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Convexity, Optimization and Geometry
of the Ball in Banach Spaces

Convexidad, Optimización y Geometría
de la Bola en Espacios de Banach

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Resumen

Esta memoria trata varios temas en el marco de la geometría de los espacios de Banach, haciendo hincapié en la estructura de conjuntos convexos y su aplicación en espacios de funciones lipschitzianas y sus preduales. A continuación resumimos el contenido de este trabajo.

Capítulo 0. Contenido preliminar

El objetivo de este capítulo introductorio es recordar ciertas propiedades geométricas que motivan varios de los resultados de este trabajo. Gran parte de los resultados aquí incluidos son bien conocidos, solo incluimos las pruebas de los menos habituales.

Tratamos en primer lugar sobre la estructura extremal de conjuntos convexos, recordando las nociones de punto extremo, expuesto, diente, etc., y las relaciones entre estos conceptos. Recordamos propiedades topológicas del conjunto de puntos extremos y exponemos algunos resultados que permiten recuperar un conjunto cerrado convexo a partir de subconjuntos distinguidos de sus puntos extremos, desde el teorema de Krein–Milman al de Bourgain–Phelps sobre conjuntos convexos con la propiedad de Radon–Nikodým.

A continuación, damos un marco general que contiene las derivaciones de conjuntos utilizadas a lo largo del texto. A grandes rasgos, una derivación de un espacio topológico consiste en eliminar cada subconjunto de una familia dada (por lo general, conjuntos abiertos o rodajas abiertas) que es pequeño con respecto a una cierta medida (por ejemplo, el diámetro). El índice ordinal asociado a la derivación es el número de pasos que se necesitan para llegar al conjunto vacío. Damos un esquema general que incluye varios índices ordinales conocidos, como los índices de Cantor–Bendixson, de Szlenk y de dentabilidad. A menudo, cuando estos índices están acotados por un cierto ordinal (normalmente, ω u ω_1), se sigue que el espacio tiene una cierta propiedad topológica o geométrica.

Finalmente, la última sección contiene algunos resultados sobre productos tensoriales y propiedades de aproximación para futura referencia.

Capítulo 1. Conjuntos compactos convexos que admiten una función estrictamente convexa e inferiormente semicontinua

Un resultado de Hervé [Her] afirma que un subconjunto compacto convexo K de un espacio localmente convexo es metrizable si y solo si existe una función $f\colon K \to \mathbb{R}$ que es

continua y estrictamente convexa. La semicontinuidad inferior resulta ser una hipótesis muy natural para una función convexa, por lo que es natural preguntarse si la existencia de una función estrictamente convexa e inferiormente semicontinua en un compacto convexo K de un espacio localmente convexo impone ciertas propiedades topológicas en K. Ribarska probó [Rib1,Rib2] que tal compacto es *fragmentable* por una métrica más fina, en particular contiene un subconjunto denso completamente metrizable. Raja probó [Raj6] que esto último también es cierto para el conjunto de sus puntos extremos, $\mathrm{ext}(K)$. Por otra parte, el argumento de Talagrand en [DGZ, Theorem 5.2.(ii)] muestra que $[0, \omega_1]$ no se embebe en tales compactos. Además, Godefroy y Li mostraron en [GL] que si el conjunto de probabilidades en un grupo compacto K admite una función estrictamente convexa inferiormente semicontinua entonces K es metrizable.

Nuestro objetivo es continuar con el estudio de la clase de conjuntos compactos convexos que admiten una función estrictamente convexa e inferiormente semicontinua. Denotaremos a esta clase $\mathscr{SC}$. El primer hecho remarcable que hemos obtenido es el siguiente resultado de representación.

Teorema A (con J. Orihuela y M. Raja)**.** Sea K un subconjunto compacto convexo de un espacio localmente convexo. Entonces $K \in \mathscr{SC}$ si y solo si K se embebe linealmente en un espacio de Banach dual estrictamente convexo dotado de la topología débil*.

Notemos que la norma estrictamente convexa del espacio de Banach dual en el enunciado es débil* inferiormente semicontinua, lo que es una condición más fuerte que ser un espacio de Banach estrictamente convexo isomorfo a un espacio dual.

Si $f \colon K \to \mathbb{R}$ es una función estrictamente convexa, entonces la *simétrica* definida por

$$\rho(x, y) = \frac{f(x) + f(y)}{2} - f\left(\frac{x + y}{2}\right)$$

proporciona un modo coherente de medir *diámetros* de subconjuntos de K. Esta idea ha sido aplicada con éxito en la teoría de renormamientos [MOTV] y será un ingrediente clave para nosotros.

Mostraremos que un compacto convexo pertenece a $\mathscr{SC}$ si y solo si tiene $(*)$ con rodajas. Esta propiedad fue introducida en [OST] para caracterizar los espacios de Banach duales que admiten una norma dual estrictamente convexa. Como consecuencia, probamos una caracterización de la clase $\mathscr{SC}$ en términos de la existencia de una simétrica con índice de dentabilidad numerable. No podemos reemplazar simétrica por métrica en ese resultado porque eso implicaría que el compacto es Gruenhage. De hecho, caracterizamos los compactos Gruenhage como aquellos en los que existe una métrica que tiene índice de Szlenk numerable.

En la última parte del capítulo analizamos la existencia de caras de continuidad de una función convexa, y de puntos expuestos de continuidad de una función estrictamente convexa. Nuestro punto de partida es un resultado de Raja [Raj6] que asegura la existencia de puntos extremos de continuidad de una función convexa inferiormente semicontinua. Para ello imitamos argumentos usados en el estudio geométrico de la propiedad de Radon–Nikodým, pero buscando rodajas que tengan diámetro pequeño con respecto a la simétrica asociada a la función. Esto conduce a resultados como el siguiente.

Teorema B (con J. Orihuela y M. Raja)**.** Sea E un espacio localmente convexo y $f\colon E \to \mathbb{R}$ una función inferiormente semicontinua, estrictamente convexa y acotada en conjuntos compactos. Entonces para cada conjunto $K \subset E$ compacto y convexo, el conjunto de puntos de K que son expuestos y de continuidad de $f|_K$ es denso en $\operatorname{ext}(K)$.

Como consecuencia, obtenemos que cada compacto convexo de la clase $\mathscr{S}\mathscr{C}$ es la envolvente convexa cerrada de sus puntos expuestos.

Este capítulo está basado en el artículo [GLOR].

Capítulo 2. Aplicaciones con la propiedad de Radon–Nikodým

La propiedad de Radon–Nikodým (RNP) desempeña un papel central en la teoría de espacios de Banach, en particular en las teorías isomorfa y no-lineal. Está relacionada con la diferenciabilidad de aplicaciones lipschitzianas, la estructura extremal de conjuntos convexos, representación sin compacidad, representación de espacios de funciones duales, optimización, etc. Dirigimos al lector interesado en la teoría y aplicaciones de la RNP a [BL, Bou4, Die, FHH$^+$].

Consideraremos la más geométrica entre las caracterizaciones de la RNP. En concreto, un subconjunto C de un espacio de Banach X tiene la RNP si y solo si cada subconjunto acotado $A \subset C$ es *dentable*, es decir, para cada $\varepsilon > 0$ existe un semiespacio abierto H tal que $\operatorname{diam}(A \cap H) < \varepsilon$.

Reĭnov [Reĭ1] y Linde [Lin] extendieron la RNP a operadores lineales. En este capítulo proponemos una definición para aplicaciones desde un subconjunto cerrado convexo de un espacio de Banach a un espacio métrico con el objetivo de generalizar la RNP a un contexto menos lineal. Nuestro punto de partida es considerar una noción de dentabilidad para aplicaciones, que surge fortaleciendo la propiedad del punto de continuidad.

Definición. Sea C un subconjunto no vacío de un espacio de Banach X y sea M un espacio métrico. Diremos que una aplicación $f\colon C \to M$ es *dentable* si para cada subconjunto no vacío acotado $A \subset C$ y cada $\varepsilon > 0$, existe un semiespacio abierto H de X tal que $A \cap H \neq \emptyset$ y $\operatorname{diam}(f(A \cap H)) < \varepsilon$.

Denotaremos $\mathscr{D}_U(C, M)$ el conjunto de las aplicaciones dentables de C a M que son además uniformemente continuas en subconjuntos acotados de C. Necesitamos esta condición técnica para llevar a cabo varias operaciones motivadas por el estudio geométrico de la RNP, que asegura buenas propiedades para esta clase de aplicaciones. Estas propiedades están resumidas en el siguiente resultado.

Teorema C (con M. Raja)**.** Sea C un subconjunto cerrado convexo de un espacio de Banach. Si M es un espacio vectorial, entonces $\mathscr{D}_U(C, M)$ es un espacio vectorial. Supongamos además que C es acotado. Entonces:

(a) si M es un espacio métrico completo, entonces $\mathscr{D}_U(C,M)$ es completo para la métrica de convergencia uniforme sobre C;
(b) si M es un espacio de Banach, entonces $\mathscr{D}_U(C,M)$ es un espacio de Banach;
(c) si M es un álgebra de Banach (resp. retículo de Banach), entonces $\mathscr{D}_U(C,M)$ es un álgebra de Banach (resp. retículo de Banach).

La clave para probar el Teorema C es el hecho de que existen muchos funcionales, en el sentido de la categoría, que definen rodajas donde la oscilación de aplicación es pequeña.

De especial interés es el caso $M=\mathbb{R}$, porque toda función convexa superiormente acotada e inferiormente semicontinua es dentable. Teniendo en cuenta que $\mathscr{D}_U(C,\mathbb{R})$ es un espacio vectorial, no es sorprendente que la diferencia de dos funciones convexas continuas acotadas sea dentable. Las funciones que son diferencia de funciones convexas, llamadas habitualmente funciones $\mathscr{DC}$, desempeñan un papel importante en análisis variacional y optimización (véase, por ejemplo [BB, HU, Tuy]). Además, la posibilidad de aproximar uniformemente una función real por funciones $\mathscr{DC}$ resulta estar fuertemente relacionada con su dentabilidad. De hecho, Cepedello Boiso [CB] caracterizó los espacios superreflexivos como aquellos espacios de Banach en los que cada función lipschitziana definida en él puede ser aproximada uniformemente sobre acotados por funciones $\mathscr{DC}$ que son lipschitzianas en conjuntos acotados. Raja probó en [Raj5] una versión localizada de este resultado que afirma que una función lipschitziana definida en un compacto convexo acotado es finitamente dentable (una versión más fuerte de la dentabilidad) si y solo si es límite uniforme de funciones $\mathscr{DC}$ lipschitzianas. Mostraremos que el resultado de Raja también vale para funciones uniformemente continuas. Además, veremos que cada función finitamente dentable uniformemente continua con rango relativamente compacto puede ser aproximada uniformemente por aplicaciones $\mathscr{DC}$. La noción de aplicación $\mathscr{DC}$ fue introducida por Veselý y Zajíček en [VZ1] como una extensión de las funciones $\mathscr{DC}$ al contexto vector-valuado. Se dice que una aplicación continua $F\colon C\to Y$ definida en un convexo $C\subset X$ es una *aplicación $\mathscr{DC}$* si existe una función continua f definida en C tal que $f+y^*\circ F$ es una función convexa y continua para cada $y^*\in S_{Y^*}$. En tal caso la función f se llama *función de control* de F. Nuestro siguiente resultado muestra que la dentabilidad de un conjunto está estrechamente relacionada con la dentabilidad de aplicaciones $\mathscr{DC}$ definidas en él.

Teorema D (con M. Raja). Sea D un subconjunto cerrado convexo de un espacio de Banach. Las siguientes afirmaciones son equivalentes:

(i) el conjunto D tiene la RNP;
(ii) para cada espacio de Banach X y cada subconjunto convexo $C\subset X$, cada aplicación $\mathscr{DC}$ acotada continua $F\colon C\to D$ que admite una función de control acotada es dentable.

A continuación nos centramos en el principio variacional de Stegall. Recordemos que este resultado afirma que una función inferiormente semicontinua y acotada inferiormente admite perturbaciones lineales arbitrariamente pequeñas de modo que la función resultante alcanza su mínimo en un sentido fuerte. Nuestro objetivo es encontrar una versión del teorema de Stegall donde la hipótesis de dentabilidad del dominio es reemplazada por la dentabilidad de la función. Veremos que esto es posible cuando la función es cerrada, utilizando para ello un marco general para principios variacionales debido a Lassonde y Revalski [LR].

Finalmente, hemos considerado el caso particular de la dentabilidad de la aplicación identidad cuando $M = C$ equipado con una métrica que es uniformemente continua respecto a la norma. En este caso, no se puede obtener mucho a menos que la métrica induzca la topología de la norma, pero esas hipótesis no son más generales que la RNP. De hecho, probaremos que si C es un cerrado convexo que es dentable con respecto a una métrica completa definida en él, y además la métrica es uniformemente continua sobre acotados con respecto a la norma e induce la topología de la norma, entonces C tiene la RNP.

Gran parte de los resultados en este capítulo han aparecido en [GLR1].

Capítulo 3. Sobre espacios fuertemente asintóticamente uniformemente suaves y convexos

El *módulo de convexidad asintótica uniforme* de un espacio de Banach X está dado por

$$\overline{\delta}_X(t) = \inf_{x \in S_X} \sup_{\dim(X/Y)<\infty} \inf_{y \in S_Y} \|x + ty\| - 1,$$

y el *módulo de suavidad asintótica uniforme* de X está dado por

$$\overline{\rho}_X(t) = \sup_{x \in S_X} \inf_{\dim(X/Y)<\infty} \sup_{y \in S_Y} \|x + ty\| - 1.$$

Se dice que el espacio X es *asintóticamente uniformemente convexo* (AUC) si $\overline{\delta}_X(t) > 0$ para cada $t > 0$, y que es *asintóticamente uniformemente suave* (AUS) si $\lim_{t\to 0} t^{-1}\overline{\rho}_X(t) = 0$. Si X es un espacio dual y solo consideramos subespacios débil* cerrados de X entonces el módulo correspondiente se denota $\overline{\delta}^*_X(t)$. El espacio X se dice que es *débil* AUC* si $\overline{\delta}^*_X(t) > 0$ para cada $t > 0$. Cabe destacar que se prueba en [DKLR] que un espacio es AUS si y solo si su dual es débil* AUC. Además, $\overline{\rho}_X$ está relacionado cuantitativamente con $\overline{\delta}^*_X$ mediante la dualidad de Young. Remitimos al lector a [JLPS] y a las referencias ahí incluidas para un estudio detallado de estas propiedades.

Lennard probó en [Len] que el espacio de los operadores traza en un espacio de Hilbert es débil* AUC. Equivalentemente, el espacio de operadores compactos $\mathscr{K}(\ell_2, \ell_2)$ es AUS. Este resultado fue extendido por Besbes en [Bes], probando que $\mathscr{K}(\ell_p, \ell_p)$ es AUS para todo $1 < p < \infty$. Además, se prueba en [DKR$^+$1] que $\mathscr{K}(\ell_p, \ell_q)$ es AUS con potencia $\min\{p', q\}$ para cada $1 < p, q < \infty$. Por otra parte, Causey ha probado recientemente en [Cau2] que el índice de Szlenk del producto tensorial inyectivo $X \widehat{\otimes}_\varepsilon Y$ es igual al máximo de los índices de Szlenk de X e Y para todos los espacios de Banach X e Y. En particular $X \widehat{\otimes}_\varepsilon Y$ admite una norma equivalente AUS si y solo si X e Y admiten una norma equivalente AUS. Además, Draga y Kochanek han probado en [DK2] que es posible obtener una norma equivalente AUS en $X\widehat{\otimes}_\varepsilon Y$ con potencia el máximo de las que tienen las normas de X e Y. Por tanto, es una pregunta natural si el producto tensorial inyectivo de espacios AUS es AUS con su norma canónica.

En este capítulo introducimos la noción de AUC fuerte y AUS fuerte para espacios de Banach con descomposiciones finito-dimensionales (FDD). Mostramos que para cada espacio de Banach con una FDD, la propiedad de ser fuertemente asintóticamente uniformemente suave es más fuerte que la de ser asintóticamente uniformemente suave, y más débil que ser uniformemente suave. Un resultado análogo se cumple reemplazando "suavidad" por "convexidad". Nuestro resultado principal es que el producto tensorial inyectivo de espacios fuertemente AUS es AUS. En particular, proporcionamos la siguiente generalización del Teorema 4.3 en [DKR$^+$1].

Teorema E (con M. Raja)**.** Sean X, Y espacios de Banach.

(a) Supongamos que X e Y tienen FDDs monótonas. Si X e Y son uniformemente suaves entonces $X\widehat{\otimes}_\varepsilon Y$ es AUS. Además, si X es uniformemente suave con potencia p e Y es uniformemente suave con potencia q entonces $X\widehat{\otimes}_\varepsilon Y$ es AUS con potencia $\min\{p, q\}$.

(b) Supongamos que X^* e Y tienen FDDs monótonas. Si X es uniformemente convexo e Y es uniformemente suave entonces $\mathscr{K}(X, Y)$ es AUS. Además, si X es uniformemente convexo con potencia p e Y es uniformemente suave con potencia q entonces $\mathscr{K}(X, Y)$ es AUS con potencia $\min\{p', q\}$.

Cabe señalar que R. Causey ha obtenido muy recientemente un resultado más general, que muestra que el producto tensorial inyectivo de espacios AUS es AUS [Cau1].

También probamos algunos resultados generales sobre espacios fuertemente AUS y fuertemente AUC. Por ejemplo, si X es fuertemente AUS (respectivamente, fuertemente AUC) con respecto a una FDD E, entonces E es una FDD shrinking (respectivamente, una FDD boundedly complete). Además, probamos un análogo a la dualidad entre el módulo de suavidad de un espacio y el módulo de convexidad de su dual. Obtenemos también que cada espacio reflexivo AUS (respectivamente AUC) que admite una FDD puede ser renormado para ser fuertemente AUS (respectivamente, fuertemente AUC).

Nuestras técnicas conducen a una caracterización de las funciones de Orlicz M, N tales que el espacio $\mathscr{K}(h_M, h_N)$ es AUS en términos de sus índices de Boyd α_M, β_M (véase la Sección 3.4 para las definiciones). En particular, obtenemos lo siguiente.

Teorema F (con M. Raja)**.** Sean M, N funciones de Orlicz. El espacio $\mathscr{K}(h_M, h_N)$ es AUS si y solo si $\alpha_M, \alpha_N > 1$ y $\beta_M < +\infty$. Además, $\min\{\beta_M', \alpha_N\}$ es el supremo de los números $\alpha > 0$ tales que el módulo de suavidad asintótica uniforme de $\mathscr{K}(h_M, h_N)$ está en potencia α.

Cabe señalar que, para la norma natural, no se puede esperar mucho más. De hecho, Ruess y Stegall mostraron en [RS1, Corollary 3.5] que ni la norma de $X\widehat{\otimes}_\varepsilon Y$ ni la norma de $\mathscr{K}(X, Y)$ son suaves cuando las dimensiones de X e Y son mayores o iguales que 2.

Por otra parte, Dilworth y Kutzarova probaron en [DK1] que $\mathscr{L}(\ell_p, \ell_q)$ no es estrictamente convexo para $1 \leq p \leq q \leq \infty$. Extendemos ese resultado, mostrando que si X e Y son espacios de Banach de dimensión mayor o igual que 2, entonces $\mathscr{K}(X,Y)$ y $X\widehat{\otimes}_\varepsilon Y$ no son estrictamente convexos.

Los resultados de este capítulo provienen del artículo [GLR2].

Capítulo 4. Dualidad de espacios de funciones lipschitzianas vector-valuadas

Es conocido que el espacio $\mathrm{Lip}_0(M)$ de las funciones lipschitzianas en un espacio métrico M que se anulan en un punto distinguido $0 \in M$ es un espacio de Banach cuando se le dota de la norma dada por la mejor constante de Lipschitz de la función. Es más, $\mathrm{Lip}_0(M)$ es un espacio dual, y su predual canónico

$$\mathscr{F}(M) = \overline{\operatorname{span}}\{\delta(m) : m \in M\},$$

donde $\langle f, \delta(m)\rangle = f(m)$, se denomina *espacio Lipschitz libre sobre M* (también *espacio de Arens–Eels sobre M*).

Los espacios $\mathrm{Lip}_0(M)$ y sus preduales canónicos $\mathscr{F}(M)$ han recibido gran atención desde el artículo [GK] de G. Godefroy y N. Kalton, remitimos al lector a [God3] para un estudio recopilatorio reciente sobre estos espacios. Un problema tradicional es determinar cuándo el espacio $\mathscr{F}(M)$ es un dual y, en tal caso, identificar un predual como subespacio de $\mathrm{Lip}_0(M)$. En particular, Weaver probó en [Wea2] que si M es compacto entonces el espacio de funciones pequeñas-Lipschitz

$$\mathrm{lip}_0(M) := \left\{ f \in \mathrm{Lip}_0(M) : \lim_{\varepsilon\to 0} \sup_{0<d(x,y)<\varepsilon} \frac{|f(x)-f(y)|}{d(x,y)} = 0 \right\},$$

es un predual de $\mathscr{F}(M)$ isométricamente siempre que *separa puntos uniformemente*, es decir, existe una constante $a > 1$ tal que para cada $x, y \in M$ existe $f \in \mathrm{lip}_0(M)$ con $f(x) - f(y) = d(x,y)$ y $\|f\|_L \leq a$.

Destacamos dos generalizaciones del resultado de Weaver. En primer lugar, si M es un espacio métrico *propio* (es decir, las bolas cerradas en M son compactas), el papel de predual es desempeñado por el espacio $S_0(M)$ introducido por Dalet en [Dal2] de aquellas funciones pequeñas-Lipschitz que son planas en el infinito en un cierto sentido.

Por otra parte, Kalton probó que si M admite una topología Hausdorff compacta τ tal que la métrica es τ-inferiormente semicontinua, entonces bajo ciertas hipótesis el espacio $\mathrm{lip}_0(M) \cap \mathscr{C}(M,\tau)$ es un predual de $\mathscr{F}(M)$. Damos una prueba diferente del resultado de Kalton, basada en el teorema de Petunīn–Plīčhko, que evita la hipótesis de metrizabilidad de la topología considerada en la prueba original de Kalton. Usamos el teorema de Kalton para deducir que ciertos espacios Lipschitz libres sobre espacios métricos uniformemente discretos son duales. La completitud de la topología de Mackey $\mu(\mathscr{F}(M), \mathrm{Lip}_0(M) \cap \mathscr{C}(M,\tau))$ y el uso de algunos resultados conocidos proporcionan un resultado de dualidad para espacios métricos uniformemente discretos en el contexto no separable.

Además, obtenemos extensiones de los resultados de Kalton y Dalet al caso vector-valuado. La versión vector-valuada de los espacios Lipschitz libres, denotada $\mathscr{F}(M,X)$, ha sido considerada recientemente en [BGLPRZ1] como un predual del espacio de funciones lipschitzianas vector-valuadas $\mathrm{Lip}_0(M,X^*)$ siguiendo el espíritu de la versión escalar. Consideramos extensiones naturales de los espacios $\mathrm{lip}_0(M)$ y $S_0(M)$ al caso vector-valuado, denotadas $\mathrm{lip}_0(M,X)$ y $S_0(M,X)$. Utilizamos las técnicas de [JVSVV] y teoría de productos tensoriales para mostrar que $S_0(M,X)$ es isométrico al producto tensorial inyectivo $S_0(M)\widehat{\otimes}_\varepsilon X$ bajo hipótesis adecuadas sobre M y X, lo que conduce a una extensión del teorema de dualidad de Dalet al caso vector-valuado. También encontramos una extensión del resultado de Kalton a este contexto, Para ello, introducimos el espacio

$$\mathrm{lip}_\tau(M,X) := \mathrm{lip}_0(M,X) \cap \{f\colon M \to X : f \text{ es } \tau\text{-}\|\ \| \text{ continua}\}$$

donde τ es una topología Hausdorff compacta en M tal que la métrica es τ-inferiormente semicontinua. Probamos que $\mathrm{lip}_\tau(M,X)$ es isométrico al espacio de operadores compactos débil*-débil continuos desde X^* a $\mathrm{lip}_\tau(M)$. Como consecuencia, bajo hipótesis adecuadas $\mathrm{lip}_\tau(M,X)$ puede ser identificado con $\mathrm{lip}_\tau(M)\widehat{\otimes}_\varepsilon X$. Esto nos lleva al siguiente resultado.

Teorema G (con C. Petitjean y A. Rueda Zoca)**.** Sea M un espacio métrico separable acotado y τ una topología Hausdorff compacta en M tal que d es τ-inferiormente semicontinua y $\mathrm{lip}_\tau(M)$ separa puntos uniformemente. Si $\mathscr{F}(M)$ o X^* tiene la propiedad de aproximación, entonces $\mathrm{lip}_\tau(M,X)^* = \mathscr{F}(M,X^*)$.

Es conocido que $\mathrm{lip}_0(M)$ (respectivamente, $S_0(M)$) es un espacio M-embebido siempre que M es compacto [Kal2] (respectivamente, M es propio [Dal2]). Por tanto, estos espacios no pueden ser duales si tienen dimensión infinita. Así que es natural preguntarse qué pasa con $\mathrm{lip}_0(M) \cap \mathscr{C}(M,\tau)$ y con la versión vector-valuada de estos espacios. Para ello, es útil encontrar una propiedad geométrica de espacios de Banach que no sea compatible con ser un espacio dual. En este sentido, la noción de espacio casi cuadrado ha sido introducida recientemente. Según [ALL], un espacio de Banach X es *casi cuadrado* (ASQ) si para cada $x_1,\ldots,x_k \in S_X$ y $\varepsilon > 0$ existe $y \in S_X$ tal que

$$\|x_i \pm y\| \leq 1+\varepsilon \text{ para todo } i \in \{1,\ldots,k\}.$$

A grandes rasgos, podemos decir que los espacios ASQ tienen un comportamiento tipo c_0 desde un punto de vista geométrico. Este comportamiento está recogido en el hecho de que un espacio de Banach admite una norma equivalente ASQ si y solo si el espacio contiene una copia isomorfa de c_0 [BGLPRZ3].

En [ALL] se pregunta si existe algún espacio dual ASQ. Proporcionamos una respuesta parcial negativa a esta pregunta considerando la noción de espacio *incondicionalmente casi cuadrado* (UASQ) y probando que un espacio UASQ no es isométrico a un espacio de Banach dual. Aplicamos esta noción para dar criterios de no dualidad en $\mathrm{lip}_0(M)$, $S_0(M)$

y sus versiones vector-valuadas. Esto conduce a resultados como el siguiente, que por lo que sabemos era desconocido incluso en el caso real-valuado.

Teorema H (con C. Petitjean y A. Rueda Zoca)**.** Sean X e Y espacios de Banach y $0 < \alpha < 1$. Entonces $\mathrm{lip}_{w^*}((B_{X^*}, \| \|^\alpha), Y)$ es UASQ. En particular, no es isométrico a un espacio de Banach dual.

Este capítulo está basado principalmente en los artículos [GLPRZ1, GLRZ, GLPPRZ].

Capítulo 5. Propiedades geométricas de espacios Lipschitz libres

La primera sección de este capítulo está dedicada al estudio de la propiedad de Daugavet en espacios de funciones lipschitzianas y espacios Lipschitz libres. Recordemos que se dice que un espacio de Banach X tiene la *propiedad de Daugavet* si $\|T + I\| = 1 + \|T\|$ para cada operador $T\colon X \to X$ de rango uno, donde I denota al operador identidad. Ejemplos de espacios con esta propiedad son los espacios $\mathscr{C}(K)$ para K compacto Hausdorff perfecto, $L_1(\mu)$ y $L_\infty(\mu)$ para una medida μ no atómica, y los preduales de espacios con la propiedad de Daugavet.

Recordemos que $\mathrm{Lip}_0([0,1])$ es isométrico a $L_\infty[0,1]$ y por tanto tiene la propiedad de Daugavet. En la sección 6 de [Wer] se pregunta si el espacio $\mathrm{Lip}_0([0,1]^2)$ tiene la propiedad de Daugavet. Este problema fue resuelto de manera positiva en [IKW1], donde se muestra, entre otros resultados, que $\mathrm{Lip}_0(M)$ tiene la propiedad de Daugavet siempre que M es un espacio *length* (es decir, para cada par de puntos $x, y \in M$ la distancia $d(x, y)$ es igual al ínfimo de la longitud de las curvas rectificables uniendo x e y). Además, en [IKW1] se caracteriza cuándo $\mathrm{Lip}_0(M)$ o $\mathscr{F}(M)$ tienen la propiedad de Daugavet para un espacio métrico compacto M, en términos de una propiedad geométrica del espacio métrico subyacente, que ellos denominan (Z). Proporcionamos una caracterización métrica de cuándo $\mathrm{Lip}_0(M)$ o $\mathscr{F}(M)$ tienen la propiedad de Daugavet para un espacio métrico M en general. Recordemos que, dado un espacio métrico M con completado $\hat{M}$, los espacios $\mathrm{Lip}_0(M)$ y $\mathrm{Lip}_0(\hat{M})$ son isométricos, así que sin pérdida de generalidad podemos suponer que el espacio métrico es completo.

Teorema I (con A. Procházka y A. Rueda Zoca)**.** Sea M un espacio métrico completo. Las siguientes afirmaciones son equivalentes:

(i) M es un espacio length;
(ii) $\mathrm{Lip}_0(M)$ tiene la propiedad de Daugavet;
(iii) $\mathscr{F}(M)$ tiene la propiedad de Daugavet.

Es conocido que los espacios completos length se caracterizan en términos de la existencia de puntos medios aproximados, así que el resultado anterior proporciona una caracterización puramente métrica de los espacios métricos M tales que $\mathrm{Lip}_0(M)$ tiene la propiedad de Daugavet.

Además, se prueba en [IKW1] que cada subconjunto compacto de un espacio de Banach suave LUR que tenga la propiedad (Z) es convexo. Obtenemos que cada subconjunto compacto de un espacio de Banach estrictamente convexo con la propiedad (Z) es convexo.

A continuación, tratamos sobre la estructura extremal de la bola unidad de un espacio Lipschitz libre. Recordemos que un punto de la bola unidad de un espacio de Banach se dice que es un *extremo preservado* si es un punto extremo de la bola bidual. Un resultado de Weaver [Wea2] asegura que todo punto extremo preservado de $B_{\mathscr{F}(M)}$ es una molécula, es decir, un elemento de la forma $\frac{\delta(x)-\delta(y)}{d(x,y)}$ para ciertos $x, y \in M$, $x \neq y$. Weaver probó también que la molécula $\frac{\delta(x)-\delta(y)}{d(x,y)}$ es un punto extremo preservado si existe una función $f \in \mathrm{Lip}_0(M)$ que se pica en el par (x, y). Mostramos que de hecho esa condición caracteriza los puntos fuertemente expuestos de $B_{\mathscr{F}(M)}$.

Teorema J (con A. Procházka y A. Rueda Zoca)**.** Sea M un espacio métrico. Dados $x, y \in M$, $x \neq y$, las siguientes afirmaciones son equivalentes:

(i) $\frac{\delta(x)-\delta(y)}{d(x,y)}$ es un punto fuertemente expuesto de $B_{\mathscr{F}(M)}$;
(ii) existe una función $f \in \mathrm{Lip}_0(M)$ que se pica en (x, y);
(iii) el par (x, y) no tiene la propiedad (Z), es decir, existe $\varepsilon > 0$ tal que

$$d(x, z) + d(y, z) \geq d(x, y) + \varepsilon \min\{d(x, z), d(y, z)\}$$

para cada $z \in M \setminus \{x, y\}$.

El resultado anterior extiende la caracterización de las parejas para las que existe una función que se pica dada en [DKP] para el caso de los $\mathbb{R}$-árboles.

Como consecuencia de los Teoremas I y J se tiene que, para un espacio métrico compacto M, el espacio $\mathscr{F}(M)$ tiene la propiedad de Daugavet si y solo si $B_{\mathscr{F}(M)}$ no tiene ningún punto fuertemente expuesto.

Mostramos además que, incluso en el contexto de los espacios libres, no es cierto que cada punto extremo de la bola unidad sea un punto extremo preservado, ni tampoco que cada punto extremo preservado sea fuertemente expuesto. Sin embargo, probamos que cada punto débil-fuertemente expuesto es fuertemente expuesto, y como consecuencia la diferenciabilidad Gâteaux y la diferenciabilidad Fréchet coinciden para la norma de $\mathrm{Lip}_0(M)$. Además, obtenemos el siguiente resultado.

Teorema K (con C. Petitjean, A. Procházka y A. Rueda Zoca)**.** Sea M un espacio métrico. Cada punto extremo preservado de $B_{\mathscr{F}(M)}$ es un punto diente.

Durante la preparación de este trabajo, Aliaga y Guirao [AG] han caracterizado métricamente los puntos extremos preservados de los espacios libres del siguiente modo: una molécula $\frac{\delta(x)-\delta(y)}{d(x,y)}$ es un punto extremo preservado de $B_{\mathscr{F}(M)}$ si y solo si para cada $\varepsilon > 0$ existe $\delta > 0$ tal que cada $z \in M$ cumple que

$$(1 - \delta)(d(x, z) + d(z, y)) < d(x, y) \Rightarrow \min\{d(x, z), d(y, z)\} < \varepsilon.$$

Proporcionamos una prueba alternativa del resultado de Aliaga y Guirao que accidentalmente prueba de nuevo nuestro Teorema K.

Dos problemas continúan abiertos en este contexto (véase [AG]). En primer lugar, no se sabe si cada punto extremo de $B_{\mathscr{F}(M)}$ es de la forma $\frac{\delta(x)-\delta(y)}{d(x,y)}$ para ciertos $x, y \in M$. Por otra parte, es fácil comprobar que si $\frac{\delta(x)-\delta(y)}{d(x,y)}$ es un punto extremo entonces $d(x,y) < d(x,z) + d(z,y)$ para cada $z \in M \setminus \{x,y\}$, pero no se conoce si esa condición es suficiente. Proporcionamos respuestas parciales positivas a esas preguntas en algunos casos particulares. El más notable entre ellos es el caso en el que $\mathscr{F}(M)$ admite un predual isométrico con ciertas propiedades adicionales. Mostramos además que la existencia de tal predual tiene consecuencias sobre el conjunto de funciones lipschitzianas que alcanzan la norma. También probamos que, para ciertos espacios métricos compactos, cada punto extremo de $B_{\mathscr{F}(M)}$ es un punto diente puesto que la norma de $\mathscr{F}(M)$ resulta ser débil* AUC.

Finalmente, la última parte del capítulo está dedicada a algunas cuestiones relativas a la teoría isomorfa de espacios Lipschitz libres. En particular, estamos interesados en la relación entre estos espacios y el espacio universal de Pełczyński $\mathbb{P}$. Recordemos que $\mathbb{P}$ es un espacio de Banach separable con base tal que cada espacio de Banach separable con la propiedad de aproximación acotada es isomorfo a un subespacio complementado de $\mathbb{P}$. En [GK] se prueba que $\mathbb{P}$ y $\mathscr{F}(\mathbb{P})$ son isomorfos.

Teorema L (con A. Procházka)**.** Existe un subconjunto compacto convexo K del espacio de Pełczyński $\mathbb{P}$ tal que $\mathbb{P}$ es isomorfo a $\mathscr{F}(K)$.

Mostramos también que si M es un retracto Lipschitz absoluto separable, entonces $\mathbb{P}$ no es isomorfo a un subespacio complementado de $\mathscr{F}(M)$. En particular, esto muestra que $\mathscr{F}(c_0)$ no es isomorfo a $\mathbb{P}$, lo que responde a una pregunta planteada en [CDW].

Los contenidos incluidos en la primera y la segunda sección del capítulo han aparecido en [GLPRZ2, GLPPRZ]. Los resultados de la tercera sección forman parte de un preprint con A. Procházka que esperamos que aparezca pronto.

Abstract

This memoir deals with several topics in the framework of geometry of Banach spaces, with a focus on the structure of convex sets and its application in spaces of Lipschitz functions and their preduals. We next summarise the content of this work.

Chapter 0. Some preliminary content

The aim of this introductory chapter is to recall a number of geometrical properties which motivate several results in this work. The results that we include here are essentially known and we only include the proofs of the less standard ones.

First we focus on the extremal structure of convex sets, recalling the notions of extreme, exposed, denting point, etc. and the relations among them. We recall some topological properties of the set of extreme points and we review some results that allow to recover a closed convex set from a distinguished subset of the extreme points, from the Krein–Milman theorem to the Bourgain–Phelps theorem on convex sets with the Radon–Nikodým property.

Next, we give a general framework which contains the set derivations used throughout the text. Roughly, a derivation of topological space consists in removing every subset in a given family (usually, open sets or open slices) which is small with respect to a certain measure (for instance, the diameter). The ordinal index associated to the derivation is number of steps needed in order to achieve the emptyset. Here we give a general scheme that includes a number of well-known ordinal indices as Cantor–Bendixson, Szlenk and dentability indices. Frequently, a certain topological or geometrical property of the space follows from the fact that these indices are bounded by a certain ordinal (usually, ω or ω_1). We review several results in this line.

Finally, the last section contains some results on tensor products and approximation properties for future reference.

Chapter 1. Compact convex sets that admit a lower semicontinuous strictly convex function

A well-known result of Hervé [Her] says that a compact convex subset K of a locally convex space is metrizable if and only if there exists a function $f\colon K \to \mathbb{R}$ which is both continuous and strictly convex. It turns out that lower semicontinuity is a very natural

hypothesis for a convex function, so it is natural to wonder if the existence of a strictly convex lower semicontinuous function on a compact convex subset K of a locally convex space enforces special topological properties on K. Ribarska proved [Rib1, Rib2] that such a compact set is *fragmentable* by a finer metric, and in particular it contains a completely metrizable dense subset. Raja proved [Raj6] that the same is true for the set of its extreme points $\operatorname{ext}(K)$. On the other hand, Talagrand's argument in [DGZ, Theorem 5.2.(ii)] shows that $[0, \omega_1]$ is not embeddable in such a compact set. In addition, Godefroy and Li showed [GL] that if the set of probabilities on a compact group K admits a strictly convex lower semicontinuous function then K is metrizable.

Our purpose here is to continue with the study of the class of compact convex sets which admit a strictly convex lower semicontinuous function. We denote this class by $\mathscr{SC}$. The first remarkable fact that we have got is a Banach representation result.

Theorem A (with J. Orihuela and M. Raja)**.** Let K be a convex compact subset of a locally convex space. Then $K \in \mathscr{SC}$ if and only if K embeds linearly into a strictly convex dual Banach space endowed with its weak* topology.

Notice that the strictly convex norm of the dual Banach space in the statement is weak* lower semicontinuous, which is a stronger condition that just being a strictly convex Banach space isomorphic to a dual space.

If $f\colon K \to \mathbb{R}$ is a strictly convex function, then the *symmetric* defined by

$$\rho(x, y) = \frac{f(x) + f(y)}{2} - f\left(\frac{x+y}{2}\right)$$

provides a consistent way to measure *diameters* of subsets of K. This idea was successfully applied in renorming theory [MOTV] and will a key ingredient for us.

We show that a compact convex set belongs to $\mathscr{SC}$ if and only if it has $(*)$ with slices. This property was introduced in [OST] in order to characterise dual Banach spaces that admit a dual strictly convex norm. As a consequence, we prove a characterisation of the class $\mathscr{SC}$ in terms of the existence of a symmetric with countable dentability index. We cannot replace symmetric by metric in that result since that would imply that the compact set is Gruenhage. Indeed, we characterise Gruenhage compacta in terms of the existence of a metric having countable Szlenk index.

In the last part of the chapter, we analyse the existence of faces of continuity of a convex function, as well as exposed points of continuity of a strictly convex function. Our starting point is a result of Raja [Raj6], which ensures the existence of extreme points of continuity of a convex lower semicontinuous function. We mimic some arguments coming from the geometric study of the Radon–Nikodým property, but looking for slices having small diameter with respect to the symmetric associated to the function. This leads to results as the following one.

Theorem B (with J. Orihuela and M. Raja)**.** Let E be a locally convex space and let $f\colon E \to \mathbb{R}$ be lower semicontinuous, strictly convex and bounded on compact sets. Then for every $K \subset E$ compact and convex, the set of points in K which are both exposed and continuity points of $f|_K$ is dense in $\mathrm{ext}(K)$.

As a consequence, we get that every compact convex set in the class $\mathscr{SC}$ is the closed convex hull of its exposed points.

This chapter is based on the paper [GLOR].

Chapter 2. Maps with the Radon–Nikodým property

The Radon–Nikodým property (RNP) plays a central role in Banach space theory, particularly in isomorphic and nonlinear theories. It is related to the differentiation of Lipschitz maps, the extremal structure of convex sets, representation theory without compactness, representation of dual function spaces, optimization theory, etc. The interested reader in RNP, theory and applications, is addressed to [BL, Bou4, Die, FHH+].

We consider the most geometrical of the characterisations of the RNP. Namely, a subset C of a Banach space X has the RNP if and only if every bounded subset $A \subset C$ is *dentable*, that is, for every $\varepsilon > 0$ there is an open half-space H such that $\mathrm{diam}(A \cap H) < \varepsilon$.

The RNP was extended to linear operators by Reĭnov [Reĭ1] and Linde [Lin]. In this chapter we propose a definition for maps from a closed convex subset of a Banach space into a metric space in order to generalise the RNP to a less linear frame. Our starting point is to consider a notion of dentability for maps, which appears as a strengthening of the point of continuity property.

Definition. Let C be a nonempty subset of a Banach space X and let M be a metric space. A map $f\colon C \to M$ is said to be *dentable* if for every nonempty bounded set $A \subset C$ and $\varepsilon > 0$, there is an open half-space H of X such that $A \cap H \neq \emptyset$ and $\mathrm{diam}(f(A \cap H)) < \varepsilon$.

By $\mathscr{D}_U(C, M)$ we denote the set of dentable maps from C to M which are moreover uniformly continuous on bounded subsets of C. That technical condition is necessary in order to perform several operations motivated by the geometrical study of the RNP, which ensures nice properties for this class of maps. These properties are summarized in the next result.

Theorem C (with M. Raja)**.** Let C be a closed convex subset of a Banach space. If M is a vector space, then $\mathscr{D}_U(C, M)$ is a vector space. Assume moreover that C is bounded. Then:

(a) if M is a complete metric space, then $\mathscr{D}_U(C, M)$ is complete for the metric of uniform convergence on C;
(b) if M is a Banach space, then $\mathscr{D}_U(C, M)$ is a Banach space;
(c) if M is a Banach algebra (resp. lattice), then $\mathscr{D}_U(C, M)$ is a Banach algebra (resp. lattice).

The key to prove Theorem C is the fact that there are many functionals, in a categorical sense, defining slices of small oscillation.

Particularly interesting is the case $M = \mathbb{R}$ because every bounded above lower semicontinuous convex function is dentable. Bearing in mind that $\mathscr{D}_U(C, \mathbb{R})$ is a vector space, it is not surprising that the difference of two bounded convex continuous functions is dentable. Differences of convex functions, usually named $\mathscr{DC}$ functions, play an important role in variational analysis and optimization (see, e.g. [BB, HU, Tuy]). Moreover, the possibility of a real function to be uniformly approximated by $\mathscr{DC}$ functions is closely related to its dentability. Indeed, Cepedello Boiso [CB] characterised super-reflexive spaces as those Banach spaces in which every Lipschitz function defined on it can be approximated uniformly on bounded sets by $\mathscr{DC}$ functions which are Lipschitz on bounded sets. Raja proved in [Raj5] a localised version of that result. Namely, a Lipschitz function defined on a bounded closed convex set is *finitely dentable* (which is a stronger notion of dentability) if and only if it is the uniform limit of $\mathscr{DC}$-Lipschitz functions. We show that Raja's result still holds for uniformly continuous functions. Moreover, we show that every finitely dentable uniformly continuous map with relatively norm-compact range can be uniformly approximated by $\mathscr{DC}$ maps. The notion of $\mathscr{DC}$ map was introduced by Veselý and Zajíček in [VZ1] as an extension of $\mathscr{DC}$ functions to the vector-valued setting. A continuous map $F\colon C \to Y$ defined on a convex subset $C \subset X$ is said to be a *$\mathscr{DC}$ map* if there exists a continuous function f on C such that $f + y^* \circ F$ is a convex continuous function on C for every $y^* \in S_{Y^*}$. The function f is called a *control function* for F. Our next result shows that the dentability of a set is closely related to the dentability of $\mathscr{DC}$ maps defined on it.

Theorem D (with M. Raja)**.** Let D be a closed convex subset of a Banach space. Then the following are equivalent:

(i) the set D has the RNP;
(ii) for every Banach space X and every convex subset $C \subset X$, every bounded continuous $\mathscr{DC}$ map $F\colon C \to D$ admitting a bounded control function is dentable.

Next we focus on Stegall's variational principle. Recall that this result ensures that a lower semicontinuous bounded below function defined on a set with the RNP admits a small linear perturbation such that the resulting function attains its minimum in a strong way. Our aim is to find a version of Stegall's theorem where the hypothesis of the dentability of the domain is replaced by the dentability of the function. By using a general approach to variational principles due to Lassonde and Revalski [LR], we show that is possible when the function is closed.

Finally, we have considered the particular case of the dentability of the identity map when $M = C$ endowed with a metric which is uniformly continuous with respect to the norm. In that case, not much can be obtained unless the metric induces the norm topology. But those hypotheses are not more general than the RNP. Indeed, we show that if C is a closed convex subset which is dentable with respect to a complete metric defined on it, and moreover the metric is uniformly continuous on bounded sets with respect to the norm and induces the norm topology, then C has the RNP.

Most of the results in this chapter have appeared in [GLR1].

Chapter 3. On strong asymptotic uniform smoothness and convexity

The *modulus of asymptotic uniform convexity* of a Banach space X is given by

$$\overline{\delta}_X(t) = \inf_{x \in S_X} \sup_{\dim(X/Y)<\infty} \inf_{y \in S_Y} \|x + ty\| - 1,$$

and the *modulus of asymptotic uniform smoothness* of X is given by

$$\overline{\rho}_X(t) = \sup_{x \in S_X} \inf_{\dim(X/Y)<\infty} \sup_{y \in S_Y} \|x + ty\| - 1.$$

The space X is said to be *asymptotically uniformly convex* (AUC for short) if $\overline{\delta}_X(t) > 0$ for each $t > 0$ and it is said to be *asymptotically uniformly smooth* (AUS for short) if $\lim_{t\to 0} t^{-1}\overline{\rho}_X(t) = 0$. If X is a dual space and we considered only weak* closed subspaces of X then the corresponding modulus is denoted by $\overline{\delta}^*_X(t)$. The space X is said to be *weak* AUC* if $\overline{\delta}^*_X(t) > 0$ for each $t > 0$. Let us highlight that it is proved in [DKLR] that a space is AUS if and only if its dual space is weak* AUC. In addition, $\overline{\rho}_X$ is quantitatively related to $\overline{\delta}^*_X$ by Young's duality. We refer the reader to [JLPS] and the references therein for a detailed study of these properties.

Lennard proved in [Len] that the space of trace class operators on a Hilbert space is weak* AUC. Equivalently, the space of compact operators $\mathscr{K}(\ell_2, \ell_2)$ is AUS. This result was extended by Besbes in [Bes], who showed that $\mathscr{K}(\ell_p, \ell_p)$ is AUS whenever $1 < p < \infty$. Moreover, in [DKR$^+$1] it is proved that $\mathscr{K}(\ell_p, \ell_q)$ is AUS with power type $\min\{p', q\}$ for every $1 < p, q < \infty$. On the other hand, Causey recently showed in [Cau2] that the Szlenk index of the injective tensor product $X\widehat{\otimes}_\varepsilon Y$ is equal to the maximum of the Szlenk indices of X and Y for all Banach spaces X and Y. In particular, $X\widehat{\otimes}_\varepsilon Y$ admits an equivalent AUS norm if and only if X and Y do. Moreover, Draga and Kochanek have proved in [DK2] that is possible to get an equivalent AUS norm in $X\widehat{\otimes}_\varepsilon Y$ with power type the maximum of the ones of the norm of X and Y. Thus, it is a natural question whether the injective tensor product of AUS spaces is an AUS space in its canonical norm.

In this chapter we introduce the notion of strongly AUC and strongly AUS spaces for Banach spaces with a finite dimensional decomposition (FDD). We show that for any Banach space with an FDD, the property of strong asymptotic uniform smoothness is stronger than the property of asymptotic uniform smoothness and weaker than the property of uniform smoothness, and the same holds replacing "smoothness" by "convexity". Our main result is that the injective tensor product of strongly AUS spaces is AUS. In particular, our result yields the following generalisation of Theorem 4.3 in [DKR$^+$1].

Theorem E (with M. Raja)**.** Let X, Y be Banach spaces.

(a) Assume that X and Y have monotone FDDs. If X and Y are uniformly smooth then $X\widehat{\otimes}_\varepsilon Y$ is AUS. Moreover, if X is uniformly smooth with power type p and Y is uniformly smooth with power type q then $X\widehat{\otimes}_\varepsilon Y$ is AUS with power type $\min\{p, q\}$.
(b) Assume that X^* and Y have monotone FDDs. If X is uniformly convex and Y is uniformly smooth then $\mathscr{K}(X, Y)$ is AUS. Moreover, if X is uniformly convex with

power type p and Y is uniformly smooth with power type q then $\mathscr{K}(X,Y)$ is AUS with power type $\min\{p',q\}$.

Let us point out that very recently R. Causey has proved a more general result, showing that the injective tensor product of AUS spaces is AUS [Cau1].

We also prove several general facts about strongly AUS and strongly AUC spaces. For example, if X is strongly AUS (respectively, strongly AUC) with respect to the FDD E, then E is a shrinking FDD (respectively, a boundedly complete FDD). Moreover, we prove analogues for the usual duality relationship between a smoothness modulus and a convexity modulus of the dual. We also get that every reflexive AUS (respectively, AUC) Banach space which admits an FDD can be renormed to be strongly AUS (respectively, strongly AUC).

Our techniques also lead to a characterisation of Orlicz functions M, N such that the space $\mathscr{K}(h_M,h_N)$ is AUS in terms of their Boyd indices α_M,β_M (see Section 3.4 for definitions). Namely, the following holds.

Theorem F (with M. Raja)**.** Let M,N be Orlicz functions. The space $\mathscr{K}(h_M,h_N)$ is AUS if and only if $\alpha_M,\alpha_N>1$ and $\beta_M<+\infty$. Moreover, $\min\{\beta'_M,\alpha_N\}$ is the supremum of the numbers $\alpha>0$ such that the modulus of asymptotic smoothness of $\mathscr{K}(h_M,h_N)$ is of power type α.

Remark that, for the natural norm, not much can be expected. Indeed, Ruess and Stegall showed in [RS1, Corollary 3.5] that neither the norm of $X\widehat{\otimes}_\varepsilon Y$ or the norm of $\mathscr{K}(X,Y)$ are smooth whenever the dimension of X and Y are greater or equal than 2. On the other hand, Dilworth and Kutzarova proved in [DK1] that $\mathscr{L}(\ell_p,\ell_q)$ is not strictly convex for $1\le p\le q\le\infty$. We extend that result by showing that if X and Y are Banach spaces with dimension greater or equal than 2, then $\mathscr{K}(X,Y)$ and $X\widehat{\otimes}_\varepsilon Y$ are not strictly convex.

The results of this chapter come from the paper [GLR2].

Chapter 4. Duality of spaces of vector-valued Lipschitz functions

It is well known that the space $\mathrm{Lip}_0(M)$ of all Lipschitz functions on a metric space M vanishing on a distinguished point $0\in M$ is a Banach space when it is endowed with the norm given by the best Lipschitz constant of the function. Moreover, $\mathrm{Lip}_0(M)$ is a dual space and its canonical predual

$$\mathscr{F}(M)=\overline{\operatorname{span}}\{\delta(m): m\in M\},$$

where $\langle f,\delta(m)\rangle=f(m)$, is called the *Lipschitz free space over M* (also the *Arens-Eels space over M*).

The space $\mathrm{Lip}_0(M)$ as well as its canonical predual $\mathscr{F}(M)$ have received much attention since the paper [GK] by G. Godefroy and N. Kalton, we refer the reader to [God3] for a recent survey on these spaces. One of the traditional problems is to determine when

$\mathscr{F}(M)$ is itself a dual Banach space and, in that case, to identify a predual as a subspace of $\mathrm{Lip}_0(M)$. In particular, Weaver proved in [Wea2] that if M is compact then the space of *little-Lipschitz functions*, that is,

$$\mathrm{lip}_0(M) := \left\{ f \in \mathrm{Lip}_0(M) : \lim_{\varepsilon\to 0} \sup_{0<d(x,y)<\varepsilon} \frac{|f(x)-f(y)|}{d(x,y)} = 0 \right\},$$

is an isometric predual of $\mathscr{F}(M)$ whenever it *separates points uniformly*, that is, there is some constant $a > 1$ such that for every $x, y \in M$ there is $f \in \mathrm{lip}_0(M)$ with $f(x) - f(y) = d(x,y)$ and $\|f\|_L \leq a$.

We highlight here two generalisations of Weaver's result. On the one hand, if M is a *proper* metric space (i.e. closed balls in M are compact sets), the role of a predual is played by the space $S_0(M)$ introduced by Dalet in [Dal2] of those little-Lipschitz functions which have an additional behaviour of flatness at infinity.

On the other hand, Kalton proved that if M admits a compact Hausdorff topology τ such that the metric is τ-lower semicontinuous, then under suitable hypotheses the space $\mathrm{lip}_0(M) \cap \mathscr{C}(M,\tau)$ is a predual of $\mathscr{F}(M)$. We give a different proof of Kalton's result, based on Petunīn–Plīčhko theorem, that avoids the metrizability assumption of the considered compact topology on M in Kalton's original proof. We use Kalton's theorem to deduce that certain Lipschitz free spaces over uniformly discrete metric spaces are dual ones. By using the completeness of Mackey topology $\mu(\mathscr{F}(M), \mathrm{Lip}_0(M) \cap \mathscr{C}(M,\tau))$ and putting together some known results, we get a duality result for uniformly discrete metric spaces in the non-separable setting.

Moreover, we get extensions the results of Kalton and Dalet to the vector-valued case. The vector-valued version of Lipschitz free spaces, denoted $\mathscr{F}(M,X)$, has been recently considered in [BGLPRZ1] as a predual of the space of vector-valued Lipschitz functions $\mathrm{Lip}_0(M,X^*)$ in the spirit of the scalar version. We consider natural vector-valued extensions of $\mathrm{lip}_0(M)$ and $S_0(M)$, denoted $\mathrm{lip}_0(M,X)$ and $S_0(M,X)$. We use the techniques of [JVSVV] and tensor product theory to show that $S_0(M,X)$ is isometric to the injective tensor product $S_0(M)\widehat{\otimes}_\varepsilon X$ under suitable assumptions on M and X, which leads to an extension of Dalet's duality theorem to the vector-valued case. We also find an extension of Kalton's duality theorem in this context. To this end, we introduce the space

$$\mathrm{lip}_\tau(M,X) := \mathrm{lip}_0(M,X) \cap \{f\colon M \to X : f \text{ is } \tau\text{-to-}\|\ \| \text{ continuous}\}$$

where τ is a compact Hausdorff topology on M such that the metric is τ-lower semicontinuous. We prove that $\mathrm{lip}_\tau(M,X)$ is isometric to the space of weak*-to-weak continuous compact operators from X^* to $\mathrm{lip}_\tau(M)$. As a consequence, under suitable hypotheses $\mathrm{lip}_\tau(M,X)$ can be identified with $\mathrm{lip}_\tau(M)\widehat{\otimes}_\varepsilon X$. This leads to the following result.

Theorem G (with C. Petitjean and A. Rueda Zoca)**.** Let M be a separable bounded metric space and let τ be a compact Hausdorff topology on M such that d is τ-lower semicontinuous and $\mathrm{lip}_\tau(M)$ separates points uniformly. If either $\mathscr{F}(M)$ or X^* has the approximation property, then $\mathrm{lip}_\tau(M,X)^* = \mathscr{F}(M,X^*)$.

It is known that $\mathrm{lip}_0(M)$ (respectively, $S_0(M)$) is an M-embedded Banach space whenever M is compact [Kal2] (respectively, M is proper [Dal2]). Consequently, these spaces cannot be dual Banach spaces whenever they are infinite dimensional. So, it is natural to wonder what happens with $\mathrm{lip}_0(M) \cap \mathscr{C}(M, \tau)$ as well as with the vector-valued version of these spaces. For this, it would be useful to find a geometrical property of Banach spaces which is not compatible with being a dual Banach space. In this line, it has been recently introduced the concept of almost squareness. According to [ALL], a Banach space X is said to be *almost square* (ASQ) if for every $x_1, \ldots, x_k \in S_X$ and $\varepsilon > 0$ there exists $y \in S_X$ such that

$$\|x_i \pm y\| \leq 1 + \varepsilon \text{ for all } i \in \{1, \ldots, k\}.$$

Roughly speaking, we can say that ASQ Banach spaces have a strong c_0 behaviour from a geometrical point of view. This c_0 behaviour is encoded by the fact that a Banach space admits an equivalent ASQ renorming if, and only if, the space contains an isomorphic copy of c_0 [BGLPRZ3].

It is asked in [ALL] whether there exists an ASQ dual Banach space. We provide a partial negative answer to this question. Namely, we introduce the notion of *unconditional almost squareness* (UASQ), and we show that an UASQ Banach space cannot be isometric to a dual one. We apply the notion of unconditional almost squareness to give some criteria on non-duality of $\mathrm{lip}_0(M)$ and $S_0(M)$ as well as their vector-valued versions. This leads to results as the following one, which to the best of our knowledge was not known even in the scalar-valued case.

Theorem H (with C. Petitjean and A. Rueda Zoca)**.** Let X and Y be Banach spaces, and let $0 < \alpha < 1$. Then $\mathrm{lip}_{w^*}((B_{X^*}, \| \|^\alpha), Y)$ is UASQ. In particular, it is not isometric to any dual Banach space.

