptyset] = 0$,
(2) $[A \cup B] = [A] + [B]$ if $A, B \in \mathbb{K}$ and $A \cap B = \emptyset$,
(3) $[s(V)] = [r(V)]$ for each open compact bisection $V$ of $G$.

With this description, it is obvious that $F(S(G))$ is the set of all the finitely additive invariant positive measures on $\mathbb{K}$.

We can now extend [31, Lemma 5.1] to groupoids as follows.

**Lemma 2.3.** *Let $G$ be an ample second countable groupoid. Then each $f \in F(S(G))$ can be extended to a Borel invariant measure $\mu_f \in UM(G)$. Moreover, the restriction of the measure $\mu_f$ to the open set $V := \bigcup U$, where $U$ ranges over all the open compact subsets of $G^{(0)}$ such that $f([1_U]) < \infty$, is unique and regular.*

*Proof.* The set $\mathbb{K}$ of all the compact open subsets of $G^{(0)}$ is a ring of subsets of $G^{(0)}$, that is, it is closed under finite unions and relative complements (meaning that $E \setminus F \in \mathbb{K}$ if $E, F \in \mathbb{K}$). Therefore the set $\mathcal{A}$ of all the subsets $A$ of $G^{(0)}$ such that either $A$ or $A^{\mathrm{c}}$ belongs to $\mathbb{K}$ is an algebra of subsets of $G^{(0)}$ (i.e. it is closed under finite unions and complements). Note that all the members of $\mathcal{A}$ are clopen sets. In particular, since $G^{(0)}$ is second countable and totally disconnected each $A \in \mathcal{A}$ can be written as $A = \bigsqcup_{i=1}^{\infty} A_i$, where $A_i$ are open compact subsets of $G^{(0)}$.

Given $f \in F(S(G)))$, we can define a premeasure $\mu$ on $\mathcal{A}$ by the rule $\mu(A) = f([1_A])$ if $A \in \mathbb{K}$ and $\mu(A) = \sum_{i=1}^{\infty} f([1_{A_i}])$ if $A = \bigsqcup_{i=1}^{\infty} A_i$ where $A_i$ are open compact subsets of $G^{(0)}$, if $A^c \in \mathbb{K}$. It is easy to check that the definition of $\mu(A)$ does not depend on the particular decomposition of $A$ into a disjoint union of a sequence of open compact subsets of $G^{(0)}$, and that $\mu$ is a premeasure on $\mathcal{A}$.

By [16, Theorem 1.14], given $f \in F(S(G))$ there exists a positive Borel measure $\mu_f$ such that $\mu_f(U) = f([1_U])$ for each $U \in \mathcal{A}$. In particualr this holds for every compact open subset $A$ of $G^{(0)}$. To show that $\mu_f$ is invariant, take an open bisection $U$. Then since $G^{(0)}$ is second countable and totally disconnected, we can write $U = \bigsqcup_{i=1}^{\infty} U_i$, where $U_i$ are compact open subsets of $U$ (and thus compact open bisections). Now we get

$$\mu_f(s(U)) = \sum_{i=1}^{\infty} \mu_f(s(U_i)) = \sum_{i=1}^{\infty} f([1_{s(U_i)}]) = \sum_{i=1}^{\infty} f([1_{r(U_i)}]) = \sum_{i=1}^{\infty} \mu_f(r(U_i)) = \mu_f(r(U)).$$

This shows invariance of $\mu_f$.

Now let $V = \bigcup \mathcal{U}$, where $\mathcal{U}$ is the family of all the open compact subsets $U$ of $G^{(0)}$ such that $f([1_U]) < \infty$. Then $V$ is $\sigma$-finite, and thus by [16, Theorem 1.14], there is a unique Borel measure $\overline{\mu}$ on $V$ such that $\overline{\mu}(U) = f([1_U])$ for all open compact subset $U$ of $V$. Hence the restriction of $\mu_f$ to $V$ is $\overline{\mu}$, and it is unique.

Now observe that every open subset of $V$ is $\sigma$-compact and that $\overline{\mu}(K) < \infty$ for each compact set $K$ of $V$. Hence, it follows from [32, Theorem 2.18] that $\overline{\mu}$ is a regular measure. $\square$

The following Lemma gives some justification that our definition of dynamical comparison is a sensible one for non-minimal groupoids.

**Lemma 2.4.** *Let $G$ be an ample groupoid. Then $G$ has dynamical comparison if and only if $G_U$ has dynamical comparison for every open $G$-invariant subset $U \subseteq G^{(0)}$.*

*Proof.* We only need to show that dynamical comparison passes to restrictions of $G$ to open $G$-invariant subsets. Let $U \subseteq G^{(0)}$ be such an open $G$-invariant subset, and let $A, B \subseteq U$ be compact open subsets of $U$ such that $A \subseteq r(G_U B)$. Moreover, assume that $\nu(A) < \nu(B)$ for every $\nu \in UM(G_U)$ such that $0 < \nu(B) < \infty$. Let $\mu \in UM(G)$ such that $0 < \mu(B) < \infty$. Then $\mu$ restricts to a measure $\mu_U \in UM(G_U)$ such that $0 < \mu_U(B) < \infty$ and so $\mu(A) = \mu_U(A) < \mu_U(B) = \mu(B)$. Since $G$ has dynamical comparison and using again that $U$ is $G$-invariant, we obtain the desired conclusion. $\square$

As mentioned in the introduction, in [8, 28] it turned out that almost unperforation of the type semigroup is a very desirable property. Recall, that a preordered monoid $S$ is called *almost unperforated* if whenever $x, y \in S$ and $n \in \mathbb{N}$ satisfy $(n+1)x \leq ny$, then $x \leq y$. One of the main goals of this paper is to relate almost unperforation of the type semigroup with certain properties of the underlying groupoid. We can now relate stable dynamical comparison with almost unperforation of $S(G)$.

**Lemma 2.5.** *Let $G$ be an ample second countable groupoid. Then $S(G)$ is almost unperforated if and only if $G$ satisfies stable dynamical comparison.*

*Proof.* Assume that $A, B$ are compact open subsets of $G^{(0)}$ such that $A \subseteq r(GB)$. Then there are open compact bisections $V_1, V_2, \ldots, V_m$ in $G$ such that $A = \bigsqcup_{i=1}^{m} r(V_i)$ and $s(V_i) \subset B$. Therefore we get $[1_A] \leq m[1_B]$. Assume in addition that $\mu(A) < \mu(B)$ for every measure

$\mu$ such that $\mu(B) = 1$. Then by Lemma 2.3 we get that $f([1_A]) < f([1_B])$ for all $f \in F(S(G), [1_B])$. Now it follows from [27, Proposition 2.1] that there is some $k \in \mathbb{N}$ such that $(k+1)[1_A] \leq k[1_B]$. Since $S(G)$ is almost unperforated, we get that $[1_A] \leq [1_B]$, as desired. The proof for $G^m$ is similar. (Note that $S(G^m) = S(G)$.)

Conversely, suppose that $G$ satisfies stable dynamical comparison, and let $x, y \in S(G)$ be such that $(k+1)x \leq ky$. Then there is some $m$ such that $x, y$ are represented by compact open subsets of $(G^m)^{(0)}$. Therefore since we are assuming that $G^m$ has dynamical comparison, we may assume that $m = 1$. With this assumption we have $x = [1_A]$ and $y = [1_B]$, where $A$ and $B$ are compact open subsets of $G^{(0)}$. Now we clearly have $A \subseteq r(GB)$ and $\mu(A) < \mu(B)$ for each $\mu \in UM(G)$ such that $0 < \mu(B) < \infty$. It follows from dynamical comparison that $[1_A] \leq [1_B]$, as desired. □

The above result begs the following natural question:

**Question 2.6.** *Is stable dynamical comparison equivalent to dynamical comparison?*

The remainder of this section is dedicated to provide an affirmative answer of this question in the minimal setting, which then leads to the proof of Theorem A. In fact, in Proposition 2.10 we show that in the minimal setting, both notions are also equivalent to the following:

**Definition 2.7** ([12])**.** Let $G$ be an ample minimal groupoid with compact unit space. We say that $G$ satisfies *weak dynamical comparison* if there exists a constant $C \geq 1$ such that whenever $A, B \subseteq G^{(0)}$ are non-empty compact open subsets satisfying $\sup_{\mu \in M(G)} \mu(A) < \frac{1}{C} \inf_{\mu \in M(G)} \mu(B)$, then $A \precsim B$.

**Remark 2.8.** For any ample groupoid $G$ with compact unit space, the set $M(G)$ of $G$-invariant regular Borel probability measures is compact in the weak*-topology. Consequently, if $A$ and $B$ are compact open sets such that $\mu(A) < \mu(B)$ for all $\mu \in M(G)$, we can use the continuity of the function $\mu \mapsto \mu(B) - \mu(A)$ to see that $\inf_{\mu \in M(G)}(\mu(B) - \mu(A)) \geq \varepsilon$ for some suitably small $\varepsilon > 0$.

Before we state and proof the result, let us make the following elementary observation which plays a crucial role in the proof:

**Lemma 2.9.** *Let $G$ be a minimal ample groupoid with compact unit space. Suppose $G$ has dynamical comparison. Then, whenever $m \in \mathbb{N}$ and $A_1, \ldots, A_m, B \subseteq G^{(0)}$ are compact open subsets such that $\sum_{i=1}^m \mu(A_i) < \mu(B)$ for all $\mu \in M(G)$, then $\bigsqcup_{i=1}^m A_i \times \{i\} \precsim B \times \{1\}$.*

*Proof.* The proof proceeds by induction on the number of levels $m$. The case $m = 1$ is immediate from the fact that $G$ has dynamical comparison. Now if $m > 1$, then we have $\mu(A_m) < \mu(B)$, so by dynamical comparison there exists $A'_m$ such that $A_m \sim A'_m \subseteq B$. Now from the assumption we have $\sum_{i=1}^{m-1} \mu(A_i) < \mu(B) - \mu(A_m) = \mu(B \setminus A'_m)$. Thus, we can apply the induction hypothesis so conclude that $\sqcup_{i=1}^{m-1} A_i \times \{i\} \precsim B \setminus A'_m \times \{1\}$. Since we also had $A_m \sim A'_m \subseteq B$, the result follows. □

**Proposition 2.10.** *Let $G$ be a $\sigma$-compact ample groupoid which is minimal and has a compact unit space. Then the following conditions are equivalent:*

(1) *$G$ satisfies stable dynamical comparison.*

*(2) $G$ satisfies dynamical comparison.*
*(3) $G$ satisfies weak dynamical comparison.*

*Proof.* It is clear that (1) $\implies$ (2) $\implies$ (3).

(3) $\implies$ (1). Suppose that $G$ satisfies weak dynamical comparison and let $C \geq 1$ be such that whenever $A, B \subseteq G^{(0)}$ are non-empty compact open subsets satisfying $\sup_{\mu \in M(G)} \mu(A) < \frac{1}{C} \inf_{\mu \in M(G)} \mu(B)$, then $A \precsim B$. Given $m \geq 1$, we show that $G^m$ has dynamical comparison.

Observe that, up to normalization, we can identify $M(G)$ and $M(G^m)$. Hence we will work with $M(G)$, with the understanding that each $\mu \in M(G)$ gives rise to the invariant measure on $(G^m)^{(0)}$ defined by

$$\mu\Big(\bigsqcup_{i=1}^{m} A_i \times \{i\}\Big) = \sum_{i=1}^{m} \mu(A_i)$$

for Borel subsets $A_i$ of $G^{(0)}$.

