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On the structure of spaces of vector-valued Lipschitz functions

by

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Abstract. We analyse the strong connections between spaces of vector-valued Lipschitz functions and spaces of continuous linear operators. We apply these links to study duality, Schur properties and norm attainment in the former class of spaces as well as in their canonical preduals.

1. Introduction. The problem of whether Lipschitz-free Banach spaces ($\mathcal{F}(M)$), which are canonical preduals of spaces of Lipschitz functions ($\mathrm{Lip}_0(M)$), are themselves dual ones has been studied for a long time (see e.g. [6, 7, 18, 28]). For instance, given a compact (respectively proper) metric space M, it is known that $\mathcal{F}(M)$ is the dual of $\mathrm{lip}_0(M)$ (respectively $S_0(M)$) under some additional assumptions on M (see formal definitions below).

The vector-valued space $\mathcal{F}(M,X)$ has recently been introduced in [3] as a predual of the space $\mathrm{Lip}_0(M,X^*)$ of vector-valued Lipschitz functions in the spirit of the scalar version. Moreover, in [3, Proposition 1.1], it is proved that $\mathcal{F}(M,X)$ is linearly isometrically isomorphic to $\mathcal{F}(M)\,\widehat{\otimes}_\pi X$. So, basic tensor product theory yields a canonical predual of vector-valued Lipschitz-free Banach spaces, namely the injective tensor product of the predual of each factor, whenever they exist and satisfy some natural conditions. On the other hand, natural vector-valued extensions of the preduals of $\mathcal{F}(M)$, namely $\mathrm{lip}_0(M,X)$ and $S_0(M,X)$, have recently been considered in [9, 16] and identified with suitable subspaces of compact operators from X^* to $\mathrm{Lip}_0(M)$. Consequently, in order to generalise the preduality results to the vector-valued setting, it is a natural question whether the equality

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$S_0(M,X) = S_0(M) \widehat{\otimes}_\varepsilon X$ holds. We will show that this is the case under suitable assumptions on M and X.

The identification of vector-valued Lipschitz-free Banach spaces with a projective tensor product is motivated not only by the problem of analysing duality but also by other properties. Some of them have even been analysed in the scalar version, such as approximation properties or the Schur property. As approximation properties are preserved by injective as well as by projective tensor products [15], it is straightforward that such properties on $\mathcal{F}(M,X)$ actually rely on the scalar case $\mathcal{F}(M)$. Nevertheless, it is an open problem if the Schur property is preserved by projective tensor product [11, Remark 6], so it is natural to wonder which conditions on M and X guarantee that $\mathcal{F}(M,X)$ enjoys the Schur property, a problem which has previously been considered in the scalar framework [13, 18].

Finally, it is a natural question how different notions of norm attainment in $\mathrm{Lip}_0(M,X)$ are related. On the one hand, in this space there is a clear notion of norm attainment for a Lipschitz function. On the other hand, the equality $\mathrm{Lip}_0(M,X) = L(\mathcal{F}(M),X)$ yields the classical notion of norm attainment considered in spaces of continuous linear operators. So we can wonder when these concepts agree and, in that case, analyse when the class of Lipschitz functions which attain their norm is dense in $\mathrm{Lip}_0(M,X)$, a problem motivated by the celebrated Bishop–Phelps theorem and recently studied in [10] and [17].

The paper is organised as follows. In Section 2 we get the two main results on the duality problem. On the one hand, in Theorem 2.4 we show that $S_0(M,X)^* = \mathcal{F}(M,X^*)$ whenever M is proper satisfying $S_0(M)^* = \mathcal{F}(M)$ and either X^* or $\mathcal{F}(M)$ has the approximation property. On the other hand, we prove in Theorem 2.9 that $\mathrm{lip}_\tau(M,X)^* = \mathcal{F}(M,X^*)$ whenever either $\mathcal{F}(M)$ or X^* has the approximation property and (M,τ) satisfies the assumptions under which $\mathcal{F}(M) = \mathrm{lip}_\tau(M)^*$ in [18, Theorem 6.2]. Those results are vector-valued extensions of the preduality results in the real case given in [7] and [18].

In Section 3 we take advantage of the theory of tensor products to study the (hereditary) Dunford–Pettis property on $S_0(M,X)$ and the (strong) Schur property on $\mathcal{F}(M,X)$. More precisely, we prove that $S_0(M,X)$ has the hereditary Dunford–Pettis property and does not contain any isomorphic copy of ℓ_1 whenever X satisfies those two conditions and M is a proper metric space such that $S_0(M)^* = \mathcal{F}(M)$ (Theorem 3.2). As a direct corollary, under the same assumptions we deduce that $\mathcal{F}(M,X^*)$ has the strong Schur property.

We end that section by extending a result of Kalton to the vector-valued setting by proving that if M is uniformly discrete and X has the Schur property, then $\mathcal{F}(M,X)$ has the Schur property (Proposition 3.4). This last result provides examples of Banach spaces with the Schur property such that their projective tensor product also enjoys this property.

Furthermore, in Section 4, we deal with the problem of norm attainment, proving in Proposition 4.4 the denseness of $\mathrm{NA}(\mathcal{F}(M), X^{**})$ in $L(\mathcal{F}(M), X^{**})$ whenever M is a proper metric space satisfying $S_0(M)^* = \mathcal{F}(M)$ and either $\mathcal{F}(M)$ or X^* has the approximation property. Finally, in Section 5 we pose some open problems and make some related comments.

NOTATION. Given a metric space M, $B(x,r)$ (respectively $\overline{B}(x,r)$) denotes the open (respectively closed) ball in M centred at $x \in M$ with radius r. Following [18], by a *gauge* we will mean a continuous, subadditive and increasing function $\omega : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ with $\omega(0) = 0$ and $\omega(t) \geq t$ for every $t \in [0,1]$. We will say that a gauge ω is *non-trivial* whenever $\lim_{t\to 0} \omega(t)/t = \infty$.

Throughout the paper we only consider real Banach spaces. Given a Banach space X, we denote by B_X (respectively S_X) the closed unit ball (respectively the unit sphere) of X. We also denote by X^* the topological dual of X. Let $X \widehat{\otimes}_\pi Y$ (respectively $X \widehat{\otimes}_\varepsilon Y$) be the projective (respectively injective) tensor product of Banach spaces. For a detailed treatment and applications of tensor products, we refer the reader to [26].

Further, $L(X,Y)$ (respectively $K(X,Y)$) will denote the space of continuous (respectively compact) linear operators from X to Y. Given topologies τ_1 on X and τ_2 on Y, we will denote by $L_{\tau_1,\tau_2}(X,Y)$ and $K_{\tau_1,\tau_2}(X,Y)$ the respective subspaces of τ_1-τ_2-continuous operators. Following [26, Proposition 4.1], a Banach space X is said to have the *approximation property* (*AP*) whenever given a compact subset K of X and $\varepsilon > 0$ there exists a finite-rank operator $S \in L(X,X)$ such that $\|x - S(x)\| < \varepsilon$ for all $x \in K$. Note that this property passes to preduals.

Given a metric space M with a designated origin 0 and a Banach space X, we denote by $\mathrm{Lip}_0(M,X)$ the Banach space of all X-valued Lipschitz functions on M which vanish at 0 under the standard Lipschitz norm

$$\|f\| := \sup\left\{ \frac{\|f(x) - f(y)\|}{d(x,y)} : x, y \in M,\, x \neq y \right\}.$$

First of all, note that with no loss of generality we can consider any point of M as the origin, because the resulting Banach spaces turn out to be isometrically isomorphic (see for instance [28, comments after Definition 1.6.1]). Moreover, $\mathrm{Lip}_0(M,X^*)$ is a dual Banach space, with a canonical predual given by

$$\mathcal{F}(M,X) := \overline{\mathrm{span}}\{\delta_{m,x} : m \in M,\, x \in X\} \subseteq \mathrm{Lip}_0(M,X^*)^*,$$

where $\delta_{m,x}(f) := f(m)(x)$ for every $m \in M$, $x \in X$ and $f \in \mathrm{Lip}_0(M,X^*)$ (see [3]). Furthermore, for every metric space M and Banach space X,

$\mathrm{Lip}_0(M,X) = L(\mathcal{F}(M),X)$ (see e.g. [16]), and $\mathcal{F}(M,X) = \mathcal{F}(M) \widehat{\otimes}_\pi X$ (see [3, Proposition 1.1]).

We will also consider the following spaces of vector-valued Lipschitz functions:

$$\mathrm{lip}_0(M,X) := \Big\{ f \in \mathrm{Lip}_0(M,X) : \lim_{\varepsilon\to 0} \sup_{0<d(x,y)<\varepsilon} \frac{\|f(x)-f(y)\|}{d(x,y)} = 0 \Big\},$$

$$S_0(M,X) := \Big\{ f \in \mathrm{lip}_0(M,X) : \lim_{r\to\infty} \sup_{\substack{x \text{ or } y\notin B(0,r)\\ x\neq y}} \frac{\|f(x)-f(y)\|}{d(x,y)} = 0 \Big\}.$$

We will omit the reference to the Banach space when it is $\mathbb{R}$ in the above definitions. Finally, we will say that a subspace $S \subset \mathrm{Lip}_0(M)$ *separates points uniformly* if there exists a constant $c \geq 1$ such that for every $x, y \in M$ there is $f \in S$ satisfying $\|f\| \leq c$ and $f(x) - f(y) = d(x,y)$. Recall that if M is a proper metric space then $S_0(M)$ separates points uniformly if, and only if, it is a predual of $\mathcal{F}(M)$ [7].