This chapter is mainly based on the papers [GLPRZ1, GLRZ, GLPPRZ].

Chapter 5. Geometrical properties of Lipschitz free spaces

The first section of this chapter is devoted to the study of the Daugavet property in spaces of Lipschitz functions and Lipschitz free spaces. Recall that Banach space X is said to have the *Daugavet property* if $\|T + I\| = 1 + \|T\|$ for every rank-one operator $T \colon X \to X$, where I denotes the identity operator. Examples of Banach spaces enjoying the Daugavet property are $\mathscr{C}(K)$ for a perfect compact Hausdorff space K, $L_1(\mu)$ and $L_\infty(\mu)$ for a non-atomic measure μ, and preduals of spaces with the Daugavet property.

Recall that $\mathrm{Lip}_0([0,1])$ is isometric to $L_\infty[0,1]$ and so it has the Daugavet property. In [Wer, Section 6] it is asked whether the space $\mathrm{Lip}_0([0,1]^2)$ enjoys the Daugavet property. A positive answer was given in [IKW1], where it was shown, among other results, that $\mathrm{Lip}_0(M)$ has the Daugavet property whenever M is a *length* metric space (that is, for every pair of points $x, y \in M$, the distance $d(x, y)$ is equal to the infimum of the length of rectifiable curves joining x and y). Moreover, in [IKW1] it is characterised when $\mathrm{Lip}_0(M)$ as well as $\mathscr{F}(M)$ have the Daugavet property for a compact metric space M in terms of a geometrical property of the underlying metric space, which they called (Z). Here we provide a metrical characterisation of the Daugavet property of $\mathrm{Lip}_0(M)$ and $\mathscr{F}(M)$ for general metric spaces. Recall that, given a metric space M with completion $\hat{M}$, the spaces $\mathrm{Lip}_0(M)$ and $\mathrm{Lip}_0(\hat{M})$ are isometric, so without loss of generality we may assume that the metric space is complete.

Theorem I (with A. Procházka and A. Rueda Zoca)**.** Let M be a complete metric space. The following assertions are equivalent:

(i) M is a length space;
(ii) $\mathrm{Lip}_0(M)$ has the Daugavet property;
(iii) $\mathscr{F}(M)$ has the Daugavet property.

It is known that complete length spaces are characterised in terms of the existence of approximated midpoints, so the above result provides a purely metrical characterisation of the metric spaces such that $\mathrm{Lip}_0(M)$ has the Daugavet property.

Moreover, it is also shown in [IKW1] that every compact subset of a smooth LUR Banach space with property (Z) is convex. We get that every compact subset of a strictly convex Banach space with property (Z) is convex.

Next, we focus on the extremal structure of the unit ball of a Lipschitz free space. Recall that a point in the unit ball of a Banach space is said to be a *preserved extreme point* if it is an extreme point of the bidual ball. A result of Weaver [Wea2] ensures that every preserved extreme point of $B_{\mathscr{F}(M)}$ is a molecule, that is, an element of the form $\frac{\delta(x)-\delta(y)}{d(x,y)}$ for some $x, y \in M$, $x \neq y$. Weaver also proved that the molecule $\frac{\delta(x)-\delta(y)}{d(x,y)}$ is a preserved extreme point whenever there is a function $f \in \mathrm{Lip}_0(M)$ peaking at the pair (x, y). We show that this condition characterises the strongly exposed points of $B_{\mathscr{F}(M)}$.

Theorem J (with A. Procházka and A. Rueda Zoca)**.** Let M be a metric space. Given $x, y \in M$, $x \neq y$, the following assertions are equivalent:

(i) $\frac{\delta(x)-\delta(y)}{d(x,y)}$ is a strongly exposed point of $B_{\mathscr{F}(M)}$;
(ii) There is $f \in \mathrm{Lip}_0(M)$ peaking at (x, y);
(iii) The pair (x, y) does not have property (Z), that is, there is $\varepsilon > 0$ such that

$$d(x,z) + d(y,z) \geq d(x,y) + \varepsilon \min\{d(x,z), d(y,z)\}$$

for all $z \in M \setminus \{x, y\}$.

The above result generalises the characterisation of peaks couples in $\mathbb{R}$-trees given in [DKP] to an arbitrary metric space.

It follows from Theorems I and J that, for a compact metric space M, the space $\mathscr{F}(M)$ has the Daugavet property if and only if $B_{\mathscr{F}(M)}$ does not have any strongly exposed point.

We show that even in the context of free spaces, it is not true that every extreme point of the ball is a preserved extreme point, nor that every denting point is a strongly exposed point. However, we prove that every weak-strongly exposed point is strongly exposed, and as a consequence Gâteaux and Fréchet differentiability coincide for the norm of $\mathrm{Lip}_0(M)$. Moreover, we get the following result.

Theorem K (with C. Petitjean, A. Procházka and A. Rueda Zoca)**.** Let M be a metric space. Every preserved extreme point of $B_{\mathscr{F}(M)}$ is a denting point.

During the preparation of the paper [GLPPRZ], Aliaga and Guirao [AG] characterised metrically the preserved extreme points of free spaces. Namely, $\frac{\delta(x)-\delta(y)}{d(x,y)}$ is a preserved extreme point of $B_{\mathscr{F}(M)}$ if and only if for every $\varepsilon > 0$ there exists $\delta > 0$ such that every $z \in M$ satisfies

$$(1-\delta)(d(x,z)+d(z,y)) < d(x,y) \Rightarrow \min\{d(x,z), d(y,z)\} < \varepsilon.$$

We provide an alternative proof of their result which accidentally reproves our Theorem K.

Two problems remain open in this context (see [AG]). First, it is not known if every extreme point of $B_{\mathscr{F}(M)}$ is of the form $\frac{\delta(x)-\delta(y)}{d(x,y)}$ for certain $x, y \in M$. Moreover, it is easy to check that if $\frac{\delta(x)-\delta(y)}{d(x,y)}$ is an extreme point then $d(x,y) < d(x,z) + d(z,y)$ for every $z \in M \setminus \{x,y\}$, but it is not known if that condition is sufficient. We provide positive answers to these questions in some particular cases. The most notable among them is the case when $\mathscr{F}(M)$ admits an isometric predual with some additional properties. Moreover, we show that the existence of such a predual has consequences on the norm-attainment of Lipschitz functions. We also show that for certain compact spaces every extreme point of $B_{\mathscr{F}(M)}$ is also a denting point since the norm of $\mathscr{F}(M)$ turns out to be weak* AUC.

Finally, the last part of the chapter deals with some questions concerning the isomorphic theory of Lipschitz free spaces. In particular, we are interested in the relation between these spaces and Pełczyński's universal space $\mathbb{P}$. Recall that $\mathbb{P}$ is a separable Banach space with a basis such that every separable Banach space with the bounded approximation property is isomorphic to a complemented subspace of $\mathbb{P}$. It is proved in [GK] that $\mathbb{P}$ and $\mathscr{F}(\mathbb{P})$ are isomorphic.

Theorem L (with A. Procházka)**.** There exists a compact convex subset K of the Pełczyński space $\mathbb{P}$ such that $\mathbb{P}$ is isomorphic to $\mathscr{F}(K)$.

We also show that if M is a separable absolute Lipschitz retract, then $\mathbb{P}$ is not isomorphic to a complemented subspace of $\mathscr{F}(M)$. In particular, this shows that $\mathscr{F}(c_0)$ is not isomorphic to $\mathbb{P}$, which answers a question posed in [CDW].

The contents included in the first and the second section of the chapter have appeared in [GLPRZ2, GLPPRZ]. The results of the third section are part of a preprint with A. Procházka that will appear soon.

Notation

Our notation is standard and will normally follow the books [FHH$^+$] and [BV].

Throughout this work we will only consider real Banach spaces. Given Banach spaces X and Y, we denote

B_X	the closed unit ball of X.
S_X	the unit sphere of X.
X^*	the topological dual of X.
$\mathscr{L}(X,Y)$	the space of bounded operators from X to Y.
$\mathscr{K}(X,Y)$	the subspace of $\mathscr{L}(X,Y)$ which consists of compact operators.

Moreover, given topologies τ_1 on X and τ_2 on Y, we denote by $\mathscr{L}_{\tau_1,\tau_2}(X,Y)$ and $\mathscr{K}_{\tau_1,\tau_2}(X,Y)$ the respective subspaces of τ_1-to-τ_2 continuous operators.

By a *slice* of a subset A of a locally convex space (E,τ) we mean the intersection of A with an open half-space. We use a specific notation for slices, namely, given f in $(E,\tau)^*$ (i.e. the topological dual of (E,τ)) and $t>0$, we denote

$$S(A,f,t)=\{x\in A : f(x)>\sup\{f,A\}-t\}.$$

Moreover, $\mathrm{conv}(A)$ denotes the convex hull of the set A.

Given a topological space (T,τ), we denote by $\mathscr{C}(T,\tau)$ (also $\mathscr{C}(T)$) the space of continuous functions on T.

Given a metric space (M,d) and a point $x\in M$, we denote $B_d(x,r)$ (also $B(x,r)$) the closed ball centred at x with radius $r>0$.

We denote ω the first infinite ordinal and ω_1 the first uncountable ordinal.

A more detailed list of notation is given at the end of this work.

Contents

Chapter 0
Some preliminary content

0.1 Extreme structure of convex sets

Given a locally convex space (E, τ) and $A \subset E$, we say that a point $x \in A$ is

an *extreme point* of A	if $x \notin \operatorname{conv}(A \setminus \{x\})$, equivalently if $x = \frac{y+z}{2}$, $y, z \in A$ imply $x = y = z$.
a *strongly extreme point of A*	if $x \notin \overline{\operatorname{conv}}(A \setminus V)$ for every neighbourhood V of x, equivalently, the slices of A containing x are neighbourhood basis of x in A.
an *exposed point* of A	if there is an affine continuous function $f\colon K \to \mathbb{R}$ such that $f(x) > f(y)$ for every $y \in A \setminus \{x\}$. In such a case we say that f *exposes* x.
a *strongly exposed point* of A	if there is an affine continuous function $f\colon K \to \mathbb{R}$ such that for every net $(x_\alpha)_\alpha$ in A we have $x_\alpha \to x$ whenever $f(x_\alpha) \to f(x)$, equivalently, the slices $\{S(A, f, t) : t > 0\}$ are a neighbourhood basis of x in A. In such a case we say that f *strongly exposes* x.

We will denote $\operatorname{ext}(A)$, $\operatorname{strext}(A)$, $\exp(A)$ and $\operatorname{strexp}(A)$ the sets of extreme, strongly extreme, exposed and strongly exposed points of a set A, respectively.

If we need to empathize which topology is considered then we will write τ-strongly extreme, τ-exposed, etc.

Commonly, the locally convex space above will be a Banach space endowed with the norm topology or the weak topology, as well as a dual Banach space endowed with the weak* topology. In such cases the terms exposed and strongly exposed point will always refer to the norm topology, moreover strongly extreme points will be called *denting points* and weak-strongly extreme points will be called *preserved extreme points*. This last terminology will be explained later.

Let us highlight that in the definition of exposed point we do not require that the exposing functional is defined in the whole locally convex space. The reason is that we want the notion of exposed point to be independent of the ambient space. Indeed, note that the point $x = \sum_{n=1}^{\infty} e_n$ is exposed in B_{ℓ_∞} by the functional $y = \sum_{n=1}^{\infty} 2^{-n} e_n \in \ell_1$. On the other hand, the identity is an affine homeomorphism between (B_{ℓ_∞}, w^*) and $[-1,1]^{\mathbb{N}} \subset (\mathbb{R}^{\mathbb{N}}, \tau_p)$. However, it is easy to check that the dual of $(\mathbb{R}^{\mathbb{N}}, \tau_p)$ can be identified with c_{00}, and there is no element in c_{00} that exposes x in $[-1,1]^{\mathbb{N}}$.

It is not difficult to check that the above concepts are related in the following way:

$$\begin{tikzcd}
 & \text{exposed} \arrow[dr] & \\
\text{strongly exposed} \arrow[ur] \arrow[dr] & & \text{extreme} \\
 & \text{strongly extreme} \arrow[ur] &
\end{tikzcd}$$

Moreover, none of these implications reverse in general. The situation is more interesting when we are dealing with a subset of a Banach space since then we have the following relations:

$$\begin{tikzcd}
\text{strongly exposed} \arrow[r] \arrow[d] & \text{weak-strongly exposed} \arrow[r] \arrow[d] & \text{exposed} \arrow[d] \\
\text{denting} \arrow[r] & \text{preserved extreme} \arrow[r] & \text{extreme}
\end{tikzcd}$$

Again, none of these implications can be reversed. This is showed by the following examples, which are essentially taken from [Bou4].

Example 0.1.1.

(a) Consider the set $K = \operatorname{conv}(\{(1,1)\} \cup B_{(\mathbb{R}^2, \|\ \|_2)}) \subset \mathbb{R}^2$. Then $x = (1,0)$ is a denting point of K which is not exposed.

(b) Let $K = \overline{\operatorname{conv}}\{e_n : n \in \mathbb{N}\} \subset \ell_2$. Then 0 is a weak-strongly exposed point of K which is not denting. Indeed, we will show that the slices given by $f = \sum_{k=1}^{\infty} 2^{-k} e_k$ are a neighbourhood basis of 0 for the weak topology. To this end, assume $(x_n)_{n=1}^{\infty} \subset K$ and $\lim_{n\to\infty} \langle f, x_n \rangle = 0$. Note that K is a weak-compact metrizable set. Moreover $\operatorname{ext}(K) \subset \overline{\{e_n : n \in \mathbb{N}\}}^{w} = \{e_n : n \in \mathbb{N}\} \cup \{0\}$ by Milman's theorem (see, e.g. [FHH$^+$, Theorem 3.41]). Thus each x_n can be expressed as $x_n = \sum_{k=1}^{\infty} a_k^n e_k$ with $a_k \geq 0$ and $\sum_{k=1}^{\infty} a_k^n \leq 1$. Therefore

$$0 = \lim_{n\to\infty} \langle f, x_n \rangle = \lim_{n\to\infty} \sum_{k=1}^{\infty} 2^{-k} a_k^n \geq \lim_{n\to\infty} 2^{-k} a_k^n.$$

This means that $\lim_{n\to\infty} \langle e_k, x_n \rangle = 0$ for each $k \in \mathbb{N}$ and so $x_n \stackrel{w}{\to} 0$. On the other hand, it is easy to check that every slice of K containing 0 has diameter bigger than $\sqrt{2}$, thus 0 is not a denting point of K.

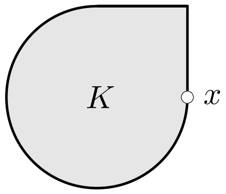

Figure 1: Example 0.1.1.(a): a denting point which is not exposed

(c) Let $D = \{\frac{1}{n}e_1 + e_n : n \in \mathbb{N}\} \cup \{\frac{1}{n}e_1 - e_n : n \in \mathbb{N}\} \subset \ell_1$ and $K = \overline{\text{conv}}^{w^*}(D)$. It is proved in [Bou4, Example 3.2.5] that 0 is an exposed point of K which is not strongly exposed. Indeed, the same argument shows that 0 is not weak-strongly exposed.

(d) An example of a preserved extreme point which is not weak-strongly exposed, and a denting point which is not strongly exposed, will be given in Example 5.2.31.

We will see in Chapter 5 that every preserved extreme point of the unit ball of a Lipschitz free space is a denting point, and that every weak-strongly exposed point is a strongly exposed point. There are more cases in which we can pass from an extremal property to a stronger one. The proof of the following result can be found in [Cho, FHH+].

Theorem 0.1.2 (Choquet's lemma). *Let K be a compact convex subset of a locally convex space and $x \in \text{ext}(K)$. Then the open slices of K containing x form a neighbourhood basis of x in K, that is, x is a strongly extreme point of K.*

Another example of strengthening of the extremal structure is the Lin–Lin–Troyanski theorem [LLT], which says that an extreme point of a closed convex subset C of a Banach space which is a continuity point (that is, there are weak open neighbourhoods in C of arbitrarily small diameter) is also a denting point.

We include now some results which will be useful in what follows. The first one is a characterisation of strongly extreme points in terms of nets. This is essentially contained in Proposition 3.4.2 of [Raj1]. Moreover, the equivalence (i)$\Leftrightarrow$(ii) is well known (see, e.g. [GMZ2, Proposition 9.1]).

Proposition 0.1.3. *Let C be a convex subset of a locally convex space such that $\overline{C}$ is compact. Let $x \in C$. The following are equivalent:*

(i) *x is an extreme point of $\overline{C}$.*

(ii) *x is a strongly extreme point.*

(iii) *For every pair of nets $(y_\alpha)_\alpha$ and $(z_\alpha)_\alpha$ in C such that $\frac{y_\alpha + z_\alpha}{2} \to x$ we have that $y_\alpha \to x$.*

Proof. (ii)$\Rightarrow$ (iii). Assume $\frac{y_\alpha + z_\alpha}{2} \to x$. Let U be a neighbourhood of x. Since $x = 2x - x \in U$, there is a neighbourhood V of x so that $2V - V \subset U$. By hypothesis, there is an open half-space H so that $C \cap H \subset V$. Now, there exists α_0 so that $\frac{y_\alpha + z_\alpha}{2} \in C \cap H$ for every

$\alpha \geq \alpha_0$, and so either y_α or z_α belongs to $C \cap H$. If $y_\alpha \in C \cap H$ then $y_\alpha \in U$. Otherwise, $z_\alpha \in C \cap H \subset V$ and so $y_\alpha = 2(\frac{y_\alpha + z_\alpha}{2}) - z_\alpha \in U$. Thus, in any case we have $y_\alpha \in U$. This shows that $y_\alpha \to x$.

(iii)⇒(i). Assume that $x = \frac{y+z}{2}$ for some $y, z \in \overline{C}$. There are nets $(y_\alpha)_\alpha$ and $(z_\alpha)_\alpha$ in C convergent to y and z, respectively. Then $\frac{y_\alpha + z_\alpha}{2} \to x$ and so $y_\alpha \to x$. This implies that $x = y = z$.

(i)⇒(ii). The Choquet's lemma ensures that the open slices are a neighbourhood basis of x in $\overline{C}$. Finally, note that the intersection of C and a slice of $\overline{C}$ is a slice of C. □

It is easy to check that conditions above are also equivalent to the following:

(iii') For every $\lambda \in (0, 1)$ and nets $(y_\alpha)_\alpha$ and $(z_\alpha)_\alpha$ in C such that $\lambda y_\alpha + (1 - \lambda) z_\alpha \to x$ we have that $y_\alpha, z_\alpha \to x$.

Indeed, suppose that $\lambda y_\alpha + (1 - \lambda) z_\alpha \to x$. We may assume that $\lambda \geq 1/2$. Then $w_\alpha = (2\lambda - 1) y_\alpha + 2(1 - \lambda) z_\alpha \in C$ and $\frac{y_\alpha + w_\alpha}{2} \to x$.

We will apply the previous proposition in the particular case in which the locally convex space is the bidual of a Banach space X endowed with the weak* topology and $C = B_X$ is the unit ball of X. By Goldstein's theorem, the above proposition characterises the extreme points of B_X which are still extreme points of $B_{X^{**}}$. This motivates to call these points preserved extreme points.

Proposition 0.1.4. *Let X be a Banach space. Let $x \in B_X$. The following are equivalent:*

(i) *x is an extreme point of $B_{X^{**}}$.*
(ii) *x is a weak-strongly extreme point (i.e. a preserved extreme point).*
(iii) *For every nets $(y_\alpha)_\alpha$ and $(z_\alpha)_\alpha$ in B_X such that $\frac{y_\alpha + z_\alpha}{2} \xrightarrow{w} x$ we have that $y_\alpha \xrightarrow{w} x$.*

We refer the reader to [GMZ2] for a survey on preserved and unpreserved extreme points in Banach spaces.

The classical Smulyan's lemma (see e.g. [DGZ]) relates strongly exposed points in B_X and points of differentiability of the dual norm. It says that the norm of a dual Banach space X^* is Fréchet differentiable at $f \in S_{X^*}$ if and only if for every sequences $(x_n)_{n=1}^\infty$, $(y_n)_{n=1}^\infty$ in S_X such that $f(x_n) \to 1$ and $f(y_n) \to 1$ we have $\|x_n - y_n\| \to 0$. Note that this means that $\operatorname{diam}(S(B_X, f, \alpha))$ tends to 0 when α tends to 0. Therefore, there is $x \in \bigcap_{\alpha > 0} S(B_X, f, \alpha)$ and x is strongly exposed by f. There is also a version of Šmulyan's lemma for Gâteaux differentiability, which says that the norm of X^* is Gâteaux differentiable at $f \in S_{X^*}$ if and only if for every sequences $(x_n)_{n=1}^\infty$, $(y_n)_{n=1}^\infty$ in S_X such that $f(x_n) \to 1$ and $f(y_n) \to 1$ we have $x_n - y_n \xrightarrow{w} 0$. If moreover the derivative of the norm at f is an element $x \in X$, then this implies that $x_n \xrightarrow{w} x$ whenever $f(x_n) \to 1$. Therefore x is weak-strongly exposed. This discussion means that we can state Šmulyan's lemma as follows:

Lemma 0.1.5 (Šmulyan)**.** *Let X a Banach space, $x \in S_X$ and $f \in S_{X^*}$. Then:*

(a) *x is strongly exposed by f in B_X if and only if the norm of X^* is Fréchet differentiable at f with derivative x;*

(b) *x is weak-strongly exposed by f in B_X if and only if the norm of X^* is Gâteaux differentiable at f with derivative x.*

A slight modification of the Šmulyan's lemma provides the following result, which will be useful in Chapter 5. Recall that a subset $V \subset S_X$ is said to be *c-norming* for X^* if

$$c\,\|x^*\| \leq \sup\{\langle x^*, v\rangle : v \in V\}$$

for every $x^* \in X^*$.

Lemma 0.1.6. *Assume that $V \subset S_X$ is a 1-norming subset for X^*. Let $v \in V$ and $f \in S_{X^*}$ be so that every sequence $(v_n)_{n=1}^{\infty}$ in V with $\lim_n f(v_n) = 1$ is norm-convergent to v. Then $\|\ \|_{X^*}$ is Fréchet-differentiable at f. Therefore, f strongly exposes v in B_X.*

Proof. We will mimic the proof of Šmulyan's lemma appearing in [DGZ, Theorem 1.4.(ii)]. Assume that $\|\ \|_{X^*}$ is not Fréchet differentiable at f. Then there exist $\varepsilon > 0$ and a sequence $(h_n)_{n=1}^{\infty}$ in X^* with $h_n \neq 0$, $\|h_n\| \xrightarrow{n} 0$ and

$$\|f + h_n\| + \|f - h_n\| \geq 2 + \varepsilon\,\|h_n\|$$

for every $n \in \mathbb{N}$. Since V is 1-norming, there exist sequences $(v_n)_{n=1}^{\infty}, (w_n)_{n=1}^{\infty} \subset V$ such that

$$\langle f + h_n, v_n\rangle \geq \|f + h_n\| - \frac{1}{n}\|h_n\|,$$
$$\langle f - h_n, w_n\rangle \geq \|f - h_n\| - \frac{1}{n}\|h_n\|.$$

Note that

$$1 \geq \langle f, v_n\rangle = \langle f + h_n, v_n\rangle - \langle h_n, v_n\rangle \geq \|f + h_n\| - \frac{1}{n}\|h_n\| - \|h_n\| \xrightarrow{n} 1,$$

since $\|h_n\| \xrightarrow{n} 0$. Thus, $\langle f, v_n\rangle \xrightarrow{n} 1$. Similarly, we have $\langle f, w_n\rangle \xrightarrow{n} 1$. It follows that both $(v_n)_{n=1}^{\infty}$ and $(w_n)_{n=1}^{\infty}$ are norm convergent to v and so $\|v_n - w_n\| \xrightarrow{n} 0$. On the other hand,

$$\langle f + h_n, v_n\rangle + \langle f - h_n, w_n\rangle \geq 2 + \varepsilon\,\|h_n\| - \frac{2}{n}\|h_n\|$$

Thus,

$$\langle h_n, v_n - w_n\rangle \geq \left(\varepsilon - \frac{2}{n}\right)\|h_n\| + 2 - \langle f, v_n\rangle - \langle f, w_n\rangle \geq \left(\varepsilon - \frac{2}{n}\right)\|h_n\|$$

This implies that $\|v_n - w_n\| \geq \frac{\varepsilon}{2}$ for large n, which is a contradiction. Therefore $\|\ \|_{X^*}$ is Fréchet differentiable at f. Finally, the classical Šmulyan's lemma yields that f strongly exposes v. $\square$

Remark 0.1.7. It is an easy exercise to check that a point x in a closed convex bounded subset C of a Banach space is a weak-strongly exposed point of C if and only if it is a weak*-exposed point of $\overline{C}^{w^*} \subset X^{**}$. Thus, weak-strongly exposed points are in some sense preserved exposed points. Moreover, it follows from this fact that x is strongly exposed in C if and only if x is a weak*-exposed point of $\overline{C}^{w^*}$ and a continuity point of C, which can be interpreted as an analogous Lin–Lin–Troyanski theorem for exposed points.

In general, the set of extreme points a compact set K is not closed, even if K is finite-dimensional. Indeed, it is easy to check that given

$$A = \{(x, y, 0) : x^2 + y^2 = 1\} \cup \{(1, 0, 1), (1, 0, -1)\} \subset \mathbb{R}^3$$

and $K = \mathrm{conv}(A)$, we have $\mathrm{ext}(K) = A \setminus (1, 0, 0)$. Moreover, there is a compact convex subset K of ℓ_2 such that $\overline{\mathrm{ext}(K)} = K$ (see [LMNS, p. 455]). If K is metrizable by a metric d, then

$$K \setminus \mathrm{ext}(K) = \bigcup_{n=1}^{\infty} \left\{ \frac{x+y}{2} : x, y \in K, d(x, y) \geq \frac{1}{n} \right\}$$

and so $\mathrm{ext}(K)$ is a $\mathscr{G}_\delta$ subset of K. However, if K is not metrizable then $\mathrm{ext}(K)$ need not to be a Borel set [BdL].

Recall that a topological space is said to be a *Baire space* if the intersection of a countable family of open dense subsets is also dense. Baire's category theorem says that complete metric spaces and compact Hausdorff spaces are Baire.

Theorem 0.1.8 (Choquet). *Let K be a compact convex subset of a locally convex space. Then* $\mathrm{ext}(K)$ *is a Baire space.*

The proof of Choquet's theorem relies on the following lemma, which is proved in [Cho].

Lemma 0.1.9. *Let K be a convex compact subset of a locally convex space. Assume that A is a nonempty convex compact subset of K such that $K \setminus A$ is also convex. Then $A \cap \mathrm{ext}(K) \neq \emptyset$.*

Proof of Theorem 0.1.8. Let $(V_n)_{n=1}^{\infty}$ be a sequence of relatively open dense subsets of $\mathrm{ext}(K)$. For each n take an open subset U_n in K so that $V_n = U_n \cap \mathrm{ext}(K)$. Let V be a relatively open subset of $\mathrm{ext}(K)$ and let us show that $V \cap \bigcap_{n=1}^{\infty} V_n \neq \emptyset$. To this end, take also U open in K so that $V = U \cap \mathrm{ext}(K)$. Fix $x_0 \in V$. By Choquet's lemma, there is an open slice S_0 of K which contains x_0 and satisfies $\overline{S_0} \subset U$. Since V_1 is dense in $\mathrm{ext}(K)$, there is $x_1 \in V_1 \cap (S_0 \cap \mathrm{ext}(K)) = U_1 \cap S_0 \cap \mathrm{ext}\, K$. Applying again Choquet's lemma we get an open slice S_1 containing x_1 and so that $\overline{S_1} \subset U_1 \cap S_0$. Proceeding inductively, we find a sequence $(S_n)_{n=1}^{\infty}$ of open slices of K such that $S_n \cap \mathrm{ext}(K) \neq \emptyset$ and $\overline{S_n} \subset U_n \cap S_{n-1}$ for each n.

Now, note that $\bigcap_{n=0}^{\infty} \overline{S_n}$ is convex, compact and non-empty as being an intersection of a decreasing sequence of compact sets. It is easy to check that $K \setminus \bigcap_{n=0}^{\infty} \overline{S_n}$ is also convex. Thus, Lemma 0.1.9 provides $x \in \bigcap_{n=0}^{\infty} \overline{S_n} \cap \mathrm{ext}(K) \neq \emptyset$. Therefore, $x \in \bigcap_{n=1}^{\infty} V_n$. Moreover, $x \in \overline{S_0} \subset U$, so $x \in V$. This shows that $V \cap \bigcap_{n=1}^{\infty} V_n \neq \emptyset$ and we are done. $\square$

Here we want to highlight that under suitable hypotheses we can give a version of Choquet's theorem for non-compact sets, where the extreme points are replaced by the strongly extreme points. We will show later an example where this corollary applies.

Corollary 0.1.10. *Let C be a convex subset of a locally convex space such that $\overline{C}$ is compact, C is a $\mathscr{G}_\delta$ subset of $\overline{C}$ and $C \subset \overline{\operatorname{conv}}(\operatorname{strext}(C))$. Then $\operatorname{strext}(C)$ is a Baire space.*

Proof. Write $C = \bigcap_{n=1}^\infty G_n \cap \overline{C}$, where each G_n is an open set. By Proposition 0.1.3, we have

$$\operatorname{strext}(C) = C \cap \operatorname{ext}(\overline{C}) = \bigcap_{n=1}^{\infty} G_n \cap \operatorname{ext}(\overline{C})$$

and so $\operatorname{strext}(C)$ is a $\mathscr{G}_\delta$ in $\operatorname{ext}(\overline{C})$. Moreover, $\operatorname{strext}(C)$ is dense in $\operatorname{ext}(\overline{C})$. Indeed, given $x \in \operatorname{ext}(\overline{C})$ and a neighbourhood U of x, by Choquet's lemma we can find an open half-space H such that $x \in \overline{C} \cap H \subset U$. Note that H must intersect $\operatorname{strext}(C)$, otherwise we would have $C \subset \overline{\operatorname{conv}}(\operatorname{strext} C) \subset \overline{C} \setminus H$, and so $\overline{C} \subset \overline{C} \setminus H$, a contradiction. This shows that $\operatorname{strext}(C)$ is a dense $\mathscr{G}_\delta$ subset of $\operatorname{ext}(\overline{C})$. Finally, it is an easy exercise to check that a dense $\mathscr{G}_\delta$ subset of a Baire space is a Baire space, so the conclusion follows from Theorem 0.1.8. $\square$

Finally, in the rest of the section we gather several results ensuring that a convex set can be recovered from a distinguished subset. The first result in this line goes back to Minkowski, who proved that every compact convex subset of $\mathbb{R}^3$ is the closed convex hull of its extreme points. This result was extended to finite dimensional spaces by Carathéodory.

Theorem 0.1.11 (Krein–Milman)**.** *Let K be a compact convex subset of a locally convex space. Then $K = \overline{\operatorname{conv}}(\operatorname{ext}(K))$.*

Let us remark that the local convexity of the topology is an essential hypothesis in the Krein–Milman theorem. Indeed, Roberts proved in [Rob] that in the complete metrizable topological vector space $L^{1/2}[0,1]$ there is a compact convex subset which does not have any extreme point. Later, Kalton [Kal1] provided an example of a complete metrizable topological vector space in which the conclusion of Krein–Milman theorem holds. To the best of our knowledge, the problem of characterising topological vector spaces such that every compact convex subset has an extreme point is still open.

One can wonder under which circumstances extreme points can be replaced by exposed points in Krein–Milman theorem. Straszewicz proved in 1935 that $K = \overline{\operatorname{conv}}(\exp(K))$ holds for every $K \subset \mathbb{R}^n$ compact and convex. Klee [Kle] analysed Straszewicz's proof and noted that it works for every compact convex subset of a space with a strictly convex smooth norm. Since every compact convex metrizable set embeds linearly into ℓ_2, it follows that the following holds:

Theorem 0.1.12 (Strascewicz–Klee)**.** *Let K be a compact convex metrizable subset of a locally convex space. Then $K = \overline{\operatorname{conv}}(\exp(K))$.*

To the best of our knowledge, it is not known a characterisation of the compact convex sets which are the closed convex hull of their exposed points. We will show in Chapter 1 that such a property holds for every compact convex set that admits a strictly convex lower semicontinuous function. It is not difficult to show that this is the case of metrizable convex compacta, so we will get an extension of Strascewicz–Klee result.

For the case of weak*-exposed points, the situation is more clear. The proof of the following result can be found in [Phe].

Theorem 0.1.13 (Phelps–Larman)**.** *Let X be a Banach space. Then the following are equivalent:*

(i) *Every weak*-compact convex subset K of X^* is the weak*-closed convex hull of its weak*-exposed points.*
(ii) *X is a* Gâteaux Differentiability Space*, that is, every convex continuous function defined on an open subset of X is Gâteaux-differentiable in a dense subset of the domain.*

Finally, one can wonder which subsets of a Banach space can be recovered from their strongly exposed points or their preserved extreme points.

Theorem 0.1.14. *Let C be a closed convex bounded subset of a Banach space X. Then the following are equivalent:*

(i) *For every vector measure τ and every scalar measure μ such that $\frac{\tau(A)}{\mu(A)} \in C$ whenever $\mu(A) \neq 0$, there is $f \in L_1(\mu, X)$ such that $\tau(A) = \int_A f d\mu$ for every measurable subset A.*
(ii) *Every martingale $(f_n)_{n=1}^{\infty}$ (in a probability space) such that $f_n(\omega) \in C$ for every n and every ω, converges almost everywhere.*
(iii) *Every subset A of C is* dentable*, that is, for every $\varepsilon > 0$ there is an open half-space H such that $A \cap H \neq \emptyset$ and $\operatorname{diam}(A \cap H) < \varepsilon$.*
(iv) *Every closed convex subset D of C is the closed convex hull of its strongly exposed points.*
(v) *Every closed convex subset D of C is the closed convex hull of its preserved extreme points.*

The equivalence of the first three properties comes from results due to Chatterji, Huff, Maynard, Rieffel, Davis and Phelps. Moreover, the equivalence between these properties and the abundance of strongly exposed points is due to Phelps and Bourgain. Finally, the relation with the existence of preserved extreme points was first noted by Stegall. For the proof of this result we refer the reader to Theorems 2.3.6, 3.5.4 and 3.7.6 in [Bou4].

Recall that a set C satisfying any of the equivalent conditions of Theorem 0.1.14 is said to have the *Radon–Nikodým Property* (RNP, for short). A Banach space is said to have the RNP if its unit ball has the RNP. It is well known that separable dual spaces and reflexive spaces have the RNP. Chapter 2 of this thesis is devoted to a version of RNP for maps.

Now we give an example where Corollary 0.1.10 applies, which is based on a result in [GM].

Corollary 0.1.15. *Let C be a closed convex bounded subset of a Banach space X. Assume that C has the RNP and that the weak* closure $\overline{C}^{w^*}$ of C in X^{**} is metrizable. Then the set of preserved extreme points of C is a Baire space.*

Proof. We are going to check that C satisfies the hypothesis of Corollary 0.1.10 for the weak* topology of X^{**}. Since C has the RNP, we know that $C = \overline{\text{conv}}(\text{strexp}(C)) \subset \overline{\text{conv}}^{w*}(\text{strexp}(C))$, and clearly every strongly exposed point of C is weak*-strongly extreme. Moreover, C is a weak*-$\mathscr{G}_\delta$ in $\overline{C}^{w*}$ by Corollary IV.3 in [GM]. Thus the set of weak*-strongly extreme points of C is a Baire space. Since the weak* and the weak topologies agree on C, we get that the set of weak-strongly extreme (that is, preserved extreme) points of C is a Baire space. □

0.2 Generalised Szlenk and dentability indices

Consider a topological space $\mathfrak{X}$, a family of open sets $\mathscr{S}$ and a function $\eta\colon 2^{\mathfrak{X}} \to [0, +\infty]$. We may define a set derivation for subsets $A \subset \mathfrak{X}$ by the rule

$$[\eta, \mathscr{S}]'_\varepsilon(A) = \{x \in A : \forall U \in \mathscr{S} \ (x \in U \Rightarrow \eta(A \cap U) \geq \varepsilon)\}$$

where $\varepsilon > 0$. When there is no confusion about η and $\mathscr{S}$ we may use a simpler notation like $[A]'_\varepsilon$.

Clearly, the set derivation defined above is monotone, that is, $[\eta, \mathscr{S}]'_\varepsilon(A) \subset A$. We may iterate the derivation to any ordinal order

$$[\eta, \mathscr{S}]^{\alpha+1}_\varepsilon(A) = [\eta, \mathscr{S}]'_\varepsilon([\eta, \mathscr{S}]^{\alpha}_\varepsilon(A))$$

and for limit ordinals

$$[\eta, \mathscr{S}]^{\alpha}_\varepsilon(A) = \bigcap_{\beta<\alpha} [\eta, \mathscr{S}]^{\beta}_\varepsilon(A).$$

A set derivation always leads to a dichotomy, only one of the following statements happens:

(i) there is a nonempty subset $A \subset \mathfrak{X}$ and $\varepsilon > 0$ such that $[\eta, \mathscr{S}]'_\varepsilon(A) = A$;
(ii) for all $\varepsilon > 0$ there is an ordinal α such that $[\eta, \mathscr{S}]^{\alpha}_\varepsilon(\mathfrak{X}) = \emptyset$.

In the second case, we define

$$\text{Sz}_{[\eta,\mathscr{S}]}(\mathfrak{X}, \varepsilon) = \min\left\{\alpha : [\eta, \mathscr{S}](\mathfrak{X})^{\alpha}_\varepsilon = \emptyset\right\}.$$

In the first case, we put $\text{Sz}_{[\eta,\mathscr{S}]}(\mathfrak{X}, \varepsilon) = \infty$, which is beyond the ordinals. Finally, the *Szlenk index of $\mathfrak{X}$ associated to η and $\mathscr{S}$* is defined as

$$\text{Sz}_{[\eta,\mathscr{S}]}(\mathfrak{X}) = \sup_{\varepsilon>0} \text{Sz}_{[\eta,\mathscr{S}]}(\mathfrak{X}, \varepsilon).$$

In many particular cases the function η is of the form

$$\eta(A) = \sup\{\rho(x,y) : x, y \in A\}$$

where $\rho\colon \mathfrak{X} \times \mathfrak{X} \to \mathbb{R}$ is a function satisfying that $\rho(x,y) = \rho(y,x) \geq 0$ for every $x, y \in \mathfrak{X}$. That is, η is a natural notion of diameter associated to ρ. In that case, for simplicity we denote $[\rho, \mathscr{S}]^{\alpha}_{\varepsilon}(A) = [\eta, \mathscr{S}]^{\alpha}_{\varepsilon}(A)$, and $\mathrm{Sz}_{\rho} = \mathrm{Sz}_{\eta}$.

Now, we recall several set derivations that can be described in the above way. Let (K, τ) be a compact topological space and ρ be the discrete metric on K. Then for every subset A of K and every $\varepsilon < 1$ we have that $[\rho, \tau](A)'_{\varepsilon}$ is the set of non-isolated points of A. Thus, $\mathrm{Sz}_{\rho}(K)$ is the *Cantor–Bendixson index* of K (see e.g. [AK2]). Moreover, we have the following:

- $\mathrm{Sz}_{\rho}(K) < \infty$ if and only if K is *scattered*, that is, every closed subset A of K contains a point which is isolated in A.
- Assume that K is metrizable. Then $\mathrm{Sz}_{\rho}(K) < \omega_1$ if and only if K is countable. Moreover, $\mathrm{Sz}_{\rho}(K) < \omega$ if and only if $\mathscr{C}(K)$ is isomorphic to c_0.

Now, let X be a Banach space and take $\mathfrak{X} = B_{X^*}$ and $\rho(x,y) = \|x - y\|$. In such a case $\mathrm{Sz}_{[\rho, w^*]}(B_{X^*})$ is the standard *Szlenk index* of X, denoted $\mathrm{Sz}(X)$. Moreover, $\mathrm{Sz}_{[\rho, \mathbb{H}]}(B_{X^*})$, where $\mathbb{H}$ denotes the family of weak* open half-spaces of X^*, is the standard *dentability index* of X, denoted $\mathrm{Dz}(X)$. Both the Szlenk and the dentability indices are isomorphic invariants which encode several geometrical properties of the Banach space. They have been used, for instance, for classifying the separable Banach spaces $\mathscr{C}(K)$ and for showing that there is not any separable reflexive Banach space universal for the class of separable reflexive Banach spaces. We refer the reader to the survey [Lan4] for a compilation of those and other applications of these indices. Here we want to highlight that X enjoys nice geometrical properties whenever these indices are bounded by a certain ordinal.

- $\mathrm{Sz}(X) < +\infty$ if and only if $\mathrm{Dz}(X) < +\infty$ if and only if X is *Asplund*, equivalently X^* has the RNP.
- If X is a separable Banach space, then $\mathrm{Sz}(X) < \omega_1$ if and only if $\mathrm{Dz}(X) < \omega_1$ if and only if X^* is separable.
- $\mathrm{Dz}(X) \leq \omega$ if and only if X is *superreflexive*. Equivalently, X admits an equivalent *uniformly convex* norm and an equivalent *uniformly smooth* norm.
- $\mathrm{Sz}(X) \leq \omega$ if and only if X admits an equivalent norm which is *asymptotically uniformly smooth*. Equivalently, X^* admits a dual norm which is *weak* asymptotically uniformly convex* (also called *weak* uniformly Kadec-Klee*) [KOS,Raj8]. These notions will be important in Chapter 3 of this work.

Note that in the above setting the underlying topological space is the compact set (B_{X^*}, w^*), and the measure of the sets is given by the norm, which is a lower semicontinuous metric. One can consider this kind of derivation for compact sets. This is closely related to the concept of *fragmentability*, we refer the reader to [Nam] for a survey on this notion. In particular, we want to recall:

- Given a metric d on K, we have $\mathrm{Sz}_d(K) < +\infty$ if and only if K is *fragmentable* with respect to d, that is, for every $A \subset K$ and every $\varepsilon > 0$ there is an open subset U such that $\operatorname{diam}(A \cap U) < \varepsilon$.
- There is a lower semicontinuous metric d on K such that $\mathrm{Sz}_d(K) < +\infty$ if and only if K is a *Radon–Nikodým compact*, that is, K embeds into a dual Banach space with the RNP endowed with the weak* topology.
- There is a lower semicontinuous symmetric ρ on K such that $\mathrm{Sz}_\rho(K) < +\infty$ if and only if K is *quasi-Radon–Nikodým compact* (by definition, due to Arvanitakis). By a *symmetric* we mean a symmetric function $\rho\colon K \times K \to [0, +\infty)$ such that $\rho(x, y) = 0$ if and only if $x = y$.
- There is a finer metric K on d such that $\mathrm{Sz}_d(K) \leq \omega$ if and only if K is *descriptive* [Raj7].

In Chapter 1 we will provide a characterisation of compact convex sets which have countable dentability index with respect to a symmetric, as well as a characterisation of compact sets which have countable Szlenk index with respect to some (not necessarily finer) metric.

Finally, we focus on derivations and indices for maps. Consider a map $f\colon (\mathfrak{X}, \tau) \to (M, d)$ with values on a metric space. Then the derivation $[d \circ f, \tau]$ consists in removing the open sets where f has small oscillation. We will denote $\mathrm{Sz}(f) = \mathrm{Sz}_{d\circ f}(\mathfrak{X})$ the associated ordinal index. In this case, $\mathrm{Sz}(f) < +\infty$ if and only if f is *fragmentable*, that is, the domain of f is fragmentable with respect to the pseudometric $d \circ f$.

Let us discuss briefly the properties of fragmentable maps. The following is essentially Lemma 2 in [Nam].

Lemma 0.2.1. *Let $f\colon \mathfrak{X} \to M$. Assume that $\mathfrak{X}$ is a hereditarily Baire space. Then the following are equivalent:*

(i) *f is fragmentable;*
(ii) *for every closed subset A of $\mathfrak{X}$ the set of points of continuity of the map $f|_A$ is a dense $\mathscr{G}_\delta$ subset of A;*
(iii) *for every closed subset A of $\mathfrak{X}$ the map $f|_A$ has a point of continuity.*

The following result shows the relation between fragmentable maps and the first Baire class. The proof can be found in [DGZ].

Theorem 0.2.2 (Baire's great theorem)**.** *Let M is a complete metric space and Y be a normed space. Then a map $f\colon M \to Y$ is fragmentable if and only if it is pointwise limit of a sequence $(f_n)_{n=1}^\infty$ of continuous maps from M to Y.*

The following result is proved in [HOR].

Theorem 0.2.3 (Haydon–Odell–Rosenthal)**.** *Let K be a compact metric space. A bounded function $f\colon K \to \mathbb{R}$ satisfies $\mathrm{Sz}(f) \leq \omega$ if and only if it can be uniformly approached by differences of bounded semicontinuous functions.*

In the case in which the domain of f is a subset of a locally convex space, it makes sense to consider the derivation $[d \circ f, \mathbb{H}]$, where $\mathbb{H}$ denotes the set of open half-spaces. The ordinal index associated to this derivation will be denoted $\mathrm{Dz}(f)$. Chapter 2 of this work deals with maps satisfying $\mathrm{Dz}(f) < +\infty$, called *dentable* maps. This notion can be regarded as a version of the Radon–Nikodým property for maps. Moreover, the analogous property to super-reflexivity for maps is to satisfy that $\mathrm{Dz}(f) \leq \omega$. These maps are called *finitely dentable* maps and studied in [Raj5]. Among other results, we will show that an analogous of Haydon–Odell–Rosenthal theorem holds for uniformly continuous finitely dentable maps.

0.3 Tensor products and approximation properties

Tensor products and approximation properties will appear frequently in Chapters 3, 4 and 5, so we include here their definitions and some useful results.

First, let us recall the definition of the approximation properties. A Banach space X is said to have the *approximation property* (*AP*) if for every norm-compact subset K of X and every $\varepsilon > 0$ there is a finite-rank operator $T\colon X \to X$ such that $\sup\{\|x - Tx\| : x \in K\} < \varepsilon$. That is, the identity can be uniformly approximated on compact sets by finite-rank operators. Moreover, given $\lambda \geq 1$, if the finite-rank operators can be taken of norm not greater than λ, then X is said to have the *λ-bounded approximation property* (*λ-BAP*). We say that X has the BAP if it has the λ-BAP for some λ. Finally, the 1-BAP is called the *metric approximation property* (*MAP*). Every Banach space with a Schauder basis (more generally, with a finite dimensional decomposition) has the BAP.

Now we recall the definition of the tensor product of Banach spaces X and Y. Given $x \in X$ and $y \in Y$, we will denote by $x \otimes y$ the rank-one operator from X^* to Y given by $x^* \mapsto x^*(x)y$. The *tensor product* of X and Y, denoted $X \otimes Y$, is the vector subspace of $\mathscr{L}(X^*, Y)$ spanned by all elements of the form $x \otimes y$ with $x \in X$ and $y \in Y$. Thus, a typical element of $X \otimes Y$ is of the form $u = \sum_{i=1}^n x_i \otimes y_i$, with $x_i \in X$ and $y_i \in Y$, although this representation is not unique.

We will consider two different norms on $X \otimes Y$. First, the *injective norm* of $u \in X \otimes Y$ is defined as:

$$\|u\|_\varepsilon = \sup\left\{\sum_{i=1}^n x^*(x_i)y^*(y_i) : x^* \in B_{X^*}, y^* \in B_{Y^*}\right\}$$

where $u = \sum_{i=1}^n x_i \otimes y_i$ is any representation of u. Note that the injective norm coincides with the norm of u as an element of $\mathscr{L}(X^*, Y)$. On the other hand, the *projective norm* of u is defined as

$$\|u\|_\pi = \inf\left\{\sum_{i=1}^n \|x_i\| \, \|y_i\| : u = \sum_{i=1}^n x_i \otimes y_i\right\}.$$

The completion of $(X \otimes Y, \| \ \|_\varepsilon)$ is called the *injective tensor product* of X and Y and denoted $X \widehat{\otimes}_\varepsilon Y$. Moreover, the completion of $(X \otimes Y, \| \ \|_\pi)$ is called the *projective tensor* *product* of X and Y and denoted $X\widehat{\otimes}_\pi Y$. Let us point out that the definition of $\| \ \|_\pi$ is maybe more intuitive from a geometric point of view, since $B_{X\widehat{\otimes}_\pi Y} = \overline{\text{conv}}(B_X \otimes B_Y)$. Moreover,

$$(X\widehat{\otimes}_\pi Y)^* = \mathscr{L}(X, Y^*)$$

for every Banach spaces X and Y.

Injective and projective tensor products turn out to be useful for describing spaces of vector-valued functions, indeed $\mathscr{C}(K, X)$ is isometric to $\mathscr{C}(K)\widehat{\otimes}_\varepsilon X$ for every compact space K and $L_1(\mu, X)$ is isometric to $L_1(\mu)\widehat{\otimes}_\pi X$ for every positive measure μ on a measure space. For a detailed treatment and applications of tensor products, we refer the reader to [Rya].

An operator $T\colon X \to Y$ is said to be *nuclear* if there exist sequences $(x_n^*)_{n=1}^\infty$ and $(y_n)_{n=1}^\infty$ such that $\sum_{n=1}^\infty \|x_n^*\| \, \|y_n\| < \infty$ and $T(x) = \sum_{n=1}^\infty x_n^*(x) y_n$ for every $x \in X$. The nuclear norm of T is defined as

$$\|T\| = \inf \left\{ \sum_{n=1}^\infty \|x_n^*\| \, \|y_n\| : T(x) = \sum_{n=1}^\infty x_n^*(x) y_n \text{ for all } x \in X \right\}.$$

We denote $\mathscr{N}(X, Y)$ the space of linear operators from X to Y, endowed with the nuclear norm. It can be showed that $X^*\widehat{\otimes}_\pi Y$ embeds linearly in $\mathscr{N}(X, Y)$. We will recall next some hypotheses which guarantee that this embedding is onto.

The following result summarises some properties of tensor products that will be important in what follows. For the proof, see Corollaries 4.8 and 4.13 and Theorem 5.33 in [Rya], and Theorem 16.30 in [FHH+].

Theorem 0.3.1 (Grothendieck)**.** *Let X and Y be Banach spaces.*

(a) *If either X^* or Y has the AP, then $X^*\widehat{\otimes}_\varepsilon Y$ is isometric to $\mathscr{K}(X, Y)$.*
(b) *If Y^* has the RNP, then $(X\widehat{\otimes}_\varepsilon Y)^*$ is isometric to $\mathscr{N}(X, Y^*)$.*
(c) *If X^* or Y has the AP, then $X^*\widehat{\otimes}_\pi Y$ is isometric to $\mathscr{N}(X, Y)$.*
(d) *If X^* has the RNP and either X^* or Y^* has the AP, then $(X\widehat{\otimes}_\varepsilon Y)^*$ is isometric to $X^*\widehat{\otimes}_\pi Y^*$.*

Chapter 1

Convex compact sets that admit a lower semicontinuous strictly convex function

Assume K is a compact convex subset of a locally convex space. If K is metrizable, then the space $\mathscr{C}(K)$ is separable, and so is the subspace of affine continuous functions $\mathscr{A}(K)$. Given a countable dense subset $\{h_n : n \in \mathbb{N}\} \subset S_{\mathscr{A}(K)}$, it is easy to check that

$$f = \sum_{n=1}^{\infty} \frac{1}{2^n} h_n^2$$

defines a continuous strictly convex function on K. The main motivation for the results in this section is the following result of Hervé [Her], which shows that the converse statement also holds.

Theorem 1.0.1 (Hervé)**.** *Let K be a compact convex subset of a locally convex space. Then K is metrizable if and only if there exists $f\colon K \to \mathbb{R}$ with is both continuous and strictly convex.*

The above result shows how the existence of a strictly convex continuous function defined on a compact set K enforces a topological property of K, namely, the metrizability. It happens that lower semicontinuity is a very natural hypothesis for a convex function. Thus, it makes sense to replace the hypothesis of continuity by lower semicontinuity in Hervé theorem and to study which properties of K follow if such a function exists. That motivates the definition of the following class of compact convex sets.

Definition 1.0.2. Given a locally convex space E, the class $\mathscr{SC}(E)$ consists of all the nonempty compact convex subsets K such that there exists a function $f\colon K \to \mathbb{R}$ which is lower semicontinuous and strictly convex. In addition, $\mathscr{SC}$ denotes the class composed of all the families $\mathscr{SC}(E)$ for any locally convex space E.

Notice that metrizable convex compact sets admit continuous strictly convex functions, so they are in the class $\mathscr{SC}$. In particular, if E is metrizable then $\mathscr{SC}(E)$ contains all the convex compact subsets of E. If E is a Banach space endowed with its weak topology, then $\mathscr{SC}(E)$ is made up of all convex weakly compact subsets as a consequence of the strictly convex renorming results for WCG spaces (see, e.g. Theorem 13.25 in [FHH$^+$]).

Let us recall previous work about the class $\mathscr{SC}$, done with a different terminology but that can be found in the literature. Assume that $K \in \mathscr{SC}$. Then

(a) K is *fragmentable* by a finer metric [Rib1, Rib2],
(b) $\operatorname{ext}(K)$ contains a completely metrizable dense subset [Raj6], and
(c) $[0, \omega_1]$ does not embed into K (Talagrand's argument in [DGZ, Theorem VII.5.2.(ii)]).

Moreover, Godefroy and Li showed in [GL] that if the set of probabilities on a compact group K admits a strictly convex lower semicontinuous function then K is metrizable.

In the first section of this chapter, we present stability properties of the class $\mathscr{SC}$ which allow us to prove that a set belongs to $\mathscr{SC}$ if and only if it embeds linearly in Banach space with a strictly convex dual ball. In the second section we give a characterisation of the class $\mathscr{SC}$ in terms of the existence of a symmetric with countable dentability index. The third section is devoted to the search of faces and exposed points of continuity of a convex function.

This chapter is based on the paper [GLOR].

1.1 The class $\mathscr{SC}$

Our first goal is to study stability properties of the class $\mathscr{SC}$. Along this section E will denote a locally convex space.

Proposition 1.1.1. *The class $\mathscr{SC}$ satisfies the following stability properties:*

(a) *$\mathscr{SC}(E)$ is stable by translations and homothetics;*
(b) *$\mathscr{SC}$ is stable by Cartesian products;*
(c) *$\mathscr{SC}$ is stable by linear continuous images;*
(d) *If $A, B \in \mathscr{SC}(E)$, then $A + B \in \mathscr{SC}(E)$.*

Proof. Statement (a) is obvious. To prove (b) suppose that f_i witnesses $A_i \in \mathscr{SC}(E_i)$ for $i = 1, \dots, n$. Then $\sum_{i=1}^n f_i \circ \pi_i$, where $\pi_i \colon E_1 \times \ldots \times E_n \to E_i$ is the coordinate projection, witnesses that $A_1 \times \ldots \times A_n \in \mathscr{SC}(E_1 \times \ldots \times E_n)$.

To prove (c) assume that $A \in \mathscr{SC}(E)$ and $T \colon E \to F$ is linear and continuous. Obviously $T(A)$ is convex and compact. Let $f \colon A \to \mathbb{R}$ be lower semicontinuous and strictly convex. It is straightforward to check that the function $g \colon T(A) \to \mathbb{R}$ defined by

$$g(y) = \inf \left\{ f(x) : x \in T^{-1}(y) \right\}$$

does the work. Finally, (d) follows by a combination of (b) and (c). $\square$

We will need a kind of external convex sum of convex compact sets.

Definition 1.1.2. Given $A, B \subset E$ convex compact, we denote

$$A \oplus B = \{(\lambda x, (1-\lambda) y, \lambda) \in E \times E \times \mathbb{R} : x \in A, y \in B, \lambda \in [0,1]\}.$$

Lemma 1.1.3. *Let $A, B \subset E$ be convex compact subsets. Then*

(a) *$A \oplus B$ is a convex compact subset of $E \times E \times \mathbb{R}$;*
(b) *if $f\colon A \to \mathbb{R}$ and $g\colon B \to \mathbb{R}$ are convex, then $h\colon A \oplus B \to \mathbb{R}$ defined by*

$$h(\lambda x, (1-\lambda)y, \lambda) = \lambda f(x) + (1-\lambda)g(y)$$

is convex as well;
(c) *if $A, B \in \mathscr{SC}(E)$, then $A \oplus B \in \mathscr{SC}(E \times E \times \mathbb{R})$.*

Proof. Compactness is clear in statement (a). Given $(\lambda_i x_i, (1-\lambda_i)y_i, \lambda_i) \in A \oplus B$ for $i = 1, 2$, just observe that

$$\begin{aligned}&\Big(\frac{\lambda_1 x_1 + \lambda_2 x_2}{2}, \frac{(1-\lambda_1)y_1 + (1-\lambda_2)y_2}{2}, \frac{\lambda_1+\lambda_2}{2}\Big)\\ &= \Big(\frac{\lambda_1+\lambda_2}{2}\frac{\lambda_1 x_1 + \lambda_2 x_2}{\lambda_1+\lambda_2}, (1-\frac{\lambda_1+\lambda_2}{2})\frac{(1-\lambda_1)y_1 + (1-\lambda_2)y_2}{(1-\lambda_1)+(1-\lambda_2)}, \frac{\lambda_1+\lambda_2}{2}\Big)\end{aligned}$$

(the case where $\lambda_1 = \lambda_2 \in \{0,1\}$ can be handed in a different way). Thus, $A \oplus B$ is convex. For the convexity of function h notice that

$$\begin{aligned}&h\Big(\frac{\lambda_1 x_1 + \lambda_2 x_2}{2}, \frac{(1-\lambda_1)y_1 + (1-\lambda_2)y_2}{2}, \frac{\lambda_1+\lambda_2}{2}\Big)\\ &= \frac{\lambda_1+\lambda_2}{2} f\Big(\frac{\lambda_1 x_1 + \lambda_2 x_2}{\lambda_1+\lambda_2}\Big) + \Big(1-\frac{\lambda_1+\lambda_2}{2}\Big) g\Big(\frac{(1-\lambda_1)y_1 + (1-\lambda_2)y_2}{(1-\lambda_1)+(1-\lambda_2)}\Big)\\ &\leq \frac{\lambda_1+\lambda_2}{2}\,\frac{\lambda_1 f(x_1) + \lambda_2 f(x_2)}{\lambda_1+\lambda_2} + \Big(1-\frac{\lambda_1+\lambda_2}{2}\Big)\frac{(1-\lambda_1)g(y_1) + (1-\lambda_2)g(y_2)}{(1-\lambda_1)+(1-\lambda_2)}\\ &= \frac{1}{2}\left(h(\lambda_1 x_1, (1-\lambda_1)y_1, \lambda_1) + h(\lambda_2 x_2, (1-\lambda_2)y_2, \lambda_2)\right).\end{aligned}$$

If f and g were strictly convex, the above inequality for h would become strict if $x_1 \neq x_2$ or $y_1 \neq y_2$. To overcome this difficulty consider the function

$$k(\lambda x, (1-\lambda)y, \lambda) = h(\lambda x, (1-\lambda)y, \lambda) + \lambda^2$$

and notice that λ^2 provides the strict inequality when $x_1 = x_2$ and $y_1 = y_2$. □

Proposition 1.1.4. *Suppose that $A, B \in \mathscr{SC}(E)$. Then* $\operatorname{conv}(A \cup B) \in \mathscr{SC}(E)$ *and* $\operatorname{aconv}(A) \in \mathscr{SC}(E)$.