Suppose $A$ and $B$ are non-empty compact open subsets of $(G^m)^{(0)}$ such that $\mu(A) < \mu(B)$ for every $\mu \in M(G)$. By the remark above, there is $1 \geq \varepsilon > 0$ such that

$$\inf_{\mu \in M(G)} (\mu(B) - \mu(A)) \geq \varepsilon.$$

Since $G^m$ is $\sigma$-compact, we can find a countable cover $(V_n)_n$ of $G^m$ by compact open bisections. Let $\alpha_n : s(V_n) \to r(V_n)$ be the corresponding homeomorphism. Now let $A_1 = A \cap \alpha_1^{-1}(B)$ and $B_1 = \alpha_1(A_1)$. For each $n > 1$, define inductively

$$A_n = (A \setminus (\bigcup_{i=1}^{n-1} A_i)) \cap (\alpha_n^{-1}(B \setminus \bigcup_{i=1}^{n-1} B_i)),$$

and $B_n = \alpha_n(A_n)$. Then all the sets $A_n$ and $B_n$ are (possibly empty) compact open disjoint subsets of $A$ and $B$ respectively. Moreover, we have $\mu(A_n) = \mu(B_n)$ for every $n \in \mathbb{N}$ and every $G$-invariant measure $\mu$. Consider the remainder sets $A_0 = A \setminus (\bigcup_{n \geq 1} A_n)$ and $B_0 = B \setminus (\bigcup_{n \geq 1} B_n)$. We clearly have $\mu(B_0) \geq \varepsilon$ for all $\mu \in M(G)$. Note also, that by construction, whenever $s(g) \in A_0$ for some $g \in G$, then $r(g)$ can not be an element of $B_0$. So $r(GA_0) \subseteq (G^m)^{(0)} \setminus B_0$, or equivalently, $B_0 \subseteq (G^m)^{(0)} \setminus r(GA_0)$. We claim that $\mu(A_0) = 0$ for all $\mu \in M(G)$. It is enough to consider ergodic measures. If we suppose $\mu(A_0) > 0$, then $\mu(r(GA_0)) > 0$ and by ergodicity $\mu(B_0) \leq \mu((G^m)^{(0)} \setminus r(GA_0)) = 0$, a contradiction. Now for a fixed $\mu$ we have

$$\lim_{n \to \infty} \mu(A \setminus (\bigcup_{i=1}^{n} A_i)) = 0.$$

Since the limit is decreasing and the above measure values viewed as functions on the set $M(G)$ are continuous, the above convergence is uniform on $M(G)$ by Dini's Theorem. Now let $\delta < \frac{\varepsilon}{mC}$. Then there exists an $n_0$ such that for all $\mu \in M(G)$ we have

$$\mu(A \setminus (\bigcup_{i=1}^{n_0} A_i)) \leq \delta < \frac{\varepsilon}{mC} \leq \frac{1}{mC}\mu(B \setminus (\bigcup_{i=1}^{n_0} B_i)).$$

Now write $A \setminus (\bigcup_{i=1}^{n_0} A_i) = \bigsqcup_{j=1}^{m} C_j \times \{j\}$, with $C_j$ a clopen subset of $G^{(0)}$. Since

$$\sup_{\mu \in M(G)} \mu(C_j) < \delta \leq 1/m$$

for all $j$, we find using Lemma 2.9 that there exists a clopen subset $C'$ of $G^{(0)}$ such that $A \setminus (\bigcup_{i=1}^{n_0} A_i) \sim C' \times \{1\}$. Now we have $B \setminus (\bigcup_{i=1}^{n_0} B_i) \sim \bigsqcup_{j=1}^{m} D_j \times \{j\}$ for clopen subsets $D_j$ of $G^{(0)}$ such that $D_1 \supseteq D_2 \supseteq \cdots \supseteq D_m$. Then we get for $\mu \in M(G)$

$$\mu(C') = \mu(A \setminus (\bigcup_{i=1}^{n_0} A_i)) \leq \delta < \frac{\varepsilon}{mC} \leq \frac{1}{mC}\mu(B \setminus (\bigcup_{i=1}^{n_0} B_i)) \leq \frac{1}{C}\mu(D_1)$$

and hence we can pass to the supremum on the left and infimum on the right to get

$$\sup_{\mu \in M(G)} \mu(C') \leq \delta < \frac{\varepsilon}{mC} \leq \frac{1}{C} \inf_{\mu \in M(G)} \mu(D_1).$$

Now we can apply the assumption to conclude $C' \precsim D_1$. Therefore

$$A \setminus (\bigcup_{i=1}^{n_0} A_i) \sim C' \times \{1\} \precsim D_1 \times \{1\} \precsim \bigsqcup_{j=1}^{m} D_j \times \{j\} \sim B \setminus (\bigcup_{i=1}^{n_0} B_i).$$

Since we clearly have

$$\bigcup_{i=1}^{n_0} A_i \precsim \bigcup_{i=1}^{n_0} B_i,$$

we obtain $A \precsim B$ as desired. □

We can finally prove Theorem A stated in the introduction.

*Proof of Theorem A.* Let us first assume that $G^{(0)}$ is compact. By Proposition 2.10, dynamical comparison implies stable dynamical comparison for $G$. Hence the result follow from Lemma 2.5.

If $G^{(0)}$ is just locally compact, we pick a compact open subset $K \subseteq G^{(0)}$. Since $G$ is minimal, $G$ and the restriction $G_{|K}$ are Morita equivalent. In particular, $G_{|K}$ is minimal itself and still has dynamical comparison. Indeed, suppose $A, B \subseteq K$ such that $\mu(A) < \mu(B)$ for all $\mu \in M(G_K)$. Then if $\nu \in UM(G, B)$, by minimality of $G$ and compactness of $K$, $0 < \nu(K) < \infty$. So the measure $\frac{1}{\nu(K)}\nu_{|K} \in M(G_K)$. Hence $\nu(A) < \nu(B)$. Since $\nu$ was arbitrary we can use dynamical comparison for $G$ to conclude $A \precsim B$ in $G$. But then the compact open bisections implementing this subequivalence have range and source in $K$ since $A, B \subseteq K$. So we actually get $A \precsim B$ in $G_K$ as desired. The result now follows from the fact that $S(G_{|K}) \cong S(G)$ [28, Corollary 5.8], and the first paragraph of this proof. □

## 2.1. **Absence of invariant measures.**

We call a measure $\mu \in UM(G)$ trivial provided that $\mu(A) \in \{0, \infty\}$ for all compact open subsets $A \subseteq G^{(0)}$. In this short section we characterize almost unperforation of the type semigroup when every measure in $UM(G)$ is trivial.

Recall that an element $x$ of a semigroup $S$ is called *properly infinite* if $2x \leq x$.

**Proposition 2.11.** *Let $G$ be an ample second countable groupoid such that every measure in $UM(G)$ is trivial. Then $G$ has dynamical comparison if and only if every element in $S(G)$ is properly infinite.*

*Proof.* Suppose first that $G$ has dynamical comparison. We first consider the case of a $G$-full compact open subset $A \subseteq G^{(0)}$. Since every measure in $UM(G)$ is trivial it follows from Lemma 2.3 that $F(S(G), [1_A]) = \emptyset$. By Tarski's Theorem we conclude that a multiple of $[1_A]$ is properly infinite. By [26, Theorem 4.3] there exist order units $u, v \in S(G)$ such that $[1_A] = u + v$. Pick representative functions $u = [f]$ and $v = [g]$ and let $U = \mathrm{supp}(f)$ and $V = \mathrm{supp}(g)$. Note, that the sets $U, V$ are both compact open and $G$-full subsets of $G^{(0)}$. Consequently, $[1_V]$ and $[1_U]$ are order units themselves by Lemma 1.2, so that $[1_A] \leq l[1_U]$ and $[1_A] \leq k[1_V]$ for some $k, l \geq 1$. This implies that $A \subseteq r(GU) \cap r(GV)$. Since there exist no $G$-invariant measures on $G^{(0)}$ which are non-trivial in $U$ or $V$, we may use dynamical comparison to conclude that in fact already $[1_A] \leq [1_U]$ and $[1_A] \leq [1_V]$. Putting everything together we compute $2[1_A] \leq [1_U] + [1_V] \leq u + v = [1_A]$ and reach our desired conclusion.

Now if $[f] \in S(G)$ is an arbitrary order unit, write $f = \sum_{i=1}^n 1_{A_i}$ and let $A = \bigcup_{i=1}^n A_i$ be its support. Then $[1_A]$ is an order unit as well. Hence we can apply the first step above (multiple times) and conclude $2[f] \leq 2n[1_A] \leq [1_A] \leq [f]$.

Suppose now that $[g]$ is an arbitrary element of $S(G)$. Let $I \subseteq S(G)$ be the order ideal generated by $[g]$. By Lemma 1.3 we have that $I \cong S(G_U)$ for some $G$-invariant open subset $U \subseteq G^{(0)}$. Note that $[g]$ is an order unit for $I = S(G_U)$, and that $G_U$ also has dynamical comparison (Lemma 2.4). Let $\mu \in UM(G_U)$. Then we can extend $\mu$ to an invariant measure $\tilde{\mu} \in UM(G)$ by the rule $\tilde{\mu}(T) = \mu(T \cap U)$ for each Borel set $T$ of $G^{(0)}$. It follows by our hypothesis that $\tilde{\mu}$ is trivial and hence so is $\mu$. Therefore every measure in $UM(G_U)$ is trivial and so it follows from the above argument that $[g]$ is properly infinite in $S(G_U)$ and hence also in $S(G)$.

Conversely, assume that every element in $S(G)$ is properly infinite. Let $A, B \subseteq G^{(0)}$ be compact open subsets, such that $A \subseteq r(GB)$. Then $[1_B]$ is a properly infinite order unit in the order ideal $S(G_{r(GB)})$ of $S(G)$. It follows that there exists an $n \in \mathbb{N}$ such that $[1_A] \leq n[1_B] \leq [1_B]$, as desired. □

Note that the equivalent properties in the previous proposition are also equivalent to every compact open subset of the base space being $(2,1)$-paradoxical in the sense of [8, Definition 4.5].

The following generalizes [24, Proposition 6.2] and gives an affirmative answer to Question 2.6 in the absence of interesting invariant measures.

**Proposition 2.12.** *Let $G$ be an ample second countable groupoid such that every measure in $UM(G)$ is trivial. Then the type semigroup $S(G)$ is almost unperforated if and only if $G$ has dynamical comparison.*

*Proof.* One implication follows from Lemma 2.5. Conversely, we assume that $G$ has dynamical comparison. Suppose we are given $[f], [g] \in S(G)$, such that $(n+1)[f] \leq n[g]$. It follows from Proposition 2.11 that $[g]$ is properly infinite in $S(G)$. We conclude that $[f] \leq (n+1)[f] \leq n[g] \leq [g]$, as desired. □

## 3. Almost Finite groupoids

In this section we study the type semigroups associated with almost finite groupoids. Our main results reveal that almost finiteness is not strong enough of a condition to prove almost unperforation of the type semigroup in the non-minimal setting. The reason for this lies in a different behaviour of the permanence properties of these two notions: almost unperforation passes to order ideals, while almost finiteness does not pass to restrictions of $G$ to arbitrary open invariant subspaces of $G^{(0)}$. Prompted by this, we will show that a strong version of almost finiteness, which basically asks for every such restriction to be almost finite, indeed provides us with an almost unperforated type semigroup.

We use this characterization of almost unperforation to clarify the relationship between the type semigroup and the positive cone of the homology group $H_0(G)$.

3.1. **Definition and Properties.** We begin by recalling the definition of almost finiteness and proving some immediate consequences.

**Definition 3.1.** [25, Definition 6.2] Let $G$ be an ample groupoid with compact unit space.

1. We say that $K \subseteq G$ is an *elementary* subgroupoid if it is a compact open principal subgroupoid of $G$ such that $K^{(0)} = G^{(0)}$.
2. Given a compact subset $C \subseteq G$ and $\varepsilon > 0$, a compact subgroupoid $K \subseteq G$ with $K^{(0)} = G^{(0)}$ is called $(C, \varepsilon)$*-invariant*, if for all $x \in G^{(0)}$ we have
$$\frac{|CKx \setminus Kx|}{|Kx|} < \varepsilon.$$
3. We say that $G$ is *almost finite* if for every compact set $C \subseteq G$ and every $\varepsilon > 0$ there exists a $(C, \varepsilon)$-invariant elementary subgroupoid $K \subseteq G$.