2. Duality results on vector-valued Lipschitz-free Banach spaces. Let (M,d) be a metric space, X be a Banach space and assume that there exists a subspace S of $\mathrm{Lip}_0(M)$ such that $S^* = \mathcal{F}(M)$. Note that basic tensor theory yields the identification

$$\mathcal{F}(M,X^*) = \mathcal{F}(M) \widehat{\otimes}_\pi X^* = (S \widehat{\otimes}_\varepsilon X)^*$$

whenever either $\mathcal{F}(M)$ or X^* has (AP) and either $\mathcal{F}(M)$ or X^* has the Radon–Nikodým property (RNP) (see [26, Theorem 5.33]). However, the natural question here is when we can give a representation of a predual of $\mathcal{F}(M,X)$ as a subspace of $\mathrm{Lip}_0(M,X^*)$.

It has recently been proved in [9, Theorem 5.2] that $S_0(M,X)$ is isometrically isomorphic to $K_{w^*,w}(X^*, S_0(M))$ whenever M is proper. Consequently, in order to prove that $S_0(M,X)^* = \mathcal{F}(M,X^*) = \mathcal{F}(M) \widehat{\otimes}_\pi X^*$ under natural assumptions on M, we begin by analysing when $K_{w^*,w}(X^*,Y) = X \widehat{\otimes}_\varepsilon Y$. In order to do that, we need to introduce two results.

Lemma 2.1. *Let X, Y be Banach spaces. Then $T \mapsto T^*$ defines an isometry from $K_{w^*,w}(X^*,Y)$ onto $K_{w^*,w}(Y^*,X)$.*

Proof. Let $T \in K(X^*,Y) \cap L_{w^*,w}(X^*,Y)$. Then $T^* \in K(Y^*,X^{**})$. Moreover, given $y^* \in Y^*$, we find that $T^*(y^*) = y^* \circ T : X^* \to \mathbb{R}$ is w^*-continuous, and thus $T^*(y^*) \in X$. Therefore $T^* \in K(Y^*,X)$. Since T^* is $\sigma(Y^*,Y)$-$\sigma(X^{**},X^*)$-continuous, we get $T^* \in L_{w^*,w}(Y^*,X)$. Conversely, if $R \in K(Y^*,X) \cap L_{w^*,w}(Y^*,X)$ then $R^* \in K(X^*,Y) \cap L_{w^*,w}(X^*,Y)$ and $R^{**} = R$. ∎

The next proposition is well-known (see [25, Remark 1.2]), although we have not found any proof in the literature. We include it for completeness.

PROPOSITION 2.2. *Let X and Y be Banach spaces and assume that either X or Y has (AP). Then $K_{w^*,w}(X^*,Y) = X \widehat{\otimes}_\varepsilon Y$.*

Proof. By the above lemma we may assume that Y has (AP). Clearly the inclusion $\supseteq$ holds, so let us prove the reverse one. To this end pick a compact operator $T : X^* \to Y$ which is w^*-w-continuous. We will approximate T in norm by a finite-rank operator, following word for word [26, proof of Proposition 4.12]. As Y has (AP), we can find a finite-rank operator $R : Y \to Y$ such that $\|x - R(x)\| < \varepsilon$ for every $x \in T(B_{X^*})$, and define $S := R \circ T$. Then S is clearly a finite-rank operator such that $\|S - T\| < \varepsilon$. We have $S = \sum_{i=1}^n x_i^{**} \otimes y_i$ for suitable $n \in \mathbb{N}$, $x_i^{**} \in X^{**}$ and $y_i \in Y$. Moreover, S is w^*-w-continuous, which means that, for every $y^* \in Y^*$,

$$y^* \circ S = \sum_{i=1}^n y^*(y_i) x_i^{**} : X^* \to \mathbb{R}$$

is a w^*-continuous functional, so $\sum_{i=1}^n y^*(y_i)x_i^{**} \in X$ for each $y^* \in Y^*$. An easy bilinearity argument allows us to assume that $\{y_1, \ldots, y_n\}$ are linearly independent. Now, a straightforward application of the Hahn–Banach theorem shows that, for every $i \in \{1, \ldots, n\}$, there exists $y_i^* \in Y^*$ such that $y_j^*(y_i) = \delta_{ij}$. Therefore, for every $j \in \{1, \ldots, n\}$,

$$X \ni y_j^* \circ S = \sum_{i=1}^n y_j^*(y_i) x_i^{**} = \sum_{i=1}^n \delta_{ij} x_i^{**} = x_j^{**}.$$

Consequently, $S \in X \otimes Y$. Summarising, we have proved that each element of $K_{w^*,w}(X^*,Y)$ can be approximated in norm by an element of $X \otimes Y$, so $K_{w^*,w}(X^*,Y) = X \widehat{\otimes}_\varepsilon Y$. ■

A consequence of Proposition 2.2 and [9, Theorem 5.2] is the following.

COROLLARY 2.3. *Let M be a proper pointed metric space. If either $S_0(M)$ or X has (AP), then $S_0(M,X)$ is linearly isometrically isomorphic to $S_0(M) \widehat{\otimes}_\varepsilon X$.*

The above corollary as well as the basic theory of tensor product spaces give us the key to proving our first duality result in the vector-valued setting.

THEOREM 2.4. *Let M be a proper pointed metric space and X a Banach space. Assume that $S_0(M)$ separates points uniformly. If either $\mathcal{F}(M)$ or X^* has (AP), then*

$$S_0(M,X)^* = \mathcal{F}(M,X^*).$$

Proof. As $S_0(M)$ separates points uniformly, we have $S_0(M)^* = \mathcal{F}(M)$ [7]. Thus $S_0(M)$ is an Asplund space. Consequently, the above corollary and [26, Theorem 5.33] imply that

$$S_0(M,X)^* = (S_0(M) \widehat{\otimes}_\varepsilon X)^* = \mathcal{F}(M) \widehat{\otimes}_\pi X^* = \mathcal{F}(M,X^*). \ ■$$

The next proposition enlarges the class of metric spaces to which the above theorem applies.

Proposition 2.5. *Let M be a proper metric space. If $(M, \omega \circ d)$ is a proper Hölder metric space where ω is a non-trivial gauge, then $S_0(M)$ separates points uniformly.*

Proof. We will adapt the technique used in [18, Proposition 3.5] in the compact case. We will show that, for every $x \neq y \in M$ and every $\varepsilon > 0$, there exists $f \in S_0(M, d_\omega)$ such that $|f(x) - f(y)| \geq d_\omega(x, y) - \varepsilon$ and $\|f\|_{\mathrm{Lip}_\omega} \leq 1$ (where d_ω denotes $\omega \circ d$). Fix $\varepsilon > 0$ and $x \neq y \in M$. Set $a = d_\omega(x, y)$ and define φ: $[0, \infty[\to [0, \infty[$ by

$$\varphi(t) = \begin{cases} t & \text{if } 0 \leq t < a - \varepsilon, \\ a - \varepsilon & \text{if } a - \varepsilon \leq t < a + \varepsilon, \\ -t + 2a & \text{if } a + \varepsilon \leq t < 2a, \\ 0 & \text{if } 2a \leq t. \end{cases}$$

Notice that $\|\varphi\|_{\mathrm{Lip}} \leq 1$. For every $n \in \mathbb{N}$ we define a new gauge

$$\omega_n(t) = \inf\{\omega(s) + n(t - s) : 0 \leq s \leq t\}.$$

Then $\omega_n(t) \to \omega(t)$ as $n \to \infty$ for every $t \in [0, \infty[$. Finally, for $n \in \mathbb{N}$, we define h_n on M by $h_n(z) = \varphi(d_{\omega_n}(z, y)) - \varphi(d_{\omega_n}(0, y))$. It is straightforward to check that, for n large enough, $|h_n(x) - h_n(y)| = a - \varepsilon = d_\omega(x, y) - \varepsilon$. Moreover, given z and z' in M, straightforward computations yield

$$\begin{aligned} |h_n(z) - h_n(z')| &= |\varphi(d_{\omega_n}(z, y)) - \varphi(d_{\omega_n}(z', y))| \\ &\leq \|\varphi\|_{\mathrm{Lip}} |d_{\omega_n}(z, y) - d_{\omega_n}(z', y)| \leq d_{\omega_n}(z, z'). \end{aligned}$$

Furthermore, from the definition of ω_n,

$$d_{\omega_n}(z, z') \leq d_\omega(z, z') \quad \text{and} \quad d_{\omega_n}(z, z') \leq n d(z, z').$$

Now the first of the above inequalities shows that $\|h_n\|_{\mathrm{Lip}} \leq 1$, while the second proves that $h_n \in \mathrm{Lip}_0(M, d) \subset \mathrm{lip}_0(M, d_\omega)$. It remains to prove that $h_n \in S_0(M)$. To this end, fix $\eta > 0$ and pick $r > 2a + d_\omega(0, y)$ such that $a/(r - 2a - d_\omega(0, y)) \leq \eta$. Now let $z, z' \in M$, and let us discuss several cases:

- If z and z' are not in $\bar{B}(0, r)$, then $|h_n(z) - h_n(z')| = 0 < \eta$.
- Now suppose that $z \notin B(0, r)$ and $z' \in B(0, r)$. We can still distinguish two more cases:
  - If $d_\omega(z', y) \geq 2a$, then $h_n(z) = h_n(z') = 0$, so trivially $|h_n(z) - h_n(z')| < \eta$.