Proof. Consider the map $T\colon E \times E \times \mathbb{R} \to E$ defined by $T\big((x,y,t)\big) = x + y$ and observe that $T(A \oplus B) = \operatorname{conv}(A \cup B)$. Since T is linear and continuous, the combination of the previous results gives us that $\operatorname{conv}(A \cup B) \in \mathscr{SC}(E)$. The application to the symmetric convex hull follows by applying it with $B = -A$. □

Lemma 1.1.5. *Let B be a symmetric compact convex subset of a locally convex space (E, τ) and let $Z = \operatorname{span}(B)$. Then the following hold:*

(a) *Z, with the norm given by the Minkowski functional of B, is isometric to a dual Banach space;*
(b) *B embeds linearly into (Z, w^*);*
(c) *if $f \colon E \to \mathbb{R}$ is convex and lower semicontinuous, then $f|_Z$ is weak* lower semicontinuous.*

Proof. Notice that $Z = \bigcup_{n=1}^{\infty} nB$, and thus the Minkowski functional of B is a norm on Z. Of course, B is the unit ball of Z endowed with this norm. By a result of Dixmier-Ng, see for instance [Ng], the space Z is isometric to the dual of the Banach space X of all linear functionals f on Z such that $f|_B$ is τ-continuous. Moreover, given a net $(x_\alpha)_\alpha \subset B$ which τ-converges to $x \in B$, we have that $f(x_\alpha) \overset{\alpha}{\to} f(x)$ for every $f \in X$ and so $(x_\alpha)_\alpha$ is also weak*-convergent to x. This shows that the identity map $I \colon (B, \tau) \to (B, w^*)$ is continuous. Hence, the τ-compactness of B yields that both topologies coincide on B.

It remains to prove statement (c). Assume $f \colon E \to \mathbb{R}$ is convex and lower semicontinuous. Then the sets $\{f \leq a\}$ are convex and closed for any $a \in \mathbb{R}$. We have $\{f|_Z \leq a\} = \{f \leq a\} \cap Z$, and thus $\{f|_Z \leq a\} \cap nB = \{f \leq a\} \cap nB$ is compact, and so it is weak* compact as subset of Z for every $n \in \mathbb{N}$. By the Banach-Dieudonné theorem, $\{f|_Z \leq a\}$ is a weak* closed subset of Z. $\square$

Several classes of compact spaces admit a characterisation in terms of an embedding into a Banach space endowed with a norm satisfying a certain geometrical condition. Namely, a compact space K is

(a) *uniform Eberlein* if and only if it embeds into a Hilbert space endowed with the weak topology [FHH+],
(b) *Namioka–Phelps* if and only if it embeds into a dual Banach space with a dual LUR norm endowed with the weak* topology [Raj2], and
(c) *descriptive* if and only if it embeds into a dual Banach space with a weak*-LUR norm endowed with the weak* topology [Raj4].

Our next result provides a characterisation of the class $\mathscr{SC}$ in the above sense. Recall that a Banach space X is said to be *strictly convex* (or *rotund*) if $\operatorname{ext}(B_X) = S_X$, equivalently, if $\left\|\frac{x+y}{2}\right\| < 1$ whenever $x, y \in S_X$, $x \neq y$.

Theorem 1.1.6. *Let K be a compact convex subset of a locally convex space. Then $K \in \mathscr{SC}$ if and only if K embeds linearly into a strictly convex dual Banach space endowed with its weak* topology.*

Proof. It is clear that if K embeds linearly into (Z, w^*) for a strictly convex dual Banach space Z then $\|\ \|^2$ is a lower semicontinuous strictly convex function on K.

Conversely, assume that $K \in \mathscr{SC}$. Let $B = \operatorname{aconv}(K)$ which is in $\mathscr{SC}$ by Proposition 1.1.4. By Lemma 1.1.5, we may assume that B is a weak* compact subset of a dual Banach space Z. We only need to renorm the dual space Z. To this end, note that the function f witnessing that $B \in \mathscr{SC}$ is weak* lower semicontinuous and strictly convex. Moreover, the function f can be taken symmetric and bounded. Indeed, for the symmetry just take $g(x) = f(x) + f(-x)$. Now apply the Baire theorem to $B = \bigcup_{n=1}^{\infty} g^{-1}((-\infty, n])$ to obtain a set of the form λB with $\lambda > 0$ where g is bounded. Then redefine f as $f(x) = g(\lambda x)$.

Without loss of generality we may assume that f takes values in $[0, 1]$. Consider the function defined on B_Z by

$$h(x) = \frac{1}{2}(3\|x\| + f(x))$$

and consider the set $C = \{x \in B_Z : h(x) \leq 1\}$. Clearly $\frac{1}{3}B_Z \subset C \subset \frac{2}{3}B_Z$, and C is convex, symmetric and weak* closed. Moreover, if $h(x) = h(y) = 1$, then $h(\frac{x+y}{2}) < 1$. Therefore, C is the unit ball of an equivalent strictly convex dual norm on Z. □

Notice that the strictly convex norm of the dual Banach space in the statement of Theorem 1.1.6 is weak* lower semicontinuous, which is a stronger condition that just being a strictly convex Banach space isomorphic to a dual space (see Theorem 5.2 in [DGZ]).

Finally, note that thanks to Theorem 1.1.6 we can improve the function witnessing that a compact set belongs to $\mathscr{SC}$.

Corollary 1.1.7. *If $K \in \mathscr{SC}$, then it is witnessed by the square of a lower semicontinuous strictly convex norm defined on* $\operatorname{span}(K)$.

1.2 Ordinal indices, (∗) property and Gruenhage compacta

We begin this section by showing the connection between the class $\mathscr{SC}$ and (∗) property. This property was introduced in [OST] in order to characterise dual Banach spaces that admit a dual strictly convex norm.

Definition 1.2.1 (Orihuela–Smith–Troyanski)**.** A compact space K is said to have (∗) if there exists a sequence $(\mathscr{U}_n)_{n=1}^{\infty}$ of families of open subsets of K such that, given any $x, y \in K$, there exists $n \in \mathbb{N}$ such that:

(a) $\{x, y\} \cap \bigcup \mathscr{U}_n$ is non-empty;
(b) $\{x, y\} \cap U$ is at most a singleton for every $U \in \mathscr{U}_n$.

If moreover K is a subset of a locally convex space and the elements of $\bigcup_{n=1}^{\infty} \mathscr{U}_n$ can be taken to be slices of K, then K is said to have (∗) *with slices*.

Here we are using the agreement that $\bigcup \mathscr{U}_n = \bigcup\{U : U \in \mathscr{U}_n\}$. It is shown in [OST, Theorem 2.7] that if X^* is a dual Banach space then (B_{X^*}, w^*) has (∗) with slices if and only if X^* admits a dual strictly convex norm.

Proposition 1.2.2. *Let K be a compact convex subset of a locally convex space (E, τ). Then $K \in \mathscr{SC}(E)$ if and only if K has (∗) with slices.*

Proof. First assume that K has (∗) with slices. By Lemma 1.1.5, $Z = \operatorname{span}(K)$ is a dual Banach space and $\operatorname{aconv}(K)$ (hence, K) embeds linearly into (Z, w^*). Moreover, given

$f \in (E,\tau)^*$ providing a slice of K, Lemma 1.1.5.(c) says that $f|_Z$ is weak* continuous and thus it provides a weak* slice of K. Therefore, K has $(*)$ with weak* slices as a subset of Z. It then follows from [OST, Proposition 2.2] that there is a lower semicontinuous strictly convex function defined on K.

On the other hand, assume that ϕ witnesses $K \in \mathscr{SC}(E)$. For $f \in (E,\tau)^*$ and $r \in \mathbb{R}$, denote $S(f,r) = \{x \in K : f(x) > r\}$. Consider the families $\{\mathscr{U}_{qr}\}_{q,r\in\mathbb{Q}}$ of open subsets given by

$$\mathscr{U}_{qr} = \{S(f,r) : f \in (E,\tau)^*, S(f,r) \cap \{x : \phi(x) \le q\} = \emptyset\} .$$

Let $x \neq y$ be in K. We may assume that $\phi(x) \le \phi(y)$. Since ϕ is strictly convex, there exists $q \in \mathbb{Q}$ such that $\phi(\frac{x+y}{2}) < q < \phi(y)$. By the Hahn–Banach theorem, there is $f \in (E,\tau)^*$ and $r \in \mathbb{Q}$ such that $\sup\{f(z) : \phi(z) \le q\} < r < f(y)$. Therefore, $S(f,r) \cap \{z : \phi(z) \le q\} = \emptyset$ and $\{x,y\} \cap \bigcup \mathscr{U}_{qr} \neq \emptyset$.

Suppose that $x, y \in S(g,r) \in \mathscr{U}_{qr}$. Then $g(x), g(y) > r$ implies $g(\frac{x+y}{2}) > r$. Hence $\frac{x+y}{2} \notin \{z : \phi(z) \le q\}$, a contradiction. So $\{x,y\} \cap S(g,q)$ is at most a singleton for each $S(g,q) \in \mathscr{U}_{qr}$. □

Our next goal is to find a characterisation of the class $\mathscr{SC}$ in terms of the dentability with respect to a symmetric. First we recall this notion.

Definition 1.2.3. A *symmetric* on a set A is a function $\rho\colon A \times A \to \mathbb{R}$ which satisfies

(a) $\rho(x,y) \ge 0$ for every $x, y \in A$.
(b) $\rho(x,y) = 0$ if and only if $x = y$.
(c) $\rho(x,y) = \rho(y,x)$ for every $x, y \in A$.

Since a symmetric does not need to satisfy the triangle inequality, its associated topology is complicated to handle. Nevertheless we have a natural notion of diameter associated to ρ defined by

$$\rho\text{-diam}(A) = \sup\{\rho(x,y) : x, y \in A\}$$

The idea of measuring diameters with respect to a symmetric was successfully applied in renorming theory in [MOTV]. We will apply this notion in order to find exposed points of continuity of a strictly convex function in the next section.

Set derivations with respect to a symmetric were introduced in [FOR] in order to characterise dual Banach spaces admitting a dual strictly convex norm. Note that if f is a strictly convex function, then

$$\rho(x,y) = \frac{f(x)+f(y)}{2} - f\left(\frac{x+y}{2}\right)$$

is a symmetric.

Now we prove the characterisation of the class $\mathscr{SC}$ announced above. We denote $\mathrm{Dz}_\rho(K) = \mathrm{Sz}_{[\rho,\mathbb{H}]}(K)$, where $\mathbb{H}$ denotes the family of open half-spaces of the locally convex space.

Proposition 1.2.4. *Let K be a convex compact subset of a locally convex space. Then the following assertions are equivalent:*

(i) $K \in \mathscr{SC}$;
(ii) *there exists a symmetric ρ on K such that* $\mathrm{Dz}_\rho(K) \leq \omega$;
(iii) *there exists a symmetric ρ on K such that* $\mathrm{Dz}_\rho(K) \leq \omega_1$.

Proof. Let f be a bounded function witnessing that $K \in \mathscr{SC}$ and assume that f takes values in $[0,1]$. For a fixed $\varepsilon > 0$, take $N > 1/\varepsilon$ and define the closed convex subsets $F_n = \{x \in K : f(x) \leq 1 - n/N\}$ for $n = 0, \dots N$. Take

$$\rho(x,y) = \frac{f(x)+f(y)}{2} - f\left(\frac{x+y}{2}\right) .$$

We claim that $[\rho, \mathbb{H}]'_\varepsilon(K) \subset F_1$. Let $x_0 \in K \setminus F_1$. By the Hahn–Banach theorem, there exists a slice S of K such that $x_0 \in S$ and $S \cap F_1 = \emptyset$. If $x, y \in S$, then $\frac{x+y}{2} \in S$ and $\rho(x,y) \leq 1 - (1 - 1/N) = 1/N$. Thus, ρ-diam$(S) < \varepsilon$ and so $x_0 \notin [\rho, \mathbb{H}]'_\varepsilon(K)$. By iteration, we get that $[\rho, \mathbb{H}]^N_\varepsilon(K) \subset F_N$ and hence $[\rho, \mathbb{H}]^{N+1}_\varepsilon(K) = \emptyset$. Therefore, $\mathrm{Dz}_\rho(K, \varepsilon) < \omega$ for each $\varepsilon > 0$.

Clearly (ii) implies (iii). Finally, suppose that $\mathrm{Dz}_\rho(K) \leq \omega_1$. Notice that indeed $\mathrm{Dz}_\rho(K) < \omega_1$ since ω_1 has uncountable cofinality. By Proposition 1.2.2, it suffices to show that K has (*) with slices. For each $n \in \mathbb{N}$ and $\alpha < \mathrm{Dz}_\rho(K, 1/n)$ consider the family

$$\mathscr{U}_{n,\alpha} = \left\{S : S \text{ is slice of } K, S \cap [\rho, \mathbb{H}]^{\alpha+1}_{1/n}(K) = \emptyset, \rho\text{-diam}(S \cap [\rho, \mathbb{H}]^{\alpha}_{1/n}(K)) < 1/n\right\} .$$

Given distinct $x, y \in K$, take n so that $\rho(x,y) > 1/n$ and let α be the least ordinal such that $\{x,y\} \cap [\rho, \mathbb{H}]^{\alpha+1}_{1/n}(K)$ is at most a singleton. Then it is clear that there is a slice in $\mathscr{U}_{n,\alpha}$ containing either x or y, and no slice in $\mathscr{U}_{n,\alpha}$ contains both points. Thus, K has (*) with slices, as desired. □

Remark 1.2.5. By using deep results of descriptive set theory, Lancien proved in [Lan3] the existence of a universal function $\psi \colon [0, \omega_1) \to [0, \omega_1)$ such that $\mathrm{Dz}(X) \leq \psi(\mathrm{Sz}(X))$ whenever X is a Banach space such that $\mathrm{Sz}(X) < \omega_1$. We do not know if a similar statement holds when the norm is replaced by a symmetric.

We will show that we cannot change symmetric by metric in Proposition 1.2.4. That would imply that K is a Gruenhage compact, which is a strictly stronger condition than being in $\mathscr{SC}$ [Smi2, Theorem 2.4]. Let us recall the definition of Gruenhage space, given in [Gru].

Definition 1.2.6 (Gruenhage)**.** A topological space T is called *Gruenhage* if there exists a sequence $(\mathscr{U}_n)_{n=1}^\infty$ of families of open subsets of T such that, given any $x, y \in T$, there exists $n \in \mathbb{N}$ and $U \in \mathscr{U}_n$ such that:

(a) $\{x,y\} \cap U$ is a singleton;
(b) either x lies in finitely many $U' \in \mathscr{U}_n$ or y lies in finitely many $U' \in \mathscr{U}$.

Rather than the original definition, we will use the following characterisation of Gruenhage spaces due to R. Smith (see Proposition 2.1 in [Smi1]). Recall that a family $\mathscr{H}$ of subsets of a topological space T is said to *separate points* if, given distinct $x, y \in T$, there is $H \in \mathscr{H}$ such that $\{x, y\} \cap H$ is a singleton.

Proposition 1.2.7 (Smith)**.** *Let T be a topological space. The following are equivalent.*

(i) *T is a Gruenhage;*

(ii) *there exist a family of closed sets $\{A_n : n \in \mathbb{N}\}$ and families $(\mathscr{U}_n)_{n=1}^{\infty}$ of open sets such that the family $\{A_n \cap U : U \in \mathscr{U}_n\}$ is pairwise disjoint for each n and the family $\{A_n \cap U : n \in \mathbb{N}, U \in \mathscr{U}_n\}$ separates points.*

(iii) *there exists a sequence $(\mathscr{U}_n)_{n=1}^{\infty}$ of families of open subsets of T and sets R_n, such that $\bigcup_{n=1}^{\infty} \mathscr{U}_n$ separates points and $U \cap V = R_n$ whenever $U, V \in \mathscr{U}_n$ are distinct.*

The characterisation above allows us to show the relation between Gruenhage and $(*)$, which we include for completeness. It comes from Proposition 4.1 in [OST].

Proposition 1.2.8 (Orihuela–Smith–Troyanski)**.** *Let T be Gruenhage. Then T has $(*)$.*

Proof. Let $(\mathscr{U}_n)_{n=1}^{\infty}$ and R_n be as in Proposition 1.2.7.(iii). Let $\mathscr{V}_n = \{R_n\}$ for each n. Given distinct $x, y \in T$, there exist n and $U \in \mathscr{U}_n$ such that $\{x, y\} \cap U$ is a singleton. If $x \in R_n$, then $y \notin R_n$ and so $\{x, y\} \cap U = \{x\}$ for every $U \in \mathscr{V}_n$. If $y \in R_n$ then we argue similarly. Otherwise, $x, y \notin R_n$ and so $\{x, y\} \cap U$ is at most a singleton for every $U \in \mathscr{U}_n$, by the definition of R_n. □

We finish the section showing that Gruenhage compacta admit a characterisation analogous to Proposition 1.2.4.

Proposition 1.2.9. *Let (K, τ) be a compact space. Then the following assertions are equivalent:*

(i) *K is Gruenhage;*

(ii) *there exists a metric d on K such that $\mathrm{Sz}_d(K) \leq \omega$;*

(iii) *there exists a metric d on K such that $\mathrm{Sz}_d(K) \leq \omega_1$.*

Proof. First assume that d is a metric on K with countable Szlenk index. Bing's metrization theorem provides a basis $\mathscr{B} = \bigcup_{m=1}^{\infty} \mathscr{B}_m$ of the metric topology such that each $\mathscr{B}_m$ is discrete. Consider the open sets $U_V^{n,\alpha} = \bigcup\{U : U \text{ open}, U \cap [d,\tau]_{2^{-n}}^{\alpha}(K) \subset V\}$ and the families $\mathscr{U}_m^{n,\alpha} = \{U_V^{n,\alpha} : V \in \mathscr{B}_m\}$. Let us consider the countable set

$$D = \{(n, m, \alpha) : n, m \in \mathbb{N}, \alpha < \mathrm{Sz}(K, 2^{-n})\}.$$

We claim that the family $\{[d,\tau]_{2^{-n}}^{\alpha}(K) : (n, m, \alpha) \in D\}$ of closed sets and the families $(\mathscr{U}_m^{n,\alpha})_{(n,m,\alpha)\in D}$ of open sets satisfy condition (ii) in Proposition 1.2.7. To see this, note first that $\{U \cap [d,\tau]_{2^{-n}}^{\alpha}(K) : U \in \mathscr{U}_m^{n,\alpha}\}$ is pairwise disjoint for each $(n, m, \alpha) \in D$ since $\mathscr{B}_m$ is discrete. Moreover, given distinct $x, y \in K$, take $V \in \mathscr{B}_m$ such that $x \in V$ and $y \notin V$. Fix n

such that $B_d(x, 2^{-n+1}) \subset V$. Let α be the least ordinal so that $x \notin [d,\tau]^{\alpha+1}_{2^{-n}}(K)$. Then there is an open subset U of K such that $x \in U \cap [d,\tau]^{\alpha}_{2^{-n}}(K)$ and $\operatorname{diam}(U \cap [d,\tau]^{\alpha}_{2^{-n}}(K)) \leq 2^{-n}$. Thus $x \in U^{n,\alpha}_V \cap [d,\tau]^{\alpha}_{2^{-n}}(K) \subset V$, so $y \notin U^{n,\alpha}_V \cap [d,\tau]^{\alpha}_{2^{-n}}(K)$. This shows that the family $\{U \cap [d,\tau]^{\alpha}_{2^{-n}}(K) : U \in \mathscr{U}^{n,\alpha}_m : (n,m,\alpha) \in D\}$ separates points. Thus, K is Gruenhage.

Finally, if K is a Gruenhage compact space, then the same construction used in the proof of [Raj7, Theorem 2.8] provides a metric on K such that $\mathrm{Sz}_d(K) \leq \omega$. We include here a sketch of the proof. Take a family $\{A_n\}$ and families $(\mathscr{U}_n)$ satisfying condition (ii) in Proposition 1.2.7. For every $n \in \mathbb{N}$, define $d_n(x,y) = 0$ if either $\{x,y\} \subset K \setminus A_n$, $\{x,y\} \subset A_n \cap \bigcup \mathscr{U}_n$ or $\{x,y\} \subset A_n \setminus \bigcup \mathscr{U}_n$ and $d_n(x,y) = 1/n$ in any other case. Define $d(x,y) = \max\{d_n(x,y) : n \in \mathbb{N}\}$. Note that d is a metric since the family

$$\{A_n \cap U : n \in \mathbb{N}, U \in \mathscr{U}_n\}$$

separates points. Now, fix $\varepsilon > 0$ and take n_0 with $n_0 \leq \varepsilon^{-1} < n_0 + 1$. We claim that $[d,\tau]^{2n_0+1}_{\varepsilon}(K) = \emptyset$. To see this, first note that $d_n(x,y) \leq \frac{1}{n_0+1} < \varepsilon$ whenever $n > n_0$ and $x, y \in K$. Thus, $\mathrm{Sz}_d(K,\varepsilon)$ coincides with $\mathrm{Sz}_{d_0}(K,\varepsilon)$ where $d_0 = \max\{d_n : n \leq n_0\}$. Moreover, if $n \leq n_0$ then the derivatives with respect to d_n satisfy $[d_n,\tau]'_{\varepsilon}(K) \subset A_n$, $[d_n,\tau]''_{\varepsilon}(K) \subset A_n \setminus \bigcup \mathscr{U}_n$ and $[d_n,\tau]'''_{\varepsilon}(K) = \emptyset$. Therefore $\mathrm{Sz}_{d_n}(K,\varepsilon) \leq 3$ for $n \leq n_0$. Now, a result of Raja (see Proposition 2.6 in [Raj7]) ensures that

$$\mathrm{Sz}_d(K,\varepsilon) = \sum_{n \leq n_0} \mathrm{Sz}_{d_n}(K,\varepsilon) - n_0 + 1 \leq 2n_0 + 1.$$

This proves that $\mathrm{Sz}_d(K) \leq \omega$. □

1.3 Faces and exposed points of continuity

In this section we will use arguments coming from the study of the Radon–Nikodým property in order to analyse the existence of faces of continuity of a convex function, as well as exposed points of continuity of a strictly convex function.

Our starting point is the following result proved in [Raj6], which ensures the existence of extreme points of continuity of a convex lower semicontinuous function.

Theorem 1.3.1 (Raja)**.** *Let K be a compact convex subset of a locally convex space. Let $f\colon K \to \mathbb{R}$ be a bounded convex lower semicontinuous function. Then $\operatorname{ext}(K)$ contains a dense subset of continuity points of f.*

The proof of Theorem 1.3.1 is not very long and relies on the topological properties of $\operatorname{ext}(K)$, so we include it here. Indeed, we prove a slightly more general result which ensures, under certain conditions, the existence of strongly extreme points of continuity. To this end we need one more result.

Proposition 1.3.2. *Let T be a Baire space and $f\colon T \to \mathbb{R}$ be a lower semicontinuous function. Then there is a dense $\mathscr{G}_\delta$ subset of continuity points of f.*

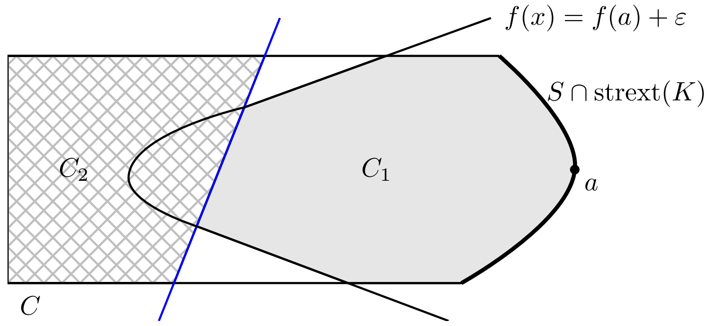

Figure 1.1: Proof of Proposition 1.3.3

Proof. For each $r \in \mathbb{Q}$ we set

$$A_r = \{x \in T : f(x) > r\} \cup \mathrm{int}\{x \in T : f(x) \leq r\}.$$

Consider $A = \bigcap_{r \in \mathbb{Q}} A_r$. Note that $\{x \in T : f(x) > r\}$ is an open set for each $r \in \mathbb{Q}$ due to the lower semicontinuity of f. Thus A is a $\mathscr{G}_\delta$ set.

Let us show that A is dense in T. Since T is a Baire space, it suffices to show that each A_r is dense in T. To this end, let $x \in T$ and U be a neighbourhood of x. Assume that $U \cap A_r = \emptyset$. Then $U \cap \{x \in T : f(x) > r\} = \emptyset$. Thus $U \subset \{x \in T : f(x) \leq r\}$ and so $U \subset \mathrm{int}\{x \in T : f(x) \leq r\} \subset A_r$, which is a contradiction. Therefore, $U \cap A_r \neq \emptyset$. This shows that A_r is dense in T for each r, as desired.

Finally, we will show that f is continuous at every point of A. It suffices to prove the upper semicontinuity. For that, take $a \in A$ and $\varepsilon > 0$. Pick $r \in \mathbb{Q}$ so that $f(a) < r < f(a) + \varepsilon$. Then $U = \mathrm{int}\{x \in T : f(x) \leq r\}$ is a neighbourhood of a satisfying that $\sup\{f, U\} \leq r < f(a) + \varepsilon$. This shows that f is upper semicontinuous on a. □

Proposition 1.3.3. *Let C be a convex subset of a locally convex space such that C is a $\mathscr{G}_\delta$ in $\overline{C}$, $\overline{C}$ is compact and $C \subset \overline{\mathrm{conv}}(\mathrm{strext}(C))$. Let $f \colon C \to \mathbb{R}$ be a bounded convex lower semicontinuous function. Then $\mathrm{strext}(C)$ contains a dense subset of continuity points of f.*

Proof. Let M be a bound for f. By Proposition 1.3.2 and Corollary 0.1.10, there is a dense subset A of $\mathrm{strext}(C)$ so that $f_{|\,\mathrm{strext}(C)}$ is continuous on A. We claim that f is actually continuous on A. Indeed, it suffices to check the upper semicontinuity. To this end, take $a \in A$ and fix $0 < \varepsilon < M^{-1}$. Note that the set $\{x \in \mathrm{strext}(C) : f(x) < f(a) + \varepsilon\}$ is open in $\mathrm{strext}(C)$. Therefore there is an open slice S of C satisfying

$$a \in S \cap \mathrm{strext}(C) \subset \{x \in \mathrm{strext}(C) : f(x) < f(a) + \varepsilon\}$$

Consider the convex sets $C_1 = \{x \in C : f(x) \leq f(a) + \varepsilon\} \cap \overline{S}$ and $C_2 = C \setminus S$. Note that

$$C \subset \overline{\mathrm{conv}}(C_1 \cup C_2) \subset \mathrm{conv}(\overline{C_1} \cup \overline{C_2}).$$

Now, consider the set

$$D = \{(1-\lambda)x_1 + \lambda x_2 : x_1 \in \overline{C_1}, x_2 \in \overline{C_2}, \lambda \in [0, \varepsilon/M]\},$$

which is closed and convex. Since $a \notin \overline{C_2}$ and $a \in \operatorname{ext}(\overline{C})$, we have $a \notin D$. Thus, $C \setminus D$ is an open neighbourhood of a in C. If $x \in C \setminus D$, then $x = (1-\lambda)x_1 + \lambda x_2$ with $x_1 \in \overline{C_1}, x_2 \in \overline{C_2}$ and $\varepsilon/M \leq \lambda \leq 1$. Take nets $(x_1^\alpha)_\alpha \subset C_1$ and $(x_2^\alpha)_\alpha \subset C_2$ converging to x_1 and x_2, respectively. The convexity and the lower semicontinuity of f yield

$$\begin{aligned} f(x) &\leq \liminf_\alpha f((1-\lambda)x_1^\alpha + \lambda x_2^\alpha) \\ &\leq \liminf_\alpha (1-\lambda)f(x_1^\alpha) + \lambda f(x_2^\alpha) \\ &\leq f(a) + \varepsilon + \frac{\varepsilon}{M}M = f(a) + 2\varepsilon. \end{aligned}$$

This shows that f is upper semicontinuous in a. □

Proof of Theorem 1.3.1. Follows readily from Proposition 1.3.3. □

Corollary 1.3.4. *Let C be a closed convex bounded subset of a Banach space X. Assume that C has the RNP and that the weak* closure $\overline{C}^{w^*}$ of C in X^{**} is metrizable. Let $f\colon C \to \mathbb{R}$ be a bounded convex lower semicontinuous function. Then the set of preserved extreme points of C contains a dense subset of continuity points of f.*

Proof. The set C satisfies the hypotheses of Proposition 1.3.3 as a subset of (X^{**}, w^*), see the proof of Corollary 0.1.15. Moreover, note that f is weak-lower semicontinuous since it is a convex lower semicontinuous function. □

Let us recall the definition of face of a convex set.

Definition 1.3.5. Let C be closed convex subset of a locally convex space. We say that a closed subset $F \subset C$ is a *face of C* if there is a continuous affine function $w\colon C \to \mathbb{R}$ such that

$$F = \{x \in C : w(x) = \sup\{w, C\}\}.$$

In that case we say that the face is produced by w.

Sometimes the face is produced by an element of the dual. Nevertheless, there may exist continuous affine functions on C that are not the restriction of an element of the dual.

Note that a point $x \in C$ is an exposed point of C if and only if $\{x\}$ is a face of C.

We will need the following lemma.

Lemma 1.3.6 (Lemma 3.3.3 of [Bou4])**.** *Let X be a Banach space. Suppose that $x^* \in X^*$ and $\|x^*\| = 1$. For $r > 0$ denote by V_r the set $rB_X \cap \ker x^*$. Assume that x_0 and y are points of X such that $x^*(x_0) > x^*(y)$ and $\|x_0 - y\| \leq r/2$. If $y^* \in X^*$ satisfies that $\|y^*\| = 1$ and $y^*(x_0) > \sup\{y^*, y + V_r\}$, then $\|x^* - y^*\| \leq \frac{2}{r}\|x_0 - y\|$.*

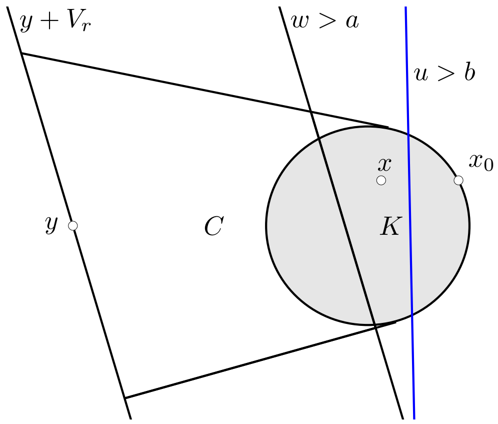

Figure 1.2: Proof of Proposition 1.3.7

First we will discuss the case in which the locally convex space is a dual Banach case X^* endowed with its weak* topology. The elements of X will be considered as functionals on X^*.

Proposition 1.3.7. *Let X be a Banach space and $f\colon X^* \to \mathbb{R}$ be a convex weak* lower semicontinuous function which is bounded on weak* compact subsets. If $K \subset X^*$ is weak* compact and convex, then there exists a $\mathcal{G}_\delta$ dense set of elements of X producing faces where $f|_K$ is constant and weak* continuous.*

Proof. Define the pseudo-symmetric ρ by the formula

$$\rho(x, y) = \frac{f(x)^2 + f(y)^2}{2} - f\left(\frac{x+y}{2}\right)^2 .$$

We claim that $\rho(x, y) = 0$ implies $f(x) = f(y) = f(\frac{x+y}{2})$ (in particular, if f were strictly convex, ρ would be a symmetric). Indeed, it follows easily from this observation

$$\rho(x, y) \geq \frac{f(x)^2 + f(y)^2}{2} - \left(\frac{f(x) + f(y)}{2}\right)^2 = \left(\frac{f(x) - f(y)}{2}\right)^2 \geq 0$$

and the strict convexity of the function $t \mapsto t^2$. Now, we claim that the set $G(K, \varepsilon)$ is open and dense in X for $K \subset X^*$ weak* compact convex and $\varepsilon > 0$, where

$$G(K, \varepsilon) = \{w \in X : \text{ there is } a < \sup\{w, K\}, \rho\text{-diam}(K \cap \{w > a\}) < \varepsilon\} .$$

Suppose that $w \in G(K, \varepsilon)$. If $w' \in X$ is close enough to w to fulfil that

$$\sup\{w', K\} > \sup\{w', K \cap \{w \leq a\}\}$$

then $w' \in G(K,\varepsilon)$ as well. Thus $G(K,\varepsilon)$ is open. In order to see that it is also dense, fix $w \in X$ and $\delta \le 1/4$. Take $x \in K$ and $y \in X^*$ with $w(x) > a > w(y)$ for some $a \in \mathbb{R}$. Take $r = \sup\{\|x'-y\|, x' \in K\}/2\delta$, consider the set $V_r = rB_{X^*} \cap \ker w$ and define the set $C = \operatorname{conv}(K \cup (y+V_r))$. By Theorem 1.3.1, the half-space $\{w > a\}$ contains a point $x_0 \in \operatorname{ext}(C)$ where $f|_C$ is weak* continuous. Notice that $x_0 \in \operatorname{ext}(K)$ and $\|x_0 - y\| \le r/2$. This fact and Choquet's lemma provide $u \in X$ and $b \in \mathbb{R}$ such that $u(x_0) > b$, $C \cap \{u > b\} \subset C \cap \{w > a\}$ and $\rho\text{-diam}(C \cap \{u > b\}) < \varepsilon$. In particular $\rho\text{-diam}(K \cap \{u > b\}) < \varepsilon$. Since $C \cap \{u > b\}$ does not meet $y + V_r$, we have $u(x_0) > \sup\{u, y+V_r\}$. Thus, $\|w-u\| \le \frac{2}{r}\|x_0 - y\| \le \delta$ by Lemma 1.3.6 applied to the Banach space X^*. That completes the proof of the density of $G(K,\varepsilon)$ in X.

By the Baire theorem, the set $G(K) = \bigcap_{n=1}^{\infty} G(K,1/n)$ is dense. If $w \in G(K)$ and $s = \sup\{w,K\}$ then

$$\lim_{t\to s^-} \rho\text{-diam}(K \cap \{w > t\}) = 0\,.$$

In particular, the face $F = K \cap \{w = s\}$ satisfies that $\rho\text{-diam}(F) = 0$. That implies that f is constant on F. Moreover, we claim that any point $x \in F$ is a point of weak* continuity of $f|_K$. If $(x_\alpha)_\alpha \subset K$ is a net with limit x, then $\lim_\alpha w(x_\alpha) = w(x)$. Therefore $\lim_\alpha \rho(x_\alpha, x) = 0$. It follows that $\lim_\alpha f(x_\alpha) = f(x)$, so $f|_K$ is weak* continuous at x. $\square$

Now the above result can be translated into a more general setting.

Proposition 1.3.8. *Let E be a locally convex space and $f\colon E \to \mathbb{R}$ be a convex lower semicontinuous function which is bounded on compact subsets. Then for every compact convex subset $K \subset E$ and every open slice $S \subset K$, there is a face $F \subset S$ of K such that $f|_K$ is constant and continuous on F.*

Proof. By Lemma 1.1.5, $Z = \bigcup_{n=1}^{\infty} n \operatorname{aconv}(K)$ is a dual Banach space and $f|_Z$ is weak* lower semicontinuous. Then we can apply the previous proposition. $\square$

Let us remark that clearly the last two results are true for countably many functions simultaneously.

Remark 1.3.9. We do not know if the function f in Propositions 1.3.7 and 1.3.8 can be assumed to be defined only on K. Notice that if $\|\ \|$ is a strictly convex norm on X^* then $f(x) = -\sqrt{1-\|x\|^2}$ is a strictly convex weak* lower semicontinuous function on (B_{X^*}, w^*) that cannot be extended to a convex function on X^*.

Note that if a strictly convex function is constant on a face of a compact K, then necessarily that face should be an exposed point of K. Having this in mind, Propositions 1.3.7 and 1.3.8 can be rewritten.

Proposition 1.3.10. *Let $f\colon X^* \to \mathbb{R}$ be a strictly convex weak* lower semicontinuous function which is bounded on compact subsets. If $K \subset X^*$ is weak* compact convex, then there exists a $\mathscr{G}_\delta$ dense set of elements of X exposing points of K at which $f|_K$ is weak* continuous.*

Proof. It follows straightforward from Proposition 1.3.7. □

Theorem 1.3.11. *Let E be a locally convex space and let $f\colon E \to \mathbb{R}$ be lower semicontinuous, strictly convex and bounded on compact sets. Then for every $K \subset E$ compact and convex, the set of points in K which are both exposed and continuity points of $f|_K$ is dense in* $\mathrm{ext}(K)$.

Proof. It follows straightforward from Proposition 1.3.8. □

We retrieve the following result, which is usually proved in the frame of Gâteaux Differentiability Spaces [Phe, Corollary 2.39 and Theorem 6.2].

Corollary 1.3.12 (Asplund, Larman–Phelps)**.** *Let X^* be a strictly convex dual Banach space. Then every convex weak* compact set is the closed convex hull of its weak* exposed points.*

We also obtain the following consequence.

Corollary 1.3.13. *Assume that $K \in \mathscr{SC}$. Then K is the closed convex hull of its exposed points.*

Proof. Thanks to Theorem 1.1.6 it can be reduced to the previous corollary. □

Notice that the previous result is far from being a characterisation. For instance, consider $X = C([0,\omega_1])^*$ and $K = (B_X, w^*)$. Then X has the RNP and thus there exist strongly exposed points of K. Nevertheless, Talagrand's argument in [DGZ, Theorem 5.2.(ii)] shows that $K \notin \mathscr{SC}(X, w^*)$. Indeed, the result of Larman and Phelps mentioned above states that Banach spaces for which each weak* compact convex subset has a weak* exposed point are exactly dual spaces of a Gâteaux Differentiability Space.

Remark 1.3.14. A point x in a subset C of a normed space $(X, \|\ \|)$ is said to be a *farthest point* in C if there exists $y \in X$ such that $\|y - x\| \geq \sup\{\|y - c\| : c \in C\}$. If $\|\ \|$ is strictly convex then every farthest point of C is exposed by a functional in X^*. In addition, it was shown in [DZ] that there exists a weak* compact subset of ℓ_1 that has no farthest points, so the existence of exposed points does not imply the existence of farthest points. On the other hand, suppose that X^* is a strictly convex dual Banach space, C is a compact subset of X^* and x is a farthest point in C with respect to $y \in X^*$. Consider the symmetric $\rho(u,v) = \frac{\|u-y\|^2+\|v-y\|^2}{2} - \|\frac{u+v}{2} - y\|^2$. Then x is a $\rho-$denting point of C, that is, admits slices with arbitrarily small ρ-diameter. Indeed, if $\delta = \frac{\varepsilon}{1+2\|x-y\|+2\|y\|}$ then every slice of C that does not meet $B(y, \|y - x\| - \delta)$ has ρ-diameter less than ε.

Typically a variational principle provides a strong minimum for certain functions after a small perturbation. But in the compact setting, a lower semicontinuous function already attains its minimum. Nevertheless, inspired by Stegall's variational principle [FHH$^+$, Theorem 11.6], we have obtained the following result.

Proposition 1.3.15. *Suppose that $K \in \mathscr{SC}$ and let $f\colon K \to \mathbb{R}$ be a lower semicontinuous function. Given $\varepsilon > 0$, there exists an affine continuous function w on K with oscillation less than ε such that $f + w$ attains its minimum exactly at one point. Moreover, if X^* is a dual Banach space then w can be taken from the predual with norm less than ε.*

Proof. By Theorem 1.1.6, we may assume that K is a subset of a dual Banach space X^*. Let m be the minimum of f and take $M > 0$ such that $K \subset MB_{X^*}$. Consider the compact set

$$H = \{(x,t) \in K \times \mathbb{R} : f(x) \leq t \leq m + \varepsilon M\}$$

and take $A = \overline{\mathrm{conv}}(H)$. By Proposition 1.1.1, $A \in \mathscr{SC}(X^* \times \mathbb{R})$. The functional on $X \times \mathbb{R}$ given by $(0,1)$ attains its minimum on A. Proposition 1.3.10 provides a small perturbation of the form $(w,1)$, with $\|w\| < \varepsilon$, attaining its minimum on A at one single point (x_0, t_0). Notice that $t_0 = f(x_0)$ and $f(x_0) + w(x_0) \leq m + \varepsilon M$. If $y \in K$, then either $f(y) \leq m + \varepsilon M$ and $(y, f(y)) \in A$, or $f(y) > m + \varepsilon M \geq f(x_0) + w(x_0)$. $\square$

Chapter 2

Maps with the Radon–Nikodým property

In this chapter we study dentable maps from a closed convex subset of a Banach space into a metric space as an attempt of generalising the Radon–Nikodým property to a less linear frame. Namely, we propose the following definition.

Definition 2.0.1. Let $C \subset X$ be a nonempty subset of a Banach space X and let M be a metric space. A map $f: C \to M$ is said to be *dentable* if for every nonempty bounded set $A \subset C$ and $\varepsilon > 0$, there is an open half-space H such that $A \cap H \neq \emptyset$ and $\operatorname{diam}(f(A \cap H)) < \varepsilon$.

Note that a closed convex set C has the Radon–Nikodým property if and only if the identity map $I: C \to (C, \| \ \|)$ is dentable thanks to Rieffel's characterization (see Theorem 0.1.14). In addition, a map $f: C \to (M, d)$ induces a pseudometric on C by the formula $\rho(x, y) = d(f(x), f(y))$ and the dentability of the map f is equivalent to the ordinary subset dentability of C with respect to ρ. Nevertheless, we prefer to consider maps from C to a metric space M since M can carry other structures, as algebraic ones. Let us remark that the notion of dentable map should be compared to two previous related concepts. First, σ-slicely continuous maps introduced in [MOTV] provide a characterisation of the existence of an equivalent LUR norm in a Banach space. On the other hand, σ-fragmentable maps were introduced in [JOPV] in order to study selection problems.

In the first section we establish that the elements of the dual which are "strongly slicing" for a given uniformly continuous dentable function form a dense $\mathscr{G}_\delta$ subset of the dual. As a consequence, the space of uniformly continuous dentable maps from a closed convex bounded set to a Banach space is a Banach space. The second section includes a characterisation of sets with the RNP in terms of dentability of continuous maps defined on them, and a characterisation of uniformly continuous finitely dentable maps. The third section is devoted to the relation between dentable maps and delta-convex maps. In the fourth section we obtain a version of Stegall's variational principle for closed dentable maps. Finally, in the fifth section we investigate sets which are dentable with respect to a metric defined on it.

This chapter is based on the paper [GLR1], although the results in Section 4 and part of Section 3 have not appeared anywhere else.

Throughout the chapter C will denote a closed convex subset of a Banach space X and M will denote a metric space with a metric d. Moreover, $\mathbb{H}$ denote the set of all the open half-spaces of X.

2.1 Properties of the dentable maps

We begin the section by studying the relation between dentable maps and RN-operators introduced by Reĭnov [Reĭ1] and Linde [Lin]. We need the following result, which should be compared with [Bou4, Proposition 2.3.2].

Proposition 2.1.1. *A map $f\colon C \to M$ is dentable if and only if for every nonempty bounded set $A \subset C$ and $\varepsilon > 0$ there exists $x \in A$ such that $x \notin \overline{\operatorname{conv}}(A \setminus f^{-1}(B(f(x), \varepsilon))$.*

Proof. First assume that f is a dentable map. Fix $\varepsilon > 0$ and $A \subset C$ nonempty and bounded. By hypothesis, there exists an open half-space H such that $A \cap H \neq \emptyset$ and $\operatorname{diam}(f(A \cap H)) < \varepsilon$. Then $A \cap H \subset f^{-1}(B(f(x), \varepsilon))$, so $\overline{\operatorname{conv}}(A \setminus f^{-1}(B(f(x), \varepsilon)) \cap H = \emptyset$ and any $x \in A \cap H$ does the work.

Conversely, fix $\varepsilon > 0$ and let $A \subset C$ be nonempty and bounded. Take $x \in A$ so that $x \notin \overline{\operatorname{conv}}(A \setminus f^{-1}(B(f(x), \varepsilon/2))$. Then the dentability condition is witnessed by any slice of A separating x from $\overline{\operatorname{conv}}(A \setminus f^{-1}(B(f(x), \varepsilon/2))$. □

Reĭnov [Reĭ2] characterised RN-operators as those bounded operators satisfying the condition in Proposition 2.1.1. Therefore, the notion of dentable function extends the class of RN-operators to the non-linear setting.

Corollary 2.1.2. *Let $T\colon X \to Y$ be a bounded linear operator between Banach spaces. Then T is an RN-operator if, and only if, T is dentable.*

Our next goal is to establish that there are many functionals defining slices of small oscillation for a dentable map. To this end, we need a version of Asplund–Bourgain–Namioka superlemma as presented in [Bou4, Theorem 3.4.1]. Let us state it for future reference.

Superlemma 2.1.3 (Asplund–Bourgain–Namioka)**.** *Let $A, B, C \subset X$ be bounded closed convex subsets and let $\varepsilon > 0$. Assume that $A \subset C \subset \overline{\operatorname{conv}}(A \cup B)$, $\operatorname{diam}(A) < \varepsilon$ and $C \setminus B \neq \emptyset$. Then there is a slice of C which contains a point of A and that is of diameter less than ε.*

Lemma 2.1.4. *Let $f\colon C \to M$ be a uniformly continuous map and $A, B \subset X$ be bounded closed convex subsets such that $A \subset C \subset \overline{\operatorname{conv}}(A \cup B)$, $\operatorname{diam}(f(A)) < \varepsilon$ and $A \setminus B \neq \emptyset$. Then there exists $H \in \mathbb{H}$ such that $A \cap H \neq \emptyset$, $B \cap H = \emptyset$ and $\operatorname{diam}(f(C \cap H)) < \varepsilon$.*

Proof. Take $\eta = 3^{-1}(\varepsilon - \operatorname{diam}(f(A)))$. Let $\delta > 0$ be such that if $x, y \in C$ satisfy $\|x - y\| < \delta$ then $d(f(x), f(y)) < \eta$. Given $r \in (0, 1]$, consider the convex set

$$D_r = \{(1 - \lambda)y + \lambda z : y \in A, z \in B, \lambda \in [r, 1]\}.$$

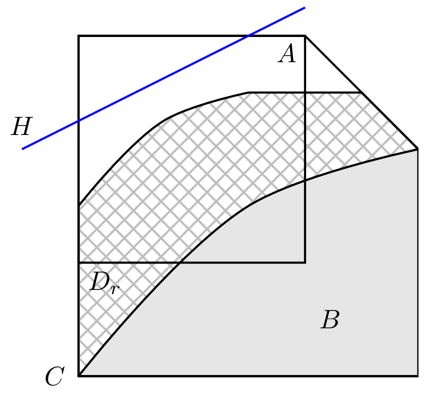

Figure 2.1: Proof of Lemma 2.1.4

First we claim that $A \setminus \overline{D_r}$ is nonempty. Indeed, let $x^* \in X^*$ be such that $\sup\{x^*, B\} < \sup\{x^*, C\}$, which exists since $A \setminus B$ is nonempty. Then $\sup\{x^*, A\} = \sup\{x^*, C\}$ and thus

$$\sup\{x^*, \overline{D_r}\} = \sup\{x^*, D_r\} \leq (1-r)\sup\{x^*, A\} + r\sup\{x^*, B\} < \sup\{x^*, A\}.$$

So $A \not\subset \overline{D_r}$.

Now note that for $x \in C \setminus D_r \subset \operatorname{conv}(A \cup B) \setminus D_r$ there are $y \in A$, $z \in B$ and $\lambda \in [0, r]$ such that $x = (1-\lambda)y + \lambda z$. Therefore, $\|x - y\| \leq r\|y - z\|$. If we take $r \in (0, 1]$ such that $r \operatorname{diam}(A - B) < \delta$, then $d(f(x), f(y)) < \eta$. That implies

$$\operatorname{diam}(f(C \setminus D_r)) < \operatorname{diam}(f(A)) + 2\eta < \varepsilon.$$

Finally, any $H \in \mathbb{H}$ separating points of A from $\overline{D_r}$ will satisfy that $\operatorname{diam}(f(C \cap H)) < \varepsilon$, as desired. □

The following result can be proved as the analogous for the RNP (see [Bou4, Proposition 3.5.2]).

Lemma 2.1.5. *Let $C, D \subset X$ be bounded closed convex subsets such that $C \setminus D \neq \emptyset$ and suppose that $f\colon C \to M$ is a uniformly continuous dentable map. Given $\varepsilon > 0$, there exists $H \in \mathbb{H}$ such that $D \cap H = \emptyset$, $C \cap H \neq \emptyset$ and* $\operatorname{diam}(f(C \cap H)) < \varepsilon$.

Proof. Take $E = \overline{\operatorname{conv}}(C \cup D)$ and let

$$F = \{x \in E : \text{ there is an } x^* \in X^* \text{ such that } x^*(x) = \sup\{x^*, E\} > \sup\{x^*, D\}\}.$$

Note that Bishop–Phelps theorem ensures the existence of sup-attaining functionals arbitrarily close to functionals separating points of C from D, so $F \neq \emptyset$. Moreover, $F \subset C$ and

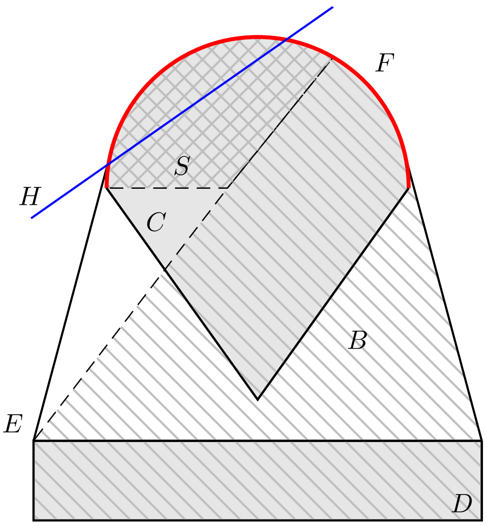

Figure 2.2: Proof of Lemma 2.1.5

$E = \overline{\text{conv}}(F \cup D)$. Indeed, the first part is standard (anyway, see [Bou4, Theorem 3.5.1]). If the second inclusion did not hold, separation with Hahn–Banach and Bishop–Phelps again would lead to a new point of F outside from F, a contradiction.

Now, find a nonempty open slice S of $\overline{\text{conv}}(F)$ such that $\text{diam}(f(\overline{S})) < \varepsilon$. Consider $B = \overline{\text{conv}}(D \cup (\overline{\text{conv}}(F) \setminus S))$. We claim that $S \setminus B \neq \emptyset$. Indeed, suppose that $S \subset B$, which clearly implies that $B = E$. There are $x \in S$ and $x^* \in X^*$ such that $x^*(x) = \sup\{x^*, E\} > \sup\{x^*, D\}$. Since we are assuming that

$$E = \overline{\text{conv}}(D \cup (\overline{\text{conv}}(F) \setminus S)),$$

we should have $x \in \overline{\text{conv}}(F) \setminus S$, which is impossible.

Finally, Lemma 2.1.4 with $A = \overline{S}$ and the same set B provides an open half-space H which does not meet D and such that $\text{diam}(f(C \cap H)) < \varepsilon$. □

We are interested in the functionals which produce slices whose images have arbitrarily small diameter.

Definition 2.1.6. Let $f\colon C \to M$ be a map, A be a bounded subset of C and $x^* \in X^*$. We say that A is *f-strongly sliced* by x^* if

$$\lim_{t \to 0^+} \text{diam}(f(S(A, x^*, t))) = 0\,,$$

and in such a case we say that x^* is *f-strongly slicing* on A. The set of all the f-strongly slicing functionals on A will be denoted $\mathscr{SS}(f, A)$.

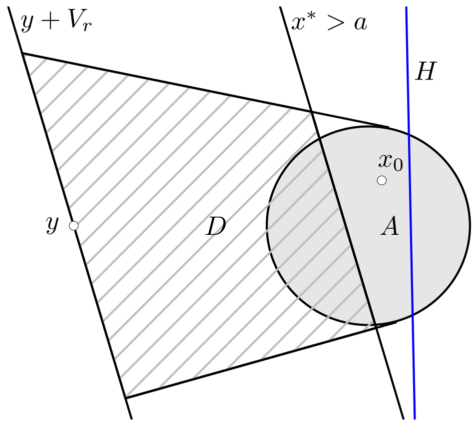

Figure 2.3: Proof of Theorem 2.1.7

Note that the notion of strongly slicing functional is similar to that of strongly exposing. However, a strongly exposing functional is always referred to a point of the set. This is not the case for a strongly slicing functional since, in general, the slices are not converging to a point. That pathology will be studied later in relation with the dentability of sets.

The set of dentable maps from C to M will be denoted $\mathscr{D}(C, M)$. By $\mathscr{D}_U(C, M)$ we denote the set of maps from $\mathscr{D}(C, M)$ which are moreover uniformly continuous on bounded subsets of C.

The next result is the analogous of Bourgain–Phelps theorem (see Theorem 0.1.14.(d)) for dentable maps.

Theorem 2.1.7. *If $f \in \mathscr{D}_U(C, M)$, then $\mathscr{S}\mathscr{S}(f, A)$ is a dense $\mathscr{G}_\delta$ subset of X^* for any nonempty bounded $A \subset C$.*

Proof. For $n \in \mathbb{N}$ consider the set

$$U_n = \{x^* \in X^* : \text{ there is } t > 0 \text{ such that } \operatorname{diam}(f(S(A, x^*, t))) < 1/n\}.$$

It is not difficult to see that U_n is open. In order to prove that it is also dense, take $x^* \in X^*$ and $0 < \varepsilon < 1$. Pick $x_0 \in A$ and $y \in X$ with $x^*(x_0) > a > x^*(y)$ for some $a \in \mathbb{R}$. Now take $r = 2\varepsilon^{-1} \sup\{\|x - y\| : x \in A\}$ and consider

$$D = \overline{\operatorname{conv}}(A \cup (y + V_r)) \cap \{x \in X : x^*(x) \leq a\},$$

where $V_r = rB_X \cap \ker x^*$.

Note that $x_0 \notin D$, so $\overline{\operatorname{conv}}(A) \setminus D \neq \emptyset$. Moreover, f is uniformly continuous on the bounded set $\overline{\operatorname{conv}}(A)$. Thus Lemma 2.1.5 provides an open half-space H such that $D \cap H = \emptyset$, $\overline{\operatorname{conv}}(A) \cap H \neq \emptyset$ and $\operatorname{diam}(f(\overline{\operatorname{conv}}(A) \cap H)) < 1/n$. Then clearly $A \cap H \neq \emptyset$

and $\operatorname{diam}(f(A\cap H)) < 1/n$, so $y^* \in U_n$ for $y^* \in S_{X^*}$ being the functional which determines H. Finally, we will show that $\|x^* - y^*\| < \varepsilon$. For this, take $x_1 \in A \cap H$. It suffices to show that $y^*(x_1) > \sup\{y^*, y + V_r\}$ and apply Lemma 1.3.6. Notice that

$$\{x^* \in X^* : x^*(x) > a\} \cap (y + V_r) = \emptyset$$

because $x^*(y) < a$. Moreover, since $D \cap H = \emptyset$ we have $x^*(x) > a$ for every $x \in (y+V_r)\cap H$. So $(y+V_r)\cap H = \emptyset$ and therefore $y^*(x_1) > \sup\{y^*, y+V_r\}$, which proves that $\|x^* - y^*\| < \varepsilon$ and finishes the proof of the density of U_n.

Finally, the set $\mathscr{SS}(f, A) = \bigcap_{n=1}^{\infty} U_n$ is dense in X^* by Baire theorem, as we want. □

As a consequence we get several corollaries.

Corollary 2.1.8. *Let M_i be metric spaces and $f_i \in \mathscr{D}_U(C, M_i)$ for $i = 1, \dots, n$. Assume that $A \subset C$ is nonempty and bounded. Given $\varepsilon > 0$, there exists an open half-space $H \subset X$ such that $A \cap H \neq \emptyset$ and*

$$\max\{\operatorname{diam}(f_i(A \cap H)) : i = 1, \dots, n\} < \varepsilon.$$

Hence, if we set $M = \prod_{i=1}^{n} M_i$ endowed with a standard product metric and $f = (f_1, \dots, f_n)$, then $f \in \mathscr{D}_U(C, M)$.