Throughout, we will always work with ample groupoids with compact unit space; hence, whenever we write that $G$ is almost finite, we also mean the above conditions.

**Definition 3.2.** [34, Definition 3.2] Let $K$ be a compact groupoid. A *clopen castle* for $K$ is a partition
$$K^{(0)} = \bigsqcup_{i=1}^{n} \bigsqcup_{j=1}^{N_i} F_j^{(i)}$$
into non-empty clopen subsets such that the following conditions hold:

1. For each $1 \leq i \leq n$ and $1 \leq j, k \leq N_i$ there exists a unique compact open bisection $V_{j,k}^{(i)}$ of $K$ such that $s(V_{j,k}^{(i)}) = F_k^{(i)}$ and $r(V_{j,k}^{(i)}) = F_j^{(i)}$.
2. $$K = \bigsqcup_{i=1}^{n} \bigsqcup_{1 \leq j,k \leq N_i} V_{j,k}^{(i)}.$$

The pair $(F_1^{(i)}, \{V_{j,k}^{(i)} \mid 1 \leq j, k \leq N_i\})$ is called the $i$-th tower of the castle and the sets $F_j^{(i)}$ are called the levels of the $i$-th tower.

**Remark 3.3.** Note that the uniqueness of the bisections in (2) above has some important consequences: If $\theta^{(i)}_{j,k} : F^{(i)}_k \to F^{(i)}_j$ denotes the partial homeomorphism corresponding to the bisection $V^{(i)}_{j,k}$, i.e. $\theta^{(i)}_{j,k} = r \circ (s_{|V^{(i)}_{j,k}})^{-1}$, then we have $(\theta^{(i)}_{j,k})^{-1} = \theta^{(i)}_{k,j}$, $\theta^{(i)}_{j,k} \circ \theta^{(i)}_{k,l} = \theta^{(i)}_{j,l}$, and $\theta^{(i)}_{j,j} = id_{F^{(i)}_j}$.

Recall that, as mentioned in [34], since compact ample principal groupoids always admit a clopen castle, Definition 3.1 is equivalent to the definition of almost finiteness given in [34, Definition 3.6]. We point out that due to this fact we will be using both equivalent notions of almost finiteness throughout the paper.

The following small lemma shows how to refine a castle as in Definition 3.2 and will be used frequently throughout the rest of this article:

**Lemma 3.4.** *Let $K$ be a compact groupoid admitting a clopen castle. Given finitely many clopen subsets $A_1, \ldots, A_r \subseteq K^{(0)}$ there exists a clopen castle for $K$ such that every level of every tower of the castle is either contained in or disjoint from $A_l$ for every $1 \leq l \leq r$.*

*Proof.* Let us consider the case that we only have one clopen subset $A \subseteq K^{(0)}$. We will replace every tower of the castle by finitely many thinner towers, such that each level of the new towers is either contained in or disjoint from $A$. Let $\theta^{(i)}_{j,k} : F^{(i)}_k \to F^{(i)}_j$ be the partial homeomorphism associated to the compact open bisection $V^{(i)}_{j,k}$. Consider the compact open subsets $\theta^{(i)}_{1,k}(A \cap F^{(i)}_k) \subseteq F^{(i)}_1$ of the base of the $i$-th tower. Taking a clopen refinement we can find a decomposition $F^{(i)}_1 = \bigsqcup_{t=1}^{L_i} X^{(i)}_{t,1}$ such that each $X^{(i)}_{t,1}$ is either contained in or disjoint from every $\theta^{(i)}_{1,k}(A \cap F^{(i)}_k)$. Let $X^{(i)}_{t,j} := \theta^{(i)}_{j,1}(X^{(i)}_{t,1}) \subseteq F^{(i)}_j$. Then we clearly have $K^{(0)} = \bigsqcup_{i=1}^n \bigsqcup_{t=1}^{L_i} \bigsqcup_{1 \leq j \leq N_i} X^{(i)}_{t,j}$. Moreover, the sets $V^{(i)}_{j,k,t} := V^{(i)}_{j,k} \cap s^{-1}(X^{(i)}_{t,k})$ are compact open bisections such that $s(V^{(i)}_{j,k,t}) = X^{(i)}_{t,k}$ and $r(V^{(i)}_{j,k,t}) = X^{(i)}_{t,j}$ and one easily checks, that $K = \bigsqcup_{i=1}^n \bigsqcup_{t=1}^{L_i} \bigsqcup_{1 \leq j,k \leq N_i} V^{(i)}_{j,k,t}$. Hence we have constructed a finer clopen castle. In this new castle, for every $k$ we have $\theta^{(i)}_{1,k}(X^{(i)}_{t,k} \cap A) = X^{(i)}_{t,1} \cap \theta^{(i)}_{1,k}(A \cap F^{(i)}_k)$. By construction, the latter set is either empty or all of $X^{(i)}_{t,1}$. Hence by applying $\theta^{(i)}_{k,1}$ we obtain that $X^{(i)}_{t,k} \cap A$ is either empty or all of $X^{(i)}_{t,k}$, as desired. Applying the above process successively to finitely many sets $A_1, \ldots, A_r$ yields the desired result. □

We continue this first part of the section showing important features and permanence properties of almost finiteness. To state them we need to recall some terminology, and well-known facts about almost finite groupoids:

1. If $G$ is almost finite, then $M(G) \neq \emptyset$ by [25, Lemma 6.5].
2. If $G$ is almost finite and minimal, then $G$ is effective [25, Remark 6.6].
3. If $G$ is an almost finite groupoid and $D \subseteq G^{(0)}$ is a closed $G$-invariant subset, then the restriction $G_D$ is almost finite [34, Lemma 3.13].

(4) If $G$ admits a proper surjective groupoid homomorphism $\pi : G \to H$ onto an almost finite groupoid $H$, such that the restriction to every source fibre $Gx \to H\pi(x)$ is bijective, then $G$ is almost finite [34, Lemma 5.1].

We would like to add another crucial and natural permanence property of almost finiteness to the above list: invariance under stable isomorphism. Recall that two étale groupoids $G$ and $G'$ are *stably isomorphic* if $G \times \mathcal{R} \cong G' \times \mathcal{R}$, where $\mathcal{R} = \mathbb{N}^2$ is the (discrete) full equivalence relation on $\mathbb{N}$. It is well-known that stable isomorphism agrees with Morita equivalence for ample (Hausdorff) groupoids with $\sigma$-compact unit spaces (see [9, Theorem 2.19]). In fact, there are a number of notions of equivalence for groupoids, and they all coincide for ample (Hausdorff) groupoids with $\sigma$-compact unit spaces (see [14, Theorem 3.12]).

**Lemma 3.5.** *Let $G$ be an almost finite groupoid and $K$ be an elementary groupoid. Then $G \times K$ is almost finite.*

*Proof.* Let $C \subseteq G \times K$ be a compact subset and $\varepsilon > 0$. Then $C$ is contained in $\widetilde{C} \times K$ for a compact subset $\widetilde{C} \subseteq G$. By almost finiteness of $G$, there exists a $(\widetilde{C}, \varepsilon)$-invariant elementary subgroupoid $\widetilde{K}$ of $G$. Then $L := \widetilde{K} \times K$ is clearly an elementary subgroupoid of $G \times K$ and for every $(x, y) \in G^{(0)} \times K^{(0)}$ we have

$$\begin{aligned} |CL(x,y) \setminus L(x,y)| &\leq |(\widetilde{C} \times K)(\widetilde{K} \times K)(x,y) \setminus (\widetilde{K} \times K)(x,y)| \\ &\leq |\widetilde{C}\widetilde{K}x \setminus \widetilde{K}x||Ky| \\ &< \varepsilon|\widetilde{K}x||Ky| = \varepsilon|L(x,y)|. \end{aligned}$$

□

**Proposition 3.6.** *Let $G$ and $G'$ be ample groupoids with compact unit spaces. Suppose that $G$ and $G'$ are stably isomorphic. Then $G$ is almost finite if and only if $G'$ is almost finite.*

*Proof.* Let us fix an isomorphism $\Phi : G \times \mathcal{R} \to G' \times \mathcal{R}$. For each $n \in \mathbb{N}$ consider the clopen subgroupoid $G_n := G \times \{1, \ldots, n\}^2$ of $G \times \mathcal{R}$. If $G$ is almost finite, then so is each $G_n$ by Lemma 3.5. Let $H_n := \Phi(G_n) \subseteq G' \times \mathcal{R}$. By definition, each $H_n$ is almost finite and $G' \times \mathcal{R} = \bigcup_n H_n$. Consider the compact open subset $W := G'^{(0)} \times \{1\} \subseteq G' \times \mathcal{R}$. Then $W$ is clearly a $G' \times \mathcal{R}$-full subset of $(G' \times \mathcal{R})^{(0)} = G'^{(0)} \times \mathbb{N}$ such that $(G' \times \mathcal{R})_{|W} \cong G'$. Hence it is enough to show that the restriction groupoid $(G' \times \mathcal{R})_{|W}$ is almost finite. But this follows from a slight adaptation of [34, Lemma 3.12]: If $C \subseteq (G' \times \mathcal{R})_{|W}$ is a compact subset and $\varepsilon > 0$, then there exists an $n \in \mathbb{N}$ such that $C \cup W \subseteq H_n$. Using the compactness of $H_n^{(0)}$ and the fact that $W$ is $G' \times \mathcal{R}$-full, there exist finitely many compact open bisections $V_1, \ldots, V_l \subseteq G' \times \mathcal{R}$ such that $\bigcup_{i=1}^{l} s(V_i) = G_n^{(0)}$ and $r(V_i) \subseteq W$. For each $1 \leq i \leq l$ we have $V_i \subseteq H_n$. Indeed, since $s(V_i), r(V_i) \subseteq H_n^{(0)}$ we have $s(\Phi^{-1}(V_i)), r(\Phi^{-1}(V_i)) \subseteq G_n^{(0)}$. But then we must have $\Phi^{-1}(V_i) \subseteq G_n$, which implies our claim.

Now let $\widetilde{C} := C \cup V_1 \cup \ldots \cup V_l \subseteq H_n$ and use almost finiteness of $H_n$ to find a $(\widetilde{C}, \frac{\varepsilon}{2l})$-invariant elementary subgroupoid $K$ of $H_n$. Then we can literally copy the argument from [34, Lemma 3.12] to show that $K_{|W}$ is a $(C, \varepsilon)$-invariant elementary subgroupoid of $(G' \times \mathcal{R})_{|W}$. This completes the proof. □

### 3.2. Almost finiteness and dynamical comparison.

In this subsection we will study the implications of almost finiteness for the type semigroup of not necessarily minimal ample groupoids. The main observation is contained in the following Lemma, which says that the algebraic preorder on $S(G)$ is witnessed by the $G$-invariant measures on the unit space $G^{(0)}$.

**Lemma 3.7.** *Let $G$ be an almost finite groupoid and let $f, g \in C(G^{(0)}, \mathbb{Z})^+$. If $\mu(f) < \mu(g)$ for all $\mu \in M(G)$, then $[f] \leq [g]$ in $S(G)$.*

*Proof.* Passing to $G^m$ for $m$ big enough, we can assume that $f = 1_A$ and $g = 1_B$ for clopen subsets $A$ and $B$ of $G^{(0)}$, with $\mu(A) < \mu(B)$ for all $\mu \in M(G)$.