– If $d_\omega(z', y) < 2a$, then $|h_n(z')| \le a$, and so

$$\frac{|h_n(z) - h_n(z')|}{d_\omega(z, z')} \le \frac{a}{d_\omega(z, 0) - d_\omega(z', y) - d_\omega(0, y)} \le \frac{a}{r - 2a - d_\omega(0, y)} \le \eta.$$

This proves that $h_n \in S_0(M)$ and concludes the proof. ∎

Now we will exhibit some examples of metric and Banach spaces to which Theorem 2.4 applies.

Corollary 2.6. *Let M be a proper metric space and X a Banach space. Then $S_0(M, X)^* = \mathcal{F}(M, X^*)$ whenever M and X satisfy one of the following assumptions:*

(1) *M is countable.*
(2) *M is ultrametric.*
(3) *$(M, \omega \circ d)$ is a Hölder metric space where ω is a non-trivial gauge, and either $\mathcal{F}(M)$ or X^* has (AP).*
(4) *M is the Cantor middle-thirds set.*

Proof. If M satisfies either (1) or (2), then $S_0(M)$ separates points uniformly and $\mathcal{F}(M)$ has the approximation property [7]. Thus Theorem 2.4 applies. If M satisfies (3) then Proposition 2.5 does the job. Finally, [28, Proposition 3.2.2] yields (4). ∎

Throughout the rest of the section we will consider a bounded metric space (M, d) and a topology τ on M such that (M, τ) is compact and d is τ-lower semicontinuous. We will consider

$$\mathrm{lip}_\tau(M) = \mathrm{lip}_0(M) \cap \mathcal{C}(M, \tau),$$

the space of little-Lipschitz functions which are τ-continuous on M. Since M is bounded, $\mathrm{lip}_\tau(M)$ is a closed subspace of $\mathrm{lip}_0(M)$, and thus it is a Banach space. Moreover, Kalton [18, Theorem 6.2] proved that $\mathrm{lip}_\tau(M)^* = \mathcal{F}(M)$ whenever M is separable and complete and the following condition holds:

(P) $\quad \forall x, y \in M\ \forall \varepsilon > 0\ \exists f \in B_{\mathrm{lip}_\tau(M)} : |f(x) - f(y)| \ge d(x, y) - \varepsilon.$

Recall that (P) holds if, and only if, $\mathrm{lip}_\tau(M)$ is 1-norming for $\mathcal{F}(M)$ [18, Proposition 3.4].

Now we can wonder whether there is a natural extension of this result to the vector-valued case. We will prove, using similar ideas to the ones of [9, Section 5], that under suitable assumptions the space

$$\mathrm{lip}_\tau(M, X) := \mathrm{lip}_0(M, X) \cap \{f : M \to X : \ f \text{ is } \tau\text{-}\|\cdot\|\text{-continuous}\}$$

is a predual of $\mathcal{F}(M, X^*)$.

We begin by characterising relative compactness in $\mathrm{lip}_\tau(M)$.

Lemma 2.7. *Let (M,d) be a metric space of radius R, and τ a topology on M such that (M,τ) is compact and d is τ-lower semicontinuous. Let $\mathcal{F} \subset \operatorname{lip}_\tau(M)$. Then $\mathcal{F}$ is relatively compact in $\operatorname{lip}_\tau(M)$ if, and only if, the following conditions hold:*

(1) *$\mathcal{F}$ is bounded.*
(2) *$\mathcal{F}$ satisfies the following uniform little-Lipschitz condition: for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\sup_{0<d(x,y)<\delta} \frac{|f(x)-f(y)|}{d(x,y)} < \varepsilon \quad \text{for every } f \in \mathcal{F}.$$

(3) *$\mathcal{F}$ is equicontinuous in $\mathcal{C}(M,\tau)$, i.e. for every $x \in M$ and every $\varepsilon > 0$ there exists a τ-neighbourhood U of x such that $y \in U$ implies $\sup_{f\in\mathcal{F}} |f(x)-f(y)| < \varepsilon$.*

Proof. In [18, Theorem 6.2] it is proved that $\operatorname{lip}_\tau(M)$ is isometrically isomorphic to a subspace of the space of continuous functions on a compact set. Indeed, let $K := \{(x,y,t) \in (M,\tau)\times(M,\tau)\times[0,2R] : d(x,y) \le t\}$. Then K is compact by τ-lower semicontinuity of d. Moreover, the map $\Phi\colon \operatorname{lip}_\tau(M) \to \mathcal{C}(K)$ defined by

$$\Phi(f)(x,y,t) := \begin{cases} (f(x)-f(y))/t, & t \neq 0, \\ 0, & \text{otherwise}, \end{cases}$$

is a linear isometry. Therefore, $\mathcal{F}$ is relatively compact if, and only if, $\Phi(\mathcal{F})$ is. By the Ascoli–Arzelà theorem, $\mathcal{F}$ is relatively compact if, and only if, $\Phi(\mathcal{F})$ is bounded and equicontinuous in $\mathcal{C}(K)$.

First, assume that (1)–(3) hold. It is clear that $\Phi(\mathcal{F})$ is bounded, so let us prove its equicontinuity. To this end pick $(x,y,t) \in K$. Now we have two possibilities:

(i) If $t \neq 0$ we can find $0 < \eta < t$ such that $t' \in \,]t-\eta, t+\eta[$ implies $|1/t - 1/t'| < \varepsilon/(4R\alpha)$, where $\alpha = \sup_{f\in\mathcal{F}} \|f\|$. As $x,y \in M$ and $\mathcal{F}$ satisfies (3), there exists a τ-neighbourhood U of x and a τ-neighbourhood V of y in M such that $x' \in U$, $y' \in V$ implies $|f(x)-f(x')| + |f(y)-f(y')| < \varepsilon t/2$ for every $f \in \mathcal{F}$. Now, given $(x',y',t') \in (U \times V \times \,]t-\eta, t+\eta[) \cap K$, one has

$$\begin{aligned} |\Phi f(x,y,t) - \Phi f(x',y',t')| &= \left| \frac{f(x)-f(y)}{t} - \frac{f(x')-f(y')}{t'} \right| \\ &\le \left| \frac{1}{t} - \frac{1}{t'} \right| |f(x')-f(y')| + \frac{1}{t}|f(x)-f(x')+f(y)-f(y')| \\ &\le \frac{\varepsilon}{4R\alpha}\|f\| d(x',y') + \frac{\varepsilon t}{2t} \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

for every $f \in \mathcal{F}$, which proves equicontinuity of $\Phi(f)$ at (x,y,t).

(ii) If $t = 0$ then $x = y$. Pick $\varepsilon > 0$. By (2) we get $\delta > 0$ such that $0 < d(x, y) < \delta$ implies $|f(x) - f(y)|/d(x, y) < \varepsilon$ for every $f \in \mathcal{F}$. Now, given $(x', y', t) \in (M \times M \times [0, \delta[) \cap K$ we have $d(x', y') \le t < \delta$, and so, given $f \in \mathcal{F}$, it follows that

$$|\Phi f(x', y', t)| \le \frac{|f(x') - f(y')|}{t} < \varepsilon \frac{d(x', y')}{t} \le \varepsilon,$$

which proves equicontinuity at $(x, x, 0)$.

Thus $\Phi(\mathcal{F})$ is equicontinuous whenever conditions (1)–(3) are satisfied.

Conversely, assume that $\Phi(\mathcal{F})$ is equicontinuous in $\mathcal{C}(K)$. It is clear that $\mathcal{F}$ is bounded, so let us prove (2) and (3).

For (3), fix $x \in M$ and $\varepsilon > 0$. Given $t \in [0, 2R]$, by equicontinuity of $\Phi(\mathcal{F})$ at (x, x, t), we can find a τ-neighbourhood U_t of x and $\eta_t > 0$ such that $x' \in U_t$ and $t' \in (t - \eta_t, t + \eta_t)$ implies $|\Phi f(x, x', t')| < \varepsilon/(2R)$ for every $f \in \mathcal{F}$. Then $[0, 2R] \subset \bigcup_t (t - \eta_t, t + \eta_t)$, and thus there exist $t_1, \ldots, t_n$ such that $[0, 2R] \subset \bigcup_{i=1}^n (t_i - \eta_{t_i}, t_i + \eta_{t_i})$. Set $U = \bigcap_{i=1}^n U_{t_i}$. We will show that U is the desired τ-neighbourhood of x. Pick $x' \in U$. Then there exists i such that $d(x, x') \in (t_i - \eta_{t_i}, t_i + \eta_{t_i})$. Since $x' \in U_{t_i}$, we get

$$|\Phi f(x, x', d(x, x'))| = \left| \frac{f(x) - f(x')}{d(x, x')} \right| < \frac{\varepsilon}{2R},$$

and thus $|f(x) - f(x')| < \varepsilon$ for every $x' \in U$ and $f \in \mathcal{F}$. This proves that $\mathcal{F}$ is equicontinuous at every $x \in M$.

Finally, to prove (2), pick $\varepsilon > 0$. For every $x \in M$, from equicontinuity of $\Phi(\mathcal{F})$ at $(x, x, 0)$, there exists a τ-open neighbourhood U_x of x in M and $\delta_x > 0$ such that $x', y' \in U_x$ and $0 < t < \delta_x$ implies $|\Phi f(x', y', t)| < \varepsilon$ for every $f \in \mathcal{F}$.