Proof. The intersection $\bigcap_{i=1}^{n} \mathscr{SS}(f_i, A)$ is non-empty by Baire theorem and Theorem 2.1.7. Moreover, every element of $\bigcap_{i=1}^{n} \mathscr{SS}(f_i, A)$ provides slices satisfying the required property. □

The following expresses the "equi-dentability" for finitely many dentable real functions. Unfortunately, our techniques require uniform continuity.

Corollary 2.1.9. *Let $C \subset X$ be a bounded closed convex set. Given $f_1, \dots, f_n \in \mathscr{D}_U(C, \mathbb{R})$ and $\varepsilon > 0$, there is $H \in \mathbb{H}$ such that $C \cap H \neq \emptyset$ and*

$$\max\{\operatorname{diam}(f_1(C \cap H)), \dots, \operatorname{diam}(f_n(C \cap H))\} < \varepsilon.$$

Let state separately the following curious result, which was observed by Bourgain in [Bou3].

Corollary 2.1.10. *Let $C \subset X$ be a bounded closed convex set. Given $x_1^*, \dots, x_n^* \in X^*$ and $\varepsilon > 0$, there is $H \in \mathbb{H}$ such that $C \cap H \neq \emptyset$ and*

$$\max\{\operatorname{diam}(x_1^*(C \cap H)), \dots, \operatorname{diam}(x_n^*(C \cap H))\} < \varepsilon.$$

Corollaries 2.1.9 and 2.1.10 follow directly from Corollary 2.1.8.

It is clear that the composition of a dentable map and a uniformly continuous map is dentable. For compositions with continuous maps we have the following result.

Corollary 2.1.11. *Let $g\colon M \to N$ be a continuous map between metric spaces M and N and $f \in \mathscr{D}_U(C, M)$. Assume that M is complete. Then $g \circ f$ is dentable.*

Proof. Take $A \subset C$ nonempty and bounded and fix $\varepsilon > 0$. Let x^* be an f-strongly slicing functional on A, which exists by Theorem 2.1.7. Since M is complete, there exists $y_0 \in \bigcap_{t>0} \overline{f(S(A, x^*, t))}$. Now, take $\delta > 0$ such that $d(g(y), g(y_0)) < \varepsilon$ whenever $y \in M$ and $d(y, y_0) < \delta$. Then there is $t > 0$ so that $\operatorname{diam}(f(S(A, x^*, t))) = \operatorname{diam}(\overline{f(S(A, x^*, t))}) < \delta$. It follows that $\operatorname{diam}((g \circ f)(S(A, x^*, t)) < 2\varepsilon$. □

Lemma 2.1.12. *Let $*$ be a binary operation on M which is uniformly continuous on bounded sets. Then $f * g \in \mathscr{D}_U(C, M)$ whenever $f, g \in \mathscr{D}_U(C, M)$.*

Proof. Note that a uniformly continuous function on a convex bounded set is bounded. This fact and the hypothesis on the operation $*$ imply that $f * g$ is uniformly continuous on bounded sets whenever f, g are. Now, if $A \subset C$ is bounded and nonempty, find $x^* \in X^*$ such that it is simultaneous f-strongly slicing and g-strongly slicing, which exists by Theorem 2.1.7. Given $\varepsilon > 0$, by using the uniform continuity of $*$ on $f(A)$ we can find $\delta > 0$ such that $\max\{\operatorname{diam}(U), \operatorname{diam}(V)\} < \delta$ for $U, V \subset f(A)$ implies that $\operatorname{diam}(U * V) < \varepsilon$. Thus, if $H \subset X$ is a half-space such that $A \cap H \neq \emptyset$ with $\operatorname{diam}(f(A \cap H)) < \delta$ and $\operatorname{diam}(g(A \cap H)) < \delta$, then $\operatorname{diam}((f * g)(A \cap H)) < \varepsilon$. □

Now we can prove the following result on the stability of dentable maps.

Theorem 2.1.13. *Let $C \subset X$ be a closed convex set. If M is a vector space, then $\mathscr{D}_U(C, M)$ is a vector space. Assume moreover that C is bounded. Then:*

(a) *if M is a complete metric space, then $\mathscr{D}_U(C, M)$ is complete for the metric of uniform convergence on C;*
(b) *if M is a Banach space, then $\mathscr{D}_U(C, M)$ is a Banach space;*
(c) *if M is a Banach algebra (resp. lattice), then $\mathscr{D}_U(C, M)$ is a Banach algebra (resp. lattice).*

Proof. Lemma 2.1.12 yields the first statement. Now, assume that C is also bounded. Note that the boundedness and convexity of C together the uniform continuity implies that every map in $\mathscr{D}_U(C, M)$ is bounded, so we may consider the uniform metric on this set.

Let $f\colon C \to M$ be a map that can be uniformly approximated by maps from $\mathscr{D}_U(C, M)$. Clearly, f is uniformly continuous. We will see that f is moreover dentable. Indeed, fix $\varepsilon > 0$ and take $g \in \mathscr{D}_U(C, M)$ such that $d_\infty(f, g) < \varepsilon/3$. If $A \subset C$ is nonempty, then there is a half-space $H \subset X$ such that $\operatorname{diam}(g(A \cap H)) < \varepsilon/3$ and $A \cap H \neq \emptyset$. The triangle inequality yields that $\operatorname{diam}(f(A \cap H)) < \varepsilon$.

Thus, $\mathscr{D}_U(C, M)$ is closed for uniform convergence, and therefore, if M is complete, then $\mathscr{D}_U(C, M)$ is complete too. From what we have proved it follows that $\mathscr{D}_U(C, M)$ is a Banach space whenever M is. Finally, the last statement is a straightforward consequence of Lemma 2.1.12. □

Let us remark that Schachermayer proved in [Sch1] that there exist sets C_1 and C_2 with the RNP such that $C_1 + C_2$ contains an isometric copy of the closed unit ball of c_0 and thus $C_1 + C_2$ does not have the RNP. This implies that the sum of two strong Radon-Nikodým operators need not be strong Radon-Nikodým (an operator is said to be strong Radon-Nikodým if the image of the closed unit ball has the RNP).

We finish the section by showing that uniformly continuous dentable maps satisfy a mixing property analogous to the one of $\mathscr{DC}$ functions (see [VZ1, Lemma 4.8]). We will need the following elementary lemma.

Lemma 2.1.14. *Let A be a connected space and let ρ be a pseudometric on A. Assume that there exist closed subsets $A_1, \ldots, A_n$ of M such that $A = \cup_{i=1}^n A_i$ and $\operatorname{diam}(A_i) \le \varepsilon$ for each i. Then $\operatorname{diam}(A) \le n\varepsilon$.*

Proof. Fix $x, y \in A$. Take $x_1 = x$ and let $\sigma(1) \in \{1, \ldots, n\}$ be such that $x \in A_{\sigma(1)}$. Since A is connected, there is $x_2 \in A_{\sigma(1)} \cap (\cup_{i \neq \sigma(1)} A_i)$, and so $\rho(x_2, x) \le \varepsilon$. Take $\sigma(2) \neq \sigma(1)$ such that $x_2 \in A_{\sigma(2)}$. Now take $x_3 \in (A_{\sigma(1)} \cup A_{\sigma(2)}) \cap (\cup_{i \in \{1,\ldots,n\} \setminus \{\sigma(1),\sigma(2)\}} A_i)$. Then either $\rho(x_3, x_2) \le \varepsilon$ or $\rho(x_3, x) \le \varepsilon$, so in any case $\rho(x_3, x) \le 2\varepsilon$. By iterating this process we get $\sigma(i)$ and x_i for each $i = 1, \ldots, n$ satisfying that $\sigma(i) \in \{1, \ldots, n\} \setminus \{\sigma(1), \ldots, \sigma(i-1)\}$, $x_i \in A_{\sigma(i)} \cap (\cup_{j<i} A_{\sigma(j)})$ and $\rho(x_i, x) \le (i-1)\varepsilon$. Thus σ defines a bijection on $\{1, \ldots, n\}$ and so there exists i such that $y \in A_{\sigma(i)}$. Therefore, $\rho(x, y) \le \rho(y, x_i) + \rho(x_i, x) \le n\varepsilon$, as desired. □

Proposition 2.1.15. *Assume that $f_1, \ldots, f_n \in \mathscr{D}_U(C, M)$ and $f\colon C \to M$ is a continuous map such that $f(x) \in \{f_1(x), \ldots, f_n(x)\}$ for every $x \in C$. Then $f \in \mathscr{D}_U(C, M)$.*

Proof. First notice that f is uniformly continuous on bounded sets. Indeed, let $A \subset C$ be bounded, fix $\varepsilon > 0$ and take $\delta > 0$ so that $d(f_i(x) - f_i(y)) \le n^{-1}\varepsilon$ for every $i = 1, \ldots, n$ whenever $x, y \in A$ and $\|x - y\| \le \delta$. Consider the closed sets $A_i = \{x \in A : f(x) = f_i(x)\}$. Now, if $x, y \in A$ satisfy that $\|x - y\| \le \delta$ then $\operatorname{diam}(A_i \cap [x, y]) \le \delta$ and thus $\operatorname{diam}(f(A_i \cap [x, y])) \le n^{-1}\varepsilon$ for every $i = 1, \ldots, n$. Now apply Lemma 2.1.14 to the connected set $[x, y] = \cup_{i=1}^n A_i \cap [x, y]$ with the pseudometric $\rho = d \circ f$ to get that $d(f(x), f(y)) \le \varepsilon$.

In order to show that f is dentable, take a nonempty bounded subset A of C and fix $\varepsilon > 0$. By Corollary 2.1.9 there is $H \in \mathbb{H}$ satisfying that

$$\max\{\operatorname{diam}(f_i(A \cap H)) : i = 1, \ldots, n\} \le n^{-1}\varepsilon.$$

The results follows by applying Lemma 2.1.14 to $A \cap H = \cup_{i=1}^n A_i \cap A \cap H$. □

2.2 Characterisations of dentability

We begin the section by showing the relation between the dentability of a set and the dentability of maps defined on it. To this end, recall that the *Kuratowski index of* *non-compactness* of a set $D \subset X$ is given by

$$\alpha(D) = \inf \left\{ \varepsilon > 0 : \exists x_1, \ldots, x_n \in X, D \subset \bigcup_{i=1}^{n} B(x_i, \varepsilon) \right\}.$$

We need the following result, which goes back to Huff and Morris [HM] (see also [GOOT] for a more general version). The proof given here is due to Bourgain [Bou2].

Lemma 2.2.1. *Let $A \subset X$ be bounded. Then A is dentable if and only if A has slices of arbitrarily small Kuratowski index of non-compactness.*

Proof. Assume that A is not dentable. Then there is $\varepsilon > 0$ so that every slice of A has diameter greater than ε. Moreover, we may assume that A is closed and bounded. Let $H \in \mathbb{H}$ be such that $A \cap H \neq \emptyset$ and let us show that $\alpha(A \cap H) \geq \varepsilon/2$. Assume $\alpha(A \cap H) < \varepsilon/2$. Then there exist closed convex subsets $A_1, \ldots, A_n$ of A with diameter at most ε so that $A \cap H \subset \cup_{i=1}^{n} A_i$. Note that

$$A = \overline{\text{conv}} \left((A \setminus H) \cup \bigcup_{i=1}^{n} A_i \right) = \overline{\text{conv}} \left(\overline{\text{conv}} \left((A \setminus H) \cup \bigcup_{i=1}^{n-1} A_i \right) \cup A_n \right)$$

If $A \not\subset \overline{\text{conv}} \left((A \setminus H) \cup \bigcup_{i=1}^{n-1} A_i \right)$, then the Superlemma 2.1.3 provides a slice of A of diameter less than ε, which is impossible. Thus $A = \overline{\text{conv}} \left((A \setminus H) \cup \bigcup_{i=1}^{n-1} A_i \right)$. By iterating this process we get $A = \overline{\text{conv}}(A \setminus H) = A \setminus H$ and so $A \cap H = \emptyset$, a contradiction.

The converse statement follows from the fact that $\alpha(D) \leq \text{diam}(D)$ for every $D \subset X$. $\square$

We denote by $\omega^{<\omega}$ the set of finite sequences of natural numbers. The length of a finite sequence s is denoted by $|s|$. Given $s, t \in \omega^{<\omega}$, we denote by $s \frown t$ the concatenation of s and t.

Proposition 2.2.2. *Let $C \subset X$ be a closed convex set. Then the following are equivalent:*

(i) *the set C has the RNP;*
(ii) *for every metric space (M, d), every continuous map $f \colon C \to M$ is dentable;*
(iii) *every Lipschitz function $f \colon C \to \mathbb{R}$ is dentable.*

Proof. Notice that C is a complete metric space as being a closed subset of X. Moreover, if C has the RNP then the identity $I \colon C \to C$ is dentable. Therefore (i)⇒(ii) follows from Corollary 2.1.11. Moreover, clearly (ii) implies (iii). Now, we use an argument from [CB] to prove (iii)⇒(ii). Assume that there is a bounded subset A of C which is not dentable. Then there exists $\varepsilon > 0$ such that any open slice of A has Kuratowski's index of non-compactness greater than ε. We will define a tree $T \subset \omega^{<\omega}$ and sequences $(x_s)_s \subset A$, $(\lambda_s)_s \subset [0, 1]$ and $(n_s)_s \in \mathbb{N}$ indexed in T satisfying

1) $\sum_{n=1}^{n_s} \lambda_{s \frown n} = 1$,

2) $\|x_s - \sum_{n=1}^{n_s} \lambda_{s\frown n} x_{s\frown n}\| < \frac{\varepsilon}{2^{|s|+2}}$, and
3) $\|x_t - x_s\| \geq \varepsilon$ for each t such that $|t| < |s|$

for each $s \in T$. In order to do that, choose $x_\emptyset \in A$ arbitrarily. Now, assume x_s has been previously defined. Then $x_s \in \overline{\mathrm{conv}}(A \setminus \cup_{|t|<|s|} B(x_t, \varepsilon))$. Indeed, otherwise there would be an open half-space H satisfying $x_s \in A \cap H \subset \cup_{|t|<|s|} B(x_t, \varepsilon)$ and thus $\alpha(A \cap H) < \varepsilon$, a contradiction. Thus, there exists $n_s \in \mathbb{N}$, $\lambda_{s\frown n} \geq 0$ and $x_{s\frown n} \in A$ satisfying above conditions.

In addition, a standard argument (see for instance Lemma 5.10 in [BL]) provides sequences $(y_s)_s \subset A$, $(\mu_s)_s \subset [0,1]$ and $(m_s)_s$ satisfying

1') $\sum_{m=1}^{m_s} \mu_{s\frown m} = 1$,
2') $y_s = \sum_{m=1}^{m_s} \mu_{s\frown m} y_{s\frown m}$, and
3') $\|x_s - y_s\| \leq \varepsilon/4$

for each $s \in T$. Thus,

$$\|y_t - y_{s\frown n}\| \geq \|x_t - x_{s\frown n}\| - \|x_t - y_t\| - \|x_{s\frown n} - y_{s\frown n}\| \geq \varepsilon/2 \tag{2.1}$$

whenever $t \leq s$.

Finally, take

$$O = \{y_s : |s| \text{ is odd}\}, \qquad E = \{y_s : |s| \text{ is even}\}$$

and notice that (2.1) implies that $d(O, E) \geq \varepsilon/2$. Consider the function $f\colon C \to \mathbb{R}$ given by $f(x) = d(x, O)$, which is a Lipschitz function. Then f is not dentable. Indeed, take $H \in \mathbb{H}$ satisfying $A \cap H \neq \emptyset$ and fix some $y_s \in A \cap H$. By condition 2' above, there is $m \leq m_s$ such that $y_{s\frown m} \in H$. Since either $y_s \in O, y_{s\frown m} \in E$ or $y_s \in E, y_{s\frown m} \in O$, it follows

$$\mathrm{diam}(f(O \cap H)) \geq |d(y_s, O) - d(y_{s\frown m}, O)| \geq \varepsilon/2$$

as we wanted. □

Next we study finitely dentable maps, which were introduced in [Raj5]. Let us recall the definition. For any dentable map $f\colon C \to M$ defined on a bounded closed convex set we may consider the derivation $[D]'_\varepsilon = [d \circ f, \mathbb{H}]'_\varepsilon(D)$. That is, $[D]'_\varepsilon$ is what remains once we have removed every slice of D where the oscillation of f is not larger than ε. The associated ordinal index is denoted by $\mathrm{Dz}(f)$. We say that f is *finitely dentable* if $\mathrm{Dz}(f) \leq \omega$, and we say that f is *countably dentable* if $\mathrm{Dz}(f) < \omega_1$ (equivalently, if $\mathrm{Dz}(f) \leq \omega_1$). Note that if C is separable then any dentable map defined on it is countably dentable.

Let us mention that any slice H which does not meet $[D]'_\varepsilon$ satisfies $\mathrm{diam}(f(D \cap H)) \leq 2\varepsilon$, that is, Lancien's midpoint argument (see, e.g. [Lan2]) applies also in the non-linear context. Indeed, assume that $x, y \in D$ satisfies that the segment $[x, y]$ does not meet $[D]'_\varepsilon$. Then we can consider the sets

$$A = \{z \in [x,y] : \exists H \in \mathbb{H}, [x,z] \subset H, \mathrm{diam}(f(D \cap H)) \leq \varepsilon\},$$
$$B = \{z \in [x,y] : \exists H \in \mathbb{H}, [z,y] \subset H, \mathrm{diam}(f(D \cap H)) \leq \varepsilon\}.$$

Note that A and B are relatively open in $[x, y]$. A connectedness argument provides a point $z \in A \cap B$, and thus $d(f(x), f(y)) \leq 2\varepsilon$. This fact was used repeatedly in [Raj5] to provide a characterisation of finitely dentable Lipschitz maps in terms of a renorming of the Banach space. Indeed, a slight modification of the proof of that theorem shows that the same result holds for uniformly continuous maps. This result will be useful in the next section.

Proposition 2.2.3. *Let $f\colon C \to M$ be a uniformly continuous map defined on a bounded closed convex set. Then the following are equivalent:*

(i) *the map f is finitely dentable;*
(ii) *there exists an equivalent norm $|||\ |||$ on X satisfying that* $\lim_n d(f(x_n), f(y_n)) = 0$ *whenever the sequences $(x_n)_{n=1}^{\infty}, (y_n)_{n=1}^{\infty} \subset C$ are such that*

$$\lim_{n\to\infty} 2|||x_n|||^2 + 2|||y_n|||^2 - |||x_n + y_n|||^2 = 0.$$

Proof. Let f be a finitely dentable uniformly continuous map. For each $\varepsilon > 0$ consider

$$\delta(\varepsilon) = \inf\{\|x - y\| : x, y \in C, d(f(x), f(y)) \geq \varepsilon\}$$

Let $N_k = \mathrm{Dz}(f, 2^{-k})$ and consider the 2-Lipschitz symmetric convex function F defined on X by the formula

$$F(x)^2 = \sum_{k=1}^{\infty}\sum_{n=1}^{N_k} \frac{2^{-k}}{N_k} d(x, [C]^n_{2^{-k}})^2 + \sum_{k=1}^{\infty}\sum_{n=1}^{N_k} \frac{2^{-k}}{N_k} d(x, -[C]^n_{2^{-k}})^2.$$

Now go through the same steps as in the proof of [Raj5, Theorem 2.2] to get that if $x, y \in C$ and $d(f(x), f(y)) > \varepsilon$, then

$$F\left(\frac{x+y}{2}\right)^2 \leq \frac{F(x)^2 + F(y)^2}{2} - \frac{\varepsilon\delta(\varepsilon/4)^2}{128\,\mathrm{Dz}(f, \varepsilon/8)^3}.$$

Therefore, an equivalent norm defined as in the proof of [Raj5, Theorem 2.2] does the work.

Conversely, assume that $|||\ |||$ is an equivalent norm satisfying the property in the statement. We may assume that $0 \in C$. Take $M = \sup\{|||x||| : x \in C\}$ and fix $\varepsilon > 0$. It is not difficult to show that there exists $\delta > 0$ such that $\mathrm{diam}(f(B_{|||\ |||}(0, r+\delta)) \cap H) < \varepsilon$ whenever $H \in \mathbb{H}$ does not intersect $B_{|||\ |||}(0, r)$. Thus, $[C]^n_{\varepsilon} \subset B_{|||\ |||}(0, M - n\delta)$ for each n, so $\mathrm{Dz}(f, \varepsilon) \leq M\delta^{-1}$. □

We will finish the section by studying the relation between the dentability of a map f with values in a normed space and the dentability of the function $\|f\|$. The following corollary was inspired by the absoluteness of difference convexity (see [BB, Theorem 2.9]).

Corollary 2.2.4. *Assume that $f\colon C \to \mathbb{R}$ is uniformly continuous on bounded sets. Then f is dentable if and only if $|f|$ is dentable.*

Proof. It is clear that $|f|$ is dentable whenever f is dentable. The converse statement follows from Proposition 2.1.15 and the fact that $f(x) \in \{|f(x)|, -|f(x)|\}$. $\square$

Notice that the above result fails when the modulus is replaced by the norm for dentable maps. Indeed, the identity map $I\colon c_0 \to c_0$ is not dentable whereas $\|\ \| \circ I = \|\ \|$ is dentable as being a continuous convex function which is bounded on bounded sets.

The next result shows that it is possible to construct such an example even if the target space is two-dimensional.

Proposition 2.2.5. *Assume that C is a bounded closed convex set which does not have the RNP. Then there exists a non-dentable Lipschitz map $f\colon C \to (\mathbb{R}^2, \|\ \|_1)$ such that $\|f(x)\|_1 = 1$ for every $x \in C$.*

Proof. Let $g\colon C \to \mathbb{R}$ be a non-dentable Lipschitz function, which exists by Proposition 2.2.2. We may assume that g is 1-Lipschitz and $g(0) = 0$. Then the function given by

$$f(x) = (\operatorname{diam}(C)^{-1} g(x), 1 - \operatorname{diam}(C)^{-1} g(x))$$

does the work. $\square$

2.3 Relation with $\mathscr{DC}$ functions and maps

Note that $\mathscr{D}(C, \mathbb{R})$ contains the bounded above lower semicontinuous convex functions. Indeed, for $A \subset C$ and $\varepsilon > 0$, any $H \in \mathbb{H}$ containing a point of A and disjoint from the convex closed set

$$D = \{x \in C : f(x) \leq \sup\{f, A\} - \varepsilon\}$$

will satisfy that $\operatorname{diam}(f(A \cap H)) \leq \varepsilon$. Thus, as a consequence of Theorem 2.1.13, the difference of two bounded convex uniformly continuous functions is dentable (indeed, a more general result holds, see Proposition 5.1 in [Raj5]).

Definition 2.3.1. A function $f\colon C \to \mathbb{R}$ is said to be $\mathscr{DC}$ (or delta-convex) if it can be represented as the difference of two convex continuous functions on C, and it is said to be *$\mathscr{DC}$-Lipschitz* (resp. *$\mathscr{DC}$-bounded*) if it is the difference of two convex Lipschitz (resp. bounded) functions.

Let us remark that an example of a Lipschitz $\mathscr{DC}$ function which is not $\mathscr{DC}$-Lipschitz is provided in [VZ3, Example 4.3]. On the other hand, the notion of $\mathscr{DC}$-bounded function only makes sense in the case in which the domain is bounded since a convex function defined on an unbounded convex set is necessarily unbounded. An example of a bounded $\mathscr{DC}$ function defined on B_{ℓ_2} which is not $\mathscr{DC}$-bounded is given in [KM, Lemma 10].

There is a large literature about $\mathscr{DC}$ functions and its applications in analysis and optimization. Indeed, optimization problems involving $\mathscr{DC}$ functions appear frequently in engineering, economics and other sciences. A typical example of $\mathscr{DC}$ function, due to

Asplund, is the square of the distance function to a closed subset A of a Hilbert space. Indeed, note that

$$d_A^2(x) = \inf_{y\in A} \|x-y\|^2 = \|x\|^2 - \sup_{y\in A}\{2\langle x,y\rangle - \|y\|^2\}.$$

The reader is referred to the surveys [BB, HU, Tuy] for relevant results, examples and applications of $\mathscr{DC}$ functions.

Let us remark that the space of $\mathscr{DC}$ functions on a convex set is not closed. Indeed, the space of $\mathscr{DC}$ functions is a lattice and so every continuous function on a norm-compact set can be uniformly approximated by $\mathscr{DC}$ functions, as a consequence of Stone–Weierstrass theorem. On the other hand, it was shown in [Fro] that for every infinite-dimensional Banach space there is a continuous function defined on the unit ball which cannot be uniformly approximated by $\mathscr{DC}$ functions uniformly on the unit ball. The situation is different if we restrict our attention to Lipschitz functions. Cepedello Boiso [CB] (see also [BV, Theorem 5.1.25] and [BL, Theorem 4.21]) characterised super-reflexive spaces as those Banach spaces in which every Lipschitz function defined on it can be approximated uniformly on bounded sets by $\mathscr{DC}$ functions which are Lipschitz on bounded sets. That result was extended by Raja in [Raj5] in the following way:

Theorem 2.3.2 (Raja). *A Lipschitz function $f\colon C \to \mathbb{R}$ can be uniformly approximated by $\mathscr{DC}$-Lipschitz functions on C if, and only if, it is finitely dentable.*

Our next aim is to generalise Theorem 2.3.2 for vector-valued maps. To this end, we will consider the notion of delta-convexity for maps which was introduced by Veselý and Zajíček in [VZ1] (see also [VZ2]). A priori, the more natural way of defining delta-convexity for a map $F\colon C \to Y$ seems to be requiring that $y^* \circ F$ is a $\mathscr{DC}$-function for every $y^* \in Y^*$. However, that notion is not suitable in order to extend some known stability results of $\mathscr{DC}$ functions to the vector-valued case. Instead, in [VZ1] it is considered the following:

Definition 2.3.3 (Veselý–Zajíček)**.** A continuous map $F\colon C \to Y$ defined on a convex subset $C \subset X$ is said to be a $\mathscr{DC}$ *map* if there exists a continuous (necesarily convex) function f on C such that $f + y^* \circ F$ is a convex continuous function on C for every $y^* \in S_{Y^*}$. The function f is called a *control function* for F.

Note that a real-valued continuous function $F\colon C \to \mathbb{R}$ is a $\mathscr{DC}$ function if and only if it is $\mathscr{DC}$ map. Indeed, if $F = g - h$, where g and h are continuous convex functions, it is clear that $g + h$ is a control function for F. On the other hand, if f is a control function for F then $f + F$ and $f - F$ are convex functions, so $F = \frac{1}{2}(f+F) - \frac{1}{2}(f-F)$ is a $\mathscr{DC}$ function.

Moreover, we say that a $\mathscr{DC}$ map is *$\mathscr{DC}$-Lipschitz* (resp. *$\mathscr{DC}$-bounded*) if it is Lipschitz (resp. bounded) and admits a Lipschitz (resp. bounded) control function. Clearly, if the domain is bounded then every $\mathscr{DC}$-Lipschitz map is also $\mathscr{DC}$-bounded. The space of $\mathscr{DC}$-bounded maps has been studied recently in [VZ2].

Our aim is to prove the following generalisation of Theorem 2.3.2 to the vector-valued case. Note also that the map is not assumed to be Lipschitz but uniformly continuous.

Theorem 2.3.4. *Let $F\colon C \to Y$ be a uniformly continuous map defined on a bounded closed convex set. The following statements are equivalent:*

(i) *F is finitely dentable and $\overline{F(C)}$ is norm-compact;*
(ii) *F is uniform limit of $\mathscr{DC}$-bounded maps with finitely-dimensional range;*
(iii) *F is uniform limit of $\mathscr{DC}$-Lipschitz maps with finitely-dimensional range.*

To prove the above result we need a number of auxiliary results. First, it is clear that every linear map is $\mathscr{DC}$, with control function 0. Moreover, it is easy to check that if $F\colon C \to Y$ is a $\mathscr{DC}$ map with control function f and $T\colon Y \to Z$ is a bounded linear operator, then $T \circ F$ is a $\mathscr{DC}$ map with control function $\|T\| \cdot f$. We also need the following result coming from [VZ1].

Lemma 2.3.5 (Veselý–Zajíček)**.** *Let $F\colon C \to \mathbb{R}^n$, $F = (F_1, \ldots, F_n)$. Then F is a $\mathscr{DC}$ (Lipschitz) map if and only if all $F_1, \ldots, F_n$ are $\mathscr{DC}$ (Lipschitz) functions.*

Proof. If F is $\mathscr{DC}$ then each F_i is also $\mathscr{DC}$ with the same control function since $F_i = T_i \circ F$, where $T_i(x_1, \ldots, x_N) = x_i$. Conversely, let $F_1, \ldots, F_N$ be controlled by $f_1, \ldots, f_N$. Given $x^* \in \mathbb{R}^N$ with $\|x^*\|_\infty = 1$, write $x^* = (x_1, \ldots, x_N)$ with $|x_i| \leq 1$. Then

$$x^* \circ F + (f_1 + \ldots f_N) = \sum_{i=1}^{N}(x_i F_i + f_i) = \sum_{i=1}^{N} |x_i|(\operatorname{sign}(x_i)F_i + f_i) + \sum_{i=1}^{N}(1 - |x_i|)f_i$$

is a continuous convex function and so F is controlled by $f_1 + \ldots + f_N$. □

The paper [VZ3] is devoted to the study of when the composition of $\mathscr{DC}$ maps is, or is not, a $\mathscr{DC}$ map. We need the following result, which is Proposition 1.3 in [VZ3]. It guarantees in particular that the composition of $\mathscr{DC}$-Lipschitz maps is $\mathscr{DC}$-Lipschitz. We denote $\|f\|_L$ the best Lipschitz constant of a map f.

Proposition 2.3.6 (Veselý–Zajíček)**.** *Let X, Y, Z be normed linear spaces, and let $A \subset X$ and $B \subset Y$ be convex sets. Let $F\colon A \to B$ and $G\colon B \to Z$ be $\mathscr{DC}$ maps with control functions $f\colon A \to \mathbb{R}$ and $g\colon B \to \mathbb{R}$, respectively. If G and g are Lipschitz on B, then $G \circ F$ is a $\mathscr{DC}$ map on A with a control function $h = g \circ F + (\|G\|_L + \|g\|_L)f$.*

It is well known that the identity map on a compact subset of a Banach space can be uniformly approximated by Lipschitz maps with finite-dimensional range (that is, every Banach space has the Lipschitz approximation property). In the next lemma we show that we can take the approximating map to be $\mathscr{DC}$.

Lemma 2.3.7. *Let K be a norm-compact subset of a Banach space Y. Then for every $\varepsilon > 0$ there is a $\mathscr{DC}$-Lipschitz map $\phi\colon K \to Y$ such that* $\operatorname{span}\phi(K)$ *is finite-dimensional and* $\sup\{\|\phi(x) - x\| : x \in K\} \leq \varepsilon$.

Proof. Let $y_1, \ldots, y_n \in K$ be such that $\{B(y_i, \varepsilon) : i = 1, \ldots, N\}$ is a finite covering of K. Consider the Lipschitz functions $\phi_i(x) := \max\{\varepsilon - \|x - y_i\|, 0\}$ for $i = 1, \ldots, n$ and

$h(x) = \sum_{j=1}^N \phi_i(x)$. Note that $h(x) \neq 0$ for every $x \in K$ and so $\inf\{h, K\} > 0$. Thus $\phi(x) \coloneqq \frac{1}{h(x)} \sum_{i=1}^N \phi_i(x) y_i$ defines a Lipschitz map with finite-dimensional range. Moreover ϕ satisfies that

$$\|\phi(x) - x\| = \left\| \sum_{\substack{i\in\{1,\dots,N\}\\ x\in B(y_i,\varepsilon)}} \phi_i(x)(y_i - x) \right\| \leq \sum_{\substack{i\in\{1,\dots,n\}\\ x\in B(y_i,\varepsilon)}} \phi_i(x)\varepsilon = \varepsilon$$

for each $x \in K$. It remains to check that ϕ is a $\mathscr{DC}$-Lipschitz map. To this end, note that $\phi = T \circ G \circ F$, where $F \colon K \to \mathbb{R}^N$ is given by $F(x) = (\phi_1(x), \dots, \phi_N(x))$, and $G \colon F(K) \to \mathbb{R}^N$ is given by

$$G(x_1, \dots, x_N) = \frac{1}{\sum_{i=1}^N x_i} (x_1, \dots, x_N)$$

and $T \colon \mathbb{R}^N \to Y$ is the linear map given by $T(x_1, \dots, x_N) = \sum_{i=1}^N x_i y_i$.

We only need to check that both F and G are $\mathscr{DC}$-Lipschitz maps and apply Proposition 2.3.6. First, note that each ϕ_i is a $\mathscr{DC}$-Lipschitz function since it is the maximum of two $\mathscr{DC}$-Lipschitz functions. Thus, Lemma 2.3.5 yields that F is $\mathscr{DC}$-Lipschitz. Moreover, again by Lemma 2.3.5, in order to show that G is a $\mathscr{DC}$-Lipschitz map we only need to check that each one of the functions $g_i(x_1, \dots, x_N) = \frac{x_i}{\sum_{j=1}^N x_j}$ is $\mathscr{DC}$-Lipschitz. Indeed, it is not difficult to check that each g_i is a convex function which is Lipschitz on every compact subset of $(0, +\infty)^N$, in particular it is a $\mathscr{DC}$-Lipschitz function on $F(K)$. This shows that F and G are $\mathscr{DC}$-Lipschitz maps, and the conclusion follows. $\square$

In [Raj5, Proposition 5.1] it is proved that every $\mathscr{DC}$-bounded function is finitely dentable. The following result says that the natural extension for vector-valued maps holds.

Proposition 2.3.8. *Let C be a closed convex set and $F \colon C \to Y$ be a $\mathscr{DC}$-bounded map such that $\overline{F(C)}$ is norm-compact. Then F is finitely dentable.*

Proof. First we prove the result for the case in which F has finite-dimensional range. Fix an isomorphism $T \colon \operatorname{span} F(C) \to (\mathbb{R}^N, \| \|_\infty)$ and write $T \circ F = (f_1, \dots, f_N)$. By Lemma 2.3.5, each f_i is a $\mathscr{DC}$-bounded function, and so finitely dentable by Proposition 5.1 in [Raj5]. Now, Proposition 3.2 in [Raj5] yields that $T \circ F$ is finitely dentable. Since $\operatorname{diam}(F(A)) \leq \|T^{-1}\| \operatorname{diam}((T \circ F)(A))$ for every $A \subset C$, it follows that F is finitely dentable.

Now, let $F \colon C \to Y$ be a $\mathscr{DC}$ map with relatively compact range and with bounded control function f. We will show that F is uniform limit of finitely dentable maps, and so it is finitely dentable. Fix $\varepsilon > 0$. By Lemma 2.3.7, we can find a $\mathscr{DC}$-Lipschitz map G such that $\operatorname{span} G(F(C))$ is finite-dimensional and $\sup\{\|(G \circ F - F)(x)\| : x \in C\} < \varepsilon$. By Proposition 2.3.6, $G \circ F$ is a $\mathscr{DC}$ map on C with control function $h = g \circ F + (\|G\|_L + \|g\|_L) f$, and thus it is $\mathscr{DC}$-bounded. Since $G \circ F$ has finite-dimensional range, it follows from what we have already proved that it is finitely dentable. Thus F is finitely dentable as being uniform limit of finitely dentable maps (Proposition 3.1 in [Raj5]). $\square$

Note that if X is not super-reflexive then the identity map $I\colon B_X \to X$ is a $\mathscr{DC}$-bounded map (with control function $f = 0$) which is not finitely dentable. This fact suggests that a hypothesis on the image of F is actually needed in Proposition 2.3.8. However, we do not know if norm-compactness can be replaced, for instance, by super-weak-compactness.

Now we are ready for the proof of Theorem 2.3.4.

Proof of Theorem 2.3.4. (iii)⇒(ii) is clear since every $\mathscr{DC}$-Lipschitz map defined on a bounded set is also $\mathscr{DC}$-bounded. Moreover, (ii)⇒(i) follows from the fact that a uniform limit of finitely dentable maps is finitely dentable and Proposition 2.3.8.

Now, assume that F is finitely dentable with relatively compact range and let us show that it is uniform limit of $\mathscr{DC}$-Lipschitz maps with finite-dimensional range. Fix $\varepsilon > 0$. By Lemma 2.3.7 there is $\phi\colon \overline{F(C)} \to Y$ Lipschitz so that $\phi(F(C))$ is finite-dimensional and $\sup\{\|(\phi \circ F - F)(x)\| : x \in C\} \leq \varepsilon$. Now fix an isomorphism $T\colon \operatorname{span}(F(C)) \to (\mathbb{R}^N, \|\ \|_\infty)$ and write $G \coloneqq T \circ \phi \circ F = (g_1, \ldots, g_N)$. Then each g_i is bounded as being a uniformly continuous function defined on a bounded set. Thus one can consider the sequence of functions given by

$$\begin{aligned} g_i^n(x) &\coloneqq \inf\{g_i(y) + n(2|||x|||^2 + 2|||y|||^2 - |||x+y|||^2) : y \in C\} \\ &= \ 2n|||x|||^2 - \sup\{n|||x+y|||^2 - 2n|||y|||^2 - g_i(x) : y \in C\} \end{aligned}$$

Note that each g_i^n is a $\mathscr{DC}$-Lipschitz function. Thus, by Lemma 2.3.5 $G_n = (g_1^n, \ldots, g_N^n)$ is a $\mathscr{DC}$ map which is also Lipschitz. We will show that the sequence $(G_n)_{n=1}^\infty$ converges to G uniformly. Given $\delta > 0$, take n_0 so that $\operatorname{diam}(G(C))/n_0 < \delta$. Fix $x \in C$, $i \in \{1, \ldots, N\}$ and $n \geq n_0$. First note that $g_i^n(x) \leq g_i(x)$. Moreover, if $y \in C$ is so that

$$g_i(y) + n(2|||x|||^2 + 2|||y|||^2 - |||x+y|||^2) \leq g_i(x)$$

then

$$2|||x|||^2 + 2|||y|||^2 - |||x+y|||^2 \leq \frac{g_i(x) - g_i(y)}{n} \leq \frac{\|G(x) - G(y)\|}{n} < \delta.$$

This yields that $g_i^n(x) \geq g_i(x) - \delta$ for each i. Therefore $\|G_n(x) - G(x)\|_\infty \leq \delta$ for each $x \in C$ and $n \geq n_0$.

Finally, it is clear from what we have done that we can find n such that $\|G_n - F\|_\infty \leq 2\varepsilon$. $\square$

We need to recall some definitions. A subset $D \subset X$ is said to be a $(\mathscr{C} \setminus \mathscr{C})_\sigma$*-set* if $D = \bigcup_{n=1}^\infty (A_n \setminus B_n)$, where A_n and B_n are convex closed subsets of X. A real function $f\colon C \to \mathbb{R}$ is said to be $(\mathscr{C} \setminus \mathscr{C})_\sigma$*-measurable* if the sets $f^{-1}(-\infty, r)$ and $f^{-1}(r, +\infty)$ are both $(\mathscr{C} \setminus \mathscr{C})_\sigma$ subsets of X for each $r \in \mathbb{R}$.

The following result summarises the connection between dentability and approximation by $\mathscr{DC}$ functions for uniformly continuous functions.

Theorem 2.3.9. *Let $f\colon C \to \mathbb{R}$ be a uniformly continuous function defined on a bounded closed convex set. Consider the following statements:*

(i) *f is finitely dentable;*
(ii) *f is uniform limit of $\mathscr{DC}$-bounded functions;*
(iii) *f is uniform limit of $\mathscr{DC}$-Lipschitz functions;*
(iv) *f is countably dentable;*
(v) *f is $(\mathscr{C} \setminus \mathscr{C})_\sigma$-measurable;*
(vi) *f is pointwise limit of $\mathscr{DC}$-Lipschitz functions.*

Then (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (vi) $\not\Rightarrow$ (v) $\not\Rightarrow$ (iv) $\not\Rightarrow$ (iii).

Proof. First we prove the positive statements. The equivalence among (i), (ii) and (iii) follows from Theorem 2.3.4. Moreover, (i)$\Rightarrow$(iv) is trivial and (v)$\Rightarrow$(vi) is [Raj3, Corollary 2.7].

Finally we will prove (iv)$\Rightarrow$(v). If $V \subset \mathbb{R}$ is open then it is not difficult to check that

$$f^{-1}(V) = \bigcup_{n=1}^{\infty} \bigcup_{\alpha<\mathrm{Dz}(f,n^{-1})} \{x \in [C]^\alpha_{n^{-1}} : \exists H \in \mathbb{H} \text{ such that } x \in H \text{ and } f([C]^\alpha_{n^{-1}} \cap H) \subset V\},$$

which is a representation of the set as countable union of $(\mathscr{C} \setminus \mathscr{C})$-sets, as union of open slices of a closed convex set.

Now we turn to the negative statements. Remark 2.2 in [Raj3] shows that the characteristic function of the Cantor set is pointwise limit of $\mathscr{DC}$-Lipschitz functions but not $(\mathscr{C} \setminus \mathscr{C})_\sigma$-measurable and thus (vi)$\not\Rightarrow$(v). Moreover, let X be a separable Banach space without the RNP and let $f\colon B_X \to \mathbb{R}$ be a uniformly continuous not dentable function, which exists by Proposition 2.2.2. As a consequence of [Raj3, Theorem 1.2], every norm open subset of X is a $(\mathscr{C} \setminus \mathscr{C})_\sigma$-set and thus f is $(\mathscr{C} \setminus \mathscr{C})_\sigma$-measurable, so (v)$\not\Rightarrow$(iv). Finally, in order to show that (iv)$\not\Rightarrow$(iii), let X be a separable non-superreflexive Banach space with the RNP, e.g. $X = \ell_1$. By Cepedello's theorem, there exists a Lipschitz function $f\colon B_X \to \mathbb{R}$ which is not finitely dentable. However, Proposition 2.2.2 and the separability of X imply that f is countably dentable. $\square$

In the next result we characterise the dentability of $\mathscr{DC}$-bounded maps in terms of the dentability of their target space. To this end we use the following fact proved in [VZ1]: a function f is a control function for a map F if, and only if,

$$\left\| \sum_{i=1}^{n} \lambda_i F(x_i) - F\left(\sum_{i=1}^{n} \lambda_i x_i\right) \right\| \leq \sum_{i=1}^{n} \lambda_i f(x_i) - f\left(\sum_{i=1}^{n} \lambda_i x_i\right)$$

whenever $x_1, \dots, x_n \in A$, $\lambda_1, \dots, \lambda_n \geq 0$ and $\sum_{i=1}^n \lambda_i = 1$. We will denote by $\mathbb{E}(f|\mathscr{F})$ the conditional expectation of a function f given a σ-algebra $\mathscr{F}$.

Theorem 2.3.10. *Let $D \subset Y$ be a closed convex set. Then the following are equivalent:*

(i) *the set D has the RNP;*

(ii) *for every Banach space X and every convex subset $C \subset X$, every $\mathscr{DC}$-bounded map $F\colon C \to D$ is dentable.*

Proof. First, assume that D does not have the RNP. Let $A \subset D$ be a non-dentable bounded subset of D and take $C = \overline{\mathrm{conv}}(D)$. Then the identity $I\colon C \to D$ is a non-dentable bounded continuous $\mathscr{DC}$ map with the zero function as control function. Thus (ii) implies (i).

Now take a bounded continuous $\mathscr{DC}$ map $F\colon C \to D$ with a bounded control function f. Given $\varepsilon > 0$ and $x \in C$, we will denote

$$\delta(x,\varepsilon) = \inf\{\|x-y\| : y \in C, \max\{\|F(x)-F(y)\|, |f(x)-f(y)|\} > \varepsilon\}.$$

Assume that F is not dentable. Then by Proposition 2.1.1 there exist $\varepsilon > 0$ and $A \subset C$ bounded such that $x \in \overline{\mathrm{conv}}(A \setminus F^{-1}(B(F(x),\varepsilon)))$ for each $x \in A$. We will construct a martingale $(h_n)_n$ with values in $F(A) \subset D$ and so that $\|h_n - h_{n+1}\| \geq \varepsilon/2$. In order to do that, we will define inductively an increasing sequence $(\mathscr{F}_n)_n$ of σ-algebras in the interval $[0,1]$ and a sequence $(g_n)_{n=1}^{\infty}$ of functions from $[0,1]$ to A satisfying the following conditions for each $n \in \mathbb{N}$:

1) $\mathscr{F}_n$ is the σ-algebra generated by a finite partition π_n of the unit interval into disjoint subintervals;

2) g_n is $\mathscr{F}_n$-measurable;

3) $\|F(g_n(t)) - F(g_{n+1}(t))\| \geq \varepsilon$ for each $n \in \mathbb{N}$ and $t \in [0,1]$;

4) $\|F \circ g_n - \mathbb{E}(F \circ g_{n+1}|\mathscr{F}_n)\|_1 \leq \int_0^1 (f \circ g_{n+1} - f \circ g_n) d\,\mathrm{m} + \frac{\varepsilon}{2^{n+3}}$, where m denotes the Lebesgue measure on $[0,1]$.

Fix $x_0 \in A$ so that $f(x_0) \geq \sup\{f, A\} - \frac{\varepsilon}{16}$, take $\pi_0 = \{[0,1]\}$ and define $g_0(t) = x_0$ for each $t \in [0,1]$. Assume we have defined $\pi_n = \{A_1, \ldots, A_p\}$, $\mathscr{F}_n = \sigma(\pi_n)$ and a $\mathscr{F}_n$-measurable map $g_n\colon [0,1] \to A$. Then there exist $x_1, \ldots, x_p \in A$ such that $g_n = \sum_{i=1}^p x_i \chi_{A_i}$. Now, the non-dentability of F on A implies that there are integers k_i, $\lambda_{ij} \geq 0$, and $x_{ij} \in A \setminus F^{-1}(B(F(x_i),\varepsilon)$ for $j \in \{1,\ldots,k_i\}$ satisfying that $\sum_{j=1}^{k_i} \lambda_{ij} = 1$ and $\left\|\sum_{j=1}^{k_i} \lambda_{ij} x_{ij} - x_i\right\| \leq \delta(x_i, 2^{-n-5}\varepsilon)$. For each i, let $\{A_{ij}\}$ be a partition of A_i into disjoint subintervals with $\mathrm{m}(A_{ij}) = \lambda_{ij}\,\mathrm{m}(A_i)$. Take $\pi_{n+1} = \{A_{ij}\}_{ij}$ and $\mathscr{F}_{n+1} = \sigma(\pi_{n+1})$. Finally, define $g_{n+1} = \sum_{i=1}^p \sum_{k=1}^{k_i} x_{ij} \chi_{A_{ij}}$. Clearly, g_{n+1} is $\mathscr{F}_{n+1}$-measurable and takes values on $F(A)$. Moreover, $\|F(g_n(t)) - F(g_{n+1}(t))\| \geq \varepsilon$ for each $t \in [0,1]$ since $x_{ij} \in A \setminus F^{-1}(B(F(x_i),\varepsilon)$ for each i,j. By using the fact that f is a control function for F and the definition of δ we get that

$$\begin{aligned}\left\|\sum_{j=1}^{k_i}\lambda_{ij}F(x_{ij})-F(x_i)\right\| &\le \left\|\sum_{j=1}^{k_i}\lambda_{ij}F(x_{ij})-F(\sum_{j=1}^{k_i}\lambda_{ij}x_{ij})\right\|+\frac{\varepsilon}{2^{n+5}}\\ &\le \left(\sum_{j=1}^{k_i}\lambda_{ij}f(x_{ij})-f(\sum_{j=1}^{k_i}\lambda_{ij}x_{ij})\right)+\frac{\varepsilon}{2^{n+5}}\\ &\le \left(\sum_{j=1}^{k_i}\lambda_{ij}f(x_{ij})-f(x_i)\right)+\frac{\varepsilon}{2^{n+4}}\,.\end{aligned}$$

In addition, it is easy to show that $\mathbb{E}(F\circ g_{n+1}|\mathscr{F}_n)=\sum_{i=1}^{n}(\sum_{j=1}^{k_i}\lambda_{ij}F(x_{ij}))\chi_{A_i}$. Thus, we can estimate $\|F\circ g_n-\mathbb{E}(F\circ g_{n+1}|\mathscr{F}_n)\|_1$ as follows:

$$\begin{aligned}\|F\circ g_n-\mathbb{E}(F\circ g_{n+1}|\mathscr{F}_n)\|_1 &\le \sum_{i=1}^{p}\mathrm{m}(A_i)\left\|F(x_i)-\sum_{j=1}^{k_i}\lambda_{ij}F(x_{ij})\right\|\\ &\le \sum_{i=1}^{p}\mathrm{m}(A_i)\left(\sum_{j=1}^{k_i}\lambda_{ij}f(x_{ij})-f(x_i)\right)+\frac{\varepsilon}{2^{n+4}}\\ &= \sum_{i=1}^{p}\sum_{j=1}^{k_i}\mathrm{m}(A_{ij})f(x_{ij})-\sum_{i=1}^{p}\mathrm{m}(A_i)f(x_i)+\frac{\varepsilon}{2^{n+4}}\\ &= \int_0^1 f\circ g_{n+1}d\,\mathrm{m}-\int_0^1 f\circ g_n d\,\mathrm{m}+\frac{\varepsilon}{2^{n+4}}\,.\end{aligned}$$

This shows that conditions 1) to 4) above are satisfied. Finally, from condition 4) and the fact that $f(x_0)\ge\sup\{f,A\}-\frac{\varepsilon}{16}$ we get that

$$\sum_{n=0}^{N}\|F\circ g_n-\mathbb{E}(F\circ g_{n+1}|\mathscr{F}_n)\|_1\le\int_0^1(f\circ g_{N+1}-f\circ g_0)d\,\mathrm{m}+\sum_{n=0}^{N}\frac{\varepsilon}{2^{n+4}}\le\frac{\varepsilon}{16}+\frac{\varepsilon}{8}$$

for each $N\in\mathbb{N}$. It follows that

$$\sum_{n=0}^{\infty}\|F\circ g_n-\mathbb{E}(F\circ g_{n+1}|\mathscr{F}_n)\|_1<\frac{\varepsilon}{4}\,.$$

Thus, one can apply Lemma 5.10 in [BL] to get a martingale $(h_n)_{n=1}^{\infty}$ with values in the bounded set $F(A)\subset D$ so that $\|h_n-F\circ g_n\|_1\le\varepsilon/4$. Therefore

$$\|h_n-h_{n+1}\|_1\ge\|F\circ g_n-F\circ g_{n+1}\|_1-\|h_n-F\circ g_n\|_1-\|h_{n+1}-F\circ g_{n+1}\|_1\ge\frac{\varepsilon}{2}\,,$$

which contradicts the assumption that D has the Radon-Nikodým property. $\square$

2.4 Stegall's variational principle for closed dentable functions

We say that a function $f\colon C \to \mathbb{R}$ has a *strong minimum* at $x_0 \in C$ if $f(x_0) = \inf\{f, C\}$ and $\|x_n - x\| \to 0$ for every sequence $(x_n)_{n=1}^{\infty} \subset C$ such that $f(x_n) \to f(x_0)$. Note that if f is a linear functional then f has a strong minimum at x_0 if and only if x_0 is strongly exposed by f.

Let us recall that Stegall's variational principle (see, e.g. [BL, Theorem 5.17]) states that a lower semicontinuous bounded below function f defined on a set with the RNP can be modified by adding an arbitrarily small linear perturbation x^* in such a way that the resulting function $f + x^*$ admits a strong minimum. More precisely,

Theorem 2.4.1 (Stegall)**.** *Let C be a closed convex bounded subset of a Banach space. Assume that C has the RNP and that $h\colon C \to \mathbb{R} \cup \{+\infty\}$ is proper lower semicontinuous bounded below function. Then for every $\varepsilon > 0$ there is $x^* \in \varepsilon B_{X^*}$ such that $h + x^*$ has a strong minimum on C.*

Our aim is to find an analogous of Stegall's theorem 2.4.1 where the dentability of the set C is replaced by the dentability of the function defined on it. To this end, we introduce the following notion of strong minimum associated to a function f.

Definition 2.4.2. Let $f\colon C \to M$ and $h\colon C \to \mathbb{R} \cup \{+\infty\}$. We say that a point $x_0 \in C$ is an *f-strong minimum* of h if

$$h(x_0) = \inf\{h, C\}$$

and for any sequence $(x_n)_{n=1}^{\infty}$ in C we have

$$\lim_{n\to\infty} f(x_n) = f(x_0)$$

whenever $\lim_{n\to\infty} h(x_n) = h(x_0)$.

The following result is the announced version of Stegall's variational principle for dentable maps. Unfortunately, we need to assume the dentable map is *closed*, that is, the image of each closed set is closed.

Theorem 2.4.3. *Let C be a closed convex bounded subset of a Banach space, (M, d) be a complete metric space and $f \in \mathscr{D}_U(C, M)$ be a closed map. Suppose that $h\colon C \to \mathbb{R} \cup \{+\infty\}$ is proper lower semicontinuous bounded below function. Then for every $\varepsilon > 0$ there is $x^* \in \varepsilon B_{X^*}$ such that $h + x^*$ has an f-strong minumum on C.*

The first part of the proof will follow the same steps as the proof of Stegall's theorem presented in [Phe]. That is, we will show that there exists an arbitrarily small perturbation x^* such that the f-diameter of the set of approximate minima is small (Proposition 2.4.5). Then we will consider a sequence of set-valued maps which contains the information about the approximate minima. The key point of the argument will be the use of a result by

Lassonde and Revalski (Theorem 2.4.7) establishing that, for a dense $\mathscr{G}_\delta$ subset of elements of X^*, such sequence of set-valued maps has non-empty intersection which corresponds to the image of a strong minimum of the perturbed functional.

Let us start with a technical lemma that is a version of Lemma 5.17 in [Phe] with identical proof.

Lemma 2.4.4. *Let C be a closed convex bounded subset of a Banach space X and let $f\colon C \to (M,d)$ be a uniformly continuous map. Suppose $(A_n)_{n=1}^\infty$ is a sequence of (eventually) nonempty subsets of C with the following property: there exist $\varepsilon > 0$ and $\lambda > 0$ such that for all $y \in C$ and $n \in \mathbb{N}$ we have*

$$A_n \subset \operatorname{conv}(A_{n+1} \setminus f^{-1}(B_d(f(y),\varepsilon))) + \frac{\lambda}{2^n} B_X.$$

Then f is not dentable.

Proof. Consider the set

$$A = \bigcap_{n=1}^\infty \overline{\bigcup_{j\geq n} \operatorname{conv}(A_j)}.$$

We will prove that $x \in \overline{\operatorname{conv}}(A \setminus f^{-1}(B(f(x),\varepsilon/2)))$ for every $x \in A$ and so f is not dentable by Proposition 2.1.1. First we show that A is nonempty and $\operatorname{conv}(A_n) \subset A + (4\lambda/2^n)B_X$. For that, fix $n \in \mathbb{N}$ so that A_n is nonempty and take $x_0 \in A_n$. Define inductively a sequence $(x_k)_{k=0}^\infty$ so that $x_k \in \operatorname{conv}(A_{n+k})$ and $\|x_k - x_{k+1}\| \leq \frac{\lambda}{2^{n+k}}$ for each k. Thus, the series $\sum_{k=0}^\infty (x_k - x_{k+1})$ converges to some $y \in X$ of norm at most $4\lambda/2^n$. Now consider $y = x_0 - z$, where $z = \lim_{k\to\infty} x_k \in C$. It follows that $z \in A$ and $x_0 \in A + \frac{4\lambda}{2^n} B_X$, which proves the claim.

Now, suppose $x \in A$. Take $\delta > 0$ so that $d(f(x), f(y)) < \varepsilon/2$ whenever $x, y \in C$, $\|x - y\| < \delta$. Fix m such that $4\lambda/2^m < \min\{\varepsilon/2, \delta\}$. Since $x \in \overline{\bigcup_{j\geq n} \operatorname{conv}(A_j)}$ for all n, for each $n \geq m$ there exists $j \geq n$ and $y_n \in \operatorname{conv}(A_j)$ such that $\|x - y_n\| \leq \lambda/2^n$. By hypothesis, there is $z_n \in \operatorname{conv}(A_{j+1} \setminus f^{-1}(B_d(f(x),\varepsilon)))$ such that $\|y_n - z_n\| \leq 2\lambda/2^j \leq 2\lambda/2^n$. We can write z_n as a finite convex combination $z_n = \sum_i \lambda_i u_i$, $u_i \in A_{j+1} \setminus f^{-1}(B_d(f(x),\varepsilon))$. Since $u_i \in A + \frac{4\lambda}{2^j} B_X \subset A + \frac{4\lambda}{2^n} B_X$ for each i, there exists $v_i \in A$ such that $\|u_i - v_i\| < 4\lambda/2^n$. Let $w_n = \sum_i \lambda_i v_i$. It follows that $\|z_n - w_n\| \leq 4\lambda/2^n$ and so $\|x - w_n\| \leq 7\lambda/2^n < \varepsilon$. Moreover, since $\|u_i - v_i\| \leq \delta$ we have

$$d(f(v_i), f(x)) \geq d(f(u_i), f(x)) - d(f(u_i), f(v_i)) > \varepsilon - \varepsilon/2 = \varepsilon/2,$$

that is, $w_n \in \operatorname{conv}(A \setminus f^{-1}(B_d(f(x),\varepsilon/2)))$. Therefore, $x \in \overline{\operatorname{conv}}(A \setminus f^{-1}(B_d(f(x),\varepsilon/2)))$ as desired. $\square$

Given a function $h\colon C \to \mathbb{R} \cup \{+\infty\}$ and $\alpha > 0$, we will denote

$$\underline{S}(C,h,\alpha) = \{x \in C : h(x) \leq \inf\{h, C\} + \alpha\}.$$

The following result is based on the proof of Theorem 5.15 in [Phe].