Given a pair $(C, \varepsilon)$ we can find a $(C, \varepsilon)$-invariant elementary subgroupoid $K$ by almost finiteness. Using Lemma 3.4 we may assume that it admits a castle $(F_1^{(i)}, (V_{j,k}^{(i)})_{1 \leq j,k \leq N_i})_{i=1}^n$, such that every level in every tower is either contained in or disjoint from each of the sets $A, B$. For each $1 \leq i \leq n$ let $E_i = \{k \mid F_k^{(i)} \subseteq A\}$ and $F_i = \{j \mid F_j^{(i)} \subseteq B\}$ be the sets counting how many levels of the $i$-th tower are contained in $A$ and $B$ respectively. Note that these sets depend on $(C, \varepsilon)$ (although we do not include this in our notation).

**Claim.** *There exists $(C, \varepsilon)$ such that for any $(C, \varepsilon)$-invariant elementary subgroupoid $K \subseteq G$ (admitting a castle for $K$ as described above), it follows that*

$$|E_i| \leq |F_i| \quad \textit{for each } 1 \leq i \leq n. \tag{3.1}$$

*Proof of Claim.* Suppose this is not the case. Then we can write $G$ as a directed union of symmetric compact subsets $C = C^{-1}$, and for each $\lambda := (C, \varepsilon)$ find $(C, \varepsilon)$-invariant compact subgroupoids $K_\lambda \subseteq G$ such that there exists a tower $\mathcal{F}_\lambda := (F_1^{(i_\lambda)}, V_{j,k}^{(i_\lambda)})_{1 \leq j,k \leq L_\lambda}$ in the corresponding clopen castle for $K_\lambda$ with the property that

$$|E_{i_\lambda}| > |F_{i_\lambda}|.$$

For each $\lambda$, let $x_\lambda$ be any element in $F_1^{(i_\lambda)}$ (the basis of $\mathcal{F}_\lambda$), and define a probability measure $\mu_\lambda$ on $G^{(0)}$ by

$$\mu_\lambda(D) = \frac{1}{L_\lambda} \sum_{j=1}^{L_\lambda} \delta_{x_\lambda}(\theta_{1,j}^{(i_\lambda)}(D \cap F_j^{(i_\lambda)})).$$

Now let $U$ be a compact open bisection such that $U \subseteq C$, and note that $r(K_\lambda x_\lambda) = \{\theta_{j,1}^{i_\lambda}(x_\lambda) \mid 1 \leq j \leq L_\lambda\}$ and $|K_\lambda x_\lambda| = L_\lambda$. Then, we get that :

$$\mu_\lambda(r(U)) = \frac{|r(U) \cap r(K_\lambda x_\lambda)|}{|K_\lambda x_\lambda|} = \frac{|U^{-1} K_\lambda x_\lambda|}{|K_\lambda x_\lambda|}.$$

Similarly, we get

$$\mu_\lambda(s(U)) = \frac{|s(U) \cap r(K_\lambda x_\lambda)|}{|K_\lambda x_\lambda|} = \frac{|U K_\lambda x_\lambda|}{|K_\lambda x_\lambda|}.$$

Now

$$|U K_\lambda x_\lambda| = |U K_\lambda x_\lambda \cap K_\lambda x_\lambda| + |U K_\lambda x_\lambda \setminus K_\lambda x_\lambda|.$$

Since $|U K_\lambda x_\lambda \cap K_\lambda x_\lambda| = |U^{-1} K_\lambda x_\lambda \cap K_\lambda x_\lambda|$, we also get

$$|U^{-1} K_\lambda x_\lambda| = |U K_\lambda x_\lambda \cap K_\lambda x_\lambda| + |U^{-1} K_\lambda x_\lambda \setminus K_\lambda x_\lambda|.$$

Putting all of this together we obtain

$$\begin{aligned}|\mu_\lambda(s(U)) - \mu_\lambda(r(U))| &= \frac{||UK_\lambda x_\lambda \setminus K_\lambda x_\lambda| - |U^{-1}K_\lambda x_\lambda \setminus K_\lambda x_\lambda||}{|K_\lambda x_\lambda|} \\ &\leq \frac{|CK_\lambda x_\lambda \setminus K_\lambda x_\lambda|}{|K_\lambda x_\lambda|} < \varepsilon.\end{aligned}$$

Now let $\mu$ be a weak-$*$ cluster point of this sequence. Then $\mu \in M(G)$. Indeed, passing to a subnet, we can assume that $\mu = \lim_\lambda \mu_\lambda$. Now if $U$ is any compact open bisection and $\varepsilon > 0$ is arbitrary, we can find $\lambda = (C, \delta)$ such that $U \subseteq C$, $\delta < \frac{\varepsilon}{3}$ and moreover $|\mu(r(U)) - \mu_\lambda(r(U))| < \varepsilon/3$ and $|\mu(s(U)) - \mu_\lambda(s(U))| < \varepsilon/3$. By the above computation, we have $|\mu_\lambda(s(U)) - \mu_\lambda(r(U))| < \delta < \varepsilon/3$.

Then

$$\begin{aligned}|\mu(s(U)) - \mu(r(U))| \leq{}& |\mu(s(U)) - \mu_\lambda(s(U))| + |\mu_\lambda(s(U)) - \mu_\lambda(r(U))| \\ &+ |\mu_\lambda(r(U)) - \mu(r(U))| < \varepsilon;\end{aligned}$$

namely $\mu$ is a $G$-invariant probability measure on $G^{(0)}$.

Now, using the fact that each level $F_j^{(i_\lambda)}$ is either contained in or disjoint from each of the sets $A, B$ (Lemma 3.4) we compute:

$$\begin{aligned}\mu(A) = \lim_\lambda \mu_\lambda(A) &= \lim_\lambda \frac{1}{L_\lambda}|E_{i_\lambda}| \\ &\geq \lim_\lambda \frac{1}{L_\lambda}|F_{i_\lambda}| \\ &= \lim_\lambda \mu_\lambda(B) = \mu(B).\end{aligned}$$

So we obtain that $\mu(A) \geq \mu(B)$, contradicting the hypothesis. ■

From the inequality (3.1) we obtain injections of sets

$$E_i \hookrightarrow F_i, \qquad (1 \leq i \leq n),$$

and from this it is straightforward to see that $[1_A] \leq [1_B]$ in $S(G)$. This concludes the proof. □

As a first immediate application of this, we obtain an easy way to identify the order units in $S(G)$:

**Corollary 3.8.** *Let $G$ be an almost finite groupoid. Then $[f] \in S(G)$ is an order unit if and only if $\mu(f) > 0$ for all $\mu \in M(G)$.*

*Proof.* If $\mu(f) > 0$ for all $\mu \in M(G)$, then by compactness of $M(G)$ there exists $N > 0$ such that $1/N < \mu(f)$ for all $\mu \in M(G)$. Thereore, if $[g] \in S(G)$ is an arbitrary element, then there exists some $n \in \mathbb{N}$ such that $\mu(g) < n\mu(f) = \mu(nf)$ for all $\mu \in M(G)$. By Lemma 3.7 we conclude that $[g] \leq [nf] = n[f]$. Conversely, if $[f] \in S(G)$ is an order unit, then $[1_{G^{(0)}}] \leq N[f]$ for some $N \in \mathbb{N}$. Hence $1 = \mu(1_{G^{(0)}}) \leq N\mu(f)$, which implies our claim. □

We can now apply this result to come back to the study of almost unperforation of type semigroup $S(G)$. In the following we will denote the algebraic preorder on $S(G)^* \cup \{0\}$ by $\leq^*$. We are now ready to prove our first main result in this section:

**Theorem 3.9.** *If $G$ is almost finite, then $S(G)^* \cup \{0\}$ is almost unperforated. In particular, if $G$ is almost finite and minimal, then $S(G)$ itself is almost unperforated.*

*Proof.* Let $[f], [g] \in S(G)^* \cup \{0\}$ such that $(n+1)[f] \leq n[g]$. We may assume $[g] \neq 0$ since the result is obvious otherwise. Then, for every $\mu \in M(G)$, we have $\mu(g) > 0$ and $(n+1)\mu(f) \leq n\mu(g)$. We conclude that $\mu(f) < \mu(g)$ and hence $[f] \leq [g]$ in $S(G)$ by Lemma 3.7. Thus, there exists some $[h] \in S(G)$ such that $[f] + [h] = [g]$. It follows that $\mu(h) = \mu(g) - \mu(f) > 0$ for all $\mu \in M(G)$ and hence $[h]$ is an order unit by the previous Lemma. It follows that in fact we have $[f] \leq^* [g]$ which completes the proof of the first statement.

For the second statement, we see that $S(G)$ is simple by Lemma 1.2. Hence, $S(G) = S(G)^* \cup \{0\}$ is almost unperforated. $\square$

This can be used to determine the groupoid homology of $G$ when combined with the following result:

**Lemma 3.10.** *If $G$ is almost finite and no restriction $G_D$, for a closed invariant set $D \subsetneq G^{(0)}$, is isomorphic to $\mathcal{R}_n$ for some $n \in \mathbb{N}$, then $S(G)^* \cup \{0\}$ is cancellative.*

*Proof.* We will use some results from [26, 6, 1]. Recall that an element $x$ of a monoid $M$ is *weakly divisible* if it can be written as $x = 2a + 3b$ for $a, b \in M$. If all order-units of $M$ are weakly divisible, then $M$ is said to have *weak divisibility for order-units* ([26, Definition 2.2]). An element $x$ of a conical monoid $M$ is said to be *irreducible* if it is nonzero and given any decomposition $x = a + b$ in $M$ we have that either $a$ or $b$ are zero. If follows easily from [6, Theorem 6.7] that an order-unit $u$ of a conical refinement monoid $M$ is weakly divisible if and only if $\overline{u}$ is not irreducible in any simple quotient $M/I$ of $M$.

Now, by Lemma 1.3, any simple quotient of $S(G)$ is of the form $S(G)/S(G_{G^{(0)}\setminus D}) \cong S(G_D)$ for a closed invariant subset $D \subsetneq G^{(0)}$. Since $G_D$ is different from $\mathcal{R}_n$ and almost finite, it follows that $D$ is the Cantor set, implying the lack of irreducible elements in the simple quotients of $S(G)$. Therefore, all the order-units of $S(G)$ are weakly divisible and thus $S(G)$ has weak divisibility for order-units.

Now, we deduce from [26, Theorem 3.4] that $S(G)^* \cup \{0\}$ is a simple refinement monoid, and, by Theorem 3.9, that $S(G)^* \cup \{0\}$ is almost unperforated. Hence, it follows from [26, Theorem 3.8] and [1, Corollary 1.8] that $S(G)^*$ is cancellative. To show that $S(G)^* \cup \{0\}$ is cancellative it is thus enough to show that for a fixed element $u \in S(G)^*$ and $a \in S(G)^* \cup \{0\}$, the relation $u + a = u$ implies $a = 0$. But this is obviously implied by the fact that $M(G) \neq \emptyset$, and the fact that $\mu(x) > 0$ for any order-unit $x$ in $S(G)$ and any $\mu \in M(G)$. $\square$

**Corollary 3.11.** *Let $G$ be a minimal almost finite groupoid. Then $S(G)$ is a cancellative monoid and $S(G) \cong H_0(G)^+$.*

*Proof.* If $G$ is elementary i.e. $G \cong \mathcal{R}_n$ for some $n \in \mathbb{N}$, we have $S(G) = \mathbb{N}_0$ which is obviously cancellative. So let us assume that $G \ncong \mathcal{R}_n$. In this case, we apply Lemma 3.10 to obtain that $S(G)$ is cancellative. In both cases the result now follows from Proposition 1.6. ☐

The results of this section so far indicate that almost finiteness itself does not lead to interesting properties of the whole type semigroup, but just to the subsemigroup of order units.

This is largely due to the following fact: In contrast to the permanence property shown in Lemma 2.4 for dynamical comparison, almost finiteness does not pass to the restrictions of $G$ to arbitrary compact open subsets of $G^{(0)}$ in general. In fact, we will build examples exhibiting this behaviour in section 4. To remedy this situation, we make the following definition:

**Definition 3.12.** We say that an ample groupoid $G$ is *strongly almost finite* if $G_{|A}$ is almost finite for all compact open subsets $A$ of $G^{(0)}$.