As $\Delta := \{(x, x) : x \in M\} \subset \bigcup_{x \in M} U_x \times U_x$, by compactness there exist $x_1, \ldots, x_n \in M$ such that $\Delta \subset \bigcup_{i=1}^n U_{x_i} \times U_{x_i}$. It follows easily that there is n_0 such that $\{(x, y) \in M \times M : d(x, y) \le n_0^{-1}\} \subset \bigcup_{i=1}^n U_{x_i} \times U_{x_i}$. Set $\delta := \min\{1/n_0, \delta_{x_1}, \ldots, \delta_{x_n}\}$. If $x, y \in M$ with $0 < d(x, y) < \delta$ then there exists $i \in \{1, \ldots, n\}$ such that $x, y \in U_{x_i}$. As $d(x, y) < \delta \le \delta_{x_i}$, we get

$$\frac{|f(x) - f(y)|}{d(x, y)} = |\Phi f(x, y, d(x, y))| < \varepsilon$$

for every $f \in \mathcal{F}$, which proves (2) and finishes the proof. ■

The previous lemma allows us to identify $\mathrm{lip}_\tau(M, X)$ as a space of compact operators from X^* to $\mathrm{lip}_\tau(M)$.

THEOREM 2.8. *Let M be a pointed metric space and let τ be a topology on M such that (M, τ) is compact and d is τ-lower semicontinuous. Then $\mathrm{lip}_\tau(M, X)$ is isometrically isomorphic to $K_{w^*,w}(X^*, \mathrm{lip}_\tau(M))$. Moreover, if* *either* $\mathrm{lip}_\tau(M)$ *or* X *has* (AP)*, then* $\mathrm{lip}_\tau(M,X)$ *is isometrically isomorphic to* $\mathrm{lip}_\tau(M)\mathbin{\widehat{\otimes}_\varepsilon} X$*.*

Proof. It is shown in [16] that $f \mapsto f^t$ defines an isometry from $\mathrm{Lip}_0(M,X)$ onto $L_{w^*,w^*}(X^*,\mathrm{Lip}_0(M)))$, where $f^t(x^*) = x^*\circ f$. Let f be in $\mathrm{lip}_\tau(M,X)$; let us prove that $f^t \in K_{w^*,w}(X^*,\mathrm{lip}_\tau(M))$. Notice that $x^*\circ f$ is τ-continuous for every $x^*\in X^*$. Moreover, for every $x\neq y\in M$ and every $x^*\in X^*$,

$$\frac{|x^*\circ f(x)-x^*\circ f(y)|}{d(x,y)} \le \|x^*\|\frac{\|f(x)-f(y)\|}{d(x,y)}, \tag{2.1}$$

thus $x^*\circ f\in \mathrm{lip}_0(M)$. Therefore $f^t(X^*)\subset \mathrm{lip}_\tau(M)$.

We claim that $f^t(B_{X^*})$ is relatively compact in $\mathrm{lip}_\tau(M)$. To see this, we need to check the conditions in Lemma 2.7. First, it is clear that $f^t(B_{X^*})$ is bounded. Moreover, (2.1) shows that the functions in $f^t(B_{X^*})$ satisfy the uniform little-Lipschitz condition. Finally, $f^t(B_{X^*})$ is equicontinuous in the sense of Lemma 2.7. Indeed, given $x\in M$ and $\varepsilon>0$, there exists a τ-neighbourhood U of x such that $\|f(x)-f(y)\|<\varepsilon$ whenever $y\in U$. That is,

$$\sup_{x^*\in B_{X^*}} |x^*\circ f(x)-x^*\circ f(y)|<\varepsilon$$

whenever $y\in U$, as desired. Now, Lemma 2.7 implies that $f^t(B_{X^*})$ is a relatively compact subset of $\mathrm{lip}_\tau(M)$, and thus $f^t\in K(X^*,\mathrm{lip}_\tau(M))\cap L_{w^*,w^*}(X^*,\mathrm{Lip}_0(M))$.

Finally, $\overline{f^t(B_{X^*})}$ is norm-compact and thus every coarser Hausdorff topology agrees on it with the norm topology. In particular, the weak topology of $\mathrm{lip}_\tau(M)$ agrees on $f^t(B_{X^*})$ with the inherited w^*-topology of $\mathrm{Lip}_0(M)$. Thus $f^t|_{B_{X^*}}: B_{X^*}\to \mathrm{lip}_\tau(M)$ is w^*-w-continuous. By [19, Proposition 3.1] we conclude that $f^t\in K_{w^*,w}(X^*,\mathrm{lip}_\tau(M))$.

It remains to prove that the isometry is onto, so take

$$T\in K_{w^*,w}(X^*,\mathrm{lip}_\tau(M)).$$

We claim that T is w^*-w^*-continuous from X^* to $\mathrm{Lip}_0(M)$. Indeed, let $\{x^*_\alpha\}$ be a net in X^* w^*-convergent to some $x^*\in X^*$. Since every $\gamma\in\mathcal{F}(M)$ is also an element in $\mathrm{lip}_\tau(M)^*$, we infer that $\langle\gamma, Tx^*_\alpha\rangle$ converges to $\langle\gamma,Tx^*\rangle$. Thus, $T\in L_{w^*,w^*}(X^*,\mathrm{Lip}_0(M)))$. By the isometry described above, there exists $f\in\mathrm{Lip}_0(M,X)$ such that $T=f^t$. Let us prove that actually $f\in\mathrm{lip}_\tau(M,X)$. As $f^t(B_{X^*})$ is relatively compact, Lemma 2.7 shows that for every $\varepsilon>0$ there exists $\delta>0$ such that

$$\sup_{0<d(x,y)<\delta}\frac{|x^*\circ f(x)-x^*\circ f(y)|}{d(x,y)}<\varepsilon$$

for each $x^* \in B_{X^*}$. By taking the supremum over $x^* \in B_{X^*}$ we see that

$$\sup_{0<d(x,y)<\delta} \frac{\|f(x)-f(y)\|}{d(x,y)} \leq \varepsilon,$$

so $f \in \mathrm{lip}_0(M,X)$.

To finish the proof, we will prove that f is τ-$\|\cdot\|$-continuous. Pick $y \in M$ and $\varepsilon > 0$. By equicontinuity of $f^t(B_{X^*})$ we can find a τ-neighbourhood U of y such that $|x^* \circ f(y') - x^* \circ f(y)| < \varepsilon$ for every $x^* \in B_{X^*}$ and $y' \in U$. Now,

$$\|f(y') - f(y)\| = \sup_{x^* \in B_{X^*}} |x^*(f(y') - f(y))| \leq \varepsilon$$

for every $y' \in U$. Consequently, f is τ-$\|\cdot\|$-continuous. So $f \in \mathrm{lip}_\tau(M,X)$, as desired.

Finally, if either $\mathrm{lip}_\tau(M)$ or X has the approximation property, then Proposition 2.2 yields $K_{w^*,w}(X^*, \mathrm{lip}_\tau(M)) = \mathrm{lip}_\tau(M) \widehat{\otimes}_\varepsilon X$. ∎

Now we get our second duality result for vector-valued Lipschitz-free Banach spaces, which extends [18, Theorem 6.2].

THEOREM 2.9. *Let M be a separable complete bounded pointed metric space. Suppose that τ is a metrizable topology on M such that (M,τ) is compact and satisfies property (P). If either $\mathcal{F}(M)$ or X^* has (AP), then $\mathrm{lip}_\tau(M,X)^* = \mathcal{F}(M,X^*)$.*

Proof. By [18, Theorem 6.2], $\mathrm{lip}_\tau(M)$ is a predual of $\mathcal{F}(M)$. Consequently, $\mathcal{F}(M)$ has (RNP). Therefore, Theorem 2.8 and [26, Theorem 5.33] imply that

$$\mathrm{lip}_\tau(M,X)^* = (\mathrm{lip}_\tau(M) \widehat{\otimes}_\varepsilon X)^* = \mathcal{F}(M) \widehat{\otimes}_\pi X^* = \mathcal{F}(M,X^*). \ ∎$$

The last result applies to the following particular case (see [18, Proposition 6.3]). Given two Banach spaces X, Y, and a non-trivial gauge ω, we will denote $\mathrm{lip}_{\omega,*}(B_{X^*}, Y) := \mathrm{lip}_{w^*}((B_{X^*}, \omega \circ \|\cdot\|), Y)$.

COROLLARY 2.10. *Let X and Y be Banach spaces, and let ω be a non-trivial gauge. Assume that X^* is separable and that either $\mathcal{F}(B_{X^*}, \omega \circ \|\cdot\|)$ or Y^* has (AP). Then $\mathrm{lip}_{\omega,*}(B_{X^*}, Y)$ is linearly isometrically isomorphic to $\mathrm{lip}_{\omega,*}(B_{X^*}) \widehat{\otimes}_\varepsilon Y$, and $\mathrm{lip}_{\omega,*}(B_{X^*}, Y)^* = \mathcal{F}((B_{X^*}, \omega \circ \|\cdot\|), Y^*)$.*

Finally, we apply the notion of unconditional almost squareness, introduced in [9], in order to prove non-duality of the space $\mathrm{lip}_{\omega,*}(B_{X^*}, Y)$ under the above hypotheses. Following [9], a Banach space X is said to be *unconditionally almost square* (UASQ) if, for each $\varepsilon > 0$, there exists a subset $\{x_\gamma\}_{\gamma \in \Gamma} \subseteq S_X$ such that

(1) For each $\{y_1,\dots,y_k\}\subseteq S_X$ and $\delta>0$ the set
$$\{\gamma\in\Gamma : \|y_i\pm x_\gamma\|\le 1+\delta\ \forall i\in\{1,\dots,k\}\}$$
is non-empty.