Proposition 2.4.5. *Let C be a closed convex bounded subset of a Banach space X and $f \in \mathscr{D}_U(C, M)$. Assume that $h\colon C \to \mathbb{R} \cup \{+\infty\}$ is a proper lower semicontinuous bounded below function. Then for every $\varepsilon > 0$ there is $x^* \in \varepsilon B_{X^*}$ and $\alpha > 0$ such that*

$$\operatorname{diam} f(\underline{S}(C, h + x^*, \alpha)) < \varepsilon.$$

Proof. Assume that is not the case. Then there is $\varepsilon > 0$ so that for every $x^* \in \varepsilon B_{X^*}$ and $\alpha > 0$ we have

$$\operatorname{diam} f(\underline{S}(C, h + x^*, \alpha)) > 2\varepsilon$$

For each n let

$$A_n = \bigcup_{\|x^*\| \leq \varepsilon - 2^{-n}} \underline{S}(C, h + x^*, 4^{-n}),$$

which is nonempty for n large enough. Let $\lambda = 5/2$. We will show that the sequence $(A_n)_{n=1}^{\infty}$ satisfies the hypothesis of Lemma 2.4.4 and so f is not dentable, a contradiction. To this end, fix n and take $x \in A_n$. Assume that there is $y \in C$ such that x does not belong to $\operatorname{conv}(A_{n+1} \setminus f^{-1}(B(f(y), \varepsilon))) + \frac{\lambda}{2^n} B_X$. By Hahn-Banach there is $y^* \in S_{X^*}$ such that

$$y^*(x) \leq \sup\{y^*, A_{n+1} \setminus f^{-1}(B(f(y), \varepsilon))\} + \frac{\lambda}{2^n}.$$

Since $x \in A_n$, there exists $x^* \in X^*$ with $\|x^*\| \leq \varepsilon - 2^{-n}$ such that $x \in \underline{S}(C, h + x^*, 1/4^n)$. Consider $z^* = x^* + 2^{-n-1} y^*$. Then

$$\|z^*\| \leq \varepsilon - 2^{-n} + 2^{-n-1} = \varepsilon - 2^{-n-1}$$

and so $\underline{S}(C, h + z^*, 4^{-n-1}) \subset A_{n+1}$. Since $\operatorname{diam} f(\underline{S}(C, h + z^*, 4^{-n-1})) > 2\varepsilon$, this slice is not contained in $f^{-1}(B(f(y), \varepsilon))$. This implies that there is $z \in A_{n+1} \setminus f^{-1}(B(f(y), \varepsilon))$ so that

$$(h + z^*)(z) \leq \inf\{h + z^*, C\} + 4^{-n-1} \leq (h + z^*)(x) + 4^{-n-1}.$$

Moreover, the choice of y^* gives that

$$y^*(x) \leq y^*(z) + \frac{\lambda}{2^n}.$$

We will show that this leads to a contradiction. First, since $x \in \underline{S}(C, h + x^*, 4^{-n})$ and $z \in C$ we have

$$(h + x^*)(x) \leq (h + x^*)(z) + 4^{-n}.$$

Therefore,

$$\begin{aligned}
(h + x^*)(z) + 2^{-n-1} y^*(z) &= (h + z^*)(z) \leq (h + z^*)(x) + 4^{-n-1} \\
&= (h + x^*)(x) + 2^{-n-1} y^*(x) + 4^{-n-1} \\
&\leq (h + x^*)(z) + 4^{-n} + 2^{-n-1} y^*(x) + 4^{-n-1},
\end{aligned}$$

and so

$$2^{-n-1} y^*(z - x) < 4^{-n} + 4^{-n-1} = 5/4^{n+1} = 5/2^{2n+2}.$$

That is, $y^*(z - x) < \lambda/2^n$, which is impossible. $\square$

The following notions were introduced in [LR] in order to find a unified proof of several variational principles.

We will consider a decreasing sequence $(T_n\colon Z \rightrightarrows M)_{n=1}^{\infty}$ of nonempty-valued mappings between a topological space Z and a metric space (M,d). That is, $\emptyset \neq T_{n+1}(z) \subset T_n(z)$ for every $z \in Z$ and $n \in \mathbb{N}$. Given such a sequence, we will denote by $T(z) \coloneqq \bigcap_{n=1}^{\infty} T_n(z)$, $z \in Z$, the limit mapping generated by $(T_n)_{n=1}^{\infty}$. Note that T may have empty values.

Definition 2.4.6. We say that the sequence $(T_n\colon Z \rightrightarrows (M,d))_{n=1}^{\infty}$ is *fragmented by the metric* d if for every nonempty open subset U of Z and $\varepsilon > 0$ there are $n \in \mathbb{N}$ and a nonempty open set $U' \subset U$ so that $\operatorname{diam} T_n(U') < \varepsilon$.

The following is part of Theorem 2.2 in [LR].

Theorem 2.4.7 (Lassonde–Revalski)**.** *Let $(T_n)_{n=1}^{\infty}$ be a decreasing sequence of mappings between a Baire topological space Z and a complete metric space (M,d) such that $T_n(z)$ is a non-empty closed subset of M for every $z \in Z$ and $n \in \mathbb{N}$. Suppose that d fragments $(T_n)_{n=1}^{\infty}$. Then there is a dense $\mathscr{G}_\delta$ subset Z_1 of Z such that for any point $z \in Z_1$ the limit mapping T satisfies that $T(z)$ is a singleton, and moreover, for any $\varepsilon > 0$ there exist an open set $U \ni z$ and $n \in \mathbb{N}$ with $T_n(U) \subset B_d(T(z),\varepsilon)$.*

Now we are ready for the proof of the version of Stegall's variational principle for dentable maps. To this end, we will mimic the proof of Proposition 3.1 in [LR].

Proof of Theorem 2.4.3. For each $x^* \in X^*$ and $n \in \mathbb{N}$, consider

$$T_n(x^*) = f(\underline{S}(C, h + x^*, 1/n)).$$

Clearly $T_n(x^*)$ is non-empty. Note that $\underline{S}(C, h + x^*, 1/n)$ is a closed subset of X^* due to the lower semicontinuity of h and so $T_n(x^*)$ is closed in M. We claim that $(T_n)_{n=1}^{\infty}$ is fragmented by d. Indeed, let U be a non-empty open subset of X^* and $\eta > 0$. Take $x_0^* \in U$ and $\gamma > 0$ with $B_{X^*}(x_0^*, \gamma) \subset U$. Consider $\eta' = \min\{\eta, \gamma/2\}$. By Proposition 2.4.5, there is $x^* \in \eta' B_{X^*}$ and $\alpha > 0$ with

$$\operatorname{diam} f(\underline{S}(C, h + x_0^* + x^*, \alpha)) < \eta'.$$

Now, we can take $\delta < \gamma/2$ so that

$$\underline{S}(C, h + y^*, \alpha/2) \subset \underline{S}(C, h + x_0^* + x^*, \alpha)$$

for any $y^* \in B_{X^*}(x_0^* + x^*, \delta)$. Thus, $\operatorname{diam}(T_n(U')) < \eta$ for $U' = B_{X^*}(x_0^* + x^*, \delta) \subset U$ and n such that $1/n < \alpha/2$. This proves that $(T_n)_{n=1}^{\infty}$ is fragmented by d. Therefore, by Theorem 2.4.7, there is $x^* \in \varepsilon B_{X^*}$ so that $T(x^*)$ is a singleton, say $f(x_0)$, and moreover, for every $\delta > 0$ there is n such that $T_n(x^*) \subset B_d(f(x_0), \delta)$. From the definition of the sequence $(T_n)_n$ we get that $h + x^*$ attains its minimum on x_0. Moreover, we claim that x_0 is an f-strong minimum of $h + x^*$. Indeed, assume that $\lim_n (h + x^*)(x_n) = (h + x^*)(x_0)$. Fix $\delta > 0$ and take n so that $T_n(x^*) \subset B_d(f(x), \delta)$. Clearly, there is m_0 such that $f(x_m) \in T_n(x^*)$ for $m \geq m_0$. Therefore, $d(f(x_m), f(x_0)) \leq \operatorname{diam}(T_n(x^*)) \leq 2\delta$ as desired. $\square$

We should point out that Theorem 2.4.3 does not provide any additional information when the map f is an injective bounded linear operator since in that case f is closed if and only if it is an isomorphism by the Open Mapping Theorem.

2.5 Dentable sets with respect to a metric

Notice that, given a uniformly continuous injective map $f\colon C \to (M, d)$, we have that f is dentable if and only if C is dentable with respect to the metric $\rho := d \circ f$. Moreover, ρ is a uniformly continuous metric on C, that is, for every $\varepsilon > 0$ there exists $\delta > 0$ so that $d(x, y) < \varepsilon$ whenever $x, y \in C$, $\|x - y\| < \delta$.

Proposition 2.5.1. *If $C \subset X$ is a bounded closed convex subset that admits a separating sequence, then it is dentable with respect to some norm-Lipschitz metric defined on it.*

Proof. Fix a sequence $(x_n^*)_{n=1}^{\infty} \subset B_{X^*}$ which is separating on C and define

$$d(x, y) = \sum_{n=1}^{\infty} 2^{-n} |x_n^*(x - y)|.$$

Clearly, d is a metric on C and it is norm Lipschitz. The dentability of C with respect to d follows from Corollary 2.1.10. Indeed, fix $A \subset C$ nonempty and $\varepsilon > 0$. Take $n \in \mathbb{N}$ such that $2^{-n} < \varepsilon$ and find a slice S of C such that $|x_k^*(x - y)| < \varepsilon/2$ for all $k \leq n$ and $x, y \in S$. Then $\operatorname{diam}(S) < \varepsilon$ with respect to d. $\square$

Geometrically, Stegall's variational principle is the consequence of the existence of strongly exposed points of the epigraph of f and of many strongly exposing functionals, as x^*. The pathological fact in our frame is that the slices associated to a strongly slicing functional could be convergent to a non strongly exposed point. Indeed, consider the following example. Let $C = B_X$, where $X = Z^*$ and Z is a separable Banach space. By Proposition 2.5.1, C is dentable for a metric d compatible with the weak* topology induced by the convergence on Z. Moreover, d is a complete metric since C is weak*-compact. Thus, if $x^* \in \mathscr{S}\mathscr{S}(I, C)$ then $S(A, x^*, t)$ converges to a point $x \in C$ when $t \to 0^+$. However, we can not ensure that $x \in \bigcap_{t>0} S(A, x^*, t)$ since in general x^* is not continuous with respect to d. Adding such a hypothesis would send us back to the classical case, as Theorem 2.5.2 and Proposition 2.5.3 show. We can deduce this result from Theorem 2.4.3 and the fact that Stegall's variational principle actually characterises sets with the RNP. However, we prefer to give a more direct proof.

Theorem 2.5.2. *Let C be a closed convex subset which is dentable with respect to a complete metric d defined on it. Assume moreover that d is uniformly continuous on bounded sets with respect to the norm and induces the norm topology. Then C has the RNP.*

Proof. Let $I: C \to (C, d)$ denote the identity map, which by assumption is uniformly continuous on bounded sets. Given $A \subset C$ be nonempty and bounded, Theorem 2.1.7 provides a I-strongly slicing functional x^*. Moreover, since d induces the norm topology, $\operatorname{diam}(\overline{S(A, x^*, t)}) = \operatorname{diam}(S(A, x^*, t))$ for each $t > 0$. Thus, the completeness of d implies that $\bigcap_{t>0} \overline{S(C, x^*, t)}$ consists exactly on one point $y \in C$. Given $\varepsilon > 0$, the coincidence of the norm topology and the one induced by d provides $\delta > 0$ such that $B_d(y, \delta) \subset B_{\| \|}(y, \varepsilon)$. Now, if $t > 0$ is small enough then $S(A, x^*, t)$ is contained in $B_d(y, \delta)$ and therefore the norm-diameter of $S(C, x^*, t)$ is less than 2ε. Thus A is norm-dentable and we are done. $\square$

Let us remark that it is possible to relax the hypothesis of coincidence of topologies in the previous result.

Proposition 2.5.3. *Assume that C is dentable with respect to a metric d which is uniformly continuous on bounded sets with respect to the norm. Consider $I: C \to (C, d)$ the identity map. If for every $A \subset C$ nonempty closed convex bounded and every $x^* \in \mathscr{SS}(I, A)$ the set $\bigcap_{t>0} \overline{S(A, x^*, t)}^{\| \|}$ is nonempty, then C has the RNP.*

Proof. Let $A \subset C$ be nonempty closed convex bounded. Given $x^* \in \mathscr{SS}(I, A)$, take $x \in \bigcap_{t>0} \overline{S(A, x^*, t)}^{\| \|}$. Then $x^*(x) \geq \sup\{x^*, A\} - t$ for each $t > 0$ and thus x^* supports A at x. It follows that the set of support functionals of A contains a dense $\mathscr{G}_\delta$ in X^*. The Bourgain–Stegall theorem, see Theorem 0.1.14.(c), implies that C has the RNP. $\square$

Finally, the following proposition provides an example of a bounded closed convex set which is not dentable with respect to any translation invariant metric. Let us recall that a theorem of Kakutani (see e.g. [Rol]) asserts that a metric on a vector space for which addition and scalar multiplication are continuous is equivalent to a translation-invariant metric.

Proposition 2.5.4. *Let C be the unit closed ball of $c_0(\Gamma)$, where Γ is uncountable, and let d be a translation-invariant metric on C. Then C is not dentable with respect to d.*

Proof. Denote by $\{e_\gamma\}_{\gamma \in \Gamma}$ the standard basis of $c_0(\Gamma)$. Fix $\varepsilon > 0$ and take $x^* = (x^*_\gamma)_{\gamma \in \Gamma} \in S_{\ell_1(\Gamma)}$ providing a slice $S = \{x \in C : x^*(x) > t\}$ with $\operatorname{diam}(S) < \varepsilon$. Fix a point $x \in S$. Then there exists a finite subset $A_\varepsilon \subset \Gamma$ such that $\sum_{\gamma \notin A_\varepsilon} |x^*_\gamma| < x^*(x) - t$. Thus, given $\eta \in \Gamma \setminus A_\varepsilon$ we have

$$x^*(x \pm e_\eta) \geq x^*(x) - |x^*_\eta| > t$$

and so $x \pm e_\eta \in S$. Therefore, $d(e_\eta, -e_\eta) = d(x + e_\eta, x - e_\eta) \leq \varepsilon$ for every $\eta \in \Gamma \setminus A_\varepsilon$. Now, take $\eta \in \Gamma \setminus \cup_n A_{1/n}$, which exists since Γ is uncountable. Then $d(e_\eta, -e_\eta) < 1/n$ for every n, which is a contradiction. $\square$

Note that such a phenomenon is impossible for a bounded closed convex subset C of $\ell_\infty(\mathbb{N})$ as a consequence of Proposition 2.5.1.

Chapter 3

On strong asymptotic uniform smoothness and convexity

Consider a real Banach space X and let S_X be its unit sphere. For $t > 0$, $x \in S_X$ we shall consider

$$\overline{\delta}_X(t,x) = \sup_{\dim(X/Y)<\infty} \inf_{y \in S_Y} \|x + ty\| - 1; \quad \overline{\rho}_X(t,x) = \inf_{\dim(X/Y)<\infty} \sup_{y \in S_Y} \|x + ty\| - 1\,.$$

The *modulus of asymptotic uniform convexity* of X is given by

$$\overline{\delta}_X(t) = \inf_{x \in S_X} \overline{\delta}_X(t,x)\,,$$

and the *modulus of asymptotic uniform smoothness* of X is given by

$$\overline{\rho}_X(t) = \sup_{x \in S_X} \overline{\rho}_X(t,x)\,.$$

Geometrically, $\overline{\rho}_X$ measures how the boundary of disc which is tangent to S_X is uniformly separated from S_X from every direction but a finite number of them, see Figure 3.1.

The space X is said to be *asymptotically uniformly convex* (AUC for short) if $\overline{\delta}_X(t) > 0$ for each $t > 0$ and it is said to be *asymptotically uniformly smooth* (AUS for short) if $\lim_{t\to 0} t^{-1}\overline{\rho}_X(t) = 0$. If X is a dual space and we consider only weak* closed subspaces of X then the corresponding modulus is denoted by $\overline{\delta}^*_X(t)$. The space X is said to be *weak* asymptotically uniformly convex* if $\overline{\delta}^*_X(t) > 0$ for each $t > 0$. Let us highlight that it is proved in [DKLR] that a space is AUS if and only if its dual space is weak* AUC. In addition, $\overline{\rho}_X$ is quantitatively related to $\overline{\delta}^*_X$ by Young's duality. We refer the reader to [JLPS] and the references therein for a detailed study of these properties.

We will consider the following preorder for functions defined on $(0,1]$. We write $f \preceq g$ if there is a constant $c > 0$ such that $f(t/c) \leq g(t)$ for all $t \in (0,1]$. If $f \preceq g$ and $g \preceq f$, then we say that f and g are *equivalent*. Given $1 \leq p < \infty$, we will say that a modulus δ of convexity is of power type p if $\delta \succeq t^p$, and that a modulus ρ of smoothness is of power type p if $\rho \preceq t^p$.

Our goal is to study asymptotic uniform smoothness of the space of compact operators between two Banach spaces, as well as for their injective tensor product, giving an estimation

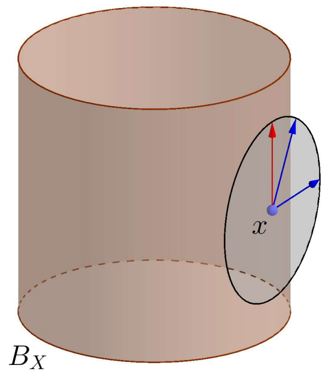

Figure 3.1: Geometrical interpretation of $\overline{\rho}_X$

of the power type when possible. To this end, we introduce a new geometrical property, which we have called strong asymptotic uniform smoothness, for Banach spaces admitting a finite dimensional decomposition (FDD).

First, we introduce a notion of asymptotic moduli with respect to a norming subspace, which includes the usual asymptotic moduli, and we give a formula for these moduli in spaces having an FDD. This formula motivates the definition of strongly AUS and strongly AUC spaces, which is given in the second section together with their basic properties. In the third section we show that the injective tensor product of strongly AUS spaces is strongly AUS, which allows us to give a sufficient condition for the asymptotic uniform smoothness of the space of compact operators. The fourth section is devoted to the study strong asymptotic uniform smoothness and convexity in the particular case of Orlicz and Lorentz sequence spaces. Finally, in Section 5 we show that neither $X\widehat{\otimes}_\varepsilon Y$ nor $\mathscr{K}(X,Y)$ is strictly convex unless X or Y are 1-dimensional.

3.1 F-AUC and F-AUS spaces

Given a norming subspace F of X^*, let us denote by $\sigma(X,F)$ the coarsest topology on X with respect to which every element of F is continuous. We shall introduce a general concept of F-AUC and F-AUS norms.

Definition 3.1.1. Let F be a norming subspace of X^*. For $t>0$ and $x\in S_X$, we define

$$\overline{\delta}_X^F(t,x) = \sup_{\substack{\dim(X/Y)<\infty\\ Y\,\sigma(X,F)\text{-closed}}} \inf_{y\in S_Y} \|x+ty\|-1;$$

$$\overline{\rho}_X^F(t,x) = \inf_{\substack{\dim(X/Y)<\infty\\ Y\,\sigma(X,F)\text{-closed}}} \sup_{y\in S_Y} \|x+ty\|-1\,.$$

The corresponding moduli are defined as follows

$$\overline{\delta}_X^F(t) = \inf_{x \in S_X} \overline{\delta}_X^F(t,x), \qquad \overline{\rho}_X^F(t) = \sup_{x \in S_X} \overline{\rho}_X^F(t,x)$$

The space X is said to be F*-asymptotically uniformly convex* if $\overline{\delta}_X^F(t) > 0$ for each $t > 0$ and it is said to be F*-asymptotically uniformly smooth* if $\lim_{t\to 0} t^{-1}\overline{\rho}_X^F(t) = 0$.

Note that $\overline{\delta}_X = \overline{\delta}_X^{X^*}$, $\overline{\rho}_X = \overline{\rho}_X^{X^*}$ and $\overline{\delta}_X^* = \overline{\delta}_{X^*}^X$. Thus, a space X is AUC (resp. AUS) if and only if it is X^*-AUC (resp. X^*-AUS), and X^* is weak* AUC if and only if it is X-AUC.

Dutrieux showed in [Dut, Lemma 37] that if X^* is separable then the modulus of asymptotic smoothness admits the following sequential expression:

$$\overline{\rho}_X(t,x) = \sup_{\substack{x_n \xrightarrow{w} 0 \\ \|x_n\| \le t}} \limsup_{n\to\infty} \|x + x_n\| - 1\,.$$

In addition, Borel-Mathurin proved in [BM] that a similar statement for the weak* modulus of asymptotic convexity of X^* holds when X is separable. Namely,

$$\overline{\delta}_X^*(t,x^*) = \inf_{\substack{x_n^* \xrightarrow{w^*} 0 \\ \|x_n^*\| \ge t}} \liminf_{n\to\infty} \|x^* + x_n^*\| - 1\,.$$

The same ideas can be used to prove the following result, which can be seen as a general version of both formulas. Note that a finite codimensional subspace Y of X is $\sigma(X,F)$-closed if and only if there are $f_1,\dots,f_n \in F$ such that $Y = \bigcap_{i=1}^n \ker f_i$. Moreover, if $(x_\alpha)_\alpha$ is a $\sigma(X,F)$-null net in X then $\lim_\alpha d(x_\alpha, Y) = 0$ for each finite codimensional $\sigma(X,F)$-closed subspace Y of X. Following [DKLR], we will consider the set $\mathscr{C}$ of finite codimensional $\sigma(X,F)$-closed subspaces of X as a directed set with the order $\preceq$ given by $E \preceq F$ if $F \subset E$.

Proposition 3.1.2. *Let F be a norming subspace of X^*. For each $x \in S_X$ and $t > 0$ we have:*

$$\overline{\delta}_X^F(t,x) = \inf_{\substack{x_\alpha \xrightarrow{\sigma(X,F)} 0 \\ \|x_\alpha\| \ge t}} \liminf_{\alpha} \|x + x_\alpha\| - 1;$$

$$\overline{\rho}_X^F(t,x) = \sup_{\substack{x_\alpha \xrightarrow{\sigma(X,F)} 0 \\ \|x_\alpha\| \le t}} \limsup_{\alpha} \|x + x_\alpha\| - 1\,.$$

If moreover F is separable then

$$\overline{\delta}_X^F(t,x) = \inf_{\substack{x_n \overset{\sigma(X,F)}{\longrightarrow} 0 \\ \|x_n\| \geq t}} \liminf_{n\to\infty} \|x+x_n\| - 1;$$

$$\overline{\rho}_X^F(t,x) = \sup_{\substack{x_n \overset{\sigma(X,F)}{\longrightarrow} 0 \\ \|x_n\| \leq t}} \limsup_{n\to\infty} \|x+x_n\| - 1\,.$$

Proof. We will prove the first formula for $\overline{\delta}_X^F(t,x)$, since the proof of the one for $\overline{\rho}_X^F(t,x)$ is similar. Let us consider

$$\theta(t,x) = \inf_{\substack{x_\alpha \overset{\sigma(X,F)}{\longrightarrow} 0 \\ \|x_\alpha\| \geq t}} \liminf_{\alpha} \|x+x_\alpha\| - 1\,.$$

Fix $\varepsilon > 0$. For each finite codimensional $\sigma(X,F)$-closed subspace Z of X, take $x_Z \in S_Z$ so that $\|x+tx_Z\| \leq \inf_{y\in S_Z} \|x+ty\| + \varepsilon$. Note that the net $(x_Z)_{Z\in\mathscr{C}}$ is $\sigma(X,F)$-convergent to 0. Indeed, given $f \in F$ we have that $f(x_Z) = 0$ whenever $Z \subset \ker f$. Thus,

$$\theta(t,x) \leq \liminf_{Z\in\mathscr{C}} \|x+tx_Z\| - 1 \leq \overline{\delta}_X^F(t,x) + \varepsilon\,.$$

Letting $\varepsilon \to 0$, we get $\theta(t,x) \leq \overline{\delta}_X^F(t,x)$. Now, take $(x_\alpha)_\alpha$ a $\sigma(X,F)$-null net such that $\|x_\alpha\| \geq t$ for each α. Fix $\varepsilon > 0$ and take Y a finite codimensional $\sigma(X,F)$-closed subspace of X. Then $\lim_\alpha d(x_\alpha, Y) = 0$, so there exist a net $(y_\alpha)_\alpha$ in Y and α_0 so that if $\alpha \geq \alpha_0$ then $\|x_\alpha - y_\alpha\| \leq \varepsilon$. Thus $\|y_\alpha\| \geq t-\varepsilon$ whenever $\alpha \geq \alpha_0$. Moreover, from the convexity of the function $t \mapsto \|x+ty_\alpha\| - 1$ we get that

$$\|x+x_\alpha\| - 1 \geq \|x+y_\alpha\| - 1 - \varepsilon \geq \frac{\|y_\alpha\|}{t-\varepsilon}\left(\left\|x + \frac{t-\varepsilon}{\|y_\alpha\|}y_\alpha\right\| - 1\right) - \varepsilon.$$

It follows that $\liminf_\alpha \|x+x_\alpha\| \geq \inf_{y\in S_Y} \|x+(t-\varepsilon)y\|$. That inequality holds for every finite codimensional $\sigma(X,F)$-closed subspace Y of X and every $\varepsilon > 0$. Since the function $t \mapsto \overline{\delta}_X^F(t,x)$ is 1-Lipschitz, we get

$$\liminf_{\alpha} \|x+x_\alpha\| - 1 \geq \overline{\delta}_X^F(t,x)\,,$$

as desired.

Finally, assume that F is separable. From what we have already proved it follows

$$\overline{\delta}_X^F(t,x) \leq \inf_{\substack{x_n \overset{\sigma(X,F)}{\longrightarrow} 0 \\ \|x_n\| \geq t}} \liminf_{n\to\infty} \|x+x_n\| - 1\,,$$

$$\overline{\rho}_X^F(t,x) \geq \sup_{\substack{x_n \overset{\sigma(X,F)}{\longrightarrow} 0 \\ \|x_n\| \leq t}} \limsup_{n\to\infty} \|x+x_n\| - 1\,.$$

Let $\{f_n : n \in \mathbb{N}\}$ be a dense sequence in F. Let us consider the finite codimensional $\sigma(X, F)$-closed subspaces of X given by $Y_n = \bigcap_{i=1}^n \ker f_i$, for each $n \in \mathbb{N}$. Fix $0 < \varepsilon < t$. For every n, take $x_n, y_n \in Y_n$ such that $\|x_n\| = \|y_n\| = 1$ and

$$\|x + tx_n\| \leq \inf_{y \in S_{Y_n}} \|x + ty\| + \varepsilon \leq \overline{\delta}_X^F(x, t) + 1 + \varepsilon\,,$$
$$\|x + ty_n\| \geq \sup_{y \in S_{Y_n}} \|x + ty\| - \varepsilon \geq \overline{\rho}_X^F(x, t) + 1 - \varepsilon\,.$$

It is easy to check that the sequences $(x_n)_n$ and $(y_n)_n$ are $\sigma(X, F)$-null. Since ε was arbitrary, this finishes the proof. □

For the norming subspaces that we will consider in the next sections, the norm will be $\sigma(X, F)$-lower semicontinuous. In view of Proposition 3.1.2, that condition guarantees that $\overline{\delta}_X^F$ and $\overline{\rho}_X^F$ are non-negative functions.

The results in the next section will apply to spaces spaces admitting an FDD. Recall that a sequence $\mathsf{E} = (E_n)_n$ of finite dimensional subspaces of X is call a *finite dimensional decomposition* (FDD for short) if every $x \in X$ has a unique representation of the form $x = \sum_{n=1}^\infty x_n$, with $x_n \in E_n$ for every n. Every FDD of X determines a sequence of uniformly bounded projections $(P_n^{\mathsf{E}})_n$ given by $P_n^{\mathsf{E}}(\sum_{i=1}^\infty x_i) = \sum_{i=1}^n x_i$. By convention, $P_n^{\mathsf{E}} = 0$. The number $K = \sup\{\left\|P_n^{\mathsf{E}}\right\|\}$ is called the decomposition constant of the FDD. An FDD is called *monotone* if $K = 1$. Moreover, an FDD is called *shrinking* if $\lim_n \|P_n^* x^* - x^*\| = 0$ for every $x^* \in X^*$, and it is called *boundedly complete* if $\sum_{n=1}^\infty x_n$ converges whenever $x_n \in E_n$ for each n and $\sup_n \left\|\sum_{i \leq n} x_i\right\| < +\infty$. We say that $\mathsf{F} = (F_n)_n$ is a *blocking* of E if there exists an increasing sequence $(m_n)_n \subset \mathbb{N}$ such that $m_1 = 0$ and $F_n = \bigoplus_{i=m_n+1}^{m_{n+1}} E_i$ for every n. For detailed treatment and applications of FDDs, we refer the reader to [LT].

It is easy to show that, if E is an FDD for X, then $F = \overline{\operatorname{span}}\{(P_n^{\mathsf{E}})^* X^* : n \in \mathbb{N}\}$ is a norming subspace of X^*. Moreover, if E is monotone then F is 1-norming and $\|\cdot\|$ is $\sigma(X, F)$-lower semicontinuous.

Proposition 3.1.3. *Let E be a monotone FDD for a Banach space X and consider $F = \overline{\operatorname{span}}\{(P_n^{\mathsf{E}})^* X^* : n \in \mathbb{N}\}$. For each $t > 0$ we have:*

$$\overline{\delta}_X^F(t) = \inf_{n \in \mathbb{N}} \sup_{m \geq n} \inf\{\|x + ty\| - 1 : x \in H_n \cap S_X, y \in H^m \cap S_X\}\,,$$
$$\overline{\rho}_X^F(t) = \sup_{n \in \mathbb{N}} \inf_{m \geq n} \sup\{\|x + ty\| - 1 : x \in H_n \cap S_X, y \in H^m \cap S_X\}$$

where $H_n = \bigoplus_{i=1}^n E_i$ and $H^n = \overline{\bigoplus_{i=n+1}^\infty E_i}$ for each $n \in \mathbb{N}$.

Proof. We prove only the statement concerning $\overline{\rho}_X^F$ as the other one is similar. Since $\cup_n H_n \cap S_X$ is dense in S_X, we have that $\overline{\rho}_X^F(t) = \sup_{n \in \mathbb{N}} \sup_{x \in H_n \cap S_X} \overline{\rho}_X^F(t, x)$ (see [GJT, Lemma 1]). Thus, it suffices to show that

$$\sup_{x \in H_n \cap S_X} \overline{\rho}_X^F(t, x) = \inf_{m \geq n} \sup\{\|x + ty\| - 1 : x \in H_n \cap S_X, y \in H^m \cap S_X\}$$

for each $n \in \mathbb{N}$. First notice that each H^m is a $\sigma(X,F)$-closed subspace of X. Indeed, since $H^m = \ker P_m^{\mathsf{E}}$, it suffices to check that P_m^{E} is $\sigma(X,F)$-continuous for every $m \in \mathbb{N}$. For that, take a net $(x_\alpha)_\alpha$ that is $\sigma(X,F)$-converging to a vector $x \in X$. Then $\lim_\alpha (P_n^{\mathsf{E}})^*(x^*)(x_\alpha) = (P_n^{\mathsf{E}})^*(x^*)(x)$ for each $x^* \in X^*$, so $(P_n^{\mathsf{E}} x_\alpha)_\alpha$ is $\sigma(X,X^*)$-convergent to $P_n^{\mathsf{E}}(x)$. Since $P_n^{\mathsf{E}}(X)$ is finite-dimensional, it follows that $(P_n^{\mathsf{E}} x_\alpha)_\alpha$ is also norm-convergent. This shows that P_m^{E} is $\sigma(X,F)$-continuous and so H^m is $\sigma(X,F)$-closed. Therefore,

$$\sup_{x \in H_n \cap S_X} \overline{\rho}_X^F(t,x) \leq \inf_{m \geq n} \sup\{\|x+ty\| - 1 : x \in H_n \cap S_X, y \in H^m \cap S_X\}$$

Now, fix $n \in \mathbb{N}$. Assume that

$$\sup_{x \in H_n \cap S_X} \overline{\rho}_X^F(t,x) < \rho < \rho + \varepsilon < \inf_{m \geq n} \sup\{\|x+ty\| - 1 : x \in H_n \cap S_X, y \in H^m \cap S_X\}$$

for some $\rho, \varepsilon > 0$. We claim that for each $x \in H_n \cap S_X$ there exists $m = m(x) > n$ such that $\|x+ty\| < 1+\rho$ for each $y \in H^m \cap S_X$. To see this, assume that there exist $x \in H_n \cap S_X$ and a sequence $(y_m)_m$ so that $y_m \in H^m \cap S_X$ and $\|x + ty_m\| \geq 1+\rho$ whenever $m \geq n$. Note that F is separable and the sequence $(y_m)_m$ is $\sigma(X,F)$-null. Therefore, the sequential formula for the modulus given in Proposition 3.1.2 yields

$$\rho \leq \limsup_{m \to \infty} \|x + ty_m\| - 1 \leq \overline{\rho}_X^F(t,x) < \rho\,,$$

which is a contradiction. This proves the claim. Now pick $\{x_i\}_{i=1}^k$ an ε-net in $H_n \cap S_X$, take $m = \max\{m(x_i) : i = 1\ldots,k\}$ and let $x \in H_n$ and $y \in H^m$ be norm-one vectors. There exists i such that $\|x - x_i\| \leq \varepsilon$. Then,

$$\|x+ty\| - 1 \leq \|x_i + ty\| - 1 + \varepsilon \leq \rho + \varepsilon\,,$$

which is a contradiction. □

Let us recall that if $\mathsf{E} = (E_n)_n$ is a monotone FDD in X with associated projections $(P_n^{\mathsf{E}})_n$, then $\mathsf{E}^* = ((P_n^{\mathsf{E}} - P_{n-1}^{\mathsf{E}})^* X^*)_n$ is an FDD for $F = \overline{\operatorname{span}}\{(P_n^{\mathsf{E}})^* X^* : n \in \mathbb{N}\}$ with associated projections given by $P_n^{\mathsf{E}^*} = (P_n^{\mathsf{E}})^*$. Note that if E is shrinking then $F = X^*$ and E^* is boundedly complete. Proposition 3.1.3 provides a formula for the asymptotic moduli in spaces admitting a monotone shrinking FDD.

Corollary 3.1.4. *Let X be a Banach space admitting a monotone shrinking FDD E. For each $t > 0$ we have:*

$$\overline{\delta}_X(t) = \inf_{n \in \mathbb{N}} \sup_{m \geq n} \inf\{\|x+ty\| - 1 : x \in P_n^{\mathsf{E}}(X) \cap S_X, y \in \ker P_m^{\mathsf{E}} \cap S_X\}\,,$$

$$\overline{\rho}_X(t) = \sup_{n \in \mathbb{N}} \inf_{m \geq n} \sup\{\|x+ty\| - 1 : x \in P_n^{\mathsf{E}}(X) \cap S_X, y \in \ker P_m^{\mathsf{E}} \cap S_X\}\,,$$

$$\overline{\delta}_X^*(t) = \inf_{n \in \mathbb{N}} \sup_{m \geq n} \inf\{\|x^*+ty^*\| - 1 : x^* \in (P_n^{\mathsf{E}^*} X^*) \cap S_X, y^* \in \ker P_m^{\mathsf{E}^*} \cap S_X\}\,.$$

3.2 Strongly AUC and strongly AUS spaces

The following definition is motivated by the formulae obtained in Proposition 3.1.3 and Corollary 3.1.4.

Definition 3.2.1. Let X a Banach space and let $\mathsf{E} = (E_n)_n$ be an FDD for X. Denote $H_n = \bigoplus_{i=1}^n E_i$ and $H^n = \overline{\bigoplus_{i=n+1}^\infty E_i}$. The space X is said to be *strongly AUC with respect to* E if the modulus defined by

$$\hat{\delta}_{\mathsf{E}}(t) = \inf_{n\in\mathbb{N}} \sup_{m\geq n} \inf\{\|x+ty\| - 1 : x \in H_m \cap S_X, y \in H^m \cap S_X\}$$

satisfies that $\hat{\delta}_{\mathsf{E}}(t) > 0$ for each $t > 0$. In addition, X is said to be *strongly AUS with respect to* E if

$$\hat{\rho}_{\mathsf{E}}(t) = \sup_{n\in\mathbb{N}} \inf_{m\geq n} \sup\{\|x+ty\| - 1 : x \in H_m \cap S_X, y \in H^m \cap S_X\}$$

satisfies $\lim_{t\to 0} t^{-1}\hat{\rho}_{\mathsf{E}}(t) = 0$. Finally, we say that X is *strongly AUS* (resp. *strongly AUC*) if X is strongly AUS (resp. strongly AUC) with respect to some FDD.

Since $\max\{\|x+y\|, \|x-y\|\} \geq \|x\|$ for each $x, y \in X$, it follows that $\hat{\rho}_{\mathsf{E}}(t) \geq 0$ for each t. Moreover, if E is monotone then $\hat{\delta}_{\mathsf{E}}(t) \geq 0$. It is clear that functions $\hat{\rho}_{\mathsf{E}}$ and $\hat{\delta}_{\mathsf{E}}$ are 1-Lipschitz functions and $\hat{\delta}_{\mathsf{E}}(t) \leq \hat{\rho}_{\mathsf{E}}(t) \leq t$ for all t. For notational convenience let us set

$$\begin{aligned}
\hat{\delta}_{\mathsf{E}}(t,m) &= \inf\{\|x+ty\| - 1 : x \in H_m \cap S_X, y \in H^m \cap S_X\}, \\
\hat{\rho}_{\mathsf{E}}(t,m) &= \sup\{\|x+ty\| - 1 : x \in H_m \cap S_X, y \in H^m \cap S_X\}.
\end{aligned}$$

Note that if F is a blocking of E then for each m there is $k_m \geq m$ so that $\hat{\delta}_{\mathsf{F}}(t,m) = \hat{\delta}_{\mathsf{E}}(t,k_m)$ and $\hat{\rho}_{\mathsf{F}}(t,m) = \hat{\rho}_{\mathsf{E}}(t,k_m)$. Thus, $\hat{\delta}_{\mathsf{F}}(t) \leq \hat{\delta}_{\mathsf{E}}(t)$ and $\hat{\rho}_{\mathsf{F}}(t) \geq \hat{\rho}_{\mathsf{E}}(t)$. In particular, X is strongly AUC (resp. strongly AUS) with respect to E whenever it is strongly AUC (resp. strongly AUS) with respect to some blocking of E.

Remark that, in the above definitions, we only compute the norms $\|x+ty\|$ for vectors x and y which belong to *complementary* subspaces, that is, $x \in H_m$ and $y \in H^m$ for a certain m. This is why we called these notions strong AUS and strong AUC. Indeed, as a consequence of Corollary 3.1.4 we obtain the following:

Corollary 3.2.2. *Let* E *be a monotone shrinking FDD for a Banach space* X. *Then* $\overline{\delta}_X(t) \geq \hat{\delta}_{\mathsf{E}}(t)$, $\overline{\rho}_X(t) \leq \hat{\rho}_{\mathsf{E}}(t)$ *and* $\overline{\delta}^*_X(t) \leq \hat{\delta}_{\mathsf{E}^*}(t)$. *Thus,* X *is AUC (resp. AUS) whenever it is strongly AUC (resp. strongly AUS) with respect to* E, *and* X^* *is weak* AUC whenever it is strongly AUC with respect to* E^*.

Example 3.2.3.

(a) Let $X = (\bigoplus_{n=1}^\infty E_n)_p$ be an ℓ_p-sum of finite dimensional spaces, $1 \leq p < \infty$, and consider $\mathsf{E} = (E_n)_{n=1}^\infty$. Then $\hat{\delta}_{\mathsf{E}}(t) = \hat{\rho}_{\mathsf{E}}(t) = (1+t^p)^{1/p} - 1$. Thus, X is strongly AUC with respect to E. If moreover $p > 1$ then it is strongly AUS with respect to E.

(b) Let $X = (\bigoplus_{n=1}^{\infty} E_n)_0$ be a c_0-sum of finite dimensional spaces, and $\mathsf{E} = (E_n)_{n=1}^{\infty}$. Then $\hat{\rho}_{\mathsf{E}}(t) = 0$ for each $t \in (0,1]$, so X is strongly AUS with respect to E.

(c) Consider the James space J endowed with the norm

$$\|(x_n)_{n=1}^{\infty}\|^2 = \sup_{1\le n_1<\ldots<n_{2m+1}} \sum_{i=1}^{m}(x_{n_{2i-1}} - x_{n_{2i}})^2 + 2x_{n_{2m+1}}^2$$

given in [Pru2] and let E be the standard basis of J. Then $\|x+y\|^2 \le \|x\|^2 + 2\|y\|^2$ whenever $x \in H_n$ and $y \in H^n$ for some n. Thus, $\hat{\rho}_{\mathsf{E}}(t) \le (1+2t^2)^{1/2} - 1$, so J is strongly AUS with respect to E.

(d) Let T be a well-founded tree in $\omega^{<\omega}$. The James Tree space JT consists of all real functions defined on T, with the norm

$$\|x\|^2 = \sup \sum_{j=1}^{n}\left(\sum_{t\in S_j} x(t)\right)^2$$

where the supremum is taken over all finite sets of pairwise disjoint segments in T. Lancien proved in [Lan1, Proposition 4.6] that there exists a basis $\mathsf{E} = (e_n)_n$ of JT and an increasing sequence $(n_k)_k$ such that if $x \in \operatorname{span}\{e_1,\ldots,e_{n_k}\}$ and $y \in \overline{\operatorname{span}}\{e_i : i > n_k\}$ then $\|x+y\|^2 \ge \|x\|^2 + \|y\|^2$. Therefore JT is strongly AUC with respect to E and $\hat{\delta}_{\mathsf{E}}(t) \ge (1+t^2)^{1/2} - 1$.

Recall that the *modulus of uniform convexity* of a Banach space X is defined by

$$\delta_X(t) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : x, y \in B_X, \|x-y\| = t\right\},$$

and the *modulus of uniform smoothness* of X is defined by

$$\rho_X(t) = \frac{1}{2}\sup\{\|x+ty\| + \|x-ty\| - 2 : x, y \in S_X\}.$$

The Banach space X is said to be *uniformly convex* if $\delta_X(t) > 0$ for every t, and it is said to be *uniformly smooth* if $\lim_{t\to 0^+} t^{-1}\rho_X(t) = 0$.

Proposition 3.2.4. *Let E be a monotone FDD for a Banach space X. Then $\delta_X(t) \le \hat{\delta}_{\mathsf{E}}(t)$ and $\hat{\rho}_{\mathsf{E}}(t) \le 2\rho_X(t)$ for each $0 < t < 1$. Thus, if X is uniformly convex (resp. uniformly smooth) then it is strongly AUC (resp. strongly AUS) with respect to E.*

Proof. We will use the same arguments that appear in Proposition 2.3.(3) in [JLPS]. From the monotony of E it follows that

$$\frac{1}{2}(\|x+ty\| - 1) \le \frac{1}{2}(\|x+ty\| + \|x-ty\|) - 1$$

whenever $x \in H_n \cap S_X$ and $y \in H^n \cap S_X$ for some $n \in \mathbb{N}$. Thus, $\hat{\rho}_{\mathsf{E}}(t) \le 2\rho_X(t)$. Now, fix $n \in \mathbb{N}$ and take $x \in H_n \cap S_X$ and $y \in H^n \cap S_X$. Let $x^* \in S_{X^*}$ be such that $x^*(x) = 1$.

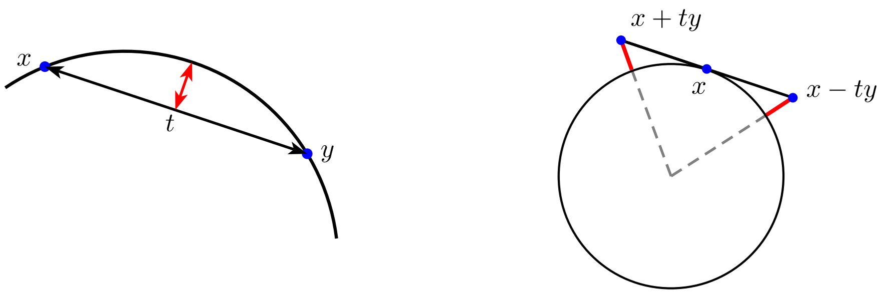

Figure 3.2: Geometrical interpretation of $\delta_X(t)$ and $\rho_X(t)$

Then $y^* = x^* \circ P_n^*$ satisfies $\|y^*\| = 1$, $y^*(x) = 1$ and $y^*(y) = 0$. Let us consider $u = \frac{x+ty}{\|x+ty\|}$ and $v = u - ty$. Then $u \in B_X$ and $\|u - v\| = t$. Moreover, v is a convex combination of x and ty and so we also have $v \in B_X$. Thus,

$$\delta_X(t) \leq 1 - \frac{1}{2}\|u+v\| \leq 1 - \frac{1}{2}y^*(u+v) = 1 - \frac{1}{\|x+ty\|} \leq \|x+ty\| - 1$$

and so $\delta_X(t) \leq \hat{\delta}_E(t)$. □

Our next result establishes the duality between strongly AUS and strongly AUC norms by using estimates similar to those in [DKLR]. Recall that, given a continuous function $f : [0,1] \to [0,+\infty)$ with $f(0) = 0$, its *dual Young function* is defined by

$$f^*(s) = \sup\{st - f(t) : 0 \leq t \leq 1\}.$$

Proposition 3.2.5. *Let $\mathsf{E} = (E_n)_n$ be a monotone FDD for a Banach space X and let E^* be the dual FDD for $F = \overline{\operatorname{span}}\{(P_n^{\mathsf{E}})^* X^* : n \in \mathbb{N}\}$ given above. Take $0 < s,t < 1$. Then:*

(a) *If $\hat{\rho}_{\mathsf{E}}(s) < st$, then $\hat{\delta}_{\mathsf{E}^*}(6t) \geq st$.*
(b) *If $\hat{\delta}_{\mathsf{E}^*}(t) > st$, then $\hat{\rho}_{\mathsf{E}}(s) \leq st$.*

Therefore, $\hat{\rho}_{\mathsf{E}}^*$ is equivalent to $\hat{\delta}_{\mathsf{E}^*}$ and $\hat{\rho}_{\mathsf{E}^*}$ is equivalent to $\hat{\delta}_{\mathsf{E}}^*$.

Proof. As usual, let us consider $H_n = \bigoplus_{i=1}^n E_i$ and $H^n = \overline{\bigoplus_{i=n+1}^\infty E_i}$. In order to prove a), assume that $\hat{\rho}_{\mathsf{E}}(s) < st$ and fix $\varepsilon > 0$ and $n \in \mathbb{N}$. Take $m \geq n$ so that $\hat{\rho}_{\mathsf{E}}(s,m) < st$. Let $f \in P_m^{\mathsf{E}^*}(X^*) \cap S_{X^*}$ and $g \in \ker P_m^{\mathsf{E}^*} \cap S_{X^*}$. We will estimate $\|f + 3tg\|$. Note that, by the monotony of E, there exists $x \in H_m \cap S_X$ such that $f(x) > 1 - \varepsilon$. Now take $y \in H^m \cap S_X$. We have $\|x + sy\| < 1 + st$. Thus,

$$\begin{aligned} \|f + 6tg\| &\geq \frac{1}{1+st}(f+6tg)(x+sy) \\ &= \frac{1}{1+st}(f(x) + 6stg(y)) \geq \frac{1 - \varepsilon + 6stg(y)}{1+st} \end{aligned}$$

From the monotony of E, it follows that $\|g\| \leq 2\sup\{g(y) : y \in H^m \cap S_X\}$. Thus,

$$\|f + 6tg\| \geq \frac{1 - \varepsilon + 3st}{1 + st}$$

Hence,

$$\hat{\delta}_{\mathsf{E}^*}(6t, m) \geq \frac{1 - \varepsilon + 3st}{1 + st} - 1\,.$$

Since ε is arbitrary, we get that for every n there exists $m \geq n$ so that $\hat{\delta}_{\mathsf{E}^*}(6t, m) \geq st$. Therefore $\hat{\delta}_{\mathsf{E}^*}(6t) \geq st$, as desired.

Now we turn to the proof of b). Assume that $\hat{\delta}_{\mathsf{E}^*}(t) > st$ and $\hat{\rho}_{\mathsf{E}}(s) > st$. Then there exist $\rho > st$ and $n \in \mathbb{N}$ such that $\inf_{m \geq n} \hat{\rho}_{\mathsf{E}}(t, m) > \rho$. Moreover, there is $m \geq n$ so that $\hat{\delta}_{\mathsf{E}^*}(t, m) > st$. Now take $x \in H_m \cap S_X$ and $y \in H^m \cap S_X$ satisfying

$$1 + st < 1 + \rho < \|x + sy\|\ .$$

Note that F is a 1-norming subspace of X^*. Thus, there is $z^* \in S_{X^*} \cap F$ such that $z^*(x + sy) > 1 + \rho$. Take $f = P_m^{\mathsf{E}^*} z^*$, $g = (I - P_m^{\mathsf{E}^*})z^*$ and $c = \|g\|$. Since E is monotone, we get that

$$1 + st < z^*(x + sy) = f(x) + g(sy) \leq 1 + cs\,.$$

Thus, $t < c$. We claim that $\|f\| \leq 1 - cs$. This clearly holds for $f = 0$, so assume that $f \neq 0$. Since $\hat{\delta}_{\mathsf{E}^*}(t, m) > st$, we get that

$$(1 + st)\,\|f\| < \left\| f + \frac{t}{c}\,\|f\|\, g \right\| \leq \frac{t}{c}\,\|f\|\,\|f + g\| + (1 - \frac{t}{c}\,\|f\|)\,\|f\|\ .$$

Hence, $1 + st < \frac{t}{c} + (1 - \frac{t}{c}\,\|f\|)$. That proves the claim. Now,

$$1 + st < z^*(x + sy) = f(x) + g(sy) \leq 1 - cs + cs = 1\,,$$

which is a contradiction.

Finally, a standard argument shows from what we have already proved that $\hat{\delta}_{\mathsf{E}^*}(t/2) \leq \hat{\rho}^*_{\mathsf{E}}(t) \leq \hat{\delta}_{\mathsf{E}^*}(6t)$, so $\hat{\rho}^*_{\mathsf{E}}$ is equivalent to $\hat{\delta}_{\mathsf{E}^*}$. On the other hand, it is easy to check that if P is a norm-one projection on X with finite-dimensional range, then $P^{**}(X^{**})$ is isometric to $P(X)$. Thus, $X = \overline{\operatorname{span}}\{(P_n^{\mathsf{E}^*})^*(X^{**})\}$ and E^{**} may be identified with E. By applying the previous formula to E^* we get that $\hat{\delta}_{\mathsf{E}}$ is equivalent to $\hat{\rho}^*_{\mathsf{E}^*}$, which finishes the proof. $\square$

Corollary 3.2.6. *Let X be a Banach space with a monotone shrinking FDD E. Then X is strongly AUS (resp. strongly AUC) with respect to E with power type p if and only if X^* is strongly AUC (resp. strongly AUS) with respect to the dual FDD E^* with power type p', the conjugate exponent of p.*

Given an FDD E for X, an element $x \in X$ is said to be a *block* of E if $x = P_n^{\mathsf{E}} x$ for some n. The interval

$$\operatorname{ran}_{\mathsf{E}} x = [\max\{n : P_n^{\mathsf{E}} x = 0\} + 1, \min\{n : P_n^{\mathsf{E}} x = x\}]$$

is called the *range* of the block x. Given $1 \le p, q \le \infty$, it is said that E *satisfies* (p,q)-*estimates* if there exists a constant $C > 0$ such that

$$\frac{1}{C}\left(\sum_{i=1}^{n} \|x_i\|^p\right)^{1/p} \le \left\|\sum_{i=1}^{n} x_i\right\| \le C\left(\sum_{i=1}^{n} \|x_i\|^q\right)^{1/q}$$

for all finite sequences $x_1, \dots, x_n$ with $\operatorname{ran}_{\mathsf{F}} x_i \cap \operatorname{ran}_{\mathsf{F}} x_j = \emptyset$ for every $i \ne j$.

The next result is based on a similar one given by Prus in [Pru1] for NUS spaces.

Proposition 3.2.7. *Let E be an FDD for a Banach space X.*

(a) *If X is strongly AUS with respect to E then there is a blocking $\mathsf{F} = (F_n)_n$ of E satisfying (∞, q)-estimates for some $1 < q < \infty$.*
(b) *If E is monotone and X is strongly AUC with respect to E then there is a blocking $\mathsf{F} = (F_n)_n$ of E satisfying $(1,p)$-estimates for some $1 < p < \infty$.*

Proof. We will mimic the proof of [Pru1, Theorem 3.3]. First assume that X is strongly AUS with respect to E and fix $t > 0$ such that $\hat{\rho}_{\mathsf{E}}(t) < t/2$. Thus, there exists an increasing sequence $(m_n)_n \subset \mathbb{N}$ so that $m_1 = 0$ and $\hat{\rho}_{\mathsf{E}}(t, m_n) < t/2$ for $n > 1$. Consider $F_n = \bigoplus_{i=m_n+1}^{m_{n+1}} E_i$ and let $q > 1$ be such that $(2 - t/2)^q < 2$. Take $\nu < 1/2$ so that $(1 + \alpha - t/2)^q < 1 + \alpha^q$ whenever $|1 - \alpha| < \nu$. Note that for such α, if $x \in \bigoplus_{i=1}^{n} F_i \cap S_X$ and $y = \overline{\bigoplus_{i=n+1}^{\infty} F_i} \cap S_X$ for some n, then

$$\|x + \alpha y\| \le \|x + ty\| + (\alpha - t)\,\|y\| \le 1 + \alpha - \frac{t}{2} \le (1 + \alpha^q)^{1/q}\,.$$

Now one can follow the same steps as in the proof of Gurarii's theorem (see, e.g. [FHH$^+$, Lemma 9.26]) to get the statement.

On the other hand, assume that E is monotone and X is strongly AUC with respect to E. We will argue as in [FHH$^+$, Lemma 9.27]. By Proposition 3.2.5, $F = \overline{\operatorname{span}}\{(P_n^{\mathsf{E}})^* X^* : n \in \mathbb{N}\}$ is strongly AUS with respect to E^*. From what we have already proved we get $q > 1$, $C > 0$ and an increasing sequence $(m_n)_n$ so that the FDD $\mathsf{F} = (F_n)_n$ given by $F_n = \bigoplus_{i=m_n+1}^{m_{n+1}} (P_{n+1}^{\mathsf{E}} - P_n^{\mathsf{E}})^* X^*$ is a blocking of E^* which satisfies (∞, q)-estimates with constant C. Now, take $p = \frac{q}{q-1}$ and $\mathsf{G} = (G_n)_n$ given by $G_n = \bigoplus_{i=m_n+1}^{m_{n+1}} E_i$. We will show that p and G do the work. For that, let $x_1, \dots, x_n \in X$ with $\operatorname{ran}_{\mathsf{G}} x_i \cap \operatorname{ran}_{\mathsf{G}} x_j = \emptyset$ for all $i \ne j$. For each i, take $f_i \in F$ such that $\|f_i\| = 1$ and $f_i(x_i) = \|x_i\|$, which exists since E is monotone and so F is a 1-norming subspace of X^*. Moreover, consider $g_i = f_i \circ (P^{\mathsf{G}}_{\max \operatorname{ran}_{\mathsf{G}}(x_i)} - P^{\mathsf{G}}_{\min \operatorname{ran}_{\mathsf{G}}(x_i) - 1})$. Then $\|g_i\| \le 2$ and $\operatorname{ran}_{\mathsf{F}}(g_i) \subset \operatorname{ran}_{\mathsf{G}}(x_i)$ for each i. Thus, $g_1, \dots, g_n$ have pairwise disjoint ranges and $g_i(x_j) = \delta_{ij}\,\|x_i\|$ for each i, j. Now let $g = \sum_{i=1}^{n} \beta_i g_i$, where $\beta_i = \|x_i\|^{1/(q-1)}$. Then

$$\left\|\sum_{i=1}^{n} x_i\right\| \ge \frac{1}{\|g\|} g\left(\sum_{i=1}^{n} x_i\right) \ge \frac{\sum_{i=1}^{n} \beta_i \,\|x_i\|}{C\left(\sum_{i=1}^{n} \|\beta_i g_i\|^q\right)^{\frac{1}{q}}} \ge \frac{1}{2C}\left(\sum_{i=1}^{n} \|x_i\|^p\right)^{\frac{1}{p}},$$

as desired. $\square$

It is well known that every FDD satisfying (∞, q)-estimates for some $q > 1$ is shrinking, and every FDD satisfying $(p, 1)$-estimates for some $p < \infty$ is boundedly complete. Moreover, an FDD is shrinking (resp. boundedly complete) if it has a shrinking (resp. boundedly complete) blocking. We will use these facts in the following result. In a previous version of the paper [GLR2], statement (b) below was proved under the assumption that the FDD was either unconditional or monotone. We thank the referee of [GLR2] for suggesting a general argument that we include below.