We remark that our notion of strong almost finiteness should not be confused with [13, Definition 1.4], which is related but ultimately different.

Clearly, every AF groupoid in the sense of [25, Definition 2.2] is strongly almost finite. If $G$ is minimal and has a compact unit space, then our notion is equivalent to almost finiteness in the usual sense by Proposition 3.6. However, in general, our notion is strictly stronger than almost finiteness. More examples of non-minimal strongly almost finite groupoids will be provided in the last section of the present article.

The remaining part of the section is dedicated to show that strong almost finiteness implies dynamical comparison of $G$ and almost unperforation of $S(G)$ (i.e. Theorem B). We need the following elementary lemma. Note that the lemma follows from [28, Corollary 5.8] in case the set $U$ in its statement is $\sigma$-compact.

**Lemma 3.13.** *Let $G$ be an ample groupoid, and let $B$ be an open compact subset of $G^{(0)}$. Then $S(G_{|B}) \cong S(G_U)$, where $U = r(GB)$ is the open invariant subset of $G$ generated by $B$.*

*Proof.* Define $\varphi\colon S(G_U) \to S(G_B)$ as follows. Let $A$ be an open compact subset of $U = r(GB)$. Then there are open compact bisections $W_1, \dots, W_n$ such that $A = \bigsqcup_{i=1}^n r(W_i)$ and $s(W_i) \subseteq B$. Then set $\varphi([A]) = \sum_{i=1}^n [s(W_i)] \in S(G_B)$. Suppose that $A = \bigsqcup_{j=1}^m r(W_j')$ for open compact bisections $W_j'$ such that $s(W_j') \subseteq B$. Then for each $1 \leq i \leq n$, we have $s(W_i) = \bigsqcup_{j=1}^m \theta_{W_i}^{-1}(r(W_i) \cap r(W_j'))$ and for each $1 \leq j \leq m$, we have $s(W_j') = \bigsqcup_{i=1}^n \theta_{W_j'}^{-1}(r(W_i) \cap r(W_j'))$. Moreover $[\theta_{W_i}^{-1}(r(W_i) \cap r(W_j'))] = [\theta_{W_j'}^{-1}(r(W_i) \cap r(W_j'))]$ in $S(G_B)$. Therefore we get

$$\sum_{i=1}^n [s(W_i)] = \sum_{i=1}^n \sum_{j=1}^m [\theta_{W_i}^{-1}(r(W_i) \cap r(W_j'))] = \sum_{j=1}^m \sum_{i=1}^n [\theta_{W_j'}^{-1}(r(W_i) \cap r(W_j'))] = \sum_{j=1}^m [s(W_j')].$$

This shows that $\varphi([A])$ does not depend of the particular decomposition of $A$. Now it is straighforward to show that $\varphi$ induces a semigroup homomorphism. Indeed, if $A \cap A' = \emptyset$, then we clearly get that $\varphi([A \cup A']) = \varphi([A]) + \varphi([A'])$. If $V$ is an open compact bisection and $s(V) \subseteq U$, then write $s(V) = \bigsqcup_{i=1}^n r(W_i)$ for open compact bisections such that $s(W_i) \subseteq B$.

Then $r(V) = \bigsqcup_{i=1}^n r(VW_i)$ and $s(VW_i) = s(W_i) \subseteq B$. Therefore we obtain

$$\varphi([r(V)]) = \sum_{i=1}^n [r(VW_i)] = \sum_{i=1}^n [s(VW_i)] = \sum_{i=1}^n [s(W_i)] = \varphi([s(V)]),$$

and so the relation $[r(V)] = [s(V)]$ is also preserved by $\varphi$.

In the other direction, we clearly can define a homomorphism $\psi \colon S(G_B) \to S(G_U)$ by $\psi([D]) = [D]$ for an open compact subset $D$ of $B$. The maps $\varphi$ and $\psi$ are clearly mutually inverses. This concludes the proof. □

**Lemma 3.14.** *If $G$ is a second countable strongly almost finite groupoid, then $G$ satisfies dynamical comparison.*

*Proof.* Let $A, B$ be open compact subsets of $G^{(0)}$ such that $A \subseteq r(GB)$, and assume that $\mu(A) < \mu(B)$ for each $\mu \in UM(G)$ such that $0 < \mu(B) < \infty$. We will show that $[1_A] \le [1_B]$ in $S(G)$.

Since $A \subseteq r(GB)$, there exist open compact bisections $V_1, V_2, \dots, V_m$ in $G$ such that $A = \bigsqcup_{i=1}^m r(V_i)$ and $s(V_i) \subseteq B$ for all $i$. Now observe that $A \times \{1\} \sim D := \bigsqcup_{i=1}^m s(V_i) \times \{i\}$ within $G^m$. Note that $D$ is an open compact subset of $B^m = ((G_B)^m)^{(0)}$, and that $(G_B)^m$ is almost finite, because $G$ is strongly almost finite and almost finiteness is Morita invariant.

We next show that $\mu(D) < \mu(B \times \{1\})$ for all $\mu \in M((G_B)^m)$. For this we will use the correspondence between $F(S(G))$ and $UM(G)$, which is valid in any ample second countable groupoid (Lemma 2.3).

Note that by using Lemmas 2.3 and 3.13, we get

$$M(G_B) \cong F(S(G_B), [B]) \cong F(S(G_U), [B]),$$

where $U := r(GB)$ is the open invariant subset generated by $B$. Now $S(G_U)$ is an order-ideal of $S(G)$, and we can extend any $f \in F(S(G|_U), [B])$ to a functional $\widetilde{f} \in F(S(G))$ by $\widetilde{f}(x) = \infty$ if $x \notin S(G_U)$. Now let $\mu \in M((G_B)^m)$, and let $\mu' \in M(G_B)$ be the invariant measure such that $\mu'(T) = m\mu(T \times \{1\})$ . Then the functional $f$ on $S(G_B)$ corresponding to $\mu'$ can be extended as seen above to a functional $\widetilde{f} \in F(S(G))$. Let $\widetilde{\mu} \in UM(G)$ be the corresponding measure, and observe that the restriction of $\widetilde{\mu}$ to $B$ coincides with $\mu'$. Since $\widetilde{\mu}$ is $G$-invariant, we have that

$$m\mu(D) = \sum_{i=1}^m \mu'(s(V_i)) = \sum_{i=1}^m \widetilde{\mu}(s(V_i)) = \sum_{i=1}^m \widetilde{\mu}(r(V_i)) = \widetilde{\mu}(A) < \widetilde{\mu}(B) = \mu'(B) = m\mu(B \times \{1\}),$$

as desired. Therefore we get that $\mu(D) < \mu(B \times \{1\})$ for all $\mu \in M((G_B)^m)$. If we show that $D \precsim B \times \{1\}$ within $(G_B)^m$, then clearly we will get that $A \precsim B$ within $G$.

Therefore, changing notation we can assume that $A, B$ are open compact subsets of $G$, that $B$ is $G$-full, and that $\mu(A) < \mu(B)$ for all $\mu \in M(G)$. In this situation, the result follows from Lemma 3.7. □

We can now obtain our second main result of this section, i.e. Theorem B.

*Proof of Theorem B.* By Lemma 2.5, it suffices to show that $G$ satisfies stable dynamical comparison. Now by Lemma 3.14, it suffices to show that $G^m$ is strongly almost finite for each $m \geq 1$. Let $B = \bigsqcup_{i=1}^{m} B_i \times \{i\}$ be an open compact subset of $(G^m)^{(0)}$, where each $B_i$ is an open compact subset of $G^{(0)}$. Let $D = \bigcup_{i=1}^{m} B_i$. Then $D$ is an open compact subset of $G^{(0)}$ and clearly $G_B^m$ and $G_D$ are stably isomorphic. Hence $G_B^m$ is almost finite by Proposition 3.6. This shows that $G^m$ is strongly almost finite, and the proof is complete. □

## 4. Coarse geometry

In this section we establish a new link between rigidity properties in topological dynamics and coarse geometry. The starting point is the following recent result on the structure of amenable groups:

**Theorem 4.1.** ([11]) Let $\Gamma$ be a countable amenable group. Then $\Gamma$ admits an exact tiling into Følner sets of arbitrary invariance, i.e. for every finite subset $K \subseteq \Gamma$ and $\varepsilon > 0$ there exist a number $n \in \mathbb{N}$, finite $(K, \varepsilon)$-invariant subsets $S_1, \dots, S_n \subseteq \Gamma$ (the shapes) and $F_1, \dots, F_n$ of $\Gamma$ (the centers), such that

$$\Gamma = \bigsqcup_{i=1}^{n} \bigsqcup_{\gamma \in S_i} \gamma F_i.$$

Amenability for groups has a straightforward generalization to more general metric spaces. For the purposes of this work we restrict ourselves to those metric spaces $(X, d)$ with *bounded geometry* (meaning that for any radius $R > 0$ we have $\sup_{x \in X} |B_R(x)| < \infty$) for reasons that will become clear shortly. To define amenability, we need the following notation: For a finite subset $F \subseteq X$ we will write

$$\partial_R^+(F) = \{x \in X \setminus F \mid d(x, F) \leq R\}$$

for what is often called the *outer $R$-boundary of $F$*.

**Definition 4.2.** Let $X$ be a bounded geometry metric space. Then $X$ is called *amenable* if for every $R > 0$ and $\varepsilon > 0$ there exists a finite set $F \subseteq X$, such that $|\partial_R^+(F)| < \varepsilon |F|$.

A set $F$ as in the definition above is often referred to as an $(R, \varepsilon)$-Følner set. Now Theorem 4.1 says that every amenable group does not just admit Følner sets of arbitrary invariance, but can be completely decomposed into Følner sets of arbitrary invariance. The following definition is a version of the latter property for arbitrary metric spaces of bounded geometry.

**Definition 4.3.** Let $X$ be a bounded geometry metric space. We say that $X$ *admits tilings of arbitrary invariance*, if for all $R > 0$ and $\varepsilon > 0$ there exists a partition $X = \bigsqcup_{i \in I} X_i$ of $X$, such that $|\partial_R^+(X_i)| < \varepsilon |X_i|$ for all $i \in I$ and $\sup_{i \in I} diam(X_i) < \infty$.

Let us illustrate this property by considering the following elementary example:

**Example 4.4.** We will show that the integers $\mathbb{Z}$ viewed as a discrete metric space with respect to the euclidean metric admits tilings of arbitrary invariance. The main point is that if $I$ is an interval in $\mathbb{Z}$ then the number $|\partial_R^+(I)|$ is at most $2R$, and hence independent of the size and position of the chosen interval. Hence, given $R > 0$, $\varepsilon > 0$, fix a natural number $N > \frac{2R}{\varepsilon}$ and partition $\mathbb{Z}$ into intervals $\mathbb{Z} = \bigsqcup_n I_n$ such that $|I_n| = N$ for all $n \in \mathbb{N}$. Then each

$I_n$ is an $(R, \varepsilon)$-Følner set by our choice of $N$ and $diam(I_n) \leq N$ since each $I_n$ is an interval, so we are done.

Clearly, admitting tilings of arbitrary invariance is a very strong form of amenability. As already explained, it was the tiling result for amenable groups that inspired the definition above. Indeed, every countable discrete group $\Gamma$ can be equipped with a proper left-invariant metric $d$ that is unique up to bijective coarse equivalence [36, Lemma 2.1]. The simplest examples are finitely generated discrete groups equipped with word metrics.

In particular, in the case of a countable discrete group equipped with any proper left-invariant metric, Theorem 4.1 tells us that admitting tilings of arbitrary invariance is in fact equivalent to amenability of the group.