(2) For every finite subset F of Γ and every choice of signs $\xi_\gamma\in\{-1,1\}$, $\gamma\in F$, we have $\|\sum_{\gamma\in F}\xi_\gamma x_\gamma\|\le 1+\varepsilon$.

It is known that there is not any dual UASQ Banach space [9, Theorem 2.5].

Proposition 2.11. *Let X and Y be Banach spaces, and let ω be a non-trivial gauge. Assume that X^* is separable. Then $\mathrm{lip}_{\omega,*}(B_{X^*},Y)$ is UASQ. In particular, it is not isometric to any dual Banach space.*

Proof. First we prove that $\mathrm{lip}_{\omega,*}(B_{X^*})$ is UASQ. By [9, Proposition 3.3], it suffices to show that there exists $x_0^*\in B_{X^*}$ and sequences r_n of positive numbers and $f_n\in\mathrm{lip}_{\omega,*}(B_{X^*})$ such that $f_n\ne 0$, $f_n(x_0^*)=0$ and f_n vanishes outside $B(x_0^*,r_n)$. Let $x_0^*\in S_{X^*}$ be a continuity point of the identity $I\colon (B_{X^*},w^*)\to(B_{X^*},\|\cdot\|)$, that is, x_0^* has relative w^*-neighbourhoods of arbitrarily small diameter. Take a sequence (W_n) of relative w^*-neighbourhoods of x_0^* and a sequence $r_n\to 0$ such that $0\notin W_n\subset B(x_0^*,r_n)\subset W_{n-1}$. For each n choose $x_n^*\in W_n\setminus\{x_0^*\}$ and define $A_n=\{x_0^*,x_n^*\}\cup(B_{X^*}\setminus W_n)$. Consider $f_n\colon A_n\to\mathbb{R}$ given by $f_n(x_n^*)=1$ and $f_n(x)=0$ otherwise. Then A_n is w^*-closed and $f_n\in\mathrm{Lip}_0(A_n,\|\cdot\|)\cap\mathcal{C}(A_n,w^*)$. By [22, Corollary 2.5], there exists $g_n\in\mathrm{Lip}_0(B_{X^*},\|\cdot\|)\cap\mathcal{C}(B_{X^*},w^*)$ extending f_n. Then g_n is a non-zero Lipschitz function which is w^*-continuous and vanishes on $B_{X^*}\setminus B(x_0^*,r_n)$. Finally, notice that $\mathrm{Lip}_0(B_{X^*},\|\cdot\|)\subset\mathrm{lip}_\omega(B_{X^*})$. Thus $\{g_n\}\subset\mathrm{lip}_{\omega,*}(B_{X^*})$, and so $\mathrm{lip}_{\omega,*}(B_{X^*})$ is UASQ.

Now, by Theorem 2.8, $\mathrm{lip}_{\omega,*}(B_{X^*},Y)$ is a subspace of $K(Y^*,\mathrm{lip}_{\omega,*}(B_{X^*}))$ which clearly contains $\mathrm{lip}_{\omega,*}(B_{X^*})\otimes Y$. Proposition 2.7 in [9] provides unconditional almost squareness of $\mathrm{lip}_{\omega,*}(B_{X^*},Y)$. Finally, the non-duality of this space follows from [9, Theorem 2.5]. ■

Remark 2.12. (1) The previous result can be strengthened if Y is a separable space with (AP). In fact, in that case $\mathrm{lip}_{\omega,*}(B_{X^*},Y)=\mathrm{lip}_{\omega,*}(B_{X^*})\,\widehat{\otimes}_\varepsilon Y$ is a separable Banach space which contains an isomorphic copy of c_0 [1, Lemma 2.6], so it cannot be even isomorphic to any dual Banach space. Moreover, to the best of our knowledge, the fact that $\mathrm{lip}_{\omega,*}(B_{X^*},Y)$ is not a dual Banach space has not been known even in the real case.

(2) The previous result has an immediate consequence in terms of octahedrality in Lipschitz-free Banach spaces. Recall that a Banach space X is said to have an *octahedral norm* if for every finite-dimensional subspace Y and for every $\varepsilon>0$ there exists $x\in S_X$ such that $\|y+\lambda x\|>(1-\varepsilon)(\|y\|+|\lambda|)$ for every $y\in Y$ and $\lambda\in\mathbb{R}$. Under the assumption of Proposition 2.11, the space $\mathcal{F}((B_{X^*},\omega\circ\|\cdot\|),Y^*)=\mathcal{F}((B_{X^*},\omega\circ\|\cdot\|))\,\widehat{\otimes}_\pi Y^*$ has an octahedral norm because of [21, Corollary 2.9]. This gives a partially positive answer

to [3, Question 2], where it is asked whether octahedrality in vector-valued Lipschitz-free Banach spaces actually relies on the scalar case.

3. Schur property on vector-valued Lipschitz-free spaces. Following [12], a Banach space X is said to have the *Schur property* whenever every weakly null sequence is actually a norm-null sequence, and X is said to have the *Dunford–Pettis property* whenever every weakly compact operator from X into a Banach space Y is *completely continuous*, i.e. carries weakly compact sets into norm-compact sets.

It is known that a dual Banach space X^* has the Schur property if, and only if, X has the Dunford–Pettis property and does not contain any isomorphic copy of ℓ_1 [12, Theorem 5.2]. So, in order to analyse the Schur property in $\mathcal{F}(M, X^*)$, it may be useful to analyse the Dunford–Pettis property in the predual in case such a predual exists. For this, in the proper case, we can go much further.

THEOREM 3.1. *Let M be a proper metric space such that $S_0(M)$ separates points uniformly. Then $S_0(M)$ does not contain any isomorphic copy of ℓ_1 and has the hereditary Dunford–Pettis property, i.e. every closed subspace of $S_0(M)$ has the Dunford–Pettis property.*

Proof. By [7], $S_0(M)$ is $(1+\varepsilon)$-isometric to a subspace of c_0, which is known to have the hereditary Dunford–Pettis property (see e.g. [4]). Consequently, $S_0(M)$ has the hereditary Dunford–Pettis property. Obviously, being isomorphic to a subspace of c_0, $S_0(M)$ cannot contain any isomorphic copy of ℓ_1. ∎

The above theorem not only applies to the scalar-valued version of $S_0(M)$ but also in the vector-valued case. Indeed, we get the following result.

THEOREM 3.2. *Let M be a proper metric space such that $S_0(M)$ separates points uniformly. Assume that X is a Banach space with the hereditary Dunford–Pettis property and X does not contain any isomorphic copy of ℓ_1. If either X or $S_0(M)$ has (AP), then $S_0(M, X)$ does not contain any isomorphic copy of ℓ_1 and has the hereditary Dunford–Pettis property.*

Proof. As $S_0(M) \widehat{\otimes}_\varepsilon X = S_0(M, X)$ by Theorem 2.4, this space contains no isomorphic copy of ℓ_1 [24, Corollary 4]. Moreover, as $S_0(M)$ is isomorphic to a subspace of c_0, $S_0(M, X) = S_0(M) \widehat{\otimes}_\varepsilon X$ is isomorphic to a subspace of $c_0 \widehat{\otimes}_\varepsilon X = c_0(X)$. As $c_0(X)$ has the hereditary Dunford–Pettis property whenever X does [20, Theorem 3.1], we conclude that $S_0(M, X)$ has the hereditary Dunford–Pettis property. ∎

We say that a Banach space X has the *strong Schur property* if there exists a constant $K > 0$ such that, given $\delta > 0$ and a sequence $(x_n)_{n\in\mathbb{N}}$ in the unit ball of X such that $\|x_n - x_m\| \geq \delta$ for all $n \neq m$, then (x_n)

contains a subsequence K/δ-equivalent to the unit vector basis of ℓ_1. As a dual Banach space X^* has the strong Schur property whenever X contains no isomorphic copy of ℓ_1 and has the hereditary Dunford–Pettis property [20, Theorem 4.1], we get the following corollary.

Corollary 3.3. *Let M be a proper metric space such that $S_0(M)$ separates points uniformly. Assume that X is a Banach space with the hereditary Dunford–Pettis property and X does not contain any isomorphic copy of ℓ_1. If either X^* or $\mathcal{F}(M)$ has (AP), then $\mathcal{F}(M, X^*)$ has the strong Schur property.*

Remark 3.4. The above corollary should be compared with [23, Proposition 17] in the real case.

Bearing in mind the identification $\mathcal{F}(M,X) = \mathcal{F}(M) \widehat{\otimes}_\pi X$, philosophically we can say that we have obtained a result about the Schur property in $\mathcal{F}(M,X)$ from tensor product theory. Now we are going to state a result in the reverse direction, i.e. we find conditions which guarantee that $\mathcal{F}(M,X)$ has the Schur property and, as a consequence, we get examples of Banach spaces with the Schur property whose projective tensor product still has that property. Note that such examples are interesting because, to the best of our knowledge, it is an open problem when projective tensor product preserves that property [11, Remark 6].

For this, we will analyse the uniformly discrete case, for which Kalton [18] proved that the scalar-valued Lipschitz-free Banach space has the Schur property. We extend this result to the vector-valued setting.