Proposition 3.2.8. *Let* E *be an FDD for a Banach space* X*.*

(a) *If* X *is strongly AUS with respect to* E *then* E *is shrinking.*
(b) *If* X *is strongly AUC with respect to* E *then* E *is boundedly complete.*

Thus, X *is reflexive whenever it is both strongly AUS and strongly AUC with respect to an FDD.*

Proof. Statement (a) follows from Proposition 3.2.7. If E is not boundedly complete, then there exists $\varepsilon > 0$ such that for any blocking F of E, there exists a block sequence $(x_i)_{i=1}^{\infty}$ with respect to F such that $\sup_{N\in\mathbb{N}} \left\|\sum_{i=1}^{N} x_i\right\| \le 1$ and $\inf_i \|x_i\| \ge \varepsilon$. To obtain a contradiction, assume $\hat{\delta}_{\mathsf{E}}(\varepsilon) > \delta > 0$. Then there exist $m_1 < m_2 < \ldots$ such that $\hat{\delta}_{\mathsf{E}}(\varepsilon, m_i) > \delta$ for all $i \in \mathbb{N}$. Let $m_0 = 0$ and $F_n = \oplus_{m=m_{n-1}+1}^{m_n} E_m$. Let $(x_i)_{i=1}^{\infty}$ be a block sequence with respect to F such that $\sup_{N\in\mathbb{N}} \left\|\sum_{i=1}^{N} x_i\right\| \le 1$ and $\inf_i \|x_i\| \ge \varepsilon$. Let $L = \limsup_N \left\|\sum_{i=1}^{N} x_i\right\| \in (0, +\infty)$. We may fix $0 < a < L < b$ such that $b/a - 1 < \delta$ and $N \in \mathbb{N}$ such that $\left\|\sum_{i=1}^{N} x_i\right\| \ge a$ and $\left\|\sum_{i=1}^{N+1} x_i\right\| \le b$. Let

$$
\begin{aligned}
x &= \left\|\sum_{i=1}^{N} x_i\right\|^{-1} \sum_{i=1}^{N} x_i, \\
y &= \|x_{N+1}\|^{-1} x_{N+1}, \\
t &= \left\|\sum_{i=1}^{N} x_i\right\|^{-1} \|x_{N+1}\| \ge \varepsilon.
\end{aligned}
$$

Since $(x_i)_{i=1}^{\infty}$ is a block sequence with respect to F, there exists $j \in \mathbb{N}$ such that $x \in H_{m_j} \cap S_X$ and $y \in H^{m_j} \cap S_X$. Now if $\alpha = \varepsilon/t \in (0, 1]$,

$$\delta < \hat{\delta}_{\mathsf{E}}(\varepsilon, m_j) \le \|x + \varepsilon y\| - 1 \le \alpha(\|x + ty\| - 1) \le \|x + ty\| - 1.$$

Here we are using the fact that $\delta < \alpha(\|x + ty\| - 1)$ implies that $\|x + ty\| - 1 > 0$. However,

$$\|x + ty\| - 1 = \frac{\left\|\sum_{i=1}^{N+1} x_i\right\|}{\left\|\sum_{i=1}^{N} x_i\right\|} - 1 \le b/a - 1 < \delta,$$

a contradiction. $\square$

From the point of view of renorming theory, the strong asymptotic properties introduced above turn out to be equivalent to the classical ones on reflexive spaces admitting an FDD. This follows from Prus' characterisation of nearly uniformly convex norms [Pru1]. Let us recall that the notions of *nearly uniformly convex* space (NUC for short) and *nearly uniformly smooth* (NUS for short) were introduced by Huff [Huf] and Prus [Pru1]. A space is NUS if and only if it is AUS and reflexive and if and only if its dual is NUC.

Proposition 3.2.9. *Let* E *be an FDD for a reflexive Banach space* X. *If* X *is AUC (respectively, AUS), then there is an equivalent norm* $|||\ |||$ *in* X *and a blocking* F *of* E *such that* $(X, |||\ |||)$ *is strongly AUC (respectively, strongly AUS) with respect to* F.

Proof. First assume that X is AUC. Since X is reflexive, it is also NUC. Theorem 4.2 in [Pru1] provides a blocking $\mathsf{F} = (F_n)_n$ of E which satisfies $(p, 1)$-estimates with constant $C > 0$ for some $p > 1$ (actually, Theorem 4.2 in [Pru1] is stated for E being a basis, but it also works for FDDs). Following [Pru1], given a block $x \in X$ we define

$$|||x|||^p := \sup \left\{ \sum_{i=1}^{n} \|x_i\|^p : x = \sum_{i=1}^{n} x_i, \operatorname{ran}_{\mathsf{F}} x_i \cap \operatorname{ran}_{\mathsf{F}} x_j = \emptyset \text{ for all } i \neq j \right\}.$$

Then $|||\ |||$ can be extended to a norm in X which satisfies $\|x\| \leq |||x||| \leq C^{-1} \|x\|$ for every $x \in X$. Moreover, $|||x|||^p + |||y|||^p \leq |||x + y|||^p$ whenever $x \in \bigoplus_{i=1}^n F_i$ and $y \in \overline{\bigoplus_{i=n+1}^\infty F_i}$. Therefore $(X, |||\ |||)$ is strongly AUC with respect to F with modulus $\hat{\delta}_{\mathsf{F}}(t) \leq (1 + t^p)^{1/p} - 1$, as desired.

Finally, assume X is AUS. Then X^* is AUC and so there is a blocking F of E^* and an equivalent norm in X^* such that X^* is strongly AUC with respect to F under this new norm. Now the result follows from the duality between strongly AUC and strongly AUS norms proved in Proposition 3.2.5. $\square$

We finish the section by providing some examples of spaces having a basis which satisfy the classical asymptotic properties but not the stronger ones.

Example 3.2.10.

(a) Johnson and Schechtman constructed in [JO] a subspace Y of c_0 with a basis such that Y^* does not have the approximation property. Thus, Y is an AUS space and it does not admit a shrinking FDD. Therefore Y is not a strongly AUS space.
(b) Girardi proved in [Gir] that JT_*, the predual of the James Tree space, is an AUC space. Since JT_* is not isomorphic to a dual space, it does not admit a boundedly complete FDD. Thus JT_* is not a strongly AUC space.

Note that the failure of strong asymptotic properties in previous examples relies on the lack of reflexivity. We do not know any example of reflexive Banach space admitting an FDD which is AUC but not strongly AUC.

3.3 AUS injective tensor products

Lennard proved in [Len] that the space of trace class operators on a Hilbert space is weak* AUC. Equivalently, $\mathscr{K}(\ell_2, \ell_2)$ is AUS. This result was extended by Besbes in [Bes], who showed that $\mathscr{K}(\ell_p, \ell_{p'})$ is AUS whenever $1 < p < \infty$. Moreover, in [DKR$^+$1] it is proved that $\mathscr{K}(\ell_p, \ell_q)$ is AUS with power type $\min\{p', q\}$ for every $1 < p, q < \infty$. On the other hand, Causey recently showed in [Cau2] that the Szlenk index of $X\widehat{\otimes}_\varepsilon Y$ is equal to the maximum of the Szlenk indices of X and Y for all Banach spaces X and Y. In particular, $X\widehat{\otimes}_\varepsilon Y$ admits an equivalent AUS norm if and only if X and Y do. Moreover, Draga and Kochanek have proved in [DK2] that is possible to get an equivalent AUS norm in $X\widehat{\otimes}_\varepsilon Y$ with power type the maximum of the ones of the norm of X and Y. Nevertheless, it seems to be an open question if the injective tensor product of AUS spaces is an AUS space in its canonical norm. Our goal here is to give a partial positive answer to this question in the case in which both X and Y are strongly AUS.

Let us recall that if $T\colon X \to X$ and $S\colon Y \to Y$ are linear operators then

$$(T \otimes S)\left(\sum_{i=1}^{n} x_i \otimes y_i\right) = \sum_{i=1}^{n} T(x_i) \otimes S(y_i)$$

defines a linear operator from $X\widehat{\otimes}_\varepsilon Y$ to $X\widehat{\otimes}_\varepsilon Y$ such that $\|T \otimes S\| = \|T\| \cdot \|S\|$.

Now we give our main result concerning stability of asymptotic uniform smoothness under injective tensor products.

Theorem 3.3.1. *Let E, F be FDDs on Banach spaces X and Y, respectively. Then there exists a constant $C > 0$ such that*

$$\overline{\rho}_{X\widehat{\otimes}_\varepsilon Y}(t) \leq (1 + \hat{\rho}_{\mathsf{E}}(Ct))(1 + \hat{\rho}_{\mathsf{F}}(Ct)) - 1$$

for every $0 < t \leq 1/C$.

Proof. Let $Q_n^E = I - P_n^E$ and $Q_n^{\mathsf{F}} = I - P_n^{\mathsf{F}}$ be the complementary projections. Take $K = \sup\{\left\|P_n^{\mathsf{E}}\right\|, \left\|P_n^{\mathsf{F}}\right\| : n \in \mathbb{N}\}$. Fix $0 < t \leq \frac{1}{4K^2}$ and $\varepsilon > 0$. There exist increasing sequences $(m_n^{\mathsf{E}})_n$ and $(m_n^{\mathsf{F}})_n$ so that $\hat{\rho}_{\mathsf{E}}(Kt, m_n^{\mathsf{E}}) \leq \hat{\rho}_{\mathsf{E}}(Kt) + \varepsilon$ and $\hat{\rho}_{\mathsf{F}}(Kt, m_n^{\mathsf{F}}) \leq \hat{\rho}_{\mathsf{F}}(Kt) + \varepsilon$.

Note that $P^{\mathsf{E}}_{m_n^{\mathsf{E}}} \otimes P^{\mathsf{F}}_{m_n^{\mathsf{F}}}$ is a linear projection on $X\widehat{\otimes}_\varepsilon Y$. Moreover, the set of all $u \in S_{X\widehat{\otimes}_\varepsilon Y}$ such that $(P^{\mathsf{E}}_{m_n^{\mathsf{E}}} \otimes P^{\mathsf{F}}_{m_n^{\mathsf{F}}})u = u$ for some $n \in \mathbb{N}$ is dense in $S_{X\widehat{\otimes}_\varepsilon Y}$. Let $u \in S_{X\widehat{\otimes}_\varepsilon Y}$ be so that $(P^{\mathsf{E}}_{m_n^{\mathsf{E}}} \otimes P^{\mathsf{F}}_{m_n^{\mathsf{F}}})u = u$ for some n. Consider the finite codimensional subspace $Z = \ker(P^{\mathsf{E}}_{m_n^{\mathsf{E}}} \otimes P^{\mathsf{F}}_{m_n^{\mathsf{F}}})$. We claim that if $v \in Z$ and $\|v\| = t$ then

$$\|u + v\| \leq \left\|u + (I_X \otimes P^{\mathsf{F}}_{m_n^{\mathsf{F}}})v\right\| (1 + \hat{\rho}_{\mathsf{F}}(4K^2 t, m_n^{\mathsf{F}}))$$

Indeed, fix $x^* \in S_{X^*}$ and consider $y = (u + (I_X \otimes P^{\mathsf{F}}_{m_n})v)(x^*)$. Note that $y \in \bigoplus_{i \le m^{\mathsf{F}}_n} F_i$. We will distinguish two cases. Assume first that $\|y\| \ge \frac{1}{2K}$. It follows that

$$\begin{aligned}\|(u+v)(x^*)\| &= \left\|(u + (I_X \otimes P^{\mathsf{F}}_{m^{\mathsf{F}}_n})(v))(x^*) + (I_X \otimes Q^{\mathsf{F}}_{m^{\mathsf{F}}_n})(v)(x^*)\right\| \\ &= \|y\| \cdot \left\| \frac{y}{\|y\|} + \frac{(I_X \otimes Q^{\mathsf{F}}_{m^{\mathsf{F}}_n})(v)(x^*)}{\|y\|} \right\| \\ &\le \left\|u + (I_X \otimes P^{\mathsf{F}}_{m^{\mathsf{F}}_n})(v)\right\| (1 + \hat{\rho}_{\mathsf{F}}(4K^2 t, m^{\mathsf{F}}_n)) ,\end{aligned}$$

since $\left\|(I_X \otimes Q^{\mathsf{F}}_{m^{\mathsf{F}}_n})(v)(x^*)\right\| \le 2Kt$. Now assume that $\|y\| \le \frac{1}{2K}$. Then

$$\begin{aligned}\|(u+v)(x^*)\| &= \left\|y + (I_X \otimes Q^{\mathsf{F}}_{m^{\mathsf{F}}_n})(v)(x^*)\right\| \\ &\le \frac{1}{2K} + 2Kt \le \frac{1}{K} \le \left\|u + (I_X \otimes P^{\mathsf{F}}_{m^{\mathsf{F}}_n})(v)\right\| (1 + \hat{\rho}_{\mathsf{F}}(4K^2 t, m^{\mathsf{F}}_n))\end{aligned}$$

since $t \le \frac{1}{4K^2}$ and $\left\|u + (I_X \otimes P^{\mathsf{F}}_{m^{\mathsf{F}}_n})v\right\| \ge \frac{1}{K}$. The claim follows by taking supremum with $x^* \in S_{X^*}$. Now, take $y \in S_{Y^*}$ and consider $x = (u + (P^{\mathsf{E}}_{m^{\mathsf{E}}_n} \otimes P^{\mathsf{F}}_{m^{\mathsf{F}}_n})v)(y^*)$. Apply the same argument and the fact that $(P^{\mathsf{E}}_{m^{\mathsf{E}}_n} \otimes P^{\mathsf{F}}_{m^{\mathsf{F}}_n})v = 0$ to get that

$$\begin{aligned}\left\|(u + (I_X \otimes P^{\mathsf{F}}_{m^{\mathsf{F}}_n})v)(y^*)\right\| &\le \left\|u + (P^{\mathsf{E}}_{m^{\mathsf{E}}_n} \otimes P^{\mathsf{F}}_{m^{\mathsf{F}}_n})v\right\| (1 + \hat{\rho}_{\mathsf{E}}(4K^2 t, m^{\mathsf{E}}_n)) \\ &= (1 + \hat{\rho}_{\mathsf{E}}(4K^2 t, m^{\mathsf{E}}_n)) ,\end{aligned}$$

as desired. Thus,

$$\|u+v\| \le (1 + \hat{\rho}_{\mathsf{E}}(4K^2 t) + \varepsilon)(1 + \hat{\rho}_{\mathsf{F}}(4K^2 t) + \varepsilon)$$

for every $\varepsilon > 0$. Taking supremum with $v \in Z$, $\|v\| = t$ we get

$$\overline{\rho}_{X \widehat{\otimes}_\varepsilon Y}(t,u) \le (1 + \hat{\rho}_{\mathsf{E}}(4K^2 t) + \varepsilon)(1 + \hat{\rho}_{\mathsf{F}}(4K^2 t) + \varepsilon) - 1 .$$

Finally, note that the above inequality holds for all u in a dense subset of $S_{X \widehat{\otimes}_\varepsilon Y}$ and for every $\varepsilon > 0$, so we are done. $\square$

From the above theorem we get a number of corollaries.

Corollary 3.3.2. *Let X, Y be strongly AUS spaces. Then $X \widehat{\otimes}_\varepsilon Y$ is AUS. If moreover X and Y are strongly AUS with power type p and q, respectively, then $X \widehat{\otimes}_\varepsilon Y$ is AUS with power type* $\min\{p, q\}$.

Recently R. Causey have improved the above corollary showing that the injective tensor product of AUS spaces is AUS, see [Cau1].

Theorem 3.3.3. *Let X and Y be Banach spaces with monotone FDDs. If X and Y are uniformly smooth then $X\widehat{\otimes}_\varepsilon Y$ is AUS. Moreover, if X is uniformly smooth with power type p and Y is uniformly smooth with power type q then $X\widehat{\otimes}_\varepsilon Y$ is AUS with power type* $\min\{p,q\}$.

Proof. It follows from Proposition 3.2.4 and Corollary 3.3.2. □

Remark 3.3.4. The injective tensor product of strongly AUC spaces need not be AUC. Indeed, $\ell_2\widehat{\otimes}_\varepsilon\ell_2$ contains a subspace isometric to c_0, namely $\overline{\operatorname{span}}\{e_n\otimes e_n : n\in\mathbb{N}\}$ where $(e_n)_n$ denotes the standard basis of ℓ_2, and so it is not AUC. On the other hand, it is proved in [DKR+1] that $\ell_p\widehat{\otimes}_\varepsilon\ell_q$ is AUC whenever $p,q<2$. We do not know if a similar statement holds for a more general class of Banach spaces.

Recall that a Banach space admits an equivalent AUS norm if and only if its Szlenk index, $\mathrm{Sz}(X)$, is less or equal than ω (see [KOS, Raj8]). By a result of Schlumprecht [Sch2], every Banach space with separable dual embeds into a Banach space with a shrinking basis and the same Szlenk index. Together with the separable determination of the Szlenk index, this provides another proof of the following particular case of Theorem 1.3 in [Cau2]: $X\widehat{\otimes}_\varepsilon Y$ admits an equivalent AUS norm whenever X and Y admit equivalent AUS norms.

Corollary 3.3.5. *Let X, Y be Banach spaces such that X^* and Y are strongly AUS. Then $\mathscr{K}(X,Y)$ is AUS. If moreover X^* is strongly AUS with power type p and Y is strongly AUS with power type q then $\mathscr{K}(X,Y)$ is AUS with power type* $\min\{p,q\}$.

Proof. Note that if Y has an FDD then $X^*\widehat{\otimes}_\varepsilon Y$ is isometric to $\mathscr{K}(X,Y)$ and apply Corollary 3.3.2. □

The previous result and the isometry between $\mathscr{K}(X,Y)$ and $X^*\widehat{\otimes}_\varepsilon Y$ yields the generalisation of Theorem 4.3 in [DKR+1].

Theorem 3.3.6. *Let X, Y be Banach spaces and assume that X^* and Y have monotone FDDs. If X is uniformly convex and Y is uniformly smooth then $\mathscr{K}(X,Y)$ is AUS. Moreover, if X is uniformly convex with power type p and Y is uniformly smooth with power type q then $\mathscr{K}(X,Y)$ is AUS with power type* $\min\{p',q\}$.

Proof. It follows readily from Proposition 3.2.4, Corollary 3.3.5 and the duality between uniform convexity and uniform smoothness. □

Corollary 3.3.7. *Let X, Y be strongly AUS spaces. Then $\mathscr{N}(X,Y^*)$ is weak* AUC. If moreover X and Y are strongly AUS with power type p and q, respectively, then $\mathscr{N}(X,Y^*)$ is weak* AUC with power type* $\max\{p',q'\}$.

Proof. Note that Y^* is separable since Y admits a shrinking FDD by Proposition 3.2.8. By a result of Grothendieck, the spaces $(X\widehat{\otimes}_\varepsilon Y)^*$ and $\mathscr{N}(X,Y^*)$ are isometric. Now the result follows from Corollary 3.3.2 and the duality between AUS and weak* AUC norms. □

By a result of Van Dulst and Sims [vDS], the weak* AUC property for a dual space X^* implies the weak* fixed point property, i.e., that every nonexpansive mapping from a weak*-closed bounded convex subset of X^* into itself has a fixed point.

Corollary 3.3.8. *Let X, Y be Banach spaces with strongly AUS norms. Then $\mathscr{N}(X, Y^*)$ has the weak* fixed point property.*

3.4 Orlicz and Lorentz sequence spaces

We recall that an *Orlicz function M* is a continuous nondecreasing convex function defined on $\mathbb{R}^+$ such that $M(0) = 0$ and $\lim_{t\to+\infty} M(t) = +\infty$. An Orlicz function is said to satisfy the Δ_2-condition at zero if

$$\limsup_{t\to 0} \frac{M(2t)}{M(t)} < +\infty$$

Every Orlicz function M such that $\lim_{t\to+\infty} M(t)/t = +\infty$ has associated another Orlicz function M^*, which is its dual Young function, i.e.

$$M^*(u) = \sup\{uv - M(v) : 0 < v < +\infty\}\,.$$

To any Orlicz function M we associate the space h_M of all sequences of scalars $(x_n)_n$ such that $\sum_{n=1}^\infty M(|x_n|/\rho) < +\infty$ for all $\rho > 0$. The space h_M endowed with the *Luxemburg norm*,

$$\|x\| = \inf\left\{\rho > 0 : \sum_{n=1}^{\infty} M(|x_n|/\rho) \le 1\right\}$$

is a Banach space. A convexity argument (see [BM, Lemma 1.2.2]) yields $\sum_{n=1}^\infty M(|x_n|) \le \|x\|$ if $\|x\| \le 1$, and $\sum_{n=1}^\infty M(|x_n|) \ge \|x\|$ if $\|x\| \ge 1$, for every $x \in h_M$.

The *Boyd indices* of an Orlicz function M are defined as follows:

$$\alpha_M = \sup\left\{q : \sup_{0<u,v\le 1} \frac{M(uv)}{u^q M(v)} < +\infty\right\}, \qquad \beta_M = \inf\left\{q : \inf_{0<u,v\le 1} \frac{M(uv)}{u^q M(v)} > 0\right\}$$

It is easy to check that $1 \le \alpha_M \le \beta_M \le +\infty$, and $\beta_M < +\infty$ if and only if M satisfies the Δ_2 condition at zero. Moreover, the space ℓ_p, or c_0 if $p = \infty$, is isomorphic to a subspace of an Orlicz space h_M if and only if $p \in [\alpha_M, \beta_M]$.

It was shown in [GJT] that the space h_M is AUS if $\alpha_M > 1$. Moreover, α_M is the supremum of the numbers $\alpha > 0$ such that the modulus of asymptotic smoothness of h_M is of power type α. In addition, Borel-Mathurin proved in [BM] that if $\beta_M < +\infty$ then h_M is AUC, and β_M is the infimum of the numbers $\beta > 0$ such that its modulus of asymptotic convexity is of power type β. A similar result was proved by Delpech in [Del]. Moreover, their proofs actually show that h_M is strongly AUC (resp. strongly AUS) whenever it is AUC (resp. AUS).

Proposition 3.4.1. *Let M be an Orlicz function. If $\alpha_M > 1$ then h_M is strongly AUS with respect to the standard basis $\mathsf{E} = (e_n)_n$. Moreover, α_M is the supremum of the numbers $\alpha > 0$ such that $\hat{\rho}_{\mathsf{E}}$ is of power type α.*

Proof. Let $1 < \alpha < \alpha_M$. Then there exists $C > 0$ such that $M(uv) \leq Cu^\alpha M(v)$ for every $0 < u, v < 1$. We will show that $\hat{\rho}_{\mathsf{E}}(t, n) \leq Ct^\alpha$ for every n and $0 < t < 1$. For that, let $x \in \operatorname{span}\{e_1, \ldots, e_n\}$ and $y \in \overline{\operatorname{span}}\{e_i : i > n\}$ with $\|x\| = \|y\| = 1$. Note that $\|x + ty\| \geq 1$ since E is monotone. Moreover, we may assume that $M(1) = 1$ and thus $|y_i| \leq \|y\|$ for each i. Therefore,

$$\begin{aligned}\|x + ty\| &\leq \sum_{i=1}^{\infty} M(|x_i + ty_i|) = \sum_{i=1}^{n} M(|x_i|) + \sum_{i=n+1}^{\infty} M(t|y_i|) \\ &\leq 1 + Ct^\alpha \sum_{i=n+1}^{\infty} M(|y_i|) = 1 + Ct^\alpha \,,\end{aligned}$$

as desired. □

Proposition 3.4.2. *Let M be an Orlicz function. If $\beta_M < +\infty$ then h_M is strongly AUC with respect to the standard basis $\mathsf{E} = (e_n)_n$. Moreover, β_M is the infimum of the numbers $\beta > 0$ such that $\hat{\delta}_{\mathsf{E}}$ is of power type β.*

Proof. Let $\beta > \beta_M$. Then there exists $C > 0$ such that $M(uv) \geq Cu^\beta M(v)$ for every $0 < u, v < 1$. Now use the monotony of E and mimic the proof of Lemma 1.3.10 in [BM] to get that $\hat{\delta}_{\mathsf{E}}(t, n) \geq Ct^\beta$ for each $n \in \mathbb{N}$ and $0 < t < 1$. □

Our techniques lead to a characterisation of Orlicz functions M, N such that the space $\mathscr{K}(h_M, h_N)$ is AUS in terms of their Boyd indices α_M, β_M (see Section 3.4 for definitions).

Theorem 3.4.3. *Let M, N be Orlicz functions. The space $\mathscr{K}(h_M, h_N)$ is AUS if and only if $\alpha_M, \alpha_N > 1$ and $\beta_M < +\infty$. Moreover, $\min\{\beta'_M, \alpha_N\}$ is the supremum of the numbers $\alpha > 0$ such that the modulus of asymptotic smoothness of $\mathscr{K}(h_M, h_N)$ is of power type α.*

Proof. First, note that $\mathscr{K}(h_M, h_N)$ contain subspaces isometric to h_M^* and h_N. Thus, if either $\alpha_M = 1$, $\alpha_N = 1$ or $\beta_M = +\infty$ then $\mathscr{K}(h_M, h_N)$ contains a quotient isomorphic to ℓ_1 or ℓ_∞ and therefore it is not even AUS renormable. Now assume that $\alpha_M > 1, \beta_M < \infty$ and $\alpha_N > 1$. First, by Proposition 3.4.2, h_M is strongly AUC with respect to the standard basis $\mathsf{E} = (e_n)_n$ with power type β for each $\beta > \beta_M$. Moreover, since M satisfies the Δ_2 condition at 0 we have that E is an unconditional basis of h_M. Note that ℓ_1 is not isomorphic to a subspace of h_M and thus, by a theorem of James, E is a shrinking basis of h_M that it is also monotone. Thus we can apply Proposition 3.2.5 to get that h_M^* is strongly AUS with power type β for each $\beta < \beta'_M$. Finally, Proposition 3.4.1 implies that h_N is strongly AUS with power type α for each $\alpha < \alpha_N$. Now it is enough to apply Corollary 3.3.5. □

Lennard proved in [Len] that the trace class operators $\mathscr{N}(\ell_2, \ell_2)$ has the weak* fixed point property. This result was extended by Besbes [Bes] to $\mathscr{N}(\ell_p, \ell_q)$ with $p^{-1} + q^{-1} = 1$. Moreover, it is shown in [DKR+1] that the same is true for $1 < p, q < \infty$.

Corollary 3.4.4. *Let M, N be Orlicz functions such that $\alpha_M, \alpha_N > 1$ and $\beta_N < \infty$. Then the space $\mathscr{N}(h_M, h_N)$ has the weak* fixed point property.*

Proof. Note that h_N is reflexive since $1 < \alpha_N, \beta_N < \infty$. Thus the canonical basis $(e_n)_n$ of h_N is shrinking and monotone. Since h_N is strongly AUC with respect to $(e_n)_n$, we get that h_N^* is strongly AUS. Thus, we can apply Corollary 3.3.8. $\square$

Next result provides a characterisation of Orlicz functions M, N such that the space $\mathscr{K}(h_M, h_N)$ is NUS. This should be compared with [DKR+1, Corollary 4.4].

Corollary 3.4.5. *Let M, N be Orlicz functions. Then the following statements are equivalent:*

(i) $1 < \alpha_N$, $\beta_N < \alpha_M$ *and* $\beta_M < \infty$.
(ii) $\mathscr{K}(h_M, h_N)$ *is reflexive.*
(iii) $\mathscr{K}(h_M, h_N)$ *is NUS.*

Proof. The equivalence between (i) and (ii) was shown in [AO]. Since each NUS space is reflexive, (iii) implies (ii). Finally, if (i) and (ii) holds then $\mathscr{K}(h_M, h_N)$ is AUS and reflexive and thus it is NUS. $\square$

Finally, we will provide a result on strong asymptotic uniform convexity in Lorentz sequence spaces. Let us recall their definition. Let $1 \leq p < \infty$ and let w be a non-increasing sequence of positive numbers such that $w_1 = 1$, $\lim_n w_n = 0$ and $\sum_{n=1}^{\infty} w_n = \infty$. The *Lorentz sequence space* $d(w, p)$ is defined as

$$d(w,p) = \left\{ x = (x_n)_n \in c_0 : \|x\| = \sup_{\sigma} \left(\sum_{n=1}^{\infty} |x_{\sigma(n)}|^p w_n \right)^{1/p} < \infty \right\}$$

where σ ranges over all permutations of the natural numbers. We refer the reader to [LT] for more information about these spaces.

Proposition 3.4.6. *Let $d(w, p)$, $1 < p < \infty$ be a Lorentz sequence space. Let $S_n = \sum_{i=1}^{n} w_i$. The following conditions are equivalent:*

(i) $d(w, p)$ *is uniformly convex.*
(ii) $d(w, p)$ *is strongly AUC.*
(iii) $d(w, p)$ *is AUC.*
(iv) $\inf_n \frac{S_{2n}}{S_n} > 1$.

Proof. The equivalence between (i) and (iv) was shown by Altshuler in [Alt]. Note that the canonical basis $\mathsf{E} = (e_n)_n$ of $d(w, p)$ is monotone. Thus, (i)⇒(ii) follows from

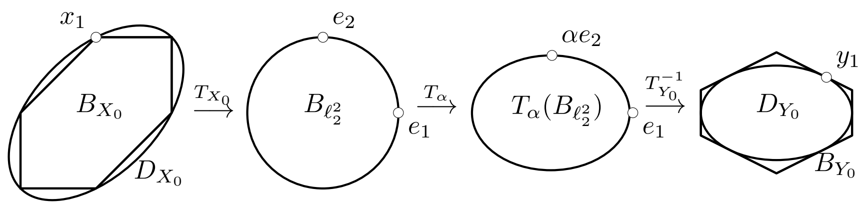

Figure 3.3: The proof of Proposition 3.5.1

Proposition 3.2.4. Moreover, $d(w,p)$ is reflexive since $p > 1$ and so (ii)⇒(iii) follows from Corollary 3.2.2. Finally, assume that $\inf_n \frac{S_{2n}}{S_n} = 1$ and let us show that $d(w,p)$ is not AUC. Fix $\varepsilon > 0$ and take $n \in \mathbb{N}$ such that $\frac{S_{2n}}{S_n} < 1 + \varepsilon$. Consider the sequence of unitary vectors given by $x_k = \sum_{i=k}^{k+n-1} \frac{1}{S_n^{1/p}} e_i$. Since E is a shrinking basis, we have that $(x_k)_k$ is weakly null. In addition,

$$\|x_1 + tx_k\|^p = \sum_{i=1}^{n} \frac{1}{S_n} w_i + \sum_{i=n+1}^{2n} \frac{1}{S_n} w_i = \frac{S_{2n}}{S_n}$$

whenever $k \geq n$. Thus

$$\overline{\delta}_{d(w,p)}(t, x_1) \leq \liminf_{k\to\infty} \|x_1 + tx_k\| - 1 \leq \varepsilon,$$

which finishes the proof. □

3.5 Strict convexity

Dilworth and Kutzarova proved in [DK1] that $\mathscr{L}(\ell_p, \ell_q)$ is not strictly convex for $1 \leq p \leq q \leq \infty$. We have obtained the following result by using John's ellipsoid theorem.

Proposition 3.5.1. *Let X, Y be Banach spaces with dimension greater or equal than 2. Then $\mathscr{K}(X,Y)$ and $X\widehat{\otimes}_\varepsilon Y$ are not strictly convex.*

Proof. First we will show that $\mathscr{K}(X,Y)$ is not strictly convex. For that, let Z be a 2-dimensional subspace of X^* and consider Q the canonical projection from X onto $X_0 := X/Z_\perp$. Note that X_0 is also 2-dimensional since X_0^* is isometric to Z. Let Y_0 be any 2-dimensional subspace of Y and denote $\iota : Y_0 \to Y$ the inclusion operator. Since $\overline{Q(B_X)} = B_{X_0}$, it follows that $T \mapsto \iota \circ T \circ Q$ defines an isometry from $\mathscr{K}(X_0, Y_0)$ onto a subspace of $\mathscr{K}(X,Y)$. Therefore, it suffices to show that $\mathscr{K}(X_0, Y_0)$ is not strictly convex. For that we will identify the vectors of X_0 and Y_0 with points in $\mathbb{R}^2$. Let D_{X_0} be the ellipsoid of minimum volume containing B_{X_0} and D_{Y_0} be the ellipsoid of maximum volume contained in B_{Y_0}. Moreover, fix $x_1 \in B_{X_0} \cap \partial D_{X_0}$ and $y_1 \in S_{Y_0} \cap D_{Y_0}$. Consider a linear map T_{X_0} which transforms D_{X_0} in $B_{\ell_2^2}$ and maps x_1 to e_1, and T_{Y_0} which transforms

D_{Y_0} in $B_{\ell_2^2}$ and maps y_1 to e_1. For each $\alpha \in [-1,1]$, let T_α be the linear map given by $T_\alpha(ae_1 + be_2) = ae_1 + b\alpha e_2$. Finally, consider $R_\alpha = T_{Y_0}^{-1} \circ T_\alpha \circ T_{X_0}$. Then

$$R_\alpha(B_{X_0}) \subset R_\alpha(D_{X_0}) \subset (T_{Y_0}^{-1} \circ T_\alpha)(B_{\ell_2^2}) \subset T_{Y_0}^{-1}(B_{\ell_2^2}) \subset D_{Y_0} \subset B_{Y_0}$$

so $\|R_\alpha\|_{\mathscr{K}(X_0,Y_0)} \leq 1$. Moreover, $R_\alpha(x_1) = (T_{Y_0}^{-1} \circ T_\alpha)(e_1) = T_{Y_0}^{-1}(e_1) = y_1$ and so R_α has norm one for each $\alpha \in [-1,1]$. Clearly $R_1 \neq R_{-1}$, so this shows that $\mathscr{K}(X_0, Y_0)$ is not strictly convex.

Finally, let X_1 be a 2-dimensional subspace of X. The injective tensor product respects subspaces isometrically and thus $X_1 \widehat{\otimes}_\varepsilon Y$ is isometric to a subspace of $X \widehat{\otimes}_\varepsilon Y$. Moreover, since X_1 is finite-dimensional we have that $X_1 \widehat{\otimes}_\varepsilon Y$ is isometric to $\mathscr{K}(X_1^*, Y)$ (see, e.g. [Rya, Corollary 4.13]), which is not strictly convex. This finishes the proof for $X \widehat{\otimes}_\varepsilon Y$. □

Remark 3.5.2. In [DK1] it is used Dvoretzky's theorem in order to show that none of the spaces $\mathscr{L}(\ell_p, \ell_q)$ or $\mathscr{L}(c_0, \ell_q)$ are super-reflexive, that is, they do not admit an equivalent uniformly convex norm. Indeed, the same argument can be used to prove that neither $\mathscr{K}(X,Y)$ nor $X \widehat{\otimes}_\varepsilon Y$ is super-reflexive whenever X and Y are infinite-dimensional.

Chapter 4

Duality of spaces of vector-valued Lipschitz functions

This chapter is organised as follows. In the first section we fix the notation and terminology about spaces of Lipschitz functions as well as Lipschitz free spaces that we are going to use in this chapter and the next one. The second section summarises the known results about the duality of Lipschitz free spaces in the scalar-valued case, namely the theorems of Weaver, Dalet and Kalton. We also include there new results, some of them coming from [GLPPRZ]. The ones involving the Mackey topology remain unpublished. The third section is devoted to the extension of the duality results of Dalet and Kalton to the vector-valued case and it is based on the papers [GLPRZ1, GLRZ]. In the fourth section we introduce a geometrical property, called unconditional almost squareness, which turns out to be useful for showing that a Banach space is not isometric to a dual one, and we apply this criterion to certain spaces of Lipschitz functions. The results of that section appear in [GLPRZ1] and [GLRZ].

4.1 Preliminaries

Given two metric spaces M and N, we denote $\mathrm{Lip}(M,N)$ the space of Lipschitz maps from M to N. Given $f \in \mathrm{Lip}(M,N)$, we denote by $\|f\|_L$ the best Lipschitz constant of f, that is,

$$\|f\|_L = \sup\left\{\frac{d(f(x),f(y))}{d(x,y)} : x,y \in M, x \neq y\right\}.$$

If the target space is a Banach space X then $\mathrm{Lip}(M,X)$ is a vector space and $\| \ \|_L$ is a seminorm on $\mathrm{Lip}(M,X)$ with $\|f\|_L = 0$ if and only if f is a constant map. In order to get a normed space, one should take the quotient of $\mathrm{Lip}(M,X)$ by the space of constant functions. Equivalently, we will fix a distinguished point in M, denoted by 0, and consider the space

$$\mathrm{Lip}_0(M,X) = \{f \in \mathrm{Lip}(M,X) : f(0) = 0\}.$$

It is well known that $(\mathrm{Lip}_0(M,X), \| \ \|_L)$ is a Banach space. Let us point out that the choice of the distinguished point in M is not relevant since it is easy to check that $f \mapsto f - f(a)$ defines a linear isometry between $\mathrm{Lip}_0(M,X)$ and $\mathrm{Lip}_a(M,X)$ for every $a \in M$.

For simplicity we will denote $\mathrm{Lip}_0(M,\mathbb{R})$ by $\mathrm{Lip}_0(M)$.

For every point $m \in M$, we will consider the evaluation functional $\delta(m) \in \mathrm{Lip}_0(M)^*$ given by $\langle f, \delta(m)\rangle = f(m)$. The *Lipschitz free space over M*, also called *Arens-Eels space over M* is defined as

$$\mathscr{F}(M) := \overline{\mathrm{span}}\{\delta(m) : m \in M\} \subset \mathrm{Lip}_0(M)^*.$$

Note that the map $\delta\colon m \mapsto \delta(m)$ defines a (non-linear) isometric embedding of M into $\mathscr{F}(M)$. The fundamental property of Lipschitz free spaces is that they linearise Lipschitz functions on M in the following sense: for every Lipschitz map $f \in \mathrm{Lip}_0(M,X)$ there exists a unique linear operator $T_f\colon \mathscr{F}(M) \to X$ such that $\|T_f\| = \|f\|_L$ and the following diagram commutes:

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, hook, "\delta"] & X \\ \mathscr{F}(M) \arrow[ur, "T_f"'] & \end{tikzcd}$$

Therefore, the operator $f \mapsto T_f$ defines a linear isometry from $\mathrm{Lip}_0(M,X)$ onto the space $\mathscr{L}(\mathscr{F}(M),X)$. In particular, taking $X=\mathbb{R}$ we get that $\mathscr{F}(M)$ is an isometric predual of $\mathrm{Lip}_0(M)$. We refer the reader to [GK, Wea2] for these and other properties of free spaces.

Example 4.1.1.

(a) Consider set of natural numbers endowed with the metric inherited from $\mathbb{R}$. Then $T\delta(n) := e_1 + \ldots + e_n$ defines an isometry from $\mathscr{F}(\mathbb{N})$ onto ℓ_1.

(b) The map $T\delta(x) := \chi_{[0,x]}$ defines an isometry from $\mathscr{F}([0,1])$ onto $L_1[0,1]$.

Note that, given a metric space (M,d) and $0<\alpha<1$, the function d^α is again a metric on M, which is usually called a *snowflacking* of d. Note that the α-Holder functions on M are precisely the Lipschitz functions with respect to the metric d^α. Indeed, one can consider a more general snowflacking in the following way. According to [Kal2], by a *gauge* we will mean a continuous, subadditive and increasing function $\omega : [0,+\infty) \to [0,+\infty)$ satisfying that $\omega(0)=0$ and that $\omega(t) \geq t$ for every $t \in [0,1]$. It is easy to check that if ω is a gauge then $\omega \circ d$ is a metric on M. Moreover, we say that a gauge ω is *non-trivial* whenever $\lim\limits_{t\to 0} \frac{\omega(t)}{t} = \infty$. Note that $\omega(t) = t^\alpha$ is a non-trivial gauge for every $0<\alpha<1$. The structure of the spaces $\mathscr{F}(M,\omega\circ d)$ is deeply analysed in [Kal2], where it is proved for instance that they have the Schur property.

Let us recall now some facts about extension of Lipschitz functions. It is well known that for every metric space M, every $A \subset M$ and every Lipschitz function $f\colon A \to \mathbb{R}$, one can find a Lipschitz function $\tilde{f}\colon M \to \mathbb{R}$ extending f such that $\left\|\tilde{f}\right\|_L = \|f\|_L$, a fact which goes back to McShane [McS]. Indeed, it suffices to take

$$\tilde{f}(x) = \sup\{f(y) - \|f\|_L\, d(x,y) : y \in A\}$$

which is known as the *McShane extension of f*. In particular, this fact allow us to

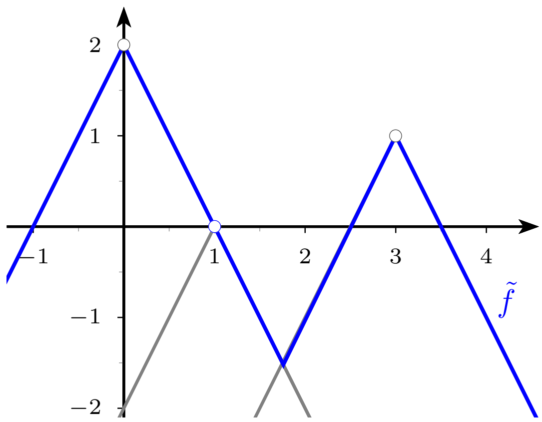

Figure 4.1: The McShane extension of the function $f\colon \{0,1,3\} \to \mathbb{R}$ given by $f(0) = 2, f(1) = 0, f(3) = 1$ to $M = \mathbb{R}$

identify $\mathscr{F}(A)$ as a subspace of $\mathscr{F}(M)$ whenever $0 \in A \subset M$. Indeed, if we consider $T\colon \mathscr{F}(A) \to \mathscr{F}(M)$ given by $T(\delta(x)) = \delta(x)$, then T^* is the restriction operator from $\mathrm{Lip}_0(M)$ to $\mathrm{Lip}_0(A)$. As a consequence of the McShane extension, T^* is an onto isometry and so T is an isometry from $\mathscr{F}(A)$ into $\mathscr{F}(M)$.

However, if the target space is other metric space N different from $\mathbb{R}$, then such an extension does not need to exist. A pair (M, N) is said to have the *contraction-extension property* (CEP) if, for every subset $A \subset M$, every Lipschitz map $f\colon A \to N$ extends to a Lipschitz map, with the same Lipschitz constant, defined on M. Examples of Banach spaces X such that (X, X) has the CEP are given in [BL, Chapter 2] and include Hilbert spaces and ℓ_∞^n.

At some moment we will deal with metric spaces in which there is another topology with respect to that the metric is lower semicontinuous. The following extension theorem of Matouskova proved in [Mat] will be very useful in that context.

Theorem 4.1.2 (Matouskova)**.** *Let (M, d) be a metric space, τ be a compact Hasdorff topology on M such that d is τ-lower semicontinuous, and N be a τ-closed subset of M. Let $f \in \mathrm{Lip}_0(N) \cap \mathscr{C}(N, \tau)$. Then there exists $g \in \mathrm{Lip}_0(M) \cap \mathscr{C}(M, \tau)$ such that $g|_N = f$, $\|g\|_L = \|f\|_L$ and $\min_N f \leq g \leq \max_N f$.*

Let us introduce some more notation. Given $x, y \in M$, $x \neq y$, the *molecule*

$$m_{x,y} := \frac{\delta(x) - \delta(y)}{d(x, y)}$$

is an element of $S_{\mathscr{F}(M)}$. The set of molecules will be denoted by V_M. Note that V_M is always 1-norming for $\mathrm{Lip}_0(M)$ and so $B_{\mathscr{F}(M)} = \overline{\mathrm{conv}}(V_M)$. This fact will be important in the next chapter.

We include here an estimation of the norm of differences of molecules that will be useful later.

Lemma 4.1.3. *Let M be a metric space and $x, y, u, v \in M$, with $x \neq y$ and $u \neq v$. Then*

$$\|m_{x,y} - m_{u,v}\| \leq 2\frac{d(x,u) + d(y,v)}{\max\{d(x,y), d(u,v)\}}$$

If moreover $\|m_{x,y} - m_{u,v}\| < 1$ then

$$\frac{\max\{d(x,u), d(y,v)\}}{\min\{d(x,y), d(u,v)\}} \leq \|m_{x,y} - m_{u,v}\|$$

Proof. First note that

$$\begin{aligned}
\|m_{x,y} - m_{u,v}\| &= \frac{\|d(u,v)(\delta(x) - \delta(y)) - d(x,y)(\delta(u) - \delta(v))\|}{d(x,y)d(u,v)} \\
&\leq \frac{\|(\delta(x) - \delta(y)) - (\delta(u) - \delta(v))\|}{d(x,y)} + \frac{|d(u,v) - d(x,y)|\,\|\delta(u) - \delta(v)\|}{d(x,y)d(u,v)} \\
&\leq 2\frac{d(x,u) + d(y,v)}{d(x,y)},
\end{aligned}$$

which proves the first inequality. Now assume that $\|m_{x,y} - m_{u,v}\| < 1$, and take $r = \min\{d(x,y), d(x,u)\}$. Consider the 1-Lipschitz functions $f(t) = \max\{r - d(t,x), 0\}$ and $g = f - f(0) \in \mathrm{Lip}_0(M)$. Then

$$\begin{aligned}
\|m_{x,y} - m_{u,v}\| \geq |\langle g, m_{x,y} - m_{u,v}\rangle| &= \left|\frac{f(x) - f(y)}{d(x,y)} - \frac{f(u) - f(v)}{d(u,v)}\right| \\
&= \left|\frac{r}{d(x,y)} + \frac{\max\{r - d(v,x), 0\}}{d(u,v)}\right| \geq \frac{r}{d(x,y)}
\end{aligned}$$

and so $r < d(x,y)$, which implies that $r = d(x,u)$ and $\frac{d(x,u)}{d(x,y)} \leq \|m_{x,y} - m_{u,v}\|$. Now we can exchange the role of the points to get the desired inequality. $\square$

Note that the above lemma implies that a sequence of molecules $(m_{x_n,y_n})_{n=1}^{\infty}$ converges to a molecule $m_{x,y}$ if and only if $x_n \to x$ and $y_n \to y$ in M. Indeed, a more general result holds, see Lemma 5.2.5.

The vector-valued version of Lipschitz free spaces has been recently considered in [BGLPRZ1] as a predual of the space of vector-valued Lipschitz functions $\mathrm{Lip}_0(M, X^*)$ in the spirit of the scalar version. Given $m \in M$ and $x \in X$, we will denote $\delta(m) \otimes x$ the element of $\mathrm{Lip}_0(M, X^*)^*$ given by $\langle f, \delta(m) \otimes x\rangle = f(m)(x)$. The *$X$-valued Lipschitz free space over M* is defined as

$$\mathscr{F}(M, X) \coloneqq \overline{\mathrm{span}}\{\delta(m) \otimes x : m \in M, x \in X\} \subset \mathrm{Lip}_0(M, X^*)^*.$$

It is proved in [BGLPRZ1] that the space $\mathscr{F}(M, X)$ is an isometric predual of $\mathrm{Lip}_0(M, X^*)$. Moreover, the operator $T\colon \mathrm{Lip}_0(M, X^*) \to \mathscr{L}(\mathscr{F}(M), X^*)$ given by $Tf(\delta(m)) = f(m)$ is $\sigma(\mathrm{Lip}_0(M, X^*), \mathscr{F}(M, X))$-to-$\sigma(\mathscr{L}(\mathscr{F}(M), X^*), \mathscr{F}(M)\widehat{\otimes}_\pi X)$-continuous and so $\mathscr{F}(M, X)$ is isometric to $\mathscr{F}(M)\widehat{\otimes}_\pi X$.

4.2 Duality results on Lipschitz free spaces

In this section we review some known results on the duality of the space $\mathscr{F}(M)$, which we will use in the next section to get some criteria on the duality of the space $\mathscr{F}(M,X)$. Moreover, we also get some slight improvements in the real-valued case. First, let us motivate the problem of studying the duality of these spaces. Kalton proved in [Kal2] that if M is uniformly discrete then $\mathscr{F}(M)$ has the RNP and the AP, and asked whether $\mathscr{F}(M)$ has the BAP for every uniformly discrete metric space M. Recall that M is *uniformly discrete* if

$$\inf\{d(x,y) : x,y \in M, x \neq y\} > 0.$$

The motivation of this question is that, if the answer is positive, then it follows that every separable Banach space is approximable, that is, the identity is the pointwise limit of a sequence of equi-uniformly continuous functions with relatively compact range. On the other hand, if M is uniformly discrete, separable and bounded, then $\mathscr{F}(M)$ is isomorphic to ℓ_1. Thus a negative answer in the separable setting would provide an equivalent norm on ℓ_1 which fails to have MAP (see [God3] for the discussion about this problem). Let us point out that by a theorem of Grothendieck (Theorem 5.50 in [Rya]), $\mathscr{F}(M)$ has the MAP whenever it is a dual space, so to determine when this happens could be useful for dealing with Kalton's question. However, there is a uniformly discrete space such that $\mathscr{F}(M)$ is not isometric to a dual Banach space, see Example 5.2.27.

Clearly, a necessary condition for a subspace of $\mathrm{Lip}_0(M)$ to be a predual of $\mathscr{F}(M)$ is that it is norming for $\mathscr{F}(M)$. The following lemma, which appears implicitly in [Wea2] and explicitly in [Kal2], is very useful to determine if that is the case.

Lemma 4.2.1 (Weaver–Kalton)**.** *Let X be a subspace of $\mathrm{Lip}_0(M)$ which is closed be taking maxima and minima. Then X is c-norming if and only if for every $x,y \in M$ and $\varepsilon > 0$ there is $f \in X$ such that $\|f\|_L \leq c+\varepsilon$ and $|f(x)-f(y)| = d(x,y)$.*

Proof. If X is c-norming then given $x,y \in M$, $x \neq y$ we can take $f \in S_X$ so that $\langle f, m_{x,y}\rangle \geq \frac{1}{c+\varepsilon}$ and the conclusion follows.

Now, assume that there is $c > 0$ satisfying the condition in the statement. Let $\gamma \in S_{\mathscr{F}(M)}$ and $\varepsilon > 0$ be fixed. Take $\eta \in S_{\mathscr{F}(M)}$ finitely supported such that $\|\gamma - \eta\| < \varepsilon$ and let $A = \mathrm{supp}(\eta) \cup \{0\}$. Now η belongs to the space $\mathscr{F}(A)$ and so there is $f \in \mathrm{Lip}_0(A)$ such that $\langle f, \eta\rangle = 1 = \|f\|_L$. For each $x,y \in A$, take $h_{x,y} \in X$ such that $\|h_{x,y}\|_L \leq c+\varepsilon$ and $|h_{x,y}(x) - h_{x,y}(y)| = d(x,y)$. One can check that

$$g = \min_{x\in A} \max_{y \in A\setminus\{x\}} \frac{|f(x)-f(y)|}{d(x,y)} h_{x,y}$$

defines a Lipschitz function in X such that $g|_A = f|_A$ and $\|g\|_L \leq c+\varepsilon$. Thus,

$$\langle g, \gamma\rangle \geq \langle g, \eta\rangle - \|\gamma - \eta\| \geq \langle f, \eta\rangle - \varepsilon = 1 - \varepsilon.$$

This shows that X is c-norming. $\square$

Lemma 4.2.1 motivates the following definition.

Definition 4.2.2. A subspace $X \subset \mathrm{Lip}_0(M)$ is said to *separate points uniformly* if there is a constant $c > 1$ such that for every $x, y \in M$ there is $f \in X$ satisfying that $\|f\|_L \leq c$ and $f(x) - f(y) = d(x, y)$.

Another necessary condition for a subspace X of $\mathrm{Lip}_0(M)$ to be a predual of $\mathscr{F}(M)$ is that X is made up of norm-attaining functionals. In the case that M is a compact metric space, this condition is clearly satisfied by the space of *little-Lipschitz* functions:

$$\mathrm{lip}_0(M) := \left\{ f \in \mathrm{Lip}_0(M) : \lim_{\varepsilon \to 0} \sup_{0 < d(x,y) < \varepsilon} \frac{|f(x) - f(y)|}{d(x, y)} = 0 \right\}.$$

That is, $\mathrm{lip}_0(M)$ is the subspace of Lipschitz functions that can be extended to a continuous function on $M \times M$ vanishing on the diagonal. Note that $\mathrm{lip}_0([0, 1]) = 0$, since if a function on $[0, 1]$ satisfies the little-Lipschitz condition then its derivative is identically 0. On the other hand, in the case of snowflawing metrics there are plenty of little-Lipschitz functions since $\mathrm{lip}_0(M, d^\alpha)$ contains $\mathrm{Lip}_0(M, d)$ for every $0 < \alpha < 1$. Indeed, given $f \in \mathrm{Lip}_0(M, d)$ and $\varepsilon > 0$, take $\delta = (\varepsilon / \|f\|_L)^{1-\alpha}$. If $d(x, y) < \delta$, then

$$|f(x) - f(y)| \leq \|f\|_L \, d(x, y) \leq \|f\|_L \, \delta^{1-\alpha} d(x, y)^\alpha = \varepsilon d(x, y)^\alpha.$$

This shows that $f \in \mathrm{lip}_0(M, d^\alpha)$.

In the compact case, Weaver proved in [Wea2, Theorem 3.3.3] that $\mathrm{lip}_0(M)$ is a predual of $\mathscr{F}(M)$ whenever it is large enough.

Theorem 4.2.3 (Weaver). *Let M be a compact metric space. The following assertions are equivalent:*

(i) *$\mathscr{F}(M)$ is isometric to $\mathrm{lip}_0(M)^*$;*

(ii) *$\mathrm{lip}_0(M)$ separates points uniformly.*

Weaver proved in [Wea2, Proposition 3.2.2] that if M is the middle-thirds Cantor set then $\mathrm{lip}_0(M)$ separates points uniformly, and that this is also the case of any metric space where the metric is of the form d^α for some $0 < \alpha < 1$.

Note that Weaver's theorem 4.2.3 can be deduced from previous comments and the following impressive result due to Petunīn and Plīčhko [PP] (also proved independently by Godefroy in [God2]). Recall that a subspace Y of a dual space X^* is *separating* if for every $x \in X \setminus \{0\}$ there is $x^* \in Y$ such that $x^*(x) \neq 0$. Clearly every norming subspace is separating.

Theorem 4.2.4 (Petunīn–Plīčhko). *Given a separable Banach space X, if a closed subspace Y of X^* is separating and consists of norm-attaining functionals, then X is isometric to Y^*.*

By using Petunīn–Plīčhko theorem, Dalet extended in [Dal2] the result of Weaver to the case of *proper* metric spaces, that is, metric spaces in which closed balls are compact sets. To this end, she introduced the space

$$S_0(M) := \left\{ f \in \mathrm{lip}_0(M) : \lim_{r\to\infty} \sup_{\substack{x \text{ or } y \notin B(0,r) \\ x \neq y}} \frac{|f(x)-f(y)|}{d(x,y)} = 0 \right\},$$

that plays the same role as $\mathrm{lip}_0(M)$ in the compact case.

Theorem 4.2.5 (Dalet)**.** *Let M be a proper metric space. The following assertions are equivalent:*

(i) *$\mathscr{F}(M)$ is isometric to $S_0(M)^*$;*
(ii) *$S_0(M)$ separates points uniformly.*

As in the compact case, it is not easy to find concrete examples of metric spaces such that $S_0(M)$ separates points uniformly. Dalet showed that this is the case if M is proper and either countable or ultrametric.

The following result of Weaver, proved in [Wea1] and apparently unnoticed, also extends Theorem 4.2.3 to the non-compact case. We state it for completeness.

Theorem 4.2.6 (Weaver)**.** *Let M be a* rigidly compact *metric space, that is, for every $k < 1$ and every $x \in M$ the closed ball $B(x, kd(x,0))$ is a compact set. Then the space*

$$X = \left\{ f \in \mathrm{Lip}_0(M) : \frac{f(\cdot)}{d(\cdot,0)} \in \mathscr{C}_0(M) \right\}$$

is a predual of $\mathscr{F}(M)$ precisely when it satisfies the following separating property: there exists $c > 0$ such that for every $x, y \in M$, $\varepsilon > 0$ and $a \in [0, d(y,0)]$, there is $f \in X$ such that $\|f\|_L \leq c$, $g(x) \leq \varepsilon + \max\{0, a - d(x,y)\}$ and $|g(y) - a| < \varepsilon$.

There is another generalisation of Theorem 4.2.3, due to Kalton, in which the hypothesis of compactness of M is replaced by the existence of a compact topology on M with respect to that the metric is lower semicontinuous. In this case the candidate for a predual is the space

$$\mathrm{lip}_\tau(M) := \mathrm{lip}_0(M) \cap \mathscr{C}(M,\tau).$$

Here we give a different proof of Kalton's result, based on Petunīn–Plīčhko theorem, that avoids the metrizability assumption of the considered compact topology on M in Kalton's original proof. Moreover, the condition

$$\forall x, y \in M \ \forall \varepsilon > 0 \ \exists f \in (1+\varepsilon) B_{\mathrm{lip}_\tau(M)} : f(x) - f(y) = d(x,y)$$

in Kalton's statement can be replaced by the (just formally) weaker hypothesis that the metric is τ-lower semicontinuous and $\mathrm{lip}_\tau(M)$ separates points uniformly. We also show that $\delta(M)$ is weak*-closed in $\mathscr{F}(M)$, a fact that will be useful in the next chapter.

Theorem 4.2.7 (Slight improvement of Theorem 6.2 in [Kal2])**.** *Let M be a separable bounded metric space and let τ be a compact Hausdorff topology on M such that d is τ-lower semicontinuous. Assume that* $\mathrm{lip}_\tau(M)$ *separates points uniformly. Then $\mathscr{F}(M)$ is isometric to* $\mathrm{lip}_\tau(M)^*$*. Moreover, $\delta(M)$ is weak*-closed in $\mathscr{F}(M)$.*

Proof. We will verify the conditions of Petunīn and Plīčhko's theorem. First, since M is bounded, it follows easily that $X = \mathrm{lip}_\tau(M)$ is a closed subspace of $\mathrm{Lip}_0(M)$. By Lemma 4.2.1 we get that X is separating since it is a lattice and separates points uniformly. Finally it remains to show that X is made of norm-attaining functionals. To this end, let $f \in S_X$ and take sequences $(x_n)_{n=1}^\infty$, $(y_n)_{n=1}^\infty$ in M such that $\lim_n \frac{f(x_n)-f(y_n)}{d(x_n,y_n)} = 1$. Note that $\inf_n d(x_n, y_n) =: \theta > 0$ since $f \in \mathrm{lip}_0(M)$. By the compactness of (M,τ) and the boundedness of d, we can find subnets $(x_\alpha)_\alpha$ of $(x_n)_n$ and $(y_\alpha)_\alpha$ of $(y_n)_n$ such that $x_\alpha \xrightarrow{\tau} x$, $y_\alpha \xrightarrow{\tau} y$ and $d(x_\alpha, y_\alpha) \to C \geq \theta$. Note that $C \geq d(x,y)$ by the lower semicontinuity. Then,

$$1 = \lim_\alpha \frac{f(x_\alpha) - f(y_\alpha)}{d(x_\alpha, y_\alpha)} = \frac{f(x) - f(y)}{C}$$

and so $x \neq y$. Moreover, it follows that $\frac{f(x)-f(y)}{d(x,y)} = 1$. Thus X is made up of norm-attaining functionals.