We now establish the connection to rigidity properties in topological dynamics. To this end we use a construction of Skandalis, Tu, and Yu in [33], which associates to every (discrete) metric space $X$ of bounded geometry a groupoid $G(X)$ over the Stone-Čech compactification $\beta X$ of $X$. Let us recall this construction: For any radius $R \geq 0$ let $\Delta_R = \{(x, y) \in X \times X \mid d(x, y) \leq R\}$ be the $R$-neighbourhood of the diagonal in $X \times X$ and let $\overline{\Delta_R}$ denote its closure in $\beta(X \times X)$. Recall that we identify any subset $S \subset X \times X$ with the corresponding set of principal ultrafilters in $\beta(X \times X)$. Then, as a set, one defines

$$G(X) = \bigcup_{R \geq 0} \overline{\Delta_R}.$$

Equip $G(X)$ with the weak topology it inherits from the union of compact open sets $\overline{\Delta_R}$ and with the groupoid structure it inherits as a subset of the pair groupoid $\beta X \times \beta X$. It was shown in [33, Proposition 3.2] that with the structure described above, $G(X)$ is a principal ample locally compact $\sigma$-compact Hausdorff groupoid with $G(X)^{(0)} = \beta X$. We call $G(X)$ the *coarse groupoid* associated to the metric space $X$.

The following is the main result of this section:

**Theorem 4.5.** *Let $X$ be a bounded geometry metric space. Then the following are equivalent:*

*(1) $G(X)$ is almost finite,*

*(2) $X$ admits tilings of arbitrary invariance.*

*In particular, $G(X)$ is strongly almost finite if and only if every subspace of $X$ admits tilings of arbitrary invariance.*

For the proof we need to recall some terminology and facts from [37] and we are indebted to Rufus Willett for pointing us towards this article. A *partial translation* is a bijection $t : dom(t) \to ran(t)$ between two subsets $dom(t)$ and $ran(t)$ of $X$ such that $\sup_{x \in dom(t)} d(x, t(x)) < \infty$. A partial translation is called *compatible* with $\omega \in \beta X$ if $\omega(domt(t)) = 1$ (i.e. $\omega \in \overline{dom(t)} \subset \beta(X)$). Given $\omega \in \beta X$, and $t : dom(t) \to ran(t) \subseteq \beta(X)$ a compatible partial translation, we use the notion of limit along the ultrafilter to define

$$t(w) := \lim_{\omega} t \in \beta(X).$$

In particular, for a fixed $\omega \in \beta(X)$, an ultrafilter $\alpha \in \beta(X)$ is called *compatible* with $\omega$ if there exists a partial translation $t$ which is compatible with $\omega$ and satisfies $t(\omega) = \alpha$. We write

$X(\omega)$ for the set of all $\alpha \in \beta X$ which are compatible with $\omega$. Note that there is a canonical bijection $F : X(\omega) \to G(X)_\omega$, given by $F(\alpha) = (\alpha, \omega)$. The set $X(\omega)$ can be equipped with a canonical metric. Let $(t_\alpha)_{\alpha \in X(\omega)}$ be a compatible family of partial translations for $\omega$, i.e. each $t_\alpha$ is compatible with $\omega$ and $t_\alpha(\omega) = \alpha$. Then one can define

$$d_\omega(\alpha, \beta) = \lim_{x \to \omega} d(t_\alpha(x), t_\beta(x)).$$

It was shown in [37, Proposition 3.7] that $d_\omega$ does indeed define a metric on $X(\omega)$ which does not depend on the choice of the compatible family. Using this freedom in choosing the compatible family we observe the following:

**Lemma 4.6.** *Let $\omega \in \beta X$ and $R \geq 0$. If $(\alpha, \omega) \in \overline{\Delta_R}$, then $d_\omega(\alpha, \omega) \leq R$.*

*Proof.* Since $\overline{\Delta_R}$ is compact and open we may choose a compatible family such that

$$\sup_{x \in X} d(t_\alpha(x), x) \leq R$$

for all $\alpha \in X(\omega)$ with $(\alpha, \omega) \in \overline{\Delta_R}$ and such that $t_\omega$ is the identity map on a suitable neighbourhood of $\omega$ in $\beta X$. It follows that

$$d_\omega(\alpha, \omega) = \lim_{x \to \omega} d(t_\alpha(x), t_\omega(x)) = \lim_{x \to \omega} d(t_\alpha(x), x) \leq R.$$

□

*Proof of Theorem 4.5.* Suppose first, that $G(X)$ is almost finite. Given $R > 0$ and $\varepsilon > 0$ we can find a $(\overline{\Delta_R}, \varepsilon)$-invariant elementary subgroupoid $K \subseteq G(X)$. Let $\{x_i \mid i \in I\}$ be a family of representatives for the action of $K$ on $X$ and $X_i := Kx_i$. Then $X = \bigsqcup X_i$ and since $K$ is compact, we must have $K \subseteq \overline{\Delta_S}$ for some $S \geq 0$, from which it follows that $\sup diam(X_i) \leq S$. Since $K$ is $(\overline{\Delta_R}, \varepsilon)$-invariant we get

$$|\partial_R^+(X_i)| = |\overline{\Delta_R} Kx_i \setminus Kx_i| < \varepsilon |Kx_i| = \varepsilon |X_i|,$$

obtaining the desired implication.

For the converse, let $C \subseteq G(X)$ be a compact subset and $\varepsilon > 0$. By compactness of $C$ there exists an $R > 0$ such that $C \subseteq \overline{\Delta_R}$. An application of condition (2), described in the statement, provides a partition $X = \bigsqcup_{i \in I} X_i$ of $X$, such that $\frac{|\partial_R^+(X_i)|}{|X_i|} < \varepsilon$ for all $i \in I$ and $\sup_{i \in I} diam(X_i) < \infty$. Define an equivalence relation $\mathcal{R} \subseteq X \times X$ by $x \mathcal{R} y$ if and only if there exists an $i \in I$ such that $x, y \in X_i$, i.e. $\mathcal{R}$ is precisely the equivalence relation which has the $X_i$ as its equivalence classes. Since the diameters of the $X_i$ are uniformly bounded, there exists an $S \geq 0$ such that $\mathcal{R} \subseteq \Delta_S$. We let $K$ be the closure of $\mathcal{R}$ in $\beta(X \times X)$. Then $K \subseteq \overline{\Delta_S} \subseteq G(X)$ is a compact open principal subgroupoid of $G(X)$ by construction. It remains to show, that $K$ is $(C, \varepsilon)$-invariant. For this we differentiate two situations:

(1) For $x \in X \subseteq G(X)^{(0)}$ fix $i \in I$ such that $Kx = X_i$. Then

$$|CKx \setminus Kx| \leq |\Delta_R Kx \setminus Kx| = |\partial_R^+(X_i)| < \varepsilon |X_i| = \varepsilon |Kx|;$$

hence, the claim follows.

(2) If $\omega \in \beta X \setminus X$ we need some more work. Let $(t_\alpha)_{\alpha \in X(\omega)}$ be a compatible family. Using that $K$ is compact and open in $G(X)$, we may (replacing finitely many $t_\alpha$, if necessary) assume that:
    - for each $\alpha \in X(\omega)$ such that $(\alpha, \omega) \in K$ we have: $U_{t_\alpha} \subseteq K$, where $U_{t_\alpha} = \{[\overline{t_\alpha}, \gamma] \mid \gamma \in \overline{dom(t_\alpha)}\}$ is a basic compact open bisection,
    - $r(U_{t_\alpha}) \cap r(U_{t_\beta}) = \emptyset$ whenever $\alpha \neq \beta$ and $(\alpha, \omega), (\beta, \omega) \in K$,
    - $t_\omega$ is the identity on a neighbourhood of $\omega$, and
    - $s(U_{t_\alpha}) = s(U_{t_\beta})$ for all $(\alpha, \omega), (\beta, \omega) \in K$.

    Using that the map $\beta X \to \mathbb{N}$, given by $\omega \mapsto |K\omega|$ is continuous (apply continuity of the Haar system on $G(X)$ to the characteristic function $1_K$), we may shrink the $U_{t_\alpha}$ further to assume that $|Ky| = |K\omega|$ for all $y \in dom(t_\alpha)$ for all $\alpha$ such that $(\alpha, \omega) \in K$.

    Now let $F : X(\omega) \to G(X)_\omega$ be the bijection from [37, Lemma C.3]. Then apply [37, Proposition 3.10] to the finite set $F^{-1}(\overline{\Delta_R} K\omega) \subseteq X(\omega)$ to find a subset $Y \subseteq X$ with $\omega(Y) = 1$, and for each $y \in Y$ an isometry $f_y : F^{-1}(\overline{\Delta_R} K\omega) \to X$ given by $f_y(\alpha) = t_\alpha(y)$. Then we claim that for all $y \in Y$ there exists a (unique) $i \in I$, such that $f_y(F^{-1}(K\omega)) = X_i$.

    *Proof of Claim.* Given $\alpha, \beta \in X(\omega)$ such that $(\alpha, \omega), (\beta, \omega) \in K$ we have that $[t_\alpha, y] \in U_{t_\alpha} \subseteq K$ and $[t_\beta, y] \in U_{t_\beta} \subseteq K$. Since $K$ is a subgroupoid, it follows that $[t_\beta \circ t_\alpha^{-1}, t_\alpha(y)] \in K$. But this means that $(t_\beta(y), t_\alpha(y)) \in \mathcal{R}$ and hence $f_y(\alpha)$ and $f_y(\beta)$ are in the same $X_i$.

    Conversely, we have $y, \omega \in \overline{dom(t_\alpha)}$ for every $\alpha \in F^{-1}(K\omega)$. Using our choice of the $t_\alpha$ we get $|K\omega| = |Ky| = |X_i|$. Since $f_y$ is an injection defined on a finite set, our claim follows. $\blacksquare$

    Using Lemma 4.6 and the fact that $f_y$ is an isometry, it is easy to check that $f_y(F^{-1}(\overline{\Delta_R} K\omega \setminus K\omega)) \subseteq \partial_R^+(X_i)$. Putting everything together we obtain

$$\begin{aligned} |CK\omega \setminus K\omega| &\leq |\overline{\Delta_R} K\omega \setminus K\omega| \\ &= |f_y(F^{-1}(\overline{\Delta_R} K\omega \setminus K\omega)| \\ &\leq |\partial_R^+(X_i)| \\ &< \varepsilon |X_i| = \varepsilon |K\omega|. \end{aligned}$$

This completes the proof of the first statement. For the second one, we first notice that $G(X)|_K = G(K \cap X)$ for every compact open subset $K$ of $\beta X$. In fact, this is the canonical one-to-one correspondence between subsets of $X$ and compact open subsets of $\beta X$. Hence, the second statement follows from the first. $\square$

We have the following immediate consequence, which indicates that admitting tilings of arbitrary invariance is a useful notion from a coarse geometric point of view.

**Corollary 4.7.** *Admitting tilings of arbitrary invariance is a coarse invariant. Moreover, if* $f \colon X \to Y$ *is a coarse equivalence between two bounded geometry metric spaces then* $G(X)$ *is strongly almost finite if and only if* $G(Y)$ *is strongly almost finite.*

*Proof.* The first statement follows from Theorem 4.5 and Proposition 3.6, once we note that coarsely equivalent metric spaces have Morita equivalent coarse groupoids (see [33, Corollary 3.6]). The second statement follows from Theorem 4.5 and the first statement, since $A$ and $f(A)$ are coarsely equivalent for every $A \subseteq X$. □

**Corollary 4.8.** *Let $\Gamma$ be a countable discrete group. Let $M \subseteq \beta\Gamma$ be the universal minimal $\Gamma$-space. Then the following are equivalent:*

*(1) $\Gamma$ is amenable.*
*(2) $G(|\Gamma|) = \Gamma \ltimes \beta\Gamma$ is almost finite.*
*(3) $\Gamma \ltimes M$ is almost finite.*
*(4) $\Gamma \ltimes (\beta\Gamma \backslash \Gamma)$ is almost finite.*

*Proof.* Suppose first that $\Gamma$ is amenable. Applying the main result of [11], we obtain an exact tiling of $\Gamma$ whose tiles are $(K, \varepsilon)$-invariant, i.e. we obtain a number $n \in \mathbb{N}$, finite subsets $S_1, \ldots, S_n \subseteq \Gamma$ and subsets $F_1, \ldots, F_n$ of $\Gamma$, such that

$$\Gamma = \bigsqcup_{i=1}^{n} \bigsqcup_{\gamma \in S_i} \gamma F_i.$$

This verifies the condition in Theorem 4.5, so $G(|\Gamma|)$ is indeed almost finite. If $G(|\Gamma|)$ is almost finite, then so is its restriction to the minimal closed $\Gamma$-invariant subset $M \subseteq \beta\Gamma$. But $G(|\Gamma|)_{|M} \cong \Gamma \ltimes M$. Similarly, (2) ⇒ (4).