Proposition 3.5. *Let (M, d) be a uniformly discrete metric space, that is, $\theta = \inf_{m_1 \neq m_2} d(m_1, m_2) > 0$, and let X be a Banach space with the Schur property. Then $\mathcal{F}(M,X)$ has the Schur property.*

Proof. We will need Kalton's decomposition [18, Lemma 4.2]: there exist a constant $C > 0$ and a sequence of operators $T_k \colon \mathcal{F}(M) \to \mathcal{F}(M_k)$, where $k \in \mathbb{Z}$ and $M_k = \overline{B}(0, 2^k)$, satisfying

$$\gamma = \sum_{k \in \mathbb{Z}} T_k \gamma \quad \text{unconditionally} \quad \text{and} \quad \sum_{k \in \mathbb{Z}} \|T_k \gamma\| \le C\|\gamma\|$$

for every $\gamma \in \mathcal{F}(M)$. Now, the map $S \colon \mathcal{F}(M) \to (\sum \mathcal{F}(M_k))_{\ell_1}$ defined by $S\gamma = (T_k\gamma)_k$ defines an isomorphism between $\mathcal{F}(M)$ and a closed subspace of $(\sum \mathcal{F}(M_k))_{\ell_1}$.

We will show that the image of $\mathcal{F}(M)$ is complemented in $(\sum \mathcal{F}(M_k))_{\ell_1}$. Define $P : (\sum \mathcal{F}(M_k))_{\ell_1} \to S(\mathcal{F}(M))$ by setting $P((\gamma_k)_k) = (T_k\gamma)_k$, where $\gamma = \sum_k \gamma_k$. This is a well defined projection. Indeed, if $(\gamma_k)_k \in (\sum \mathcal{F}(M_k))_{\ell_1}$ then $P(P((\gamma_k)_k)) = P((T_k\gamma)_k)$. Now, if we define $\gamma := \sum_{k \in \mathbb{Z}} T_k\gamma$, it follows that $P((T_k\gamma)_k) = (T_k\gamma)_k$, which proves $P \circ P = P$. Notice that P is continuous since, given $(\gamma_k) \in (\sum \mathcal{F}(M_k))_{\ell_1}$, if we define $\gamma := \sum_{k\in\mathbb{Z}} \gamma_k$, then

$$\|P((\gamma_k))\| = \Big\|\sum_{k\in\mathbb{Z}} T_k\gamma\Big\| \le \sum_{k\in\mathbb{Z}} \|T_k\gamma\| \le C\|\gamma\| = C\Big\|\sum_{k\in\mathbb{Z}} \gamma_k\Big\| \le C\sum_{k\in\mathbb{Z}} \|\gamma_k\|.$$

Thus $\mathcal{F}(M) \mathbin{\widehat{\otimes}_\pi} X$ is isomorphic to a subspace of $(\sum \mathcal{F}(M_k))_{\ell_1} \mathbin{\widehat{\otimes}_\pi} X$ [26, Proposition 2.4]. It is not difficult to prove that the latter is isometrically isomorphic to $(\sum \mathcal{F}(M_k) \mathbin{\widehat{\otimes}_\pi} X)_{\ell_1}$. Consequently, $\mathcal{F}(M, X)$ is isomorphic to a subspace of $(\sum \mathcal{F}(M_k, X))_{\ell_1}$.

To finish the proof, we will show that $\mathcal{F}(M_k, X)$ has the Schur property for every k, which will be enough since the Schur property is stable under ℓ_1-sums [27] and passes to subspaces. To do so, we will show that $\mathcal{F}(M_k, X)$ is isomorphic to $\ell_1(M_k, X)$ (the space of all absolutely summable families in X indexed by M_k), which enjoys the Schur property since X does. Let F be a finite set, $(a_i)_{i\in F}$ a finite sequence of scalars and $\gamma = \sum_{i\in F} a_i\delta_{m_i,y_i} \in \mathcal{F}(M_k, X)$. Using the triangle inequality we have $\|\gamma\| \le \sum_{i\in F} |a_i|\,\|\delta_{m_i}\|\,\|y_i\| \le 2^k \sum_{i\in F} |a_i|\,\|y_i\|$. Moreover, for each $i \in F$, pick $x_i^* \in X^*$ such that $x_i^*(y_i) = \operatorname{sign}(a_i)\|x_i\|$ and define $f\colon M_k \to X^*$ by

$$f(m) := \begin{cases} x_i^* & \text{if } m = m_i \text{ for some } i \in F,\\ 0 & \text{otherwise.}\end{cases}$$

Since $2^{-k}\|f\|_\infty \le \|f\|_{\mathrm{Lip}} \le 2\theta^{-1}\|f\|_\infty$, we get $\|f\|_{\mathrm{Lip}} \le 2\theta^{-1}$. Thus $\|\gamma\| \ge \langle(\theta/2)f, \gamma\rangle = (\theta/2)\sum_{i\in F} |a_i|\|x_i\|$. This proves that the linear operator T: $\mathcal{F}(M_k, X) \to \ell_1(M_k, X)$ defined by $T(\sum_{i\in F} a_i\delta_{m_i,x_i}) = (z_m)_{m\in M_k}$, where $z_{m_i} = a_i x_i$ and $z_m = 0$ otherwise, is an isomorphism. ∎

Remark 3.6. Since $\mathcal{F}(M_k, X^*)$ is isomorphic to $\ell_1(X^*)$, we find that $\mathcal{F}(M, X^*)$ has the strong Schur property whenever X^* does. Indeed, this follows from the two next propositions.

Proposition 3.7. *Let $(X_k)_{k=1}^N$ be a finite family of Banach spaces. Assume that all X_k have the strong Schur property with the same constant K. Then $X = (\sum_{k=1}^N X_k)_{\ell_1}$ has the strong Schur property with constant $K + \varepsilon$ for every $\varepsilon > 0$.*

Proof. Fix $\delta > 0$ and let $(x_n)_n$ be a δ-separated sequence in the unit ball of X. We write $(x'_n)_n \prec (x_n)_n$ to mean that $(x'_n)_n$ is a subsequence of $(x_n)_n$. Set

$$\varDelta = \sup\Big\{\sum_{k=1}^N \delta_k : \exists (x'_n)_n \prec (x_n)_n, \forall k = 1, \dots, N, (x'_n(k))_n \text{ is } \delta_k\text{-separated}\Big\}.$$

We will show that $\varDelta \ge \delta$. Let $\varepsilon > 0$, and assume that $\varDelta < \delta - \varepsilon$. Then there exist $(x'_n)_n \prec (x_n)_n$ and $(\delta_1, \dots, \delta_N) \in [0, 2]^N$ such that, for every k, $(x'_n(k))_n$ is δ_k-separated and $\sum_{k=1}^N \delta_k > \varDelta - \varepsilon/N$. The Ramsey theorem provides $(x''_n)_n \prec (x'_n)_n$ such that one of the following holds:

(1) $\|x''_n(1) - x''_m(1)\| > \delta_1 + \varepsilon/N$ for every $n \neq m$.
(2) $\|x''_n(1) - x''_m(1)\| \leq \delta_1 + \varepsilon/N$ for every $n \neq m$.

Let us see that (1) does not hold. Indeed, if it does then $(x''_n(1))$ is $(\delta_1+\varepsilon/N)$-separated, and so $\Delta \geq \sum_{k=1}^N \delta_k + \varepsilon/N > \Delta$, which is impossible. Therefore (2) holds. By iterating this argument we can assume that, for every $n \neq y$ and every k, $\|x_n(k) - x_m(k)\| \leq \delta_k + \varepsilon/N$. Thus $\sum_{k=1}^N \|x_n(k) - x_m(k)\| \leq \sum_{k=1}^N \delta_k + \varepsilon < \delta$, which contradicts the δ-separation of the original sequence. Consequently, $\Delta \geq \delta$.

We now consider $(x'_n)_n \prec (x_n)_n$ such that, for every k, $(x'_n(k))_n$ is δ_k-separated and $\sum_{k=1}^N \delta_k \geq \delta - \varepsilon$. Since each X_k has the strong Schur property, for every k with $\delta_k > 0$ there exists a subsequence of $(x'_n(k))_n$, still denoted by $(x'_n(k))_n$, such that $(x'_n(k))_n$ is (K/δ_k)-equivalent to the ℓ_1-basis. Next, by a diagonal argument, there exists $(x''_n)_n \prec (x'_n)_n$ such that, for every k with $\delta_k > 0$, $(x''_n(k))_n$ is (K/δ_k)-equivalent to the ℓ_1-basis. Then a simple computation shows that, for every $(a_i)_{i=1}^m \in \mathbb{R}^m$,

$$\Big\|\sum_{i=1}^m a_i x''_i\Big\| = \sum_{k=1}^N \Big\|\sum_{i=1}^m a_i x''_i(k)\Big\| \geq \sum_{k=1}^N \frac{\delta_k}{K} \sum_{i=1}^m |a_i| \geq \frac{\delta - \varepsilon}{K} \sum_{i=1}^m |a_i|.$$

This proves that $(x''_n)_n$ is $\frac{K}{\delta-\varepsilon}$-equivalent to the ℓ_1-basis. ∎

Now we extend the previous result to infinite ℓ_1-sums. To achieve this we need to assume that the spaces X_k are dual ones.