Finally, we show that $\delta(M)$ is weak*-closed. To this end, note that the weak* topology of $\mathscr{F}(M)$ and the topology τ coincide on $\delta(M)$. Indeed, every weak*-open set in $\delta(M)$ is also τ-open since X is made up of τ-continuous functions, so the weak* topology is weaker than τ on $\delta(M)$. Since τ is compact, we have that they agree on $\delta(M)$. □

As far as we know, the theorems of Weaver, Dalet and Kalton mentioned above are the only known positive results on the duality of $\mathscr{F}(M)$.

Kalton's theorem 4.2.7 can be used to deduce that certain Lipschitz free spaces over uniformly discrete metric spaces are dual ones. Note that in this case $\mathrm{Lip}_0(M) = \mathrm{lip}_0(M)$.

Corollary 4.2.8. *Let (M,d) be a uniformly discrete bounded separable metric space. Assume that there is a compact Hausdorff topology τ on M such that d is τ-lower semicontinuous. Then* $\mathrm{Lip}_0(M,d) \cap \mathscr{C}(M,\tau)$ *is an isometric predual of $\mathscr{F}(M)$.*

Proof. Given $x, y \in M$, $x \neq y$, define $f\colon \{x,y\} \to \mathbb{R}$ by $f(x) = 0$ and $f(y) = d(x,y)$. By Matouskova's extension theorem 4.1.2, there is $\tilde{f} \in \mathrm{Lip}_0(M) \cap \mathscr{C}(M,\tau)$ extending f such that $\left\|\tilde{f}\right\|_L = 1$. Thus, the hypotheses of Proposition 4.2.7 are satisfied. □

However, Corollary 4.2.8 does not provide an answer to Kalton's problem, even in the separable case, since there exists a uniformly discrete bounded countable metric space which does not admit a compact topology such that the metric is lower semicontinuous (see Example 5.2.26).

Let us show an easy example where Corollary 4.2.8 applies.

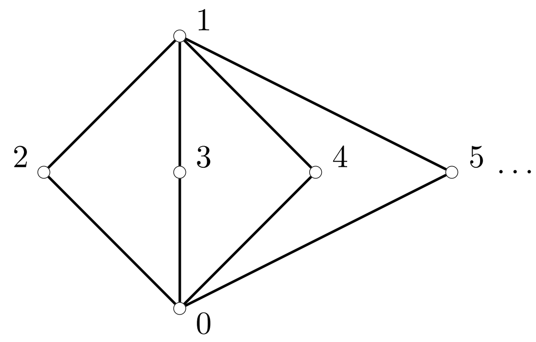

Figure 4.2: The metric space of Example 4.2.9

Example 4.2.9. Let $M := \{0\} \cup \mathbb{N}$. First, we define a graph structure on M. The edges are exactly the pairs $\{0, n\}$ or $\{n, 1\}$ where $n \notin \{0, 1\}$. We define the metric d on M as the shortest path distance in this graph. Now we define the topology τ by declaring all the points except 2 isolated. Clearly (M, τ) is compact and one can easily check that d is τ-lower semicontinuous.

Finally, we are going to give an extension of Corollary 4.2.8 in the non-separable setting. The idea is to replace the separability assumption by the completeness of the Mackey topology. Given a dual pair $\langle X, Y\rangle$, the *Mackey topology* $\mu(X, Y)$ is the topology on X of uniform convergence on all $\sigma(Y, X)$-compact convex balanced sets in Y. It turns out that $\mu(X, Y)$ is the strongest locally convex topology τ on X such that $(X, \tau)^* = Y$, recall also that $\sigma(X, Y)$ is the weakest topology with this property, see e.g. Corollary 3.44 in [FHH$^+$].

The following result proved in [GMZ1] provides a criterion for the completeness of the Mackey topology.

Theorem 4.2.10 (Guirao–Montesinos–Zizler)**.** *Let X be a Banach space and Y be a norm-closed separating subspace of X^*. Assume that (Y, w^*) is* Mazur*, that is, every w^*-sequentially continuous linear form $\phi\colon Y \to \mathbb{R}$ is w^*-continuous. Then $\mu(X, Y)$ is complete.*

In particular, if X is a separable Banach space and $Y \subset X^*$, then $\mu(X, Y)$ is complete. Indeed, (B_{X^*}, w^*) is compact and metrizable and so it is hereditarily separable. This implies that $Y = \cup_{n=1}^{\infty} nB_Y$ is w^*-separable too. It follows from this that (Y, w^*) is Mazur, so Theorem 4.2.10 yields that $\mu(X, Y)$ is complete.

The following result, apparently unpublished, shows that in the non-separable setting the completeness of the Mackey topology is crucial to ensure that a certain subspace of the dual is an isometric predual.

Theorem 4.2.11 (Rossi, [Ros])**.** *Let X be a Banach space and Y be a 1-norming norm-closed subspace of X^* which consists of norm-attaining functionals. The following are equivalent:*

(i) $\mu(X,Y)$ *is complete;*
(ii) X *is isometric to* Y^*.

Proof. That $\mu(Y^*,Y)$ is complete is well known, see e.g. Exercise 3.41 in [FHH+]. The converse is an application of a general version of James' characterisation of weakly compact sets in the setting of complete locally convex spaces, which says that a bounded $\sigma(X,(X,\tau)^*)$-closed subset of a complete locally convex space (X,τ) is $\sigma(X,(X,\tau)^*)$-compact provided every $\phi \in (X,\tau)^*$ attains its supremum on it (see [Flo], p. 59). Moreover, recall that $(X,\mu(X,Y))^* = Y$ and so $\sigma(X,Y) = \sigma(X,(X,\mu(X,Y))^*)$.

Assume that $\mu(X,Y)$ is complete. First we show that B_X is $\sigma(X,Y)$-compact by applying the James' compactness theorem to the locally convex space $(X,\mu(X,Y))$. To this end, note that B_X is $\mu(X,Y)$-bounded, since it is norm-bounded and $\mu(X,Y)$ is coarser that the norm topology. Moreover, since Y is 1-norming we have

$$B_X = \{x \in X : |f(x)| \leq 1 \text{ for all } f \in Y\}$$

and so B_X is $\sigma(X,Y)$-closed. Finally, since Y consists of norm-attaining functionals, every $f \in Y$ attains its supremum on B_X. This shows that B_X is $\sigma(X,Y)$-compact.

Now consider $\phi\colon X \to Y^*$ given by $\phi(x)(y) = \langle y, x\rangle$. Since Y is 1-norming we have that $\|\phi(x)\| = x$ for every $x \in X$. Moreover, $\overline{\phi(X)}^{w^*} = (\phi(X)_\perp)^\perp = \{0\}^\perp = Y^*$. Therefore, it only remains to check that $\phi(X)$ is w^*-closed. By Banach-Dieudonné theorem, it suffices to show that $\phi(X) \cap nB_{Y^*} = \phi(nB_X)$ is w^*-compact for every n. This follows from the $\sigma(X,Y)$-compactness of B_X and the $\sigma(X,Y)$-to-w^*-continuity of ϕ. $\square$

We need one more lemma for our non-separable extension of Corollary 4.2.8.

Lemma 4.2.12. *Let* (M,d) *be a uniformly discrete bounded metric space. Let* τ *be a topology* τ *on* M *such that* (M,τ) *is compact Hausdorff scattered and* d *is* τ*-lower semicontinuous. Consider* $Y = \mathrm{Lip}_0(M) \cap \mathscr{C}(M,\tau)$*. Then* (Y,w^*) *is Mazur.*

Proof. First note that Y is a closed subspace of $\mathrm{Lip}_0(M)$ since M is bounded. Moreover, by Matouskova extension theorem 4.1.2 and Lemma 4.2.1 we have that Y is 1-norming. Now, let $\phi\colon Y \to \mathbb{R}$ be w^*-sequentially continuous linear form and we will show that ϕ is continuous. To this end, it suffices to show that $\phi^{-1}(0)$ is w^*-closed in Y. Since $\phi^{-1}(0)$ is convex, the Banach-Dieudonné theorem says that it suffices to check that $\phi^{-1}(0) \cap nB_{\mathrm{Lip}_0(M)} = \phi^{-1}(0) \cap nB_Y$ is w^*-closed for each n. Clearly, it suffices to do this for $n=1$. Since ϕ is w^*-sequentially continuous, we have that $(\phi^{-1}(0) \cap B_Y, w^*)$ is sequentially closed. It is easy to check that the topologies w^* and τ_p agree on bounded subsets of $\mathrm{Lip}_0(M)$ and so $\phi^{-1}(0) \cap B_Y$ is τ_p-sequentially closed. Moreover, since (M,τ) is scattered, a theorem of Gerlits and Nagy [GN, Corollary, p. 158] ensures that $(\mathscr{C}(M,\tau),\tau_p)$ is Fréchet-Urysohn, that is, the sequential closure and the closure of every subset of $(\mathscr{C}(M,\tau),\tau_p)$ agree. Thus we get that $\phi^{-1}(0) \cap B_Y$ is τ_p-closed, hence w^*-closed. This shows that ϕ is w^*-continuous. $\square$

Note that every compact Hausdorff countable space is scattered. Thus, the next proposition can be seen as a slight generalisation of Corollary 4.2.8 to the non-separable case.

Proposition 4.2.13. *Let (M, d) be a uniformly discrete bounded metric space. Let τ be a compact Hausdorff topology τ on M such that (M, τ) is scattered and d is τ-lower semicontinuous. Then $\mathscr{F}(M)$ is isometric to $(\mathrm{Lip}_0(M) \cap \mathscr{C}(M, \tau))^*$.*

Proof. The Mackey topology $\mu(\mathscr{F}(M), \mathrm{Lip}_0(M) \cap \mathscr{C}(M, \tau))$ is complete as a consequence of Lemma 4.2.12 and Theorem 4.2.10. Moreover, given $x, y \in M$, $x \neq y$, define $f\colon \{x, y\} \to \mathbb{R}$ by $f(x) = 0$ and $f(y) = d(x, y)$. By Matouskova's extension theorem 4.1.2, there is $\tilde{f} \in \mathrm{Lip}_0(M) \cap \mathscr{C}(M, \tau)$ extending f such that $\left\|\tilde{f}\right\|_L = 1$. By Lemma 4.2.1 we get that $\mathrm{Lip}_0(M) \cap \mathscr{C}(M, \tau)$ is 1-norming. Now the result follows from Theorem 4.2.11. □

4.3 Duality of vector-valued Lipschitz free spaces

Our next aim is to provide a vector-valued version of Theorem 4.2.5 and Theorem 4.2.7. First, note that if S is a subspace of $\mathrm{Lip}_0(M)$ such that $S^* = \mathscr{F}(M)$, tensor product theory yields the following identification

$$\mathscr{F}(M, X^*) = \mathscr{F}(M) \widehat{\otimes}_\pi X^* = (S \widehat{\otimes}_\varepsilon X)^*$$

whenever either $\mathscr{F}(M)$ or X^* has the approximation property and either $\mathscr{F}(M)$ or X^* has the Radon-Nikodým property. This means that, under suitable hypotheses, the existence of a predual of $\mathscr{F}(M, X^*)$ relies on the scalar-valued case. Therefore, we will not concentrate on the existence but on giving a representation of a predual of $\mathscr{F}(M, X^*)$, in the case that exists, as a subspace of $\mathrm{Lip}_0(M, X^{**})$. To this end, we consider the following spaces of vector-valued Lipschitz functions.

$$\mathrm{lip}_0(M, X) := \left\{ f \in \mathrm{Lip}_0(M, X) : \lim_{\varepsilon \to 0} \sup_{0 < d(x,y) < \varepsilon} \frac{\|f(x) - f(y)\|}{d(x, y)} = 0 \right\},$$

$$S_0(M, X) := \left\{ f \in \mathrm{lip}_0(M, X) : \lim_{r \to \infty} \sup_{\substack{x \text{ or } y \notin B(0,r) \\ x \neq y}} \frac{\|f(x) - f(y)\|}{d(x, y)} = 0 \right\}.$$

We will study whether $S_0(M, X)$ is isometric to $S_0(M) \widehat{\otimes}_\varepsilon X$. We will show that it is the case under suitable assumptions on M and X. First, we analyse when $\mathscr{K}_{w^*,w}(X^*, Y)$ can be identified with $X \widehat{\otimes}_\varepsilon Y$.

Lemma 4.3.1. *Let X and Y be Banach spaces. Then $T \mapsto T^*$ defines an isometry from $\mathscr{K}_{w^*,w}(X^*, Y)$ onto $\mathscr{K}_{w^*,w}(Y^*, X)$.*

Proof. Let $T \in \mathscr{K}(X^*, Y) \cap \mathscr{L}_{w^*,w}(X^*, Y)$. Then $T^* \in \mathscr{K}(Y^*, X^{**})$. Moreover, given $y^* \in Y^*$ we have that $T^*(y^*) = y^* \circ T : X^* \to \mathbb{R}$ is weak*-continuous and thus $T^*(y^*) \in X$. Therefore $T^* \in \mathscr{K}(Y^*, X)$. Since T^* is $\sigma(Y^*, Y)$-to-$\sigma(X^{**}, X^*)$-continuous, we get $T^* \in \mathscr{L}_{w^*,w}(Y^*, X)$. Conversely, if $R \in \mathscr{K}(Y^*, X) \cap \mathscr{L}_{w^*,w}(Y^*, X)$ then $R^* \in \mathscr{K}(X^*, Y) \cap \mathscr{L}_{w^*,w}(X^*, Y)$ and $R^{**} = R$. □

The next proposition is well known (see Remark 1.2 in [RS2]), although we have not found a proof in the literature. We include it here for the sake of completeness.

Proposition 4.3.2. *Let X and Y be Banach spaces and assume that either X or Y has the AP. Then $\mathscr{K}_{w^*,w}(X^*, Y) = X \widehat{\otimes}_\varepsilon Y$.*

Proof. By Lemma 4.3.1 we may assume that Y has the AP. Clearly the inclusion $\supseteq$ holds, so let us prove the reverse one. To this end, let $T \colon X^* \to Y$ be a weak*-to-weak continuous compact operator. We will approximate T in norm by a finite-rank operator following the proof of [Rya, Proposition 4.12]. As Y has the approximation property, we can find a finite-rank operator $R \colon Y \to Y$ such that $\|x - R(x)\| < \varepsilon$ for every $x \in T(B_{X^*})$. Define $S := R \circ T$, which clearly is a finite-rank operator such that $\|S - T\| < \varepsilon$. Write $S = \sum_{i=1}^n x_i^{**} \otimes y_i$ for suitable $n \in \mathbb{N}, x_i^{**} \in X^{**}$ and $y_i \in Y$. Moreover, since S is weak*-to-weak continuous, the functional

$$y^* \circ S = \sum_{i=1}^n y^*(y_i) x_i^{**} \colon X^* \to \mathbb{R}$$

is weak*-continuous for every $y^* \in Y^*$. Thus $\sum_{i=1}^n y^*(y_i) x_i^{**} \in X$ for each $y^* \in Y^*$. Now, an easy argument of bilinearity allows us to assume that $y_1, \ldots, y_n$ are linearly independent. A straightforward application of Hahn-Banach theorem yields that, for every $i \in \{1, \ldots, n\}$, there exists $y_i^* \in Y^*$ such that $y_j^*(y_i) = \delta_{ij}$. Therefore, for every $j \in \{1, \ldots, n\}$, one has

$$x_j^{**} = \sum_{i=1}^n \delta_{ij} x_i^{**} = \sum_{i=1}^n y_j^*(y_i) x_i^{**} = y_j^* \circ S \in X.$$

Consequently, $S \in X \otimes Y$. To sum up, we have proved that each element of $\mathscr{K}_{w^*,w}(X^*, Y)$ can be approximated in norm by an element of $X \otimes Y$, so $\mathscr{K}_{w^*,w}(X^*, Y) = X \widehat{\otimes}_\varepsilon Y$ and we are done. □

In [JVSVV, Proposition 3.7] it is proved that if M is compact then $\mathrm{lip}_0(M, X)$ is linearly isometric to the space of compact operators from X^* to $\mathrm{lip}_0(M)$ which are continuous for the bounded weak* topology. We will extend that result to the case of proper metric spaces.

The following result, which is a slight generalisation of Theorem 3.2 in [Joh], gives us a criterion for compactness in $S_0(M)$.

Lemma 4.3.3. *Let M be a proper metric space and $\mathscr{F}$ be a subset of $S_0(M)$. Then the following are equivalent:*

(a) $\mathscr{F}$ *is relatively compact in* $S_0(M)$.
(b) $\mathscr{F}$ *is bounded and satisfies the* $S_0(M)$*-condition uniformly, that is, for each* $\varepsilon > 0$ *there exist* $\delta > 0$ *and* $r > 0$ *such that*

$$\sup_{\substack{x \text{ or } y \notin B(0,r) \\ x \neq y}} \frac{|f(x) - f(y)|}{d(x,y)} < \varepsilon, \quad \sup_{0<d(x,y)<\delta} \frac{|f(x) - f(y)|}{d(x,y)} < \varepsilon$$

holds for every $f \in \mathscr{F}$.

Proof. Notice that $(M \times M) \setminus \Delta$ is a locally compact space, where $\Delta = \{(x,x) : x \in M\}$. Let $K = ((M \times M) \setminus \Delta) \cup \{\infty\}$ be its one-point compactification. Given $f \in S_0(M)$, consider $\tilde{f} \colon K \to \mathbb{R}$ defined by $\tilde{f}(x,y) = \frac{f(x)-f(y)}{d(x,y)}$ and $\tilde{f}(\infty) = 0$. Clearly $\tilde{f}$ is continuous at each $(x,y) \in (M \times M) \setminus \Delta$. Moreover, given $\varepsilon > 0$ there exist $r > 0$ and $\delta > 0$ such that $\tilde{f}(x,y) < \varepsilon$ whenever (x,y) belongs to the complement of the compact set $(B(0,r) \times B(0,r)) \cap \{(u,v) : d(u,v) \geq \delta\}$. So $\tilde{f}$ is continuous at ∞. Thus $f \mapsto \tilde{f}$ defines a linear isometry, say Φ, from $S_0(M)$ into $\mathscr{C}(K)$. By the Ascoli-Arzelà theorem, a subset $\mathscr{F}$ of $S_0(M)$ is relatively compact if, and only if, $\Phi(\mathscr{F})$ is equicontinuous and bounded in $\mathscr{C}(K)$. Clearly $\mathscr{F}$ is bounded in $S_0(M)$ if, and only if, $\Phi(\mathscr{F})$ is bounded in $\mathscr{C}(K)$. Moreover, for every $x, y, u, v \in M$ with $x \neq y$ and $u \neq v$, Lemma 4.1.3 yields that

$$|\tilde{f}(x,y) - \tilde{f}(u,v)| \leq \|f\|_L \|m_{x,y} - m_{u,v}\| \leq 2\|f\|_L \frac{d(x,u) + d(y,v)}{d(x,y)}$$

for every $f \in \mathscr{F}$. Thus, if $\mathscr{F}$ is bounded then $\Phi(\mathscr{F})$ is equicontinuous at each $(x,y) \in (M \times M) \setminus \Delta$. Therefore it suffices to show that $\mathscr{F}$ satisfies the $S_0(M)$-condition uniformly if, and only if, $\Phi(\mathscr{F})$ is equicontinuous at ∞. That follows from the fact that the family $\mathscr{U} = \{U_{r,\delta}\}_{r>0,\delta>0}$ is a neighbourhood basis of ∞ in K, where

$$U_{r,\delta} = K \setminus (B(0,r) \times B(0,r)) \cap \{(u,v) : d(u,v) \geq \delta\}),$$

which finishes the proof. $\square$

Given a Lipschitz map $f \colon M \to X$, one can consider the linear operator $f^t \colon X^* \to \mathrm{Lip}_0(M)$ given by $f^t(x^*) = x^* \circ f$. It is shown in [JVSVV] that $f \mapsto f^t$ defines a linear isometry from $\mathrm{Lip}_0(M,X)$ onto $\mathscr{L}_{w^*,w^*}(X^*, \mathrm{Lip}_0(M))$. This identification and the characterisation of relative compactness in $S_0(M)$ given by Lemma 4.3.3 will be the key to proving the following result.

Theorem 4.3.4. *Let* M *be a proper metric space. Then* $S_0(M,X)$ *is linearly isometrically isomorphic to* $\mathscr{K}_{w^*,w}(X^*, S_0(M))$.

Proof. Let $f \in S_0(M,X)$. Notice that for any $x \neq y \in M$ and $x^* \in X^*$ we have

$$\frac{|x^* \circ f(x) - x^* \circ f(y)|}{d(x,y)} \leq \|x^*\| \frac{\|f(x) - f(y)\|}{d(x,y)},$$

thus $f^t(x^*) \in S_0(M)$. Moreover, the previous inequality proves that the functions in $f^t(B_{X^*})$ satisfy the $S_0(M)$-condition uniformly. By Lemma 4.3.3 we get that $f^t(B_{X^*})$ is a relatively compact subset of $S_0(M)$ and thus $f^t \in \mathscr{K}(X^*, S_0(M)) \cap \mathscr{L}_{w^*,w^*}(X^*, \mathrm{Lip}_0(M))$. The set $\overline{f^t(B_{X^*})}$ is norm-compact and thus every coarser Hausdorff topology agrees on it with the norm topology. In particular, the weak topology of $S_0(M)$ agrees on $f^t(B_{X^*})$ with the inherited weak* topology of $\mathrm{Lip}_0(M)$. Thus $f^t|_{B_{X^*}} : B_{X^*} \to S_0(M)$ is weak*-to-weak continuous. By [Kim, Proposition 3.1] we have that $f^t \in \mathscr{K}_{w^*,w}(X^*, S_0(M))$.

It only remains to show that the isometry is onto. For this take $T \in \mathscr{K}_{w^*,w}(X^*, S_0(M))$. We claim that T is weak*-to-weak* continuous from X^* to $\mathrm{Lip}_0(M)$. Indeed, assume that $(x^*_\alpha)_\alpha$ is a net in X^* weak* convergent to some $x^* \in X^*$. As every $\gamma \in \mathscr{F}(M)$ can be interpreted as an element in $S_0(M)^*$ then $(\langle \gamma, Tx^*_\alpha \rangle)_\alpha$ converges to $\langle \gamma, Tx^* \rangle$. Thus, $T \in \mathscr{L}_{w^*,w^*}(X^*, \mathrm{Lip}_0(M))$. By the isometry described above, there exists $f \in \mathrm{Lip}_0(M, X)$ such that $T = f^t$. Since $f^t(B_{X^*})$ is relatively compact, given $\varepsilon > 0$ there exist $r > 0$ and $\delta > 0$ such that

$$\sup_{0<d(x,y)<\delta} \frac{|x^* \circ f(x) - x^* \circ f(y)|}{d(x,y)} < \varepsilon, \quad \sup_{\substack{x \text{ or } y \notin B(0,r) \\ x \neq y}} \frac{|x^* \circ f(x) - x^* \circ f(y)|}{d(x,y)} < \varepsilon$$

for each $x^* \in B_{X^*}$. Taking supremum with $x^* \in B_{X^*}$, we get that

$$\sup_{0<d(x,y)<\delta} \frac{\|f(x) - f(y)\|}{d(x,y)} \leq \varepsilon, \quad \sup_{\substack{x \text{ or } y \notin B(0,r) \\ x \neq y}} \frac{\|f(x) - f(y)\|}{d(x,y)} \leq \varepsilon,$$

so $f \in S_0(M, X)$. Consequently, the isometry is onto and we are done. □

From Proposition 4.3.2 and Theorem 4.3.4 we get the following.

Corollary 4.3.5. *Let M be a proper metric space. If either $S_0(M)$ or X has the AP, then $S_0(M, X)$ is linearly isometrically isomorphic to $S(M)\widehat{\otimes}_\varepsilon X$.*

As a consequence we get our first duality result in the vector-valued setting.

Corollary 4.3.6. *Let M be a proper metric space and let X be a Banach space. Assume that $S_0(M)$ separates points uniformly. If either $\mathscr{F}(M)$ or X^* has the AP, then*

$$S_0(M, X)^* = \mathscr{F}(M, X^*).$$

Proof. As $S_0(M)$ separates points uniformly, $S_0(M)^* = \mathscr{F}(M)$ by Dalet's theorem 4.2.5. Thus $S_0(M)$ is an Asplund space. Consequently, we get from Corollary 4.3.5 and Theorem 0.3.1 that

$$S_0(M, X)^* = (S_0(M)\widehat{\otimes}_\varepsilon X)^* = \mathscr{F}(M)\widehat{\otimes}_\pi X^* = \mathscr{F}(M, X^*),$$

so we are done. □

Now we exhibit some examples in which the above result applies.

Corollary 4.3.7. *Let M be a proper metric space and X be a Banach space. Then $S_0(M,X)^* = \mathscr{F}(M,X^*)$ whenever M and X satisfy one of the following assumptions:*

(a) *M is countable.*
(b) *M is ultrametric, that is, $d(x,y) \leq \max\{d(x,z), d(z,y)\}$ for every $x,y,z \in M$.*
(c) *$d = \omega \circ d'$ for a non-trivial gauge ω and a metric d', and either $\mathscr{F}(M)$ or X^* has the AP.*
(d) *M is the middle-third Cantor set.*

Proof. If M satisfies either (a) or (b), then $S_0(M)$ separates points uniformly and $\mathscr{F}(M)$ has the approximation property [Dal2]. Thus Corollary 4.3.6 applies. Moreover, if M satisfies (c) then Proposition 2.5 in [GLPRZ1] does the work. Finally, [Wea2, Proposition 3.2.2] yields (d). □

Throughout the rest of the section we will consider a bounded metric space (M,d) and a compact Hausdorff topology τ on M such that d is τ-lower semicontinuous. Our next goal is to find a natural extension of Kalton's duality theorem 4.2.7 to the vector-valued case. We will prove that, under suitable assumptions, the space

$$\mathrm{lip}_\tau(M,X) \coloneqq \mathrm{lip}_0(M,X) \cap \{f\colon M \to X : f \text{ is } \tau\text{-to-}\|\ \| \text{ continuous}\}$$

is a predual of $\mathscr{F}(M,X^*)$. For this, we begin by characterising relative compactness in $\mathrm{lip}_\tau(M)$.

Lemma 4.3.8. *Let (M,d) be a metric space of radius R and τ be a compact Hausdorff topology on M such that d is τ-lower semicontinuous. Let $\mathscr{F}$ be a subset of $\mathrm{lip}_\tau(M)$. Then $\mathscr{F}$ is relatively compact in $\mathrm{lip}_\tau(M)$ if, and only if, the following three conditions hold:*

(a) *$\mathscr{F}$ is bounded.*
(b) *$\mathscr{F}$ satisfies the following uniform little-Lipschitz condition: for every $\varepsilon > 0$ there exists $\delta > 0$ such that*
$$\sup_{0<d(x,y)<\delta} \frac{|f(x) - f(y)|}{d(x,y)} < \varepsilon$$
for every $f \in \mathscr{F}$.
(c) *$\mathscr{F}$ is equicontinuous in $\mathscr{C}(M,\tau)$, i.e. for every $x \in M$ and every $\varepsilon > 0$ there exists a τ-neighbourhood U of x such that $y \in U$ implies $\sup_{f\in\mathscr{F}} |f(x) - f(y)| < \varepsilon$.*

Proof. In [Kal2, Theorem 6.2] it is proved that $\mathrm{lip}_\tau(M)$ is isometrically isomorphic to a subspace of a space of continuous functions on a compact set. Indeed, let

$$K \coloneqq \{(x,y,t) \in (M,\tau) \times (M,\tau) \times [0,2R] : d(x,y) \leq t\}.$$

Then K is compact by the τ-lower semicontinuity of d. Moreover, the map $\Phi\colon \operatorname{lip}_\tau(M) \to \mathscr{C}(K)$ defined by

$$\Phi(f)(x,y,t) := \begin{cases} \frac{f(x)-f(y)}{t} & t \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

is a linear isometry. Therefore, we have that $\mathscr{F}$ is relatively compact if, and only if, $\Phi(\mathscr{F})$ is relatively compact. By Ascoli-Arzelà theorem we get that $\mathscr{F}$ is relatively compact if, and only if, $\Phi(\mathscr{F})$ is bounded and equicontinuous in $\mathscr{C}(K)$. We will first assume that conditions (a), (b) and (c) hold. It is clear that $\Phi(\mathscr{F})$ is bounded, so let us prove the equicontinuity of $\Phi(\mathscr{F})$. To this end pick $(x,y,t) \in K$. Now we have two possibilities:

(i) If $t \neq 0$ we can find a positive number $\eta < t$ such that $t' \in (t-\eta, t+\eta)$ implies $\left|\frac{1}{t} - \frac{1}{t'}\right| < \frac{\varepsilon}{4R\alpha}$, where $\alpha = \sup_{f\in\mathscr{F}} \|f\|_L$. Now, as x and y are two points of M and $\mathscr{F}$ satisfies condition (c), we conclude the existence of a τ-neighbourhood U of x and a τ-neighbourhood V of y in M satisfying

$$|f(x) - f(x')| + |f(y) - f(y')| < \frac{\varepsilon t}{2}$$

for every $f \in \mathscr{F}$, $x' \in U$ and $y' \in V$. Now, given $(x', y', t) \in (U \times V \times (t-\eta, t+\eta)) \cap K$, one has

$$\begin{aligned} |\Phi f(x,y,t) - \Phi f(x',y',t')| &= \left| \frac{f(x)-f(y)}{t} - \frac{f(x')-f(y')}{t'} \right| \\ &\leq \left|\frac{1}{t} - \frac{1}{t'}\right| |f(x') - f(y')| + \frac{1}{t}|f(x) - f(x') + f(y) - f(y')| \\ &\leq \frac{\varepsilon}{4R\alpha} \|f\|_L \, d(x',y') + \frac{\varepsilon t}{2t} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

for every $f \in \mathscr{F}$, which proves equicontinuity of $\Phi(f)$ at (x,y,t).

(ii) If $t = 0$ then $x = y$. Take an arbitrary $\varepsilon > 0$. By (b) we get $\delta > 0$ such that $0 < d(x,y) < \delta$ implies $\frac{|f(x)-f(y)|}{d(x,y)} < \varepsilon$ for every $f \in \mathscr{F}$. Now, given $(x', y', t) \in (M \times M \times [0,\delta)) \cap K$ we have $d(x',y') \leq t < \delta$ and so, given $f \in \mathscr{F}$, it follows

$$|\Phi f(x',y',t)| \leq \frac{|f(x') - f(y')|}{t} < \varepsilon \frac{d(x',y')}{t} \leq \varepsilon,$$

which proves equicontinuity at $(x,x,0)$.

This shows that $\Phi(\mathscr{F})$ is equicontinuous whenever conditions (a), (b) and (c) are satisfied.

Conversely, assume that $\Phi(\mathscr{F})$ is equicontinuous in $\mathscr{C}(K)$. It is clear that $\mathscr{F}$ is bounded, so let us prove that conditions (b) and (c) are satisfied. We shall begin by proving (c), for which we fix $x \in M$ and $\varepsilon > 0$. Given $t \in [0, 2R]$, by equicontinuity of $\Phi(\mathscr{F})$ at the point (x,x,t) we can find a τ-neighbourhood U_t of x and $\eta_t > 0$ such that $x' \in U_t$ and $t' \in (t-\eta_t, t+\eta_t)$ implies $|\Phi f(x,x',t')| < \frac{\varepsilon}{2R}$ for every $f \in \mathscr{F}$. Then $[0,2R] \subset \bigcup_t (t-\eta_t, t+\eta_t)$ and thus there exist $t_1, \ldots, t_n$ such that $[0,2R] \subset \bigcup_{i=1}^n (t_i - \eta_{t_i}, t_i + \eta_{t_i})$. Now take

$U = \bigcap_{i=1}^n U_{t_i}$. We will show that U is the desired τ-neighbourhood of x. Pick $x' \in U$. Then there exists t_i such that $d(x,x') \in (t_i - \eta_{t_i}, t_i + \eta_{t_i})$. Since $x' \in U_{t_i}$ we get

$$|\Phi f(x,x',d(x,x'))| = \left|\frac{f(x)-f(x')}{d(x,x')}\right| < \frac{\varepsilon}{2R}$$

and thus $|f(x)-f(x')| < \varepsilon$ for every $x' \in U$ and $f \in \mathscr{F}$. This proves that $\mathscr{F}$ is equicontinuous at every $x \in M$.

Finally, let us prove condition (b). To this end pick $\varepsilon > 0$. For every $x \in M$ we have, from equicontinuity of $\Phi(\mathscr{F})$ at $(x,x,0)$, the existence of a τ-open neighbourhood U_x of x in M and $\delta_x > 0$ such that $x',y' \in U_x$ and $0 < t < \delta_x$ implies $|\Phi f(x',y',t)| < \varepsilon$ for every $f \in \mathscr{F}$.

As $\Delta := \{(x,x) : x \in M\} \subset \bigcup_{x\in M} U_x \times U_x$, by compactness there exist $x_1,\dots,x_n \in M$ such that $\Delta \subseteq \bigcup_{i=1}^n U_{x_i} \times U_{x_i}$. Note that we can write Δ as the intersection of a decreasing sequence of τ-compact sets,

$$\Delta = \bigcap_{n\geq 1} \{(x,y) \in M\times M : d(x,y) \leq n^{-1}\}.$$

It follows easily from this that there is n_0 such that

$$\{(x,y) \in M\times M : d(x,y) \leq n_0^{-1}\} \subset \bigcup_{i=1}^n U_{x_i} \times U_{x_i}.$$

Set $\delta := \min\{1/n_0, \delta_{x_1},\dots,\delta_{x_n}\}$. Now, if $x,y \in M$ verify that $0 < d(x,y) < \delta$ then there exists $i \in \{1,\dots,n\}$ such that $x,y \in U_{x_i}$. As $d(x,y) < \delta \leq \delta_{x_i}$ we get

$$\frac{|f(x)-f(y)|}{d(x,y)} = |\Phi f(x,y,d(x,y))| < \varepsilon$$

for every $f \in \mathscr{F}$, which proves (b) and finishes the proof. $\square$

The previous lemma allows us to identify $\mathrm{lip}_\tau(M,X)$ as a space of compact operators from X^* to $\mathrm{lip}_\tau(M)$.

Theorem 4.3.9. *Let M be a metric space and let τ be a topology on M such that (M,τ) is compact and d is τ-lower semicontinuous. Then $\mathrm{lip}_\tau(M,X)$ is isometrically isomorphic to $\mathscr{K}_{w^*,w}(X^*, \mathrm{lip}_\tau(M))$. Moreover, if either $\mathrm{lip}_\tau(M)$ or X has the AP, then $\mathrm{lip}_\tau(M,X)$ is isometrically isomorphic to $\mathrm{lip}_\tau(M)\widehat{\otimes}_\varepsilon X$.*

Proof. It is shown in [JVSVV] that $f \mapsto f^t$ defines an isometry from $\mathrm{Lip}_0(M,X)$ onto $\mathscr{L}_{w^*,w^*}(X^*, \mathrm{Lip}_0(M)))$, where $f^t(x^*) = x^* \circ f$. Let f be in $\mathrm{lip}_\tau(M,X)$ and let us prove that $f^t \in \mathscr{K}_{w^*,w}(X^*, \mathrm{lip}_\tau(M))$. Notice that $x^*\circ f$ is τ-continuous for every $x^* \in X^*$. Moreover, for every $x \neq y \in M$ and every $x^* \in X^*$, we have

$$\frac{|x^*\circ f(x) - x^*\circ f(y)|}{d(x,y)} \leq \|x^*\| \frac{\|f(x)-f(y)\|}{d(x,y)} \tag{4.1}$$

thus $x^* \circ f \in \mathrm{lip}_0(M)$. Therefore $f^t(X^*) \subset \mathrm{lip}_\tau(M)$. We claim that $f^t(B_{X^*})$ is relatively compact in $\mathrm{lip}_\tau(M)$. In order to show that, we need to check the conditions in Lemma 4.3.8. First, it is clear that $f^t(B_{X^*})$ is bounded. Moreover, it follows from (4.1) that the functions in $f^t(B_{X^*})$ satisfy the little-Lipschitz condition uniformly. Finally, $f^t(B_{X^*})$ is equicontinuous in the sense of Lemma 4.3.8. Indeed, given $x \in M$ and $\varepsilon > 0$, there exists a τ-neighbourhood U of x such that $\|f(x) - f(y)\| < \varepsilon$ whenever $y \in M$. That is,

$$\sup_{x^* \in B_{X^*}} |x^* \circ f(x) - x^* \circ f(y)| < \varepsilon$$

whenever $y \in U$, as we wanted. Now, Lemma 4.3.8 implies that $f^t(B_{X^*})$ is a relatively compact subset of $\mathrm{lip}_\tau(M)$ and thus $f^t \in \mathscr{K}(X^*, \mathrm{lip}_\tau(M)) \cap \mathscr{L}_{w^*,w^*}(X^*, \mathrm{Lip}_0(M))$. Finally, the set $\overline{f^t(B_{X^*})}$ is norm-compact and thus every coarser Hausdorff topology agrees on it with the norm topology. In particular, the weak topology of $\mathrm{lip}_\tau(M)$ agrees on $f^t(B_{X^*})$ with the inherited weak* topology of $\mathrm{Lip}_0(M)$. Thus $f^t|_{B_{X^*}} : B_{X^*} \to \mathrm{lip}_\tau(M)$ is weak*-to-weak continuous. By [Kim, Proposition 3.1] we have that $f^t \in \mathscr{K}_{w^*,w}(X^*, \mathrm{lip}_\tau(M))$.

It remains to show that the isometry is onto. For this take $T \in \mathscr{K}_{w^*,w}(X^*, \mathrm{lip}_\tau(M))$. We claim that T is weak*-to-weak* continuous from X^* to $\mathrm{Lip}_0(M)$. Indeed, assume that $(x^*_\alpha)_\alpha$ is a net in X^* weak* convergent to some $x^* \in X^*$. Since every $\gamma \in \mathscr{F}(M)$ is also an element of $\mathrm{lip}_\tau(M)^*$, we get that $(\langle \gamma, Tx^*_\alpha \rangle)_\alpha$ converges to $\langle \gamma, Tx^* \rangle$. Thus, $T \in \mathscr{L}_{w^*,w^*}(X^*, \mathrm{Lip}_0(M)))$. By the isometry described above, there exists $f \in \mathrm{Lip}_0(M, X)$ such that $T = f^t$. Let us prove that f actually belongs to $\mathrm{lip}_\tau(M, X)$. As $f^t(B_{X^*})$ is relatively compact, then Lemma 4.3.8 we have that for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\sup_{0 < d(x,y) < \delta} \frac{|x^* \circ f(x) - x^* \circ f(y)|}{d(x,y)} < \varepsilon$$

for each $x^* \in B_{X^*}$. By taking supremum with $x^* \in B_{X^*}$ we get that

$$\sup_{0 < d(x,y) < \delta} \frac{\|f(x) - f(y)\|}{d(x,y)} \leq \varepsilon,$$

so $f \in \mathrm{lip}_0(M, X)$. We will prove, to finish the proof, that f is τ-to-$\|\ \|$ continuous. To this end take $y \in M$ and $\varepsilon > 0$. By equicontinuity of $f^t(B_{X^*})$ we can find a τ-neighbourhood U of y such that $|x^* \circ f(y') - x^* \circ f(y)| < \varepsilon$ for every $x^* \in B_{X^*}$ and $y' \in U$. Now,

$$\|f(y') - f(y)\| = \sup_{x^* \in B_{X^*}} |x^*(f(y') - f(y))| \leq \varepsilon$$

for every $y' \in U$. Consequently, f is τ-to-$\|\ \|$ continuous. So $f \in \mathrm{lip}_\tau(M, X)$, as desired.

Finally, if either $\mathrm{lip}_\tau(M)$ or X has the AP, then Proposition 4.3.2 yields the equality $\mathscr{K}_{w^*,w}(X^*, \mathrm{lip}_\tau(M)) = \mathrm{lip}_\tau(M) \widehat{\otimes}_\varepsilon X$. $\square$

Now we get our duality result for vector-valued Lipschitz free Banach spaces, which extends [Kal2, Theorem 6.2].

Corollary 4.3.10. *Let M be a separable bounded metric space. Suppose that τ is a compact Hausdorff topology on M such that the metric is τ-lower semicontinous and* $\mathrm{lip}_\tau(M)$ *separates points uniformly. If either $\mathscr{F}(M)$ or X^* has the AP, then* $\mathrm{lip}_\tau(M,X)^* = \mathscr{F}(M,X^*)$.

Proof. By Theorem 4.2.7 we have that $\mathrm{lip}_\tau(M)$ is a predual of $\mathscr{F}(M)$. Consequently, $\mathscr{F}(M)$ has the RNP. Therefore, we get from Theorem 4.3.9 and Theorem 0.3.1.(d) that

$$\mathrm{lip}_\tau(M,X)^* = (\mathrm{lip}_\tau(M)\widehat{\otimes}_\varepsilon X)^* = \mathscr{F}(M)\widehat{\otimes}_\pi X^* = \mathscr{F}(M,X^*),$$

which finishes the proof. □

The last result applies to the following particular case (see Proposition 6.3 in [Kal2]). Given two Banach spaces X, Y, and a non-trivial gauge ω, we will denote $\mathrm{lip}_{\omega,*}(B_{X^*}, Y) := \mathrm{lip}_{w^*}((B_{X^*}, \omega \circ \| \ \|), Y)$.

Corollary 4.3.11. *Let X and Y be Banach spaces, and let ω be a non-trivial gauge. Assume that X^* is separable and that $\mathscr{F}(B_{X^*}, \omega \circ \| \ \|)$ or Y^* has the AP. Then $\mathrm{lip}_{\omega,*}(B_{X^*}, Y)$ is isometric to $\mathrm{lip}_{\omega,*}(B_{X^*})\widehat{\otimes}_\varepsilon Y$ and $\mathrm{lip}_{\omega,*}(B_{X^*}, Y)^*$ is isometric to $\mathscr{F}((B_{X^*}, \omega \circ \| \ \|), Y^*)$.*

The identification of vector-valued Lipschitz free Banach spaces with a projective tensor product is motivated not only by the problem of analysing duality but also by other properties. Indeed, some results on the (hereditary) Dunford–Pettis property on $S_0(M,X)$ and the (strong) Schur property on $\mathscr{F}(M,X)$ appear in [GLPRZ1].

4.4 Unconditional almost squareness and applications

Kalton proved in [Kal2] that if M is compact then $\mathrm{lip}_0(M)$ is $(1+\varepsilon)$-isometric to a subspace of c_0 for every $\varepsilon > 0$. Therefore, $\mathrm{lip}_0(M)$ is an M-embedded space (see [HWW] for this notion) and so it is not isometric to any dual Banach space whenever it is infinite-dimensional. This fact was extended by Dalet, who showed in [Dal2] that an analogous result holds for $S_0(M)$ whenever M is a proper metric space. So, it is natural to wonder if the previous spaces can be dual ones in a more general setting. For this, it would be useful to find a geometrical property of Banach spaces which is not compatible with being a dual Banach space. In this line, it has been recently introduced the concept of almost squareness.

Definition 4.4.1 (Abrahamsen–Langemets–Lima, [ALL])**.** A Banach space X is said to be *almost square* (ASQ) if for every $x_1, \ldots, x_k \in S_X$ and $\varepsilon > 0$ there exists $y \in S_X$ such that

$$\|x_i \pm y\| \leq 1 + \varepsilon \quad \text{for all } i \in \{1, \ldots, k\}.$$

Roughly speaking, we can say that ASQ Banach spaces have a strong c_0 behaviour from a geometrical point of view. This c_0 behaviour is encoded by the fact that a Banach space admits an equivalent ASQ renorming if, and only if, the space contains an isomorphic copy of c_0 [BGLPRZ3].

It is asked in [ALL] whether there exists an ASQ dual Banach space. We will provide a partial negative answer to this question by showing that if a Banach space satisfies the ASQ condition in a stronger unconditional way, then it cannot be isometric to a dual space. Moreover, we will apply this fact to give some criteria on non-duality of $\mathrm{lip}_0(M)$ and $S_0(M)$ as well as their vector-valued versions.

4.4.1 An unconditional sense of almost square Banach space

We begin by introducing a stronger notion of almost squareness in Banach spaces.

Definition 4.4.2. Let X be a Banach space. We say that X is *unconditionally almost square* (UASQ) if, for each $\varepsilon > 0$, there exists a subset $\{x_\gamma\}_{\gamma\in\Gamma} \subseteq S_X$ such that:

1. for each $y_1, \ldots, y_k \in S_X$ and $\delta > 0$ there exists $\gamma \in \Gamma$ such that
$$\|y_i \pm x_\gamma\| \leq 1 + \delta \text{ for all } i \in \{1, \ldots, k\};$$
2. for every finite subset F of Γ and every choice of signs $\xi_\gamma \in \{-1, 1\}$, $\gamma \in F$, we have $\|\sum_{\gamma\in F} \xi_\gamma x_\gamma\| \leq 1 + \varepsilon$.

First of all, we provide some examples of such spaces.

Example 4.4.3.

(a) The space $c_0(\Gamma)$ is UASQ, where $\{e_\gamma\}_{\gamma\in\Gamma}$ works for every $\varepsilon > 0$.

(b) Consider an uncountable set Γ and $\ell_\infty^c(\Gamma) := \{x \in \ell_\infty(\Gamma) : \mathrm{supp}(x) \text{ is countable}\}$. Then $\ell_\infty^c(\Gamma)$ is UASQ, where the set $\{e_\gamma\}_{\gamma\in\Gamma}$ works for every $\varepsilon > 0$.

(c) Given an infinite set Γ and a free ultrafilter $\mathscr{U}$ over Γ, the space
$$X := \{x \in \ell_\infty(\Gamma) : \lim_{\mathscr{U}}(x) = 0\}$$
is UASQ, where the set $\{e_\gamma : \gamma \in \Gamma\}$ works for every $\varepsilon > 0$.

Let us exhibit a result which will give a wide class of examples of UASQ spaces. We need the following fact proved in [ALL, Lemma 2.2]: if $x, y \in S_X$ and $\|x \pm y\| \leq 1 + \varepsilon$, then
$$(1 - \varepsilon)\max\{|\alpha|, |\beta|\} \leq \|\alpha x + \beta y\| \leq (1 + \varepsilon)\max\{|\alpha|, |\beta|\} \tag{4.2}$$
holds for all scalars α and β.

Proposition 4.4.4. *Let X be a Banach space. Assume that there exists a sequence $(x_n)_{n=1}^\infty$ in S_X such that, for each $\varepsilon > 0$ and every $y \in S_X$, there exists $m \in \mathbb{N}$ such that $n \geq m$ implies $\|y \pm x_n\| \leq 1 + \varepsilon$. Then X is UASQ.*

Proof. First, note that every subsequence of $(x_n)_{n=1}^{\infty}$ satisfies condition (1) in the definition of UASQ. Therefore, it suffices to show that for every $\varepsilon > 0$ there is a subsequence of $(x_n)_{n=1}^{\infty}$ satisfying condition (2).

Fix $\varepsilon > 0$ and pick a sequence of positive numbers $(\varepsilon_n)_{n=1}^{\infty}$ such that $\prod_{n=1}^{\infty}(1+\varepsilon_n) < 1+\varepsilon$. Consider $\sigma(1) := 1$. From the hypothesis on the sequence $(x_n)_{n=1}^{\infty}$ we get $\sigma(2) > 1$ such that $\|\xi_1 x_{\sigma(1)} + \xi_2 x_{\sigma(2)}\| \leq 1 + \varepsilon_1$ for every $\xi_1, \xi_2 \in \{-1, 1\}$. Again by assumptions, we can find $\sigma(3) > \sigma(2)$ such that

$$\left\| \frac{\xi_1 x_{\sigma(1)} + \xi_2 x_{\sigma(2)}}{\|\xi_1 x_{\sigma(1)} + \xi_2 x_{\sigma(2)}\|} + \xi_3 x_{\sigma(3)} \right\| \leq 1 + \varepsilon_2$$

for all $\xi_1, \xi_2, \xi_3 \in \{-1, 1\}$. Now, by (4.2), we get that

$$\left\| \sum_{i=1}^{3} \xi_i x_{\sigma(i)} \right\| \leq (1+\varepsilon_1)(1+\varepsilon_2).$$

By this procedure, we get a subsequence $(x_{\sigma(n)})_{n=1}^{\infty}$ of $(x_n)_{n=1}^{\infty}$ such that, given $n \in \mathbb{N}$ and $\xi_1, \ldots, \xi_n \in \{-1, 1\}$, we have

$$\left\| \sum_{i=1}^{n} \xi_i x_{\sigma(i)} \right\| \leq \prod_{i=1}^{n-1} (1+\varepsilon_i) \leq 1 + \varepsilon.$$

This proves condition (2) in the definition of UASQ, so we are done. $\square$

From Proposition 4.4.4 we conclude that ASQ and UASQ are equivalent properties for separable spaces.

Corollary 4.4.5. *Let X be a separable Banach space. If X is ASQ, then X is UASQ.*

Proof. Let $(y_n)_{n=1}^{\infty}$ be a dense sequence in S_X. For each $n \in \mathbb{N}$ pick, from the ASQ condition, an element $x_n \in S_X$ such that

$$\|y_i \pm x_n\| \leq 1 + \frac{1}{n} \text{ for all } i \in \{1, \ldots, n\}.$$

We will prove that the sequence $(x_n)_{n=1}^{\infty}$ satisfies the conditions of Proposition 4.4.4. To this end let $\varepsilon > 0$ and $y \in S_X$. Now we can find a natural number k such that $\|y - y_k\| < \frac{\varepsilon}{2}$. Pick m large enough to ensure $m > k$ and $\frac{1}{m} < \frac{\varepsilon}{2}$. Given $n \geq m$ one has

$$\|y \pm x_n\| \leq \|y - y_k\| + \|y_k \pm x_n\| \leq \frac{\varepsilon}{2} + 1 + \frac{1}{n} < 1 + \varepsilon,$$

which finishes the proof. $\square$

The next result is our main motivation for considering UASQ spaces.

Theorem 4.4.6. *Let X be a Banach space. Then X^* is not UASQ.*

Proof. Assume that X^* is UASQ. Let $\varepsilon > 0$, and let $\{x^*_\gamma\}_{\gamma\in\Gamma}$ be the set of the definition of UASQ for X^*. Given $x \in X$ and $\gamma \in \Gamma$, take $\xi^x_\gamma \in \{-1, 1\}$ such that $\xi^x_\gamma x^*_\gamma(x) = |x^*_\gamma(x)|$. For every $x \in S_X$, given a finite subset F of Γ, we have

$$1 + \varepsilon \geq \left| \sum_{\gamma\in F} \xi^x_\gamma x^*_\gamma(x) \right| = \sum_{\gamma\in F} |x^*_\gamma(x)|, \tag{4.3}$$

and it follows that the family $\{x^*_\gamma(x)\}$ is absolutely summable. Thus the functional $x^*(x) = \sum_{\gamma\in\Gamma} x^*_\gamma(x)$, $x \in X$, is well defined. Obviously $x^* \in X^*$ with $\|x^*\| \leq 1 + \varepsilon$. Moreover, $\|x^*\| \geq 1-\varepsilon$. Indeed, let $\delta > 0$ and $\gamma_0 \in \Gamma$. Since $\|x^*_{\gamma_0}\| = 1$, there exists $x \in S_X$ such that $x^*_{\gamma_0}(x) > 1 - \delta$. Pick a finite subset $F_0 \subseteq \Gamma$ so that $\gamma_0 \in F_0$ and that, whenever F is a finite subset of Γ such that $F \supseteq F_0$, one has $\left|x^*(x) - \sum_{\gamma\in F} x^*_\gamma(x)\right| < \delta$. Now

$$\begin{aligned}\|x^*\| \geq |x^*(x)| &\geq \left|\sum_{\gamma\in F} x^*_\gamma(x)\right| - \left|x^*(x) - \sum_{\gamma\in F} x^*_\gamma(x)\right| \\ &> |x^*_{\gamma_0}(x)| - \sum_{\gamma\in F\setminus\{\gamma_0\}} |x^*_\gamma(x)| - \delta > 1 - 3\delta - \varepsilon,\end{aligned}$$

where the last inequality follows from (4.3). This shows that $\|x^*\| \geq 1-\varepsilon$. Define $u^* := \frac{x^*}{\|x^*\|}$. By condition (2) of the definition of unconditional almost squareness we can find $\gamma_0 \in \Gamma$ such that

$$\|u^* \pm x^*_{\gamma_0}\| \leq 1 + \varepsilon.$$

Let $x \in S_X$ be such that $|x^*_{\gamma_0}(x)| > 1 - \varepsilon$. Let also F_0 be a finite subset of Γ containing γ_0 such that, given a finite subset F of Γ such that $F_0 \subseteq F$, then $\left|x^*(x) - \sum_{\gamma\in F} x^*_\gamma(x)\right| < \varepsilon$. Now

$$\begin{aligned}1 + \varepsilon \geq |u^*(x) + x^*_{\gamma_0}(x)| &= \left| \frac{x^*(x) + x^*_{\gamma_0}(x)}{\|x^*\|} + x^*_{\gamma_0}(x) - \frac{x^*_{\gamma_0}(x)}{\|x^*\|}\right| \\ &\geq \frac{|x^*(x) + x^*_{\gamma_0}(x)| - |x^*_{\gamma_0}(x)|\, |\|x^*\| - 1|}{\|x^*\|} > \frac{|x^*(x) + x^*_{\gamma_0}(x)| - \varepsilon}{\|x^*\|}.\end{aligned}$$

Moreover,

$$\begin{aligned}|x^*(x) + x^*_{\gamma_0}(x)| &\geq \left|\sum_{\gamma\in F} x^*_\gamma(x) + x^*_{\gamma_0}(x)\right| - \left|x^*(x) - \sum_{\gamma\in F} x^*_\gamma(x)\right| \\ &> 2|x^*_{\gamma_0}(x)| - \sum_{\gamma\in F\setminus\{\gamma_0\}} |x^*_\gamma(x)| - \varepsilon \\ &> 2(1-\varepsilon) - 2\varepsilon - \varepsilon = 2 - 5\varepsilon,\end{aligned}$$

where the last inequality follows from (4.3). Summarising one has

$$1+\varepsilon \geq |u^*(x) + x^*_{\gamma_0}(x)| > \frac{2-6\varepsilon}{1+\varepsilon},$$

which does not hold for small enough values of ε. Consequently, there is not any UASQ dual Banach space. $\square$

The following is a stability result for unconditional almost squareness which will be used later. The proof is a straightforward adaptation of the one given for ASQ Banach spaces in [LLRZ1, Theorem 2.6].

Proposition 4.4.7. *Let X and Y be non-zero Banach spaces and assume that H is a subspace of $\mathscr{K}(Y^*, X)$ such that $X \otimes Y \subseteq H$. If X is UASQ, then so is H.*

Proof. Let $\varepsilon > 0$. As X is UASQ we can find a set $\{x_\gamma\}_{\gamma\in\Gamma} \subseteq S_X$ satisfying the conditions of Definition 4.4.2. Now, letting $y \in S_Y$ be fixed, for each $\gamma \in \Gamma$ define $T_\gamma \coloneqq x_\gamma \otimes y$, which is a norm-one element of H. We will prove that $\{T_\gamma\}_{\gamma\in\Gamma}$ satisfies the conditions of Definition 4.4.2. On the one hand, let $F \subseteq \Gamma$ be a finite subset. Now, for every $\gamma \in F$ and every $\xi_\gamma \in \{-1, 1\}$, one has

$$\left\| \sum_{\gamma\in F} \xi_\gamma T_\gamma \right\| = \left\| \sum_{\gamma\in F} (\xi_\gamma x_\gamma) \otimes y \right\| = \left\| \sum_{\gamma\in F} \xi_\gamma x_\gamma \right\| \|y\| \leq 1+\varepsilon.$$

On the other hand, let $S_1, \ldots, S_k \in S_H$ and $\delta > 0$. Consider the relatively norm-compact set $K \coloneqq \bigcup_{i=1}^{k} S_i(B_{Y^*})$. Now we can find $x_1, \ldots, x_n \in B_X$ such that $K \subseteq \bigcup_{i=1}^{n} B\left(x_i, \frac{\delta}{2}\right)$.

Define $E \coloneqq \operatorname{span}\{x_i : i \in \{1, \ldots, n\}\}$, which is a finite-dimensional subspace of X. Let $\{y_i\}$ be a finite $\delta/4$-net in S_E and take $\gamma \in \Gamma$ such that $\|y_i \pm x_\gamma\| \leq 1 + \delta/4$ holds for every i. Thus, $\|x \pm x_\gamma\| \leq 1 + \delta/2$ holds for every $x \in S_E$. By (4.2),

$$\|x + \lambda x_\gamma\| \leq \left(1+\frac{\delta}{2}\right) \max\{\|x\|, |\lambda|\} \quad \text{for all } \lambda \in \mathbb{R} \text{ and } x \in E.$$

It remains to prove that $\|S_i \pm T_\gamma\| \leq 1+\delta$ for every $i \in \{1, \ldots, k\}$. To this end, let $i \in \{1, \ldots, k\}$ and $y^* \in B_{Y^*}$. From the condition on $x_1, \ldots, x_n$ we conclude the existence of $j \in \{1, \ldots, n\}$ such that $\|S_i y^* - x_j\| < \frac{\delta}{2}$. Now

$$\begin{aligned}\|(S_i \pm T_\gamma)y^*\| &\leq \|S_i(y^*) - x_j\| + \|x_j \pm y^*(y)x_\gamma\| \\ &\leq \frac{\delta}{2} + \left(1+\frac{\delta}{2}\right) \max\{\|x_j\|, |y^*(y)|\} \leq 1+\delta.\end{aligned}$$

Now, taking supremum in $y^* \in B_{Y^*}$, we get

$$\|S_i \pm T_\gamma\| = \sup_{y^*\in B_{Y^*}} \|(S_i \pm T_\gamma)y^*\| \leq 1+\delta,$$

so we are done. $\square$

4.4.2 Unconditional almost squareness in $\mathrm{lip}_0(M,X)$ and $S_0(M,X)$

In this section we provide several examples of little-Lipschitz UASQ Banach spaces. We begin with a result for proper metric spaces in the scalar-valued case.