The implications (3) ⇒ (1) and (4) ⇒ (1) now follow from [23, Proposition 4.7]. □

Let us now use the above characterization to treat another class of bounded geometry metric spaces that has attracted a lot of attention in geometric group theory, namely the so-called *box spaces* associated to any countable discrete residually finite group. Let us recall the relevant definitions: Suppose $\Gamma$ is a countable discrete residually finite group and $\sigma = (N_i)_{i \in \mathbb{N}}$ is a decreasing sequence of finite index normal subgroups of $\Gamma$ whose intersection $\bigcap_{i \in \mathbb{N}} N_i$ is trivial. Equip $\Gamma$ with a proper right-invariant metric $d$. For each $i \in \mathbb{N}$ let $\pi_i : \Gamma \to \Gamma/N_i$ be the canonical quotient map, and equip $\Gamma/N_i$ with the quotient metric. Then the box space $\Box_\sigma \Gamma$ is defined as the coarse disjoint union $\bigsqcup_i \Gamma/N_i$ (see e.g. [40, Definition 6.3.2]). In this setting, the following is our main result.

**Proposition 4.9.** *Let $\Gamma$ be a countable discrete residually finite group with any nested decreasing sequence $\sigma = (N_i)_{i \in \mathbb{N}}$ of finite index normal subgroups of $\Gamma$. Then the following are equivalent:*

*(1) $\Gamma$ is amenable;*
*(2) $\Box_\sigma \Gamma$ admits tilings of arbitrary invariance;*
*(3) $G(\Box_\sigma \Gamma)$ is almost finite.*

*Proof.* ((1)⇒(2)) Fix an arbitrary radius $R > 0$, a tolerance $\varepsilon > 0$ and a nested decreasing sequence $\sigma = (N_i)_{i \in \mathbb{N}}$. By a classical result of Weiss [39] (see also [10, Proposition 5.5] for the version we are using), we can find a (large) number $i_0 \in \mathbb{N}$ and a finite subset $T \subseteq \Gamma$ such that

(i) $\Gamma = \bigsqcup_{\gamma \in N_{i_0}} T\gamma$ (i.e. $T$ is a monotile and $N_{i_0}$ is the set of tiling centers), and

(ii) $|\partial_R^+(T)| < \varepsilon |T|$.

Moreover, by the definition of a box space and [35, Lemmas 2.7, 2.11], we may choose $i_1 \geq i_0$ such that

(iii) $d(\Gamma/N_{i-1}, \Gamma/N_i) > R$ for all $i \geq i_1$, and

(iv) for every $i \geq i_1$ the quotient map $\pi_i : \Gamma \to \Gamma/N_i$ has large isometry radii, in the sense that each $\pi_i$ is isometric on $B_{R+L}(\gamma)$ for all $\gamma \in N_{i_0}$, where $L := \max\{d(t,e) \mid t \in T\}$ and $e$ the identity.

Now for each $i \geq i_1$, let $C_i$ be a complete family of representatives for the quotient $N_{i_0}/N_i$. Set $X_0 := \bigsqcup_{i<i_1} \Gamma/N_i$. Then we have a decomposition

$$\Box_\sigma \Gamma = X_0 \sqcup \bigsqcup_{i \geq i_1} \bigsqcup_{c \in C_i} \pi_i(Tc). \tag{4.1}$$

Note that the latter union is indeed disjoint by our choice of $C_i$ and property (i) above.

We claim that for every $i \geq i_1$ and every $c \in C_i$, the quotient map $\pi_i$ restricts to an isometric bijection $\partial_R^+(Tc) \to \partial_R^+(\pi_i(Tc))$. Indeed, using right-invariance of the metric on $\Gamma$, we have $\partial_R^+(Tc) \subseteq B_{R+L}(c)$. It follows from item (iv) that $\pi_i$ is isometric on $B_{R+L}(c)$; hence, one has that $\pi_i(\partial_R^+(Tc)) \subseteq \partial_R^+(\pi_i(Tc))$. To see the converse inclusion, let $xN_i \in \partial_R^+(\pi_i(Tc))$ and observe that $\partial_R^+(\pi_i(Tc)) \subseteq \Gamma/N_i$ by item (iii). Also, let $t \in T$ such that $d(xN_i, tcN_i) = d(xN_i, \pi_i(Tc))$. Then we have

$$R \geq d(xN_i, tcN_i) = \inf_{m,n \in N_i} d(xn, tcm) = \inf_{m,n \in N_i} d(xnm^{-1}, tc).$$

Therefore, there exists a $y \in \Gamma$ such that $xN_i = yN_i$ and $d(y, Tc) \leq R$. Clearly, we have $y \notin Tc$ and hence $y \in \partial_R^+(Tc)$ such that $\pi_i(y) = xN_i$.

Combining the above, that the metric on $\Gamma$ is right-invariant and item (ii), we obtain

$$|\partial_R^+(\pi_i(Tc))| = |\partial_R^+(Tc)| = |\partial_R^+(T)| < \varepsilon|T| = \varepsilon|\pi_i(Tc)|.$$

Notice that by (iii) we also have $\partial_R^+(X_0) = \emptyset$, so every set in the decomposition (4.1) is $(R,\varepsilon)$-invariant, as desired.

Finally, combining item (iv) and the fact that the metric on $\Gamma$ is right-invariant, we deduce $diam(\pi_i(Tc)) = diam(Tc) = diam(T)$. So

$$S := \max\{diam(T), diam(X_0)\}$$

is a uniform bound on the diameters of the sets appearing in the decomposition (4.1).

((2)⇒(3)) Provided by Theorem 4.5.

((3)⇒(1)) For ease of notation, we set $X := \Box_\sigma \Gamma$. Then $G(X)|_{\beta X \setminus X} = (\beta X \setminus X) \rtimes \Gamma$ by [15, Proposition 2.50 and Example 2.6]. Since $\beta X \setminus X$ is a closed $G(X)$-invariant subset of $\beta X$, it follows that $(\beta X \setminus X) \rtimes \Gamma$ is almost finite as well. The claim now follows from [23, Proposition 4.7]. $\square$

We conclude from the above Proposition that admitting tilings of arbitrary invariance is indeed a much stronger property than amenability if one considers metric spaces beyond groups: box spaces are always (supr)amenable for rather trivial reasons. However, there exist many examples of finitely generated residually finite groups which are not amenable (e.g. the free groups $\mathbb{F}_n$ or $SL_n(\mathbb{Z})$ for $n \geq 2$).

We will now proceed to present a construction that starting from any bounded geometry metric space $X$ produces another bounded geometry metric space $Y$ containing $X$, such that $Y$ admits tilings of arbitrary invariance. This will be very useful later in order to exhibit our examples.

**Proposition 4.10.** *Let $X$ be a discrete metric space with bounded geometry. Then the metric space $Y := X \times \mathbb{N}$ with the graph metric (i.e. $d_Y((x,n),(y,m)) = n + m + d_X(x,y)$ whenever $x \neq y$ and $d_Y((x,n),(x,m)) = |n - m|$) has bounded geometry and admits tilings of arbitrary invariance.*

*As a consequence, admitting tilings of arbitrary invariance does not imply Yu's property A.*

*Proof.* We start by showing that $Y$ has bounded geometry. Indeed if $R \geq 0$ we have $C := \sup_{x\in X}|B_R(x)| < \infty$ and then $|B_R(x,k)| \leq 2R|B_R(x)| \leq 2RC$. Next we will prove that $Y$ admits tilings of arbitrary invariance. Let $R > 0$ and $\varepsilon > 0$ be given. Since $Y$ has bounded geometry, there exists and $S \geq 0$ such that $\sup_{x\in X}|B_R(x,0)| \leq S$. Now let $N > \frac{max\{S,R\}+R}{\varepsilon}$. For $x \in X$ and $k \in \mathbb{N}$ write $Y_{x,k} := \{x\} \times \{kN, \dots, ((k+1)N) - 1\}$. Then we obtain a partition

$$Y = \bigsqcup_{x\in X} \bigsqcup_{k=0}^{\infty} Y_{x,k}.$$

The cardinality of each $Y_{x,k}$ is precisely $N$ and its diameter is $N - 1$ independent of $x$ and $k$. It remains to show that the outer boundary of each of the sets in this partition is small relative to its cardinality. If $k \neq 0$ then $|\partial_R^+(Y_{x,k})| \leq 2R < \varepsilon N = \varepsilon|Y_{x,k}|$. If $k = 0$, then we have $\partial_R^+(Y_{x,0}) \subseteq B_R(x,0) \cup \{x\} \times \{N, \dots, N + R - 1\}$. It follows that $|\partial_R^+(Y_{x,0})| \leq S + R < \varepsilon N = \varepsilon|Y_{x,0}|$.

The last statement follows from the fact that Yu's property A passes to subspaces (see [36, Proposition 4.2]). □

Note that the above shows in particular that the property of admitting tilings of arbitrary invariance suffers the same shortcoming as amenability: It does not pass to arbitrary subspaces. Combined with Theorem 4.5 and using the identification $G(X \times \mathbb{N})_{|\overline{X}^{\beta(X\times\mathbb{N})}} \cong G(X)$, our constructions show that almost finiteness for groupoids does not pass to restrictions to arbitrary compact open subsets.

Moreover, we can use it to produce a lot of examples which show that admitting tilings of arbitrary invariance is independent from other notions frequently studied in coarse geometry.

**Example 4.11.**
(1) Let $X$ be a bounded geometry metric space without Yu's property A. Then $Y = X \times \mathbb{N}$ defined as in Proposition 4.10 contains $X$ as a subspace by the construction. Hence, $Y$ admits tilings of arbitrary invariance and cannot have Yu's property A. Conversely, the free group on two generators $\mathbb{F}_2$ has Yu's property A, but can not admit tilings of arbitrary invariance, since it is non-amenable. Recall that $\mathbb{F}_2$ has asymptotic dimension one, so even finite asymptotic dimension does not imply tilings of arbitrary invariance.
(2) Let $X$ be a bounded geometry metric space which does not coarsely embed into a Hilbert space. Then $Y = X \times \mathbb{N}$ does not coarsely embed into a Hilbert space as well, but $Y$ admits tilings of arbitrary invariance.

The above examples are also very interesting when combined with Theorem 4.5. Most examples of almost finite groupoids known so far are amenable. In fact, for a transformation groupoid $\Gamma \ltimes X$ associated to a topologically free action of a discrete group $\Gamma$ acting on a totally disconnected compact space $X$, almost finiteness implies amenability of the acting group and a posteriori amenability of $\Gamma \ltimes X$ by [23, Proposition 4.7]. Our results yield new examples of almost finite groupoids which lack other desirable properties like amenability or a-T-menability. In particular, this shows that almost finiteness for general ample groupoids behaves very differently from the transformation groupoid case.