Proposition 3.8. *Let $(X_k)_{k\in\mathbb{N}}$ be a family of Banach spaces. Assume that all X_k^* have the strong Schur property with the same constant K. Consider $X = (\sum_{k\in\mathbb{N}} X_k)_{c_0}$ and its dual space $X^* = (\sum_{k\in\mathbb{N}} X_k^*)_{\ell_1}$. Then X^* has the strong Schur property with constant* $\max\{2K + \varepsilon, 4 + \varepsilon\}$ *for every $\varepsilon > 0$.*

Proof. For $N \in \mathbb{N}$, we denote by $P_N\colon X^* \to (\sum_{k=1}^N X_k^*)_{\ell_1}$ the norm-one projection on the first N coordinates. Fix $\delta > 0$ and let $(x_n)_n$ be a δ-separated sequence in the unit ball of X^*. Fix also $\varepsilon > 0$. Now two cases may occur.

Case 1: *There exists $N \in \mathbb{N}$ such that there is $(x'_n)_n \prec (x_n)_n$ satisfying $d(x'_n, \sum_{k=1}^N X_k^*) \leq \delta/4$.* Then a straightforward computation using the triangle inequality shows that $(P_N(x'_n))_n$ is $(\delta/2)$-separated. Thus, according to Proposition 3.7, $(P_N(x'_n))_n$ admits a subsequence (not relabelled) $\frac{2K+\varepsilon}{\delta}$-equivalent to the ℓ_1-basis. Now for $(a_i)_{i=1}^m \in \mathbb{R}^m$, we have

$$\Big\|\sum_{i=1}^m a_i x'_i\Big\| \geq \Big\|\sum_{i=1}^m a_i P_N(x'_i)\Big\| \geq \frac{\delta}{2K + \varepsilon} \sum_{i=1}^m |a_i|.$$

This ends the first case.

CASE 2: *For every $N \in \mathbb{N}$ and every subsequence $(x'_n)_n$, there exists n such that $d(x'_n, \sum_{k=1}^{N} X_k^*) > \delta/4$.* Passing to a subsequence we can assume that $(x_n)_n$ is w^*-convergent to 0 and $\|x_n\| \geq \delta/2$ for every n. We will construct by induction a subsequence with the desired property.

To achieve this, fix a sequence $(\varepsilon_i)_i$ of positive real numbers smaller than $\delta/4$ such that $\prod_{i=1}^{\infty}(1-\varepsilon_i) \geq 1-\varepsilon$, and set $C := 4\sum_{k=1}^{\infty} \varepsilon_k/\delta < \varepsilon$. We begin with the construction of a sequence in X very close to $(x_n)_n$ which is equivalent to the ℓ_1-basis; then we will use the principle of small perturbations (see for example [2, Theorem 1.3.9]). More precisely, we will construct a sequence $(P_{K_i}(x_{n_i}))_i$ which is $\frac{4}{\delta(1-\varepsilon)}$-equivalent to the ℓ_1-basis and such that $\|P_{K_i}(x_{n_i}) - x_{n_i}\| \leq \varepsilon_i$.

First of all, we set $n_1 = 1$ and choose $N_1 \in \mathbb{N}$ such that $\|P_{N_1} x_{n_1}\| \geq \|x_{n_1}\| - \varepsilon_1$.

Construction of $n_2 > n_1$. Since P_{N_1} is w^*-continuous, $(P_{N_1}(x_n))_n$ is w^*-null. By [2, Lemma 1.5.1], there exists $m > n_1$ such that for all $n \geq m$ and all $(\lambda_1, \lambda_2) \in \mathbb{R}^2$,

$$\|\lambda_1 P_{N_1}(x_{n_1}) + \lambda_2 P_{N_1}(x_n)\| \geq (1-\varepsilon_2)\|\lambda_1 P_{N_1}(x_{n_1})\|.$$

Now, the assumption of Case 2 implies that there exists $n_2 \geq m$ such that $\|x_{n_2} - P_{K_1}(x_{n_2})\| > \delta/4$. We then pick $N_2 > N_1$ such that $\|P_{N_2}(x_{n_2}) - P_{N_1}(x_{n_2})\| > \delta/4$ and $\|P_{N_2}(x_{n_2}) - x_{n_2}\| < \varepsilon_2$. Then

$$\begin{aligned}
&\|\lambda_1 P_{N_1}(x_{n_1}) + \lambda_2 P_{N_2}(x_{n_2})\| \\
&\quad = \|\lambda_1 P_{N_1}(x_{n_1}) + \lambda_2 P_{N_1}(x_{n_2})\| + \|\lambda_2[P_{N_2} - P_{N_1}](x_{n_2})\| \\
&\quad > (1-\varepsilon_2)\|\lambda_1 P_{N_1}(x_{n_1})\| + |\lambda_2|\frac{\delta}{4} \\
&\quad > (1-\varepsilon_1)(1-\varepsilon_2)\frac{\delta}{4}|\lambda_1| + |\lambda_2|\frac{\delta}{4} > (1-\varepsilon_1)(1-\varepsilon_2)\frac{\delta}{4}(|\lambda_1| + |\lambda_2|).
\end{aligned}$$

We continue by induction to get a sequence $(P_{N_i}(x_{n_i}))_i$ which is $\frac{4}{\delta(1-\varepsilon)}$-equivalent to the ℓ_1-basis and $\|P_{N_i}(x_{n_i}) - x_{n_i}\| \leq \varepsilon_i$. By our choice we have $C < \varepsilon$. Thus we can apply the principle of small perturbations to get

$$\begin{aligned}
\Big\|\sum_{i=1}^{m} a_i x_{n_i}\Big\| &\geq \frac{1-C}{1+C}\Big\|\sum_{i=1}^{m} a_i P_{N_i}(x_{n_i})\Big\| \\
&\geq \frac{1-\varepsilon}{1+\varepsilon}(1-\varepsilon)\frac{\delta}{4}\sum_{i=1}^{m}|a_i| \geq \frac{\delta}{4}\,\frac{(1-\varepsilon)^2}{1+\varepsilon}\sum_{i=1}^{m}|a_i|
\end{aligned}$$

for every $(a_i)_{i=1}^m \in \mathbb{R}^m$. This ends Case 2 and finishes the proof. ∎

4. Norm attainment. Given a metric space M and a Banach space X, the equality $\mathrm{Lip}_0(M,X) = L(\mathcal{F}(M),X)$ yields two natural definitions of norm attainment for $f \in \mathrm{Lip}_0(M,X)$. On the one hand, if we view f as a linear operator, we can consider the classical definition of norm attainment. On the other hand, considering $f \in \mathrm{Lip}_0(M,X)$, we say that f *strongly attains its norm* if there are distinct $x, y \in M$ such that $\|f(x)-f(y)\| = \|f\| d(x,y)$. A natural question here is when the two concepts agree and, relatedly, whether the class of Lipschitz functions which strongly attain their norm is dense in $\mathrm{Lip}_0(M,X)$.

We denote by $\mathrm{Lip}_{\mathrm{SNA}}(M,X)$ (respectively $\mathrm{NA}(\mathcal{F}(M),X)$) the class of all functions in $\mathrm{Lip}_0(M,X)$ which strongly attain their norm (respectively, which attain their norm as a continuous linear operator from $\mathcal{F}(M)$ to X). Nice related results have recently appeared. On the one hand, negative results can be found in [17], where it is proved that $\mathrm{Lip}_{\mathrm{SNA}}(X)$ is not dense in $\mathrm{Lip}_0(X)$ whenever X is a Banach space [17, Theorem 2.3]. On the other hand, positive results are found in [10], where it is proved that if M is a compact metric space such that $\mathrm{lip}_0(M)$ separates points uniformly, and if E is finite-dimensional, then $\mathrm{Lip}_{\mathrm{SNA}}(M,E)$ is norm-dense in $\mathrm{Lip}_0(M,E)$. In the following we will use tensor product theory to generalise this result by considering proper metric spaces and more general target spaces.

We begin by stating the scalar case. This can be seen as a generalisation of [10, Proposition 5.3], though we will actually use the same ideas.

Proposition 4.1. *Let M be a proper metric space such that $S_0(M)$ separates points uniformly. Then every $f \in \mathrm{Lip}_0(M)$ which attains its norm on $\mathcal{F}(M)$ also strongly attains it. In other words,*

$$\mathrm{NA}(\mathcal{F}(M),\mathbb{R}) = \mathrm{Lip}_{\mathrm{SNA}}(M,\mathbb{R}).$$

Therefore, $\overline{\mathrm{Lip}_{\mathrm{SNA}}(M,\mathbb{R})}^{\|\cdot\|} = \mathrm{Lip}_0(M)$.

Proof. Since the inclusion $\mathrm{Lip}_{\mathrm{SNA}}(M,\mathbb{R}) \subseteq \mathrm{NA}(\mathcal{F}(M),\mathbb{R})$ always holds, we just prove the reverse one. For this, pick $f \in \mathrm{Lip}_0(M)$ which attains its norm as an element of $L(\mathcal{F}(M),\mathbb{R})$. By [7, Lemma 3.9], for every $\varepsilon > 0$, $S_0(M)$ is $(1+\varepsilon)$-isomorphic to a subspace of c_0. Therefore, since the property of being an M-ideal in its bidual is invariant under almost isometric isomorphism and under taking subspaces (see [14, Theorem 3.1.6 and Remark 1.7]), the space $S_0(M)$ is an M-ideal in its bidual.

Consequently, by [10, Lemma 5.2], f attains its norm on some $\gamma \in \mathcal{F}(M) \cap \mathrm{Ext}(\mathrm{Lip}_0(M)^*)$. But [28, Corollary 2.5.4] implies that γ is of the form $\gamma = \lambda(\delta(x)-\delta(y))/d(x,y)$ where $x \neq y \in M$ and $\lambda \in \mathbb{R}$ is such that $|\lambda| = 1$, so clearly f strongly attains its norm. The final assertion is a consequence of the Bishop–Phelps theorem. ■

We now turn to the study of vector-valued Lipschitz functions. The next result proves that, under the assumption of M being a proper metric space, the two concepts of norm attainment in $\mathrm{Lip}_0(M,X)$ are the same whenever $S_0(M)$ is a predual of $\mathcal{F}(M)$.

Proposition 4.2. *Let M be a proper metric space such that $S_0(M)$ separates points uniformly, and let X be a Banach space. Then, for a Lipschitz function $f\colon M \to X$, the following are equivalent:*

(1) *f strongly attains its norm.*
(2) *$f\colon \mathcal{F}(M) \to X$ attains its operator norm.*

Thus $\mathrm{NA}(\mathcal{F}(M), X) = \mathrm{Lip}_{\mathrm{SNA}}(M, X)$.

Proof. We just have to prove (2)⇒(1) since the other direction is always true and trivial. Assume that $\gamma \in \mathcal{F}(M)$, $\|\gamma\| \le 1$ and $\|f(\gamma)\| = \|f\|_{\mathrm{Lip}}$. Then, by the Hahn–Banach theorem, there exists $x^* \in S_{X^*}$ such that $\langle x^*, f(\gamma)\rangle = \|f(\gamma)\|$. But $x^* \circ f\colon M \to \mathbb{R}$ is a real-valued Lipschitz function which attains its operator norm on γ. Thus Proposition 4.1 gives the conclusion. ∎

Since the Bishop–Phelps theorem fails in the vector-valued case, we cannot deduce directly the same density result as in Proposition 4.1. However, we can state such a result for a quite large class of Banach spaces using tensor product theory under natural assumptions. First of all we prove the following lemma, which will be useful to get the desired density result for vector-valued Lipschitz functions.

Lemma 4.3. *Let X and Y be Banach spaces such that either X^* or Y^* has (AP) and both X^* and Y^* have (RNP). If $T \in L(X^*, Y^{**}) = (X^* \hat{\otimes}_\pi Y^*)^*$ attains its norm as a linear form on $X^* \hat{\otimes}_\pi Y^*$, then it also attains its norm as an operator on X^*. The converse is true when Y is reflexive.*

Proof. Assume that T attains its norm as a linear form on $X^* \hat{\otimes}_\pi Y^*$. Since $X^* \hat{\otimes}_\pi Y^*$ has (RNP) [8, Theorem VIII.4.7], T attains its norm at some extreme point of the unit ball of this space. As $\mathrm{Ext}(B_{X^* \hat{\otimes}_\pi Y^*}) = \mathrm{Ext}(B_{X^*}) \otimes \mathrm{Ext}(B_{Y^*})$ [25], there exist $x^* \in \mathrm{Ext}(B_{X^*})$ and $y^* \in \mathrm{Ext}(B_{Y^*})$ such that

$$\langle y^*, T(x^*)\rangle = \langle T, x^* \otimes y^*\rangle = \|T\|_{(X^* \hat{\otimes}_\pi Y^*)^*} = \|T\|_{L(X^*, Y^{**})},$$

and obviously $\|T(x^*)\|_Y = \|T\|_{L(X^*, Y^{**})}$.

Conversely, assume that Y is reflexive, so $L(X^*, Y^{**}) = L(X^*, Y)$. Then, if T attains its norm as an operator on X^*, there exists $x^* \in S_{X^*}$ such that $\|T(x^*)\|_Y = \|T\|_{L(X^*, Y)}$. Now, by the Hahn–Banach theorem, there exists $y^* \in B_{Y^*}$ such that $\langle y^*, T(x^*)\rangle = \|T(x^*)\|_Y$. Therefore T attains its norm on $x^* \otimes y^* \in X^* \hat{\otimes}_\pi Y^*$. ∎

We now state our density result for vector-valued Lipschitz functions.

Theorem 4.4. *Let (M, d) be a proper metric space such that $S_0(M)$ separates points uniformly. Let X be a Banach space such that X^* has*

(RNP). Assume that either $\mathcal{F}(M)$ or X^* has (AP). Then

$$\overline{\mathrm{NA}(\mathcal{F}(M), X^{**})}^{\|\cdot\|} = L(\mathcal{F}(M), X^{**}).$$

Equivalently, $\overline{\mathrm{Lip}_{\mathrm{SNA}}(M, X^{**})}^{\|\cdot\|} = \mathrm{Lip}_0(M, X^{**})$.

Proof. The Bishop–Phelps theorem applied to

$$(\mathcal{F}(M) \hat{\otimes}_\pi X^*)^* = L(\mathcal{F}(M) \hat{\otimes}_\pi X^*, \mathbb{R})$$

provides the norm-denseness of the set of those linear forms which attain their norm. But according to Lemma 4.3, if f attains its norm as a linear form on $\mathcal{F}(M) \hat{\otimes}_\pi X^*$, then it also attains its norm as an operator defined on $\mathcal{F}(M)$. This yields the result. Finally, the last assertion follows from Proposition 4.2. ∎

5. Some remarks and open questions. Note that in [7] it is proved that, given a proper metric space M, $S_0(M)$ is $(1+\varepsilon)$-isometric to a subspace of c_0 for every $\varepsilon > 0$. In a previous version of this paper we wondered whether $\mathrm{lip}_\tau(M)$ is $(1+\varepsilon)$-isometric to a subspace of c_0 for every $\varepsilon > 0$ whenever M and τ satisfy the hypothesis of Theorem 2.9. Aude Dalet and Antonín Procházka showed us that the answer is negative. Indeed, they have found such a metric space with $\mathcal{F}(M)$ having points of Fréchet differentiability. This implies that the norm of $\mathcal{F}(M)$ is not octahedral, which provides a counterexample to our question. We are deeply grateful to A. Dalet and A. Procházka for letting us include their proof.

PROPOSITION 5.1. *There is a metric space (M, d) and a topology τ on M such that (M, τ) is compact, d is τ-lower semicontinuous, and:*

(1) $\mathrm{lip}_\tau(M)^* = \mathcal{F}(M)$.
(2) *$\mathcal{F}(M)$ has points of Fréchet differentiability.*

For the proof we will need the following lemma.

LEMMA 5.2. *Let (M, d) be a uniformly discrete bounded separable pointed metric space. Let τ be a Hausdorff topology on M such that (M, τ) is compact. If d is τ-lower semicontinuous, then $\mathcal{F}(M) = \mathrm{lip}_\tau(M)^*$.*

Proof. First, since M is bounded, $\mathrm{lip}_\tau(M)$ is a closed subspace of $\mathrm{Lip}_0(M)$. Second, $\mathrm{lip}_\tau(M)$ separates points uniformly. Indeed, let $x \neq y$ in M. Since M is countable (by separability and uniform discreteness), the space (M, τ) is scattered, and thus y has a clopen neighbourhood U_y such that $x \notin U_y$. We set $f(z) = 0$ for $z \in M \setminus U_y$ and $f(z) = d(x, y)$ for $z \in U_y$ (or vice versa if $0 \in U_y$). We have $|f(z) - f(z')|/d(z, z') \leq D/d$, where D is the diameter of M and d is the separation of M. This shows that $\mathrm{lip}_\tau(M)$ separates points uniformly. Finally, $\mathrm{lip}_\tau(M) \in \mathrm{NA}(\mathcal{F}(M))$. Indeed, let $f \in \mathrm{lip}_\tau(M)$. The function $(i, j) \mapsto |f(i) - f(j)|/d(i, j)$ if $i \neq j$

and $(i,i) \mapsto 0$ is τ-upper semicontinuous. Thus it attains its maximum, as desired. It follows from [5, Proposition 28] that $\mathcal{F}(M) = \mathrm{lip}_\tau(M)^*$. ∎

Proof of Proposition 5.1. Let $M := \{0\} \cup \mathbb{N}$. First, we define a graph structure on M. The edges are exactly the pairs $\{0,n\}$ or $\{n,1\}$ where $n \notin \{0,1\}$. We define the metric d on M as the shortest path distance in this graph. Now we define the topology τ by declaring all the points except 2 isolated. Clearly (M,τ) is compact and one can easily check that d is τ-lower semicontinuous. Lemma 5.2 now implies that $\mathrm{lip}_\tau(M)^* = \mathcal{F}(M)$. Finally, similar estimates to the ones of [3, Proposition 2.5] prove that $\mathcal{F}(M)$ has a point of Fréchet differentiability, so we are done. ∎

A question from Section 3 is the following.

Question 5.3. Let M be a metric space and X a Banach space. If both $\mathcal{F}(M)$ and X have the Schur property, can we deduce that $\mathcal{F}(M,X)$ has the Schur property?

Note that an affirmative answer holds for $X = \ell_1(I)$, for any set I, since

$$\mathcal{F}(M,\ell_1(I)) = \mathcal{F}(M) \widehat{\otimes}_\pi \ell_1(I) = \ell_1(I,\mathcal{F}(M))$$

has the Schur property if, and only if, $\mathcal{F}(M)$ does [27, Proposition, Section 2]. The same conclusion holds whenever M is a proper ultrametric space because in this case $\mathcal{F}(M)$ is isomorphic to ℓ_1 [7].

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