**Corollary 4.12.** *There exist almost finite ample principal groupoids $G$ which lack at least one of the following properties:*

1. *amenability,*
2. *a-T-menability,*

*Moreover, there exist ample groupoids with finite dynamic asymptotic dimension which are not almost finite.*

*Proof.* To obtain the desired examples just take the coarse groupoid for the metric spaces described in Example 4.11 and combine them with the following facts:

1. $G(Y)$ is amenable if and only if $Y$ has property A [33, Theorem 5.3];
2. $G(Y)$ is a-T-menable if and only if $Y$ coarsely embeds into a Hilbert space [33, Theorem 5.4]; and

For the final statement, we consider $\beta\mathbb{F}_2 \rtimes \mathbb{F}_2 = G(\mathbb{F}_2)$. From [18, Theorem 6.4] we know that the dynamic asymptotic dimension of $G(\mathbb{F}_2)$ equals one, but as seen above $G(\mathbb{F}_2)$ is not almost finite. □

We should mention that Gabor Elek has independently found examples of non-amenable almost finite groupoids using a different approach (see [13] for further details).

Finally, prompted by the results in section 3, we want to give some examples of strongly almost finite groupoids.

**Example 4.13.** The coarse groupoids $G(\mathbb{Z})$ and $G(\mathbb{N})$ are strongly almost finite. Let us focus on the case of the integers $\mathbb{N}$ (the result for $\mathbb{Z}$ follows the same line of argument by doing everything in two "directions"). In view of Theorem 4.5 it is enough to show that every subspace $A \subseteq \mathbb{N}$ admits tilings of arbitrary invariance. If $A \subseteq \mathbb{N}$ is bounded, it is finite and hence there is nothing to do. So let us assume that $A$ is unbounded. Write $A = \{a_n \mid n \in \mathbb{N}\}$ as an increasing sequence. Then there are two options: If $\sup_{n\in\mathbb{N}}|a_n - a_{n+1}| < \infty$, then $A$ is coarsely equivalent to $\mathbb{N}$ itself and hence admits tilings of arbitrary invariance. We can deal with the remaining case $\sup_{n\in\mathbb{N}}|a_n - a_{n+1}| = \infty$ by hand: Let $R > 0$ and $\varepsilon > 0$ be given. Let $N > \frac{2R}{\varepsilon}$. First, since the above supremum is infinite, we can find a subsequence $(a_{n_m})_m$ in $A$ such that $|a_{n_m+1} - a_{n_m}| > R$ for all $m \in \mathbb{N}$ and $|a_{n+1} - a_n| \leq R$ for all $n \notin \{n_m \mid m \in \mathbb{N}\}$. Now let $A_1 = \{a_1, \ldots, a_{n_1}\}$ and for $m > 1$ we let $A_m := \{a_{n_{m-1}+1}, \ldots, a_{n_m}\}$. These sets form a disjoint partition of $A$ into $(R, \varepsilon)$-Følner sets such that $diam(A_m) \leq R|A_m|$. So if $\sup_m |A_m| < \infty$ we are done. This need not be the case however, so assuming that the sequence $(|A_m|)_m$ is unbounded, we need to refine our partition further. Now pick the subsequence consisting of all $A_{m_k}$ such that $|A_{m_k}| \geq N \geq \frac{2R}{\varepsilon}$. We may assume that the maximal element of $A_{m_k}$ is strictly smaller than the smallest element of $A_{m_{k+1}}$ for all $k \in \mathbb{N}$. Now, writing $A_{m_k}$ as an increasing sequence we can easily partition each $A_{m_k}$ as $A_{m_k} = \bigsqcup_{l=1}^{L_{m_k}} B_{m_k,l}$, where $B_{m_k,1}$ consists of the first $N$ elements of $A_{m_k}$, $B_{m_k,2}$ of the next $N$ elements and so on, such that $N = |B_{m_k,l}|$ for all $1 \leq l < L_{m_k}$ and $N \leq |B_{m_k,L_{m_k}}| \leq 2N$. Then, we clearly have

$$\frac{|\partial_R(B_{m_k,l})|}{|B_{m_k,l}|} \leq \frac{2R}{N} < \varepsilon.$$

Thus,

$$A = \bigsqcup_{m:|A_m|\leq N} A_m \sqcup \bigsqcup_{k=1}^{\infty} \bigsqcup_{l=1}^{L_{m_k}} B_{m_k,l}$$

is a partition of $A$ into $(R,\varepsilon)$-Følner sets of diameter at most $2RN$.

We finish this subsection by slightly improving upon the above example. Recall the following definition:

**Definition 4.14.** Let $X$ be a metric space. We say that the *asymptotic dimension* of $X$ does not exceed $n$ and write $\operatorname{asdim}(X) \leq n$ provided for every $R > 0$ there exist $R$-disjoint families $\mathcal{U}^0, \dots, \mathcal{U}^n$ of uniformly bounded subsets of $X$ such that $\cup_i \mathcal{U}^i$ is a cover of $X$.

It is trivial to see that every bounded geometry metric space $X$ with $\operatorname{asdim}(X) = 0$ admits tilings of arbitrary invariance. Hence, its coarse groupoid $G(X)$ is strongly almost finite by Theorem 4.5.

**Proposition 4.15.** *Let $\Gamma$ be a finitely presented amenable group with asdim$(\Gamma) = 1$. Then $G(\Gamma)$ is strongly almost finite.*

*Proof.* By [17, Theorem 2] $\Gamma$ must be virtually cyclic. In particular, $\Gamma$ and $\mathbb{Z}$ are coarsely equivalent. From Corollary 4.7 we only have to show that $G(\mathbb{Z})$ is strongly almost finite, which is done in Example 4.13. ☐

### 4.1. **Non-amenable spaces.**

Non-amenable metric spaces are well-studied in terms of their connections with properly infinite Roe-algebras [4, 5]. Using the type semigroup of the coarse groupoid and the dichotomy between amenability and paradoxicality for discrete metric spaces, we will in this section recover a celebrated Theorem by Block and Weinberger by a rather easy and conceptual proof.

**Proposition 4.16.** *Let $X$ be a bounded geometry metric space and $G(X)$ be the coarse groupoid of $X$. Then the following are equivalent:*

*(1) $X$ is non-amenable;*
*(2) every order unit in $S(G(X))$ is properly infinite;*
*(3) $H_0(G(X)) = 0$.*

*Proof.* (1) $\Rightarrow$ (2) Since a non-amenable space admits a paradoxical decomposition $[1_{\beta X}]$ is properly infinite in $S(G(X))$. Now if $K \subseteq \beta X$ is $G(X)$-full, then $K \cap X$ is cobounded in $X$ and hence coarsely equivalent to $X$ itself. Since paradoxicality is a coarse invariant, $K \cap X$

is paradoxical itself and hence $[1_K]$ is properly infinite in $S(G(K \cap X)) = S(G(X)_{|K}) \cong S(G(X))$.

(2) ⇒ (3): First of all $[1_{\overline{A}}]_0 = 0$ in $H_0(G(X))$ for all cobounded $A \subseteq X$, since a properly infinite element in $S(G(X))$ actually satisfies $2[1_A] = [1_A]$ and so we can just cancel in $H_0(G(X))$. Now if $A \subseteq X$ is arbitrary follow the arguments in the proof of [22, Lemma 5.4] to get $[1_{\overline{A}}]_0 = 0$. The claim follows.

(3) ⇒ (1): Let $H_0(G(X)) = 0$ and suppose $X$ is amenable. Then there exists a $G(X)$-invariant Borel probability measure $\mu \in M(G(X))$. Denoting by $\widehat{\mu}$ the corresponding functional on $H_0(G(X))$, it follows that $0 = \widehat{\mu}(0) = \widehat{\mu}([1_{\beta X}]) = \mu(\beta X) = 1$, which is a contradiction. □

The remaining step is the identification of the 0-th groupoid homology group of the coarse groupoid with the 0-th uniformly finite homology group of $X$. Recall that $H_n^{uf}(X)$ is obtained from the chain complex $(C_n^{uf}(X,\mathbb{Z}), \partial_n^{uf})_n$, where $C_n^{uf}(X,\mathbb{Z})$ consists of formal linear combinations $c = \sum_n c_{\overline{x}}\overline{x}$, where $\overline{x}$ denotes an $(n+1)$-tuple $(x_0,\dots,x_n) \in X^{n+1}$, $c_{\overline{x}} \in \mathbb{Z}$ such that

1. $c$ has finite propagation, in the sense that there exists a constant $P_c > 0$ such that $c_{\overline{x}} = 0$, provided that $\max d(x_i,x_j) \geq P_c$,
2. and $c$ is bounded, meaning that $\sup_{\overline{x} \in X^{n+1}} |c_{\overline{x}}| < \infty$.

The boundary map $\partial_n^{uf} : C_n^{uf}(X,\mathbb{Z}) \to C_{n-1}^{uf}(X,\mathbb{Z})$ is defined on simplices by $\partial_n^{uf}(x_0,\dots,x_n) = \sum_{i=0}^n (-1)^i (x_0,\dots,\widehat{x_i},\dots,x_n)$, where hat denots omission of the term. One extends $\partial_n^{uf}$ to the whole of $C_n^{uf}(X,\mathbb{Z})$ by linearity.

**Lemma 4.17.** *There is a canonical isomorphism* $H_0^{uf}(X,\mathbb{Z}) \cong H_0(G(X))$.

*Proof.* Using the universal property of the Stone-Čech compactification it is easy to see that there is a canonical linear bijection

$$\Phi_0 : C_0^{uf}(X,\mathbb{Z}) \to C(\beta X, \mathbb{Z})$$

given by extension of functions. Indeed, every element $c \in C_0^{uf}(X,\mathbb{Z})$ can be viewed as a bounded (continuous) function $c : X \to \mathbb{Z}$ and can hence be extended to a bounded continuous function $\beta X \to \mathbb{Z}$. Restriction of functions clearly gives an inverse to $\Phi$. To complete the proof we need to check that the respective boundary maps are compatible. Similarly to the above observation, we can view a chain $c \in C_1^{uf}(X,\mathbb{Z})$ as a bounded function $c : X \times X \to \mathbb{Z}$. Since $c$ has finite propagation, it is supported on $\Delta_{P_c}$. Again, we can extend $c$ continuously to a (compactly supported) function on $\overline{\Delta_{P_c}} \subseteq G(X)$, thus obtaining a well-defined linear map $\Phi_1 : C_1^{uf}(X,\mathbb{Z}) \to C_c(G(X),\mathbb{Z})$. Conversely, every function $f \in C_c(G(X),\mathbb{Z})$ is bounded and its support is contained in $\overline{\Delta_R}$ for some $R > 0$. Hence, restricting it to to $\Delta_R$ (and extending by zero on $X \times X \setminus \Delta_R$) gives rise to a chain in $C_1^{uf}(X,\mathbb{Z})$. One easily verifies that these constructions are inverse to each other. For $c \in C_1^{uf}(X,\mathbb{Z})$ let $c_i = \sum_{\overline{x}} c_{\overline{x}} x_i$, $i = 0,1$. Then we compute

$$\begin{aligned}\Phi_0(\partial_n^{uf}(c)) &= \Phi_0(c_1) - \Phi_0(c_0) \\ &= s_*(\Phi_1(c)) - r_*(\Phi_1(c)) = \partial_1(\Phi_1(c)),\end{aligned}$$

where the second equation clearly holds when checking only for elements in $X \subseteq \beta X$ and hence by continuity on the whole of $\beta X$. Thus we have verified $\Phi(im(\partial_1^{uf})) \subseteq im(\partial_1)$, and a similar computation using the inverses of the $\Phi_i$ shows equality. ☐

The following corollary was first proved by Block and Weinberger in [7, Theorem 3.1].

**Corollary 4.18.** *Let $X$ be a bounded geometry metric space. Then $X$ is non-amenable if and only if $H_0^{uf}(X, \mathbb{Z}) = 0$.*

*Proof.* It follows directly from Proposition 4.16 and Lemma 4.17. ☐

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