Proposition 4.4.8. *Let M be a proper metric space. Then $S_0(M)$ is either finite-dimensional or UASQ. In particular, when M is compact, the same conclusion holds for* $\mathrm{lip}_0(M)$.

Proof. By [Dal2, Lemma 3.9] it is known that $S_0(M)$ is $(1+\varepsilon)$-isometric to a subspace of c_0 for every $\varepsilon > 0$. Then, either $S_0(M)$ is finite dimensional or $S_0(M)$ is not reflexive. In the non-reflexive case, we get that $S_0(M)$ is $(1+\varepsilon)$-isometric to an M-embedded Banach space for every $\varepsilon > 0$, so $S_0(M)$ is a non-reflexive M-embedded space. By [ALL, Corollary 4.3], $S_0(M)$ is ASQ. As $S_0(M)$ is separable then it is UASQ by Corollary 4.4.5. □

Notice that $X \otimes S_0(M)$ is a subspace of $\mathscr{K}_{w^*,w}(X^*, S_0(M))$. By Propositions 4.4.7 and 4.4.8 we get the desired result on non-duality in the vector-valued case.

Theorem 4.4.9. *Let M be a proper metric space and let X be a Banach space. If $S_0(M)$ is infinite dimensional, then $S_0(M,X)$ is UASQ. Consequently, it is not a dual Banach space. In particular, the same conclusion holds for $\mathrm{lip}_0(M,X)$ when M is compact.*

The following result provides a useful criterion for determining when a certain subspace of $\mathrm{lip}_0(M)$ is UASQ without assuming compactness on M.

Proposition 4.4.10. *Let M be a metric space and let W be a closed subspace of $\mathrm{lip}_0(M)$. Assume that there exist sequences $(f_n)_{n=1}^\infty \subset W \setminus \{0\}$, $(x_n)_{n=1}^\infty \subset M$ and $(r_n)_{n=1}^\infty$ of positive numbers such that $r_n \to 0$, $f_n(x_n) = 0$ and $f_n(t) = 0$ for each $t \in M \setminus B(x_n, r_n)$. Then W is UASQ.*

Proof. Replacing f_n by $f_n/\|f_n\|$ for each n, we may assume that $(f_n)_{n=1}^\infty \subset S_W$. We will show that the sequence $(f_n)_{n=1}^\infty$ satisfies the assumptions of Proposition 4.4.4. To this end let $\varepsilon > 0$ and $g \in S_W$. By the little-Lipschitz condition, there is $\delta > 0$ such that

$$0 < d(x,y) \leq 2\delta \text{ implies } \frac{|g(x)-g(y)|}{d(x,y)} < \varepsilon.$$

Pick $m \in \mathbb{N}$ such that $n \geq m$ implies $r_n < \delta$ and $\frac{r_n}{\delta - r_n} < \varepsilon$. Consider $n \geq m$ and let us estimate $\|g \pm f_n\|_L$. To this end let $x, y \in M, x \neq y$. One has

$$\frac{|g(x) \pm f_n(x) - (g(y) \pm f_n(y))|}{d(x,y)} \leq \underbrace{\frac{|g(x)-g(y)|}{d(x,y)}}_{A} + \underbrace{\frac{|f_n(x)-f_n(y)|}{d(x,y)}}_{B} =: C$$

Now we distinguish the following cases, depending on the position of x and y.

1. If $x, y \notin B(x_n, \delta)$, then $B = 0$ and so $C \leq 1$.

2. If $x, y \in B(x_n, \delta)$, then $A \leq \varepsilon$ and so $C \leq 1 + \varepsilon$.
3. If $x \in B(x_n, \delta), y \notin B(x_n, \delta)$, then $f_n(y) = 0$. We distinguish two more cases here. First, if $x \notin B(x_n, r_n)$, then $f_n(x) = f_n(y) = 0$, so $B = 0$ and thus $C \leq 1$. On the other hand, if $x \in B(x_n, r_n)$ then $d(x, y) > \delta - r_n$. Consequently
$$B \leq \frac{|f_n(x)|}{d(x,y)} \leq \frac{r_n}{\delta - r_n} < \varepsilon,$$
so $C \leq 1 + \varepsilon$.

Therefore, we get that $\|g \pm f_n\|_L \leq 1 + \varepsilon$ as desired. Thus Proposition 4.4.4 implies that W is UASQ. □

Proposition 4.4.10 is applied in [GLRZ] to prove that both $\mathrm{lip}_0(M)$ and $S_0(M)$ are UASQ under some topological assumptions on M, for instance if M is locally compact and totally disconnected metric space which is not uniformly discrete. Here we want to highlight the following application, which appears in [GLPRZ1].

Theorem 4.4.11. *Let X and Y be Banach spaces, and let ω be a non-trivial gauge. Then $\mathrm{lip}_{\omega,*}(B_{X^*}, Y)$ is UASQ. In particular, it is not isometric to any dual Banach space.*

Proof. First we prove that $\mathrm{lip}_{\omega,*}(B_{X^*})$ is UASQ. By Proposition 4.4.10, it suffices to show that there exists a point $x_0^* \in B_{X^*}$ and sequences $(r_n)_{n=1}^\infty$ of positive numbers and $(f_n)_{n=1}^\infty \subset \mathrm{lip}_{\omega,*}(B_{X^*})$ such that $f_n \neq 0$, $f_n(x_0^*) = 0$ and f_n vanishes out of $B(x_0^*, r_n)$. Note that the norm $\|\ \| : (B_{X^*}, w^*) \to \mathbb{R}$ is a lower semicontinuous map and so by Proposition 1.3.2 it has some point of continuity, say x_0^*. That is, x_0^* has weak* neighbourhoods of arbitrarily small diameter. Clearly, x_0^* belongs to S_{X^*}. Now, take a sequence $(W_n)_{n=1}^\infty$ of weak* neighbourhoods of x_0^* and a sequence $r_n \to 0$ such that $W_n \subset B(x_0^*, r_n) \subset W_{n-1}$. For each n choose $x_n^* \in W_n \setminus \{x_0^*\}$ and define $A_n = \{x_0^*, x_n^*\} \cup (B_{X^*} \setminus W_n)$. Consider $f_n : A_n \to \mathbb{R}$ given by $f_n(x_n^*) = 1$ and $f_n(x) = 0$ otherwise. Then A_n is weak*-closed and $f_n \in \mathrm{Lip}_0(A_n, \|\ \|) \cap \mathscr{C}(A_n, w^*)$. By the extension theorem of Matouskova 4.1.2, there exists $g_n \in \mathrm{Lip}_0(B_{X^*}, \|\ \|) \cap \mathscr{C}(B_{X^*}, w^*)$ extending f_n. Then g_n is a non-zero Lipschitz function which is weak*-continuous and vanishes on $B_{X^*} \setminus B(x_0^*, r_n)$. Finally note that $\mathrm{Lip}_0(B_{X^*}, \|\ \|)$ is a subset of $\mathrm{lip}_0(B_{X^*}, \omega \circ \|\ \|)$. Thus $\{g_n : n \in \mathbb{N}\} \subset \mathrm{lip}_{\omega,*}(B_{X^*})$ and so $\mathrm{lip}_{\omega,*}(B_{X^*})$ is UASQ.

By Theorem 4.3.9, we have that $\mathrm{lip}_{\omega,*}(B_{X^*}, Y)$ is a subspace of $\mathscr{K}(Y^*, \mathrm{lip}_{\omega,*}(B_{X^*}))$ which clearly contains $\mathrm{lip}_{\omega,*}(B_{X^*}) \otimes Y$. Therefore Proposition 4.4.7 provides unconditional almost squareness of $\mathrm{lip}_{\omega,*}(B_{X^*}, Y)$. Finally, the non-duality of this space follows from Theorem 4.4.6. □

Let us point out that, by Lemma 3.2 in [Kim], the space $\mathscr{K}_{w^*,w}(X^*, Y)$ is isometric to a subspace of $\mathscr{C}((B_{X^*}, w^*) \times (B_{Y^*}, w^*))$. Therefore, $\mathscr{K}_{w^*,w}(X^*, Y)$ is separable whenever X and Y are separable Banach spaces, since in such a case $(B_{X^*}, w^*) \times (B_{Y^*}, w^*)$ is a metrizable compact space. As a consequence, the previous result can be strengthened under separability assumptions.

Corollary 4.4.12. *Let X and Y be separable Banach spaces and let ω be a non-trivial gauge. Then* $\mathrm{lip}_{\omega,*}(B_{X^*}, Y)$ *is a separable Banach space which contains a copy of* c_0*. Thus* $\mathrm{lip}_{\omega,*}(B_{X^*}, Y)$ *is not isomorphic to a dual Banach space.*

Proof. We have shown that $\mathrm{lip}_{\omega,*}(B_{X^*}, Y)$ is isometric to $\mathscr{K}_{w^*,w}(Y^*, \mathrm{lip}_{\omega,*}(B_{X^*}))$. Note that $\mathrm{lip}_{\omega,*}(B_{X^*})$ is separable since it can be identified with a subspace of $\mathscr{C}(K)$, where K is the metrizable compact space given by

$$K = \{(x^*, y^*, t) \in (B_{X^*}, w^*) \times (B_{X^*}, w^*) \times [0, \omega(2)] : \omega(\|x - y\|) \leq t\},$$

see the proof of Lemma 4.3.8. Thus $\mathrm{lip}_{\omega,*}(B_{X^*}, Y)$ is separable too. Moreover, it contains an isomorphic copy of c_0 since it is ASQ [ALL, Lemma 2.6]. Thus, it does not have the RNP and so the separability yields that it is not isomorphic to any dual Banach space. $\square$

Remark 4.4.13. The previous result has an immediate consequence in terms of octahedrality in Lipschitz free Banach spaces. Recall that a Banach space X is said to have an *octahedral norm* if for every finite-dimensional subspace Y and for every $\varepsilon > 0$ there exists $x \in S_X$ satisfying that $\|y + \lambda x\| > (1 - \varepsilon)(\|y\| + |\lambda|)$ for every $y \in Y$ and $\lambda \in \mathbb{R}$. Notice that, given a Banach X space under the assumption of Proposition 4.4.11, it follows that $\mathscr{F}((B_{X^*}, \omega \circ \|\cdot\|), Y^*) = \mathscr{F}((B_{X^*}, \omega \circ \|\cdot\|)) \widehat{\otimes}_\pi Y^*$ has an octahedral norm for every non-zero Banach space Y because of [LLRZ1, Corollary 2.9]. Notice that this gives a partially positive answer to [BGLPRZ2, Question 2], where it is wondered whether octahedrality in vector-valued Lipschitz free Banach spaces actually relies on the scalar case.

Finally, let us remark that Dalet and Procházka showed that the metric space considered in Example 4.2.9 is a uniformly discrete metric space M admitting a compact topology τ such that the metric is τ-lower semicontinuous and $\mathrm{lip}_\tau(M)$ separates points uniformly, but $\mathrm{lip}_\tau(M)$ is not ASQ (see Proposition 5.1 in [GLPRZ1]).

Chapter 5

Geometrical properties of Lipschitz free spaces

This chapter is mainly focused on the isometric structure of Lipschitz free spaces. Let us recall that Godard proved in [God1] that $\mathscr{F}(M)$ is isometric to a subspace of an L_1-space if and only if M embeds isometrically into an $\mathbb{R}$-tree (that is, a metric space where any couple of points is connected by a unique curve isometric to a compact interval of $\mathbb{R}$). This property is equivalent to the following geometrical condition on M, called *the four-point condition*:

$$d(x,y) + d(u,v) \leq \max\{d(x,u) + d(y,v), d(x,v) + d(y,u)\} \text{ for all } x,y,u,v \in M.$$

Moreover, in [DKP] it is characterised when $\mathscr{F}(M)$ is isometric to ℓ_1. Other geometrical properties of $\mathscr{F}(M)$ has been characterised in terms of a geometrical property of the underlying metric space. Namely, it is showed in [PRZ] that the norm of $\mathscr{F}(M)$ is octahedral if and only if M has the *Long Trapezoid Property* (LTP), that is, for each finite subset $N \subset M$ and $\varepsilon > 0$, there exist $u, v \in M$, $u \neq v$, such that

$$(1-\varepsilon)(d(x,y) + d(u,v)) \leq d(x,u) + d(y,v) \text{ for all } x,y \in N.$$

Moreover, for compact metric spaces, it is proved in [IKW1] that $\mathscr{F}(M)$ has the Daugavet property if and only if M has property (Z), we define this notion below. In the first section of this chapter we take advantage of the arguments of [IKW1] to provide a general characterisation of Lipschitz free spaces as well as spaces of Lipschitz functions which enjoy the Daugavet property. This is based on the paper [GLPRZ2]. In the second section we focus on the extremal structure of $B_{\mathscr{F}(M)}$, proving for instance a characterisation of the strongly exposed points of $B_{\mathscr{F}(M)}$. During the preparation of the paper [GLPPRZ], Aliaga and Guirao [AG] characterised preserved extreme points in $B_{\mathscr{F}(M)}$, we provide a different proof of their result. We also prove that every preserved extreme point of $B_{\mathscr{F}(M)}$ is also a denting point. That section is based on the papers [GLPRZ2] and [GLPPRZ].

The last section of the chapter is devoted to a different topic, namely to isomorphic questions around the Lipschitz free space over Pełczyński's universal space. These results are part of a preprint with A. Procházka that will appear soon.

Throughout the chapter M will denote a metric space.

5.1 The Daugavet property in spaces of Lipschitz functions

A Banach space X is said to have the *Daugavet property* if every rank-one operator $T\colon X \to X$ satisfies the equality

$$\|T + I\| = 1 + \|T\|,$$

where I denotes the identity operator. The previous equality is known as the *Daugavet equation* because I. Daugavet proved in [Dau] that it is satisfied by every compact operator on $\mathscr{C}([0,1])$. Since then, many examples of Banach spaces enjoying the Daugavet property have appeared such as $\mathscr{C}(K)$ for a perfect compact Hausdorff space K, and $L_1(\mu)$ and $L_\infty(\mu)$ for a non-atomic measure μ. Moreover, it follows readily from the definition that a Banach space has the Daugavet property provided its dual has it, a fact that will be important in what follows. We refer the reader to [Wer] for an excellent survey on the Daugavet property.

Recall that $\mathrm{Lip}_0([0,1])$ is isometric to $L_\infty[0,1]$ and so it has the Daugavet property. In [Wer, Section 6] it is asked whether the space $\mathrm{Lip}_0([0,1]^2)$ enjoys or not the Daugavet property. A positive answer was given in [IKW1], where it was shown, among other results, that $\mathrm{Lip}_0(M)$ has the Daugavet property whenever M is a length metric space. In what follows we spend some time recalling the notions considered in [IKW1] and the results that they obtain.

Definition 5.1.1. We will say that a metric space (M,d) is a *length space* if, for every pair of points $x, y \in M$, the distance $d(x,y)$ is equal to the infimum of the length of rectifiable curves joining them. Moreover, if that infimum is always attained then we will say that M is a *geodesic space*.

These definitions are standard, for more details see e.g. [BH]. Geodesic spaces and length spaces were considered in [IKW1], where they are called metrically convex spaces and almost metrically convex spaces, respectively.

The following notions were introduced in [IKW1].

Definition 5.1.2. Let M be a metric space.

(a) M is said to be *local* if, for every $\varepsilon > 0$ and every Lipschitz function $f\colon M \to \mathbb{R}$ there exist $u, v \in M$, $u \neq v$, such that $d(u,v) < \varepsilon$ and $\frac{f(u)-f(v)}{d(u,v)} > \|f\|_L - \varepsilon$.

(b) M is said to be *spreadingly local* if for every $\varepsilon > 0$ and every Lipschitz function $f\colon M \to \mathbb{R}$ the set

$$\left\{x \in M : \inf_{\delta>0} \left\|f\restriction_{B(x,\delta)}\right\|_L > \|f\|_L - \varepsilon\right\}$$

is infinite.

(c) M has *property (Z)* if, for every $x, y \in M$ and $\varepsilon > 0$, there is $z \in M \setminus \{x, y\}$ satisfying

$$d(x,z) + d(z,y) \leq d(x,y) + \varepsilon \min\{d(x,z), d(z,y)\}.$$

We will show in Section 5.2 that property (Z) characterises the absence of strongly exposed points in $B_{\mathscr{F}(M)}$.

The above notions are closely related to the Daugavet property of $\mathrm{Lip}_0(M)$. The following result gathers the main theorems proved in [IKW1] (see also [IKW2]).

Theorem 5.1.3 (Ivakhno–Kadets–Werner)**.** *Let M be a complete metric space.*

(a) *If M is a length space then M is spreadingly local.*
(b) *If M is local then M has property (Z).*
(c) *If M is spreadingly local, then* $\mathrm{Lip}_0(M)$ *has the Daugavet property.*
(d) *If M is a compact and has property (Z), then M is spreadingly local.*
(e) *If M is compact and $\mathscr{F}(M)$ has the Daugavet property, then M is local.*

These properties are summarised in the next diagram, where the dashed implications hold for compact metric spaces.

$$\begin{tikzcd}
\text{geodesic} \arrow[r] & \text{length} \arrow[l, dashed, bend right] \arrow[r] & \text{spreadingly local} \arrow[r] \arrow[d] & \text{local} \arrow[r] & \text{(Z)} \arrow[ll, dashed, bend right] \\
 & & \mathrm{Lip}_0(M)\ \text{Daugavet P.} \arrow[d] & & \\
 & & \mathscr{F}(M)\ \text{Daugavet P.} \arrow[uur, dashed, bend right] & &
\end{tikzcd}$$

Clearly $[0,1]^2$ is a geodesic metric space, so Theorem 5.1.3 gives a positive answer to the problem posed in [Wer] mentioned above. Moreover, Theorem 5.1.3 provides a metric characterisation of compact metric spaces M such that $\mathrm{Lip}_0(M)$ has the Daugavet property: they are exactly the spaces which have property (Z). Our goal here is to provide a metric characterisation for complete metric spaces. To this end, we will show that two more implications in this diagram hold. Namely, we are going to show that complete local spaces are length spaces and that if $\mathscr{F}(M)$ has the Daugavet property then M is local (without assuming compactness of M).

Let us point out that, given a metric space M and its completion $\hat{M}$, every Lipschitz function on M extends uniquely to a Lipschitz function on $\hat{M}$ with the same norm. It follows easily from this fact that $\mathrm{Lip}_0(M)$ is isometric to $\mathrm{Lip}_0(\hat{M})$ and $\mathscr{F}(M)$ is isometric to $\mathscr{F}(\hat{M})$. For that reason, in order to analyse when those spaces enjoy the Daugavet property we can assume that the underlying metric space is complete.

The following lemma is well known and easy to prove, see [BBI]. It says that, for complete spaces, being length is characterised in terms of the existence of approximate midpoints and so it is a purely metrical property.

Lemma 5.1.4. *Let (M,d) be a complete metric space. Then*

(a) *M is a geodesic space if and only if for every $x, y \in M$ there is $z \in M$ such that $d(x,z) = d(y,z) = \frac{1}{2}d(x,y)$.*
(b) *M is a length space if and only if for every $x, y \in M$ and for every $\delta > 0$ the set*

$$\mathrm{Mid}(x,y,\delta) := B\left(x, \frac{1+\delta}{2}d(x,y)\right) \cap B\left(y, \frac{1+\delta}{2}d(x,y)\right)$$

is non-empty.

Proposition 5.1.5. *Let M be a complete metric space. The following are equivalent:*

(i) *M is a length space.*
(ii) *M is spreadingly local.*
(iii) *M is local.*

Proof. (ii)$\Rightarrow$(iii) is trivial and (i)$\Rightarrow$(ii) was proved in [IKW1], see the remark after Proposition 2.3. For the reader's convenience we sketch the main idea. For a given $f \in B_{\mathrm{Lip}_0(M)}$ and $\varepsilon > 0$ let $x, y \in M$ be such that $f(x) - f(y) \geq (1 - \frac{\varepsilon^2}{4})d(x,y)$. Let $\varphi : [0, d(x,y)(1+\frac{\varepsilon}{2})] \to M$ be a 1-Lipschitz map such that $\varphi(0) = x$ and $\varphi(d(x,y)(1+\frac{\varepsilon}{2})) = y$. Then $f(y) = f(x) + \int_0^{d(x,y)(1+\varepsilon/2)} (f \circ \varphi)'(t)\, dt$ and the integrand has to be larger than $1 - \frac{\varepsilon}{2}$ in a non-negligible subset A of $[0, (1+\varepsilon/2)d(x,y)]$. It is immediate to check $\varphi(A)$ satisfies the definition of spreading locality for ε.

Finally, assume that M is not a length space. Then there exist $x, y \in M$ and $\delta > 0$ such that $\mathrm{Mid}(x,y,2\delta) = \emptyset$. Let us denote $r := \frac{d(x,y)}{2}$. Notice by passing that

$$\mathrm{dist}(B(x,(1+\delta)r), B(y,(1+\delta)r)) \geq \delta r.$$

Let $f_i \colon M \to \mathbb{R}$ be defined by

$$f_1(t) = \max\left\{r - \frac{1}{1+\delta}d(x,t), 0\right\} \text{ and } f_2(t) = \min\left\{-r + \frac{1}{1+\delta}d(y,t), 0\right\}.$$

Clearly $\|f_i\|_L \leq \frac{1}{1+\delta}$ so $f = f_1 + f_2$ is a Lipschitz function. Since $f(x) - f(y) = d(x,y)$ we have that $\|f\|_L \geq 1$. Moreover we have that $\{z : f_1(z) \neq 0\} \subset B(x,(1+\delta)r)$ and $\{z : f_2(z) \neq 0\} \subset B(y,(1+\delta)r)$. It follows that if $\frac{f(u)-f(v)}{d(u,v)} > \frac{1}{1+\delta}$ then $u \in B(x,(1+\delta)r)$ and $v \in B(y,(1+\delta)r)$. But then $d(u,v) \geq \delta r$ and so M is not local. This shows that (iii)$\Rightarrow$(i). $\square$

It is also shown in [IKW1] that every compact subset of a smooth LUR Banach space with property (Z) is convex. As a consequence of Proposition 5.1.5 we have the following:

Corollary 5.1.6. *Let M be a compact metric space with property (Z). Then M is a geodesic space. If moreover M is a subset of a strictly convex Banach space then M is convex.*

Proof. It has been proved in [IKW1, Proposition 2.8] that a compact metric space with property (Z) is local. Thus the first statement above follows from Proposition 5.1.5 and the fact that every compact length space is geodesic. Finally, it is easy to show that every complete geodesic subset of a strictly convex Banach space is convex. □

The main result of this section is the following theorem. It improves [IKW1, Theorem 3.3] where it is proved that $\mathrm{Lip}_0(M)$ has the Daugavet property if and only if M is local for M compact.

Theorem 5.1.7. *Let M be a complete metric space. The following assertions are equivalent:*

(i) *M is a length space.*
(ii) *$\mathrm{Lip}_0(M, X)$ has the Daugavet property for every Banach space X such that (M, X) has the CEP.*
(iii) *$\mathrm{Lip}_0(M)$ has the Daugavet property.*
(iv) *$\mathscr{F}(M)$ has the Daugavet property.*

For the proof of Theorem 5.1.7 we need a number of auxiliary results. First, we recall a geometric characterisation of the Daugavet property in terms of the slices of the unit ball due to Kadets, Shvidkoy, Sirotkin and Werner. We refer the reader to [KSSW, Wer] for a detailed proof.

Theorem 5.1.8 (Kadets–Shvidkoy–Sirotkin–Werner)**.** *Let X be a Banach space. The following assertions are equivalent:*

(i) *X has the Daugavet property.*
(ii) *For every $x \in S_X$, every slice S of B_X and every $\varepsilon > 0$ there exists another slice T of the unit ball such that $T \subseteq S$ and such that*
$$\|x + y\| > 2 - \varepsilon$$
for every $y \in T$.
(iii) *For every $x \in S_X$ and every $\varepsilon > 0$, the following holds:*
$$B_X \subset \overline{\mathrm{conv}}(\{y \in (1 + \varepsilon)B_X : \|y - x\| > 2 - \varepsilon\}).$$

Note that (iii) is particularly useful in those Banach spaces in which there is not a complete description of the dual space.

The following auxiliary result is inspired by [PRZ, Theorem 3.1].

Lemma 5.1.9. *Let M be a metric space. Assume that $\mathscr{F}(M)$ has the Daugavet property. Then for every $x, y \in M$, every $f \in S_{\mathrm{Lip}_0(M)}$ and every $\varepsilon > 0$ there are $u, v \in M$, $u \neq v$, such that $\langle f, m_{u,v}\rangle > 1 - \varepsilon$ and*
$$(1 - \varepsilon)(d(x, y) + d(u, v)) \leq \min\{d(x, u) + d(y, v), d(x, v) + d(y, u)\}.$$

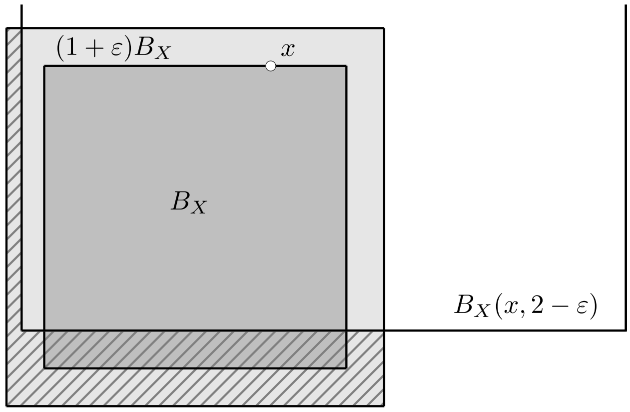

Figure 5.1: Condition (iii) in Theorem 5.1.8

Proof. Since $\mathscr{F}(M)$ has the Daugavet property we can find, using Theorem 5.1.8, a slice $T \subset S(B_{\mathscr{F}(M)}, f, \varepsilon)$ such that $\|m_{x,y} + \mu\| > 2 - \varepsilon$ for every $\mu \in T$. Now, since $B_{\mathscr{F}(M)} = \overline{\text{conv}}(V_M)$, there are two different elements $u, v \in M$ such that $m_{u,v} \in T \subset S(B_{\mathscr{F}(M)}, f, \varepsilon)$ and so

$$\|m_{x,y} + m_{u,v}\| > 2 - \varepsilon.$$

Now we mimic the argument given in the proof of [PRZ, Theorem 3.1]. There exists $g \in S_{\mathrm{Lip}_0(M)}$ such that

$$\langle g, m_{x,y}\rangle + \langle g, m_{u,v}\rangle > 2 - \varepsilon.$$

Then $\langle g, m_{x,y}\rangle > 1 - \varepsilon$ and $\langle g, m_{u,v}\rangle > 1 - \varepsilon$. Now, we have,

$$\begin{aligned} d(x,v) &\geq g(x) - g(v) = g(x) - g(y) + g(u) - g(v) + g(y) - g(u) \\ &\geq (1-\varepsilon)d(x,y) + (1-\varepsilon)d(u,v) - d(u,y) \end{aligned}$$

Therefore,

$$(1-\varepsilon)(d(x,y) + d(u,v)) < d(x,v) + d(u,y).$$

A similar argument shows that

$$(1-\varepsilon)(d(x,y) + d(u,v)) < d(x,u) + d(v,y).$$

$\square$

Given $x, y \in M$, we denote $[x, y]$ the *metric segment* between x and y, that is,

$$[x,y] = \{z \in M : d(x,z) + d(z,y) = d(x,y)\}.$$

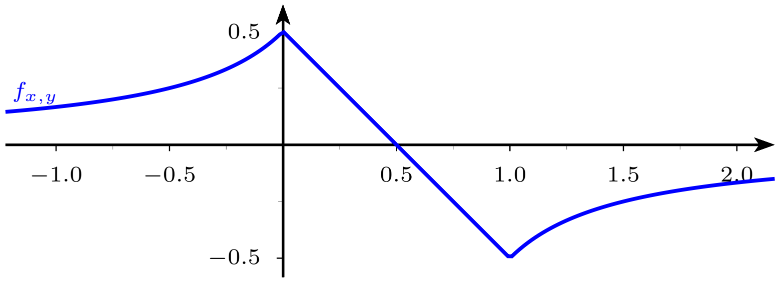

Figure 5.2: The function $f_{x,y}$ for $M = \mathbb{R}$, $x = 0$ and $y = 1$

We will consider for every $x, y \in M$, $x \neq y$, the function

$$f_{x,y}(t) := \frac{d(x,y)}{2} \frac{d(t,y) - d(t,x)}{d(t,y) + d(t,x)}.$$

The properties collected in the next lemma were proved in [IKW2]. They make of $f_{x,y}$ a useful tool for studying the geometry of $B_{\mathscr{F}(M)}$.

Lemma 5.1.10. *Let $x, y \in M$ with $x \neq y$.*

(a) *For all $u, v \in M$, $u \neq v$, we have*

$$\frac{f_{x,y}(u) - f_{x,y}(v)}{d(u,v)} \leq \frac{d(x,y)}{\max\{d(x,u) + d(u,y), d(x,v) + d(v,y)\}}.$$

(b) *$f_{x,y}$ is Lipschitz and $\|f_{x,y}\|_L \leq 1$.*

(c) *Let $u \neq v \in M$ and $\varepsilon > 0$ be such that $\frac{f_{x,y}(u) - f_{x,y}(v)}{d(u,v)} > 1 - \varepsilon$. Then*

$$(1 - \varepsilon) \max\{d(x,v) + d(y,v), d(x,u) + d(y,u)\} < d(x,y).$$

(d) *If $u, v \in M$, $u \neq v$ and $\frac{f_{x,y}(u) - f_{x,y}(v)}{d(u,v)} = 1$, then $u, v \in [x,y]$.*

Proof. Statement (a) follows from the next easily proved fact (see [IKW2]): given numbers $u_1, v_1, u_2, v_2 > 0$, we have

$$\left| \frac{u_1 - v_1}{u_1 + v_1} - \frac{u_2 - v_2}{u_2 + v_2} \right| \leq 2 \frac{\max\{|u_1 - u_2|, |v_1 - v_2|\}}{\max\{u_1 + v_1, u_2 + v_2\}}.$$

Finally, the statements (b),(c) (resp. (d)) are a straightforward consequence of (a) (resp. (c)). □

We need one more lemma, which is an extension of Lemma 3.2 in [IKW1].

Lemma 5.1.11. *Assume that $\mathscr{F}(M)$ has the Daugavet property. Then for every $\varepsilon > 0$, $x, y \in M$ and every function $f \in S_{\mathrm{Lip}_0(M)}$ such that $f(x) - f(y) > (1 - \varepsilon) d(x,y)$ there exist $u, v \in M$ such that $f(u) - f(v) > (1 - \varepsilon) d(u,v)$ and $d(u,v) < \frac{\varepsilon}{(1-\varepsilon)^2} d(x,y)$.*

Proof. Let us consider the following functions:

$$f_1 = f, \quad f_2(t) = d(y,t), \quad f_3(t) = -d(x,t), \quad f_4(t) = f_{x,y}(t).$$

We have $f_1(x) - f_1(y) > (1-\varepsilon)d(x,y)$ and $f_i(x) - f_i(y) = d(x,y)$ for $i = 2,3,4$. Moreover, clearly $\|f_i\|_L = 1$ for $i = 1,2,3$, and $\|f_4\|_L = 1$ as a consequence of Lemma 5.1.10. Consider the function $g = \frac{1}{4}\sum_{i=1}^4 f_i$. First notice that

$$1 \geq \|g\|_L \geq \frac{1}{4}\sum_{i=1}^4 \frac{f_i(x) - f_i(y)}{d(x,y)} > 1 - \frac{\varepsilon}{4}.$$

Now, Lemma 5.1.9 provides u, v in M such that

$$(1-\varepsilon)(d(x,y) + d(u,v)) \leq \min\{d(x,u) + d(y,v), d(x,v) + d(y,u)\} \tag{5.1}$$

and $g(u) - g(v) > (1 - \frac{\varepsilon}{4})d(u,v)$, that is,

$$\frac{1}{4}\sum_{i=1}^4 (f_i(u) - f_i(v)) > \left(1 - \frac{\varepsilon}{4}\right) d(u,v).$$

Notice that each of these summands is less than or equal to $d(u,v)$. Thus, we get

$$\min\{f_i(u) - f_i(v) : i \in \{1,2,3,4\}\} > (1-\varepsilon)d(u,v).$$

The case $i = 1$ gives $f(u) - f(v) > (1-\varepsilon)d(u,v)$. Moreover, the cases $i = 2,3$ yield

$$\min\{d(y,u) - d(y,v), d(x,v) - d(x,u)\} > (1-\varepsilon)d(u,v). \tag{5.2}$$

By Lemma 5.1.10 and the case $i = 4$ we have

$$(1-\varepsilon)\max\{d(x,v) + d(y,v), d(x,u) + d(y,u)\} < d(x,y). \tag{5.3}$$

The above inequalities yield

$$\begin{aligned} \frac{d(x,y)}{1-\varepsilon} &\overset{(5.3)}{>} d(x,u) + d(y,u) \\ &\overset{(5.2)}{>} d(x,u) + d(y,v) + (1-\varepsilon)d(u,v) \\ &\overset{(5.1)}{\geq} (1-\varepsilon)(d(x,y) + d(u,v)) + (1-\varepsilon)d(u,v) \end{aligned}$$

and so

$$2(1-\varepsilon)d(u,v) < \left(\frac{1}{1-\varepsilon} - (1-\varepsilon)\right) d(x,y) = \frac{\varepsilon(2-\varepsilon)}{1-\varepsilon}d(x,y) < \frac{2\varepsilon}{1-\varepsilon}d(x,y)$$

as desired. $\square$

Proof of Theorem 5.1.7. (i)⇒(iii) was proved in [IKW1, Theorem 3.1]. Indeed, the same argument can be used to prove (i)⇒(ii), let us include a sketch of the proof. Assume that M is a length space and that (M,X) has the CEP. Then by Proposition 5.1.5 M is spreadingly local. In order to prove that $\mathrm{Lip}_0(M,X)$ has the Daugavet property we will apply Theorem 5.1.8.(iii), so we will prove that, given $f,g\in S_{\mathrm{Lip}_0(M,X)}$ and $\varepsilon>0$ we have

$$g\in\overline{\mathrm{conv}}\left\{u\in(1+\varepsilon)B_{\mathrm{Lip}_0(M,X)}:\|f+u\|_L>2-\varepsilon\right\}.$$

It is easy to find $x^*\in S_{X^*}$ such that $\|x^*\circ f\|_L>1-\varepsilon$. Fix $n\in\mathbb{N}$. Since M is spreadingly local we can find $r>0$ and $\delta_0>0$ such that, for every $0<\delta<\delta_0$, there are $x_1,y_1,\ldots,x_n,y_n\in M$ such that $d(x_i,y_i)<\delta$, $\langle x^*\circ f,m_{x_i,y_i}\rangle>1-\varepsilon/2$ holds for each i and such that $B(x_i,r)\cap B(x_j,r)=\emptyset$ for all $i\neq j$. Now, for every $i\in\{1,\ldots,n\}$ and for δ small enough, the contraction-extension property ensures the existence of a $(1+\varepsilon)$-Lipschitz function $f_i\colon M\to X$ such that $f_i=f$ in $\{x_i,y_i\}$ and $f_i=g$ in $M\setminus B(x_i,r)$. Note that

$$\|f+f_i\|_L\geq\frac{\|f(x_i)+f_i(x_i)-f(y_i)-f_i(y_i)\|}{d(x_i,y_i)}\geq 2\langle x^*\circ f,m_{x_i,y_i}\rangle>2-\varepsilon$$

and so

$$f_i\in\left\{u\in(1+\varepsilon)B_{\mathrm{Lip}_0(M,X)}:\|f+u\|_L>2-\varepsilon\right\}$$

for every $i\in\{1,\ldots,n\}$. On the other hand notice that, given $x\in M$, the set $\{i\in\{1,\ldots,n\}:f_i(x)\neq g(x)\}$ is at most a singleton. From the definition of the Lipschitz norm we deduce that

$$\left\|g-\frac{1}{n}\sum_{i=1}^n f_i\right\|_L\leq\frac{4+2\varepsilon}{n}.$$

Since n was arbitrary we get

$$g\in\overline{\mathrm{conv}}\left(\left\{u\in(1+\varepsilon)B_{\mathrm{Lip}_0(M,X)}:\|f+u\|_L>2-\varepsilon\right\}\right)$$

as desired.

Note that $(M,\mathbb{R})$ has the CEP and so (ii)⇒(iii). Moreover, the Daugavet property passes to preduals and so (iii)⇒(iv).

Finally, we prove (iv)⇒(i). Assume that $\mathscr{F}(M)$ has the Daugavet property and let us prove that M is a length space. By Proposition 5.1.5 it is enough to show that M is local.

To this end, let $0<\varepsilon<\frac{1}{4}$ and $f\in S_{\mathrm{Lip}_0(M)}$ be given. Pick $x,y\in M$, $x\neq y$, such that $\langle f,m_{x,y}\rangle>1-\varepsilon$. From Lemma 5.1.11 we can find $x_1\neq y_1\in M$ such that $\langle f,m_{x_1,y_1}\rangle>1-\varepsilon$ and that $d(x_1,y_1)<\frac{\varepsilon}{(1-\varepsilon)^2}d(x,y)$. A new application of Lemma 5.1.11 yields $x_2,y_2\in M$, $x_2\neq y_2$, such that $\langle f,m_{x_2,y_2}\rangle>1-\varepsilon$ and that

$$d(x_2,y_2)\leq\frac{\varepsilon}{(1-\varepsilon)^2}d(x_1,y_1)<\left(\frac{\varepsilon}{(1-\varepsilon)^2}\right)^2d(x,y).$$

Continuing in this fashion we get a pair of sequences $(x_n)_{n=1}^{\infty}, (y_n)_{n=1}^{\infty}$ in M such that $\langle f, m_{x_n,y_n}\rangle > 1-\varepsilon$ and that

$$d(x_n, y_n) < \left(\frac{\varepsilon}{(1-\varepsilon)^2}\right)^n d(x,y)$$

holds for each $n \in \mathbb{N}$. Thus M is local. $\square$

Remark 5.1.12. The question of whether the Daugavet property is preserved by projective tensor products from both factors was posed in [Wer]. It remains, to the best of our knowledge, unsolved. It is known, however, that the Daugavet property is not preserved by projective tensor products from one factor. Indeed, in [KKW, Corollary 4.3] is given an example of a complex 2-dimensional Banach space X so that $L_\infty^{\mathbb{C}}([0,1])\widehat{\otimes}_\pi X$ fails to have the Daugavet property (see also [LLRZ2, Remark 3.13] for real counterexamples failing to fulfil weaker requirements than the Daugavet property). In spite of the previous fact, we get from Theorem 5.1.7 that the following holds:

(a) If the pair (M, X) has the contraction-extension property and $\mathrm{Lip}_0(M)$ has the Daugavet property then $\mathrm{Lip}_0(M, X) = \mathscr{L}(\mathscr{F}(M), X)$ has the Daugavet property.
(b) If the pair (M, X^*) has the contraction-extension property and $\mathscr{F}(M)$ has the Daugavet property, then $\mathscr{F}(M)\widehat{\otimes}_\pi X$ has the Daugavet property since its dual is the space $\mathscr{L}(\mathscr{F}(M), X^*)$.

In particular if H is a Hilbert space then $\mathscr{F}(H)\widehat{\otimes}_\pi H$ has the Daugavet property.

Remark 5.1.13. The proof of (i)⇒(iii) in Theorem 5.1.7 actually shows that $\mathrm{Lip}_0(M)$ satisfies a stronger version of the Daugavet property whenever M is a complete length space. Let us introduce some notation, coming from [BKSW]. Given $A \subset X$, we denote by $\mathrm{conv}_n(A)$ the set of all convex combinations of n elements of A. Given $x \in S_X$ and $\varepsilon > 0$, we denote

$$l^+(x, \varepsilon) = \{y \in (1+\varepsilon)B_X : \|x+y\| > 2-\varepsilon\}.$$

The space X is said to have the *uniform Daugavet property* if

$$\lim_{n\to\infty} \sup_{x,y\in S_X} d(y, \mathrm{conv}_n(l^+(x,\varepsilon)) = 0$$

for every $\varepsilon > 0$. In [BKSW] is proved that X has the uniform Daugavet property if and only if the ultrapower $X_{\mathscr{U}}$ has Daugavet property for every free ultrafilter $\mathscr{U}$ on $\mathbb{N}$. They also showed that $\mathscr{C}(K)$ with K perfect and $L_1[0,1]$ have the uniform Daugavet property. Moreover, Becerra and Martín proved in [BGM] that the Daugavet and the uniform Daugavet properties are equivalent for Lindenstrauss spaces. That is also the case for spaces of Lipschitz functions. Indeed, the proof of (i)⇒(iii) in Theorem 5.1.7 yields that, given $f, g \in S_{\mathrm{Lip}_0(M)}$, $n \in \mathbb{N}$ and $\varepsilon > 0$, we have

$$d(g, \mathrm{conv}_n(l^+(f,\varepsilon)) \leq \frac{4+2\varepsilon}{n}$$

which goes to 0 as $n \to \infty$. As a consequence, we get that $\mathrm{Lip}_0(M)$ has the Daugavet property if and only if the ultrapower $\mathrm{Lip}_0(M)_{\mathscr{U}}$ has the Daugavet property for every free ultrafilter $\mathscr{U}$ on $\mathbb{N}$.

Now we are going to focus on the relation between geodesic and length spaces. It is clear from Lemma 5.1.4 that every compact length space is geodesic. But the compactness is not always needed for this implication to hold. Indeed, in some particular cases, being a length space automatically implies being a geodesic space. For instance, this is the case for weak*-closed length subsets of dual Banach spaces as a consequence of Lemma 5.1.4 and the weak*-lower semicontinuity of the norm. In what follows we study geometric properties of a Banach space X that ensure that every complete length subset is geodesic. Recall that $\alpha(D)$ denotes the Kuratowski index of non-compactness of a set $D \subset X$.

Proposition 5.1.14. *Assume that* $\lim_{\delta\to 0} \alpha(\mathrm{Mid}(x, -x, \delta)) = 0$ *for every* $x \in S_X$*. Let* M *be a complete subset of* X*. Then if* M *is a length space, it is a geodesic space.*

Proof. Let $x, y \in M$ be given, by scaling and shifting we may assume that $x \in S_X$ and $y = -x$. Using Lemma 5.1.4 there is, for every $n \in \mathbb{N}$, a point $x_n \in \mathrm{Mid}(x, y, \frac{1}{n})$. It follows by our hypothesis and by $\mathrm{Mid}(x, y, \frac{1}{n+1}) \subset \mathrm{Mid}(x, y, \frac{1}{n})$ that $\lim_{n\to\infty} \alpha(\{x_k : k \geq n\}) = 0$. Therefore for every $\varepsilon > 0$ there is $n > 0$ such that $\{x_k : k \geq n\}$ can be covered by finitely many balls of radius ε. This suffices for selecting a Cauchy subsequence. Since M is complete, we have that its limit z belongs to M. It is now clear that $d(x, z) \leq 1$ and $d(y, z) \leq 1$ hence z is a metric midpoint between x and y. Now Lemma 5.1.4 gives that M is geodesic. $\square$

The hypothesis of Proposition 5.1.14 admits the following reformulation in terms of an asymptotic property of the Banach space X.

Proposition 5.1.15. *Let* $x \in S_X$*. The following are equivalent:*

(i) $\lim_{\delta\to 0} \alpha(\mathrm{Mid}(x, -x, \delta)) = 0$.

(ii) *For every* $0 < t < 1$ *there is* $\delta > 0$ *and a finite codimensional subspace* $Y \subset X$ *such that*

$$\inf_{y\in S_Y} \max\{\|x + ty\|, \|x - ty\|\} > 1 + \delta.$$

Proof. Follow the same arguments as in [DKR$^+$2, Theorem 2.1]. Let us sketch the main idea for reader's convenience. If (ii) fails, then for some $t > 0$ and every $\delta > 0$ it is easy to construct inductively a t-separated sequence in $\mathrm{Mid}(x, -x, \delta)$ showing that $\alpha(\mathrm{Mid}(x, -x, \delta)) \geq t/2$.

Conversely, let $t > 0$ be given and let Y and $\delta > 0$ be as in (ii). Since $\mathrm{Mid}(x, -x, \delta)$ is a ball of an equivalent norm on X, Lemma 2.13 of [JLPS] shows that there is a finite-dimensional subspace $Z \subset X$ so that

$$\mathrm{Mid}(x, -x, \delta) \subset 2(Z \cap \mathrm{Mid}(x, -x, \delta)) + 3(Y \cap \mathrm{Mid}(x, -x, \delta)).$$

Since we have for every $y \in S_Y$ that $ty \notin \mathrm{Mid}(x, -x, \delta)$, it follows by convexity that $Y \cap \mathrm{Mid}(x, -x, \delta) \subset tB_X$. Therefore $\alpha(\mathrm{Mid}(x, -x, \delta)) \leq 6t$. $\square$

In [DKR+2] the *asymptotically midpoint uniformly convex* spaces (AMUC, for short) were introduced as those Banach spaces in which $\lim_{\delta\to 0}\alpha(\operatorname{Mid}(x,-x,\delta))=0$ uniformly in $x\in S_X$, or, in other words, the same δ works for all $x\in S_X$ in the condition (ii) above. That is, for every $0<t<1$ there is $\delta>0$ such that

$$\inf_{x\in S_X}\sup_{\dim X/Y<\infty}\inf_{y\in S_Y}\max\{\|x+ty\|,\|x-ty\|\}\geq 1+\delta.$$

In particular, every AUC space is also AMUC.

It is clear that if

$$\lim_{\delta\to 0}\operatorname{diam}(\operatorname{Mid}(x,-x,\delta))=0 \text{ for every } x\in S_X \tag{5.4}$$

then the hypothesis of Proposition 5.1.14 is satisfied. The norms which satisfy (5.4) are called *midpoint locally uniformly rotund* (MLUR). For example, one can easily see that LUR norms are MLUR (see [Meg, Proposition 5.3.27]).

We are going to resume these comments into the following corollary.

Corollary 5.1.16. *A complete length subset M of a Banach space X is geodesic if any of the following conditions is satisfied:*

(a) *$X=Y^*$ for some Banach space Y and M is w^*-closed (in particular if M is a compact).*
(b) *X is AMUC (in particular if X is AUC, for example $X=\ell_p$, $1\leq p<\infty$).*
(c) *X is MLUR (in particular if X is LUR).*

We finish this section by focusing on Lipschitz free spaces with octahedral norms. These spaces have been recently characterised in [PRZ] in terms of a geometrical property of the underlying metric space. Namely, the norm of $\mathscr{F}(M)$ is octahedral if and only if M has the *Long Trapezoid Property* (LTP), that is, for each finite subset $N\subset M$ and $\varepsilon>0$, there exist $u,v\in M$, $u\neq v$, such that

$$(1-\varepsilon)(d(x,y)+d(u,v))\leq d(x,u)+d(y,v)$$

holds for all $x,y\in N$.

It is shown in [PRZ] that every infinite subset of a Banach space X has the LTP whenever $\overline{\delta}_X(t)=t$ for every $t>0$. Let us notice that in general $\overline{\delta}_X(t)\leq\overline{\rho}_X(t)\leq t$ so in such a case we also have $\overline{\rho}_X(t)=t$ for every $t>0$. The following result shows in particular that if X is a AUS space then it contains an infinite subset failing the LTP, which extends Proposition 4.5 in [PRZ].

Proposition 5.1.17. *Assume that there is $0<t\leq 1$ such that $\overline{\rho}_X(t)<t/2$. Then X contains an infinite subset failing the LTP.*

Proof. First notice that X is Asplund as a consequence of Proposition 2.4 in [JLPS]. Therefore, S_X contains a weakly-null sequence. Indeed, let Y be any separable infinite-dimensional subspace of X. Then $0\in\overline{S_Y}^{\sigma(Y,Y^*)}$. Since Y^* is separable, $(B_Y,\sigma(Y,Y^*))$ is metrizable and so there is a $\sigma(Y, Y^*)$-null sequence $(x_n)_{n=1}^\infty \subset S_Y$. Then $(x_n)_{n=1}^\infty$ is also $\sigma(X, X^*)$-convergent to 0, as desired.

Now, fix $\overline{\rho}_X(t) < \rho < t/2$ and $0 < \varepsilon < 1/3$ such that $2\rho < t(1-\varepsilon)^2 - 2\varepsilon$. Take $x \in S_X$ and fix $x^* \in S_{X^*}$ such that $x^*(x) = 1$. Then there is a finite codimensional subspace Y_0 of X such that $\sup_{y \in S_{Y_0}} \|x + ty\| < 1 + \rho$. Notice that $Y_0 \cap \ker x^*$ is also of finite codimension and so $\lim_{n\to\infty} d(x_n, Y_0 \cap \ker x^*) = 0$. Therefore, we may assume that $x_n \in Y_0 \cap \ker x^*$ for every n. Thus,

$$1 \leq x^*(x \pm tx_n) \leq \|x \pm tx_n\| < 1 + \rho$$

for every n. By passing to a subsequence, we may also assume that $\|x_n - x_m\| > 1 - \varepsilon$ for every $n \neq m$. Indeed, otherwise we could use Ramsey's theorem to get a subsequence, still denoted $(x_n)_{n=1}^\infty$, such that $\|x_n - x_m\| \leq 1 - \varepsilon$ for every $n \neq m$. Then, for a fixed n we would have

$$1 = \|x_n\| \leq \lim_{m\to\infty} \|x_n - x_m\| \leq 1 - \varepsilon$$

which is impossible.

We will prove that $M = \{x, -x\} \cup \{tx_n\}_{n=1}^\infty$ fails the LTP. For that, let $N = \{x, -x\}$. We will show that

$$(1-\varepsilon)(\|x - (-x)\| + \|u - v\|) > \min\{\|x - u\| + \|x + v\|, \|x + u\| + \|x - v\|\}$$

for every $u, v \in M$, $u \neq v$. We distinguish two cases.

1. If $u = tx_n$ and $v = tx_m$ for some $n \neq m$, then

$$\begin{aligned}(1-\varepsilon)(2\|x\| + t\|x_n - x_m\|) &\geq (1-\varepsilon)(2 + t(1-\varepsilon)) \\ &= 2 + t(1-\varepsilon)^2 - 2\varepsilon \\ &> 2 + 2\rho \\ &> \|x - tx_n\| + \|x + tx_m\|.\end{aligned}$$

2. If $u = \pm x$ and $v = tx_n$ for some n, then

$$(1-\varepsilon)(2\|x\| + \|x \mp tx_n\|) \geq 3(1-\varepsilon) > 2 > 1 + \rho > \|x \pm tx_n\|.$$

Therefore, M is an infinite subset of X which does not have the LTP, as desired. □

5.2 Extremal structure of Lipschitz free spaces

The goal of this section is to study the notions of extreme, preserved extreme, exposed and strongly exposed point of the unit ball of a Lipschitz free space. The study of the extremal structure of these spaces probably started in [Wea2], where it is proved for instance that preserved extreme points of $B_{\mathscr{F}(M)}$ are always molecules. Very recently, Aliaga and Guirao pushed this work further (see [AG]). In particular, answering a question of Weaver, they showed in the compact case that a molecule $m_{x,y}$ is an extreme point of $B_{\mathscr{F}(M)}$ if and only if there are no points except x and y in the metric segment $[x,y]$, and in that case $m_{x,y}$ is also a preserved extreme point. Note that if there exists $z \in [x,y] \setminus \{x,y\}$ then

$$m_{x,y} = \frac{d(x,z)}{d(x,y)} m_{x,z} + \frac{d(z,y)}{d(x,y)} m_{z,y}$$

and so $m_{x,y}$ is not an extreme point of $B_{\mathscr{F}(M)}$. To the best of our knowledge, in the non-compact case it is not known if every extreme point of $B_{\mathscr{F}(M)}$ is a molecule, and if the condition $[x,y] = \{x,y\}$ is sufficient for the molecule $m_{x,y}$ to be an extreme point.

5.2.1 Strongly exposed points of the unit ball of $\mathscr{F}(M)$

In what follows we will characterise the strongly exposed points of $B_{\mathscr{F}(M)}$. This will allow us to characterise the metric spaces M such that the unit ball of the free space $\mathscr{F}(M)$ has a strongly exposed point. In a general Banach space X the property that $\operatorname{strexp}(B_X) \neq \emptyset$ is extremely opposite to the Daugavet property. Our results below yield in particular that in the class of free spaces of compact metric spaces these properties are plainly complementary.

Let us introduce a bit of notation which will play a central role in the sequel.

Definition 5.2.1. Let $x, y \in M$, $x \neq y$. A function $f \in \operatorname{Lip}(M)$ is *peaking at* (x,y) if $\frac{f(x)-f(y)}{d(x,y)} = 1$ and for every open subset U of $M^2 \setminus \Delta$ containing (x,y) and (y,x), there exists $\delta > 0$ such that $\frac{|f(z)-f(t)|}{d(z,t)} \leq 1-\delta$ whenever $(z,t) \notin U$.

This definition is equivalent to the following assertion: $\frac{f(x)-f(y)}{d(x,y)} = 1$ and for every pair of sequences $(u_n)_{n=1}^{\infty}, (v_n)_{n=1}^{\infty} \subset M$ such that $\lim_{n\to+\infty} \frac{f(u_n)-f(v_n)}{d(u_n,v_n)} = 1$ we have $\lim_{n\to+\infty} u_n = x$ and $\lim_{n\to+\infty} v_n = y$. We say that $(x,y) \in M^2$ is a *peak couple* if there is a function peaking at (x,y).

Moreover in [Wea2, Proposition 2.4.2] it is proved that if a pair of points (x,y) is a peak couple then the molecule $m_{x,y}$ is a preserved extreme point of $B_{\mathscr{F}(M)}$. Below we give an alternative proof of this fact, showing that every peak couple corresponds to a strongly exposed point of $B_{\mathscr{F}(M)}$.

In [DKP, Proposition 2] a characterization of peak couples $(x,y) \in M^2$ is given when M is a subset of an $\mathbb{R}$-tree. We generalise that characterisation to an arbitrary metric space M. We shall need the following classical notation. Given $x, y, z \in M$ the *Gromov product of x and y at z* is defined as

$$(x,y)_z := \frac{1}{2}(d(x,z) + d(y,z) - d(x,y)).$$

It corresponds to the distance of z to the unique closest point b on the unique geodesic between x and y in any $\mathbb{R}$-tree into which $\{x,y,z\}$ can be isometrically embedded (such a tree, tripod really, always exists). Notice that $(x,z)_y + (y,z)_x = d(x,y)$ and that $0 \leq (x,y)_z \leq d(x,z)$ which we will use without further comment.

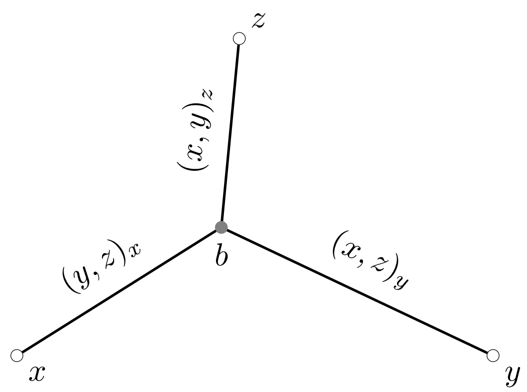

Figure 5.3: The Gromov product

Definition 5.2.2. We say that a pair (x, y) of points in M, $x \neq y$ has *property (Z)* if for every $\varepsilon > 0$ there is $z \in M \setminus \{x, y\}$ such that $(x, y)_z \leq \varepsilon \min\{d(x, z), d(y, z)\}$.

Clearly, M has property (Z) if, and only if, each pair of distinct points in M has property (Z).

We are now ready to give the characterisation of strongly exposed points in $B_{\mathscr{F}(M)}$. Let us recall that every strongly exposed point of $B_{\mathscr{F}(M)}$ is also a preserved extreme point, and so it is a molecule (see Corollary 2.5.4 in [Wea2]).

Theorem 5.2.3. *Let $x, y \in M$, $x \neq y$. The following assertions are equivalent:*

(i) *$m_{x,y}$ is a strongly exposed point of $B_{\mathscr{F}(M)}$.*
(ii) *There is $f \in \mathrm{Lip}_0(M)$ peaking at (x, y), i.e. (x, y) is a peak couple.*
(iii) *For every $\varepsilon > 0$*

$$\inf_{u \in M \setminus (\{x\} \cup B(y,\varepsilon))} \frac{(y,x)_u}{(u,y)_x} > 0 \text{ and } \inf_{u \in M \setminus (\{y\} \cup B(x,\varepsilon))} \frac{(y,x)_u}{(u,x)_y} > 0 \tag{5.5}$$

(with the convention that $\frac{\alpha}{0} = +\infty$).
(iv) *The pair (x, y) does not have the property (Z).*

Proof. (i)⇒(ii) is clear. Now we prove (ii)⇒(i). Assume that there is $f \in S_{\mathrm{Lip}_0(M)}$ peaking at (x, y). Assume that $\lim_{n\to\infty} \langle f, m_{u_n,v_n}\rangle = 1$. Since f peaks at (x, y), we have $\lim_{n\to\infty} d(u_n, x) = \lim_{n\to\infty} d(v_n, y) = 0$ and so $\lim_{n\to\infty} m_{u_n v_n} = m_{x,y}$ by Lemma 4.1.3. Thus, recalling that $V_M = \{m_{u,v} : u \neq v \in M\}$ is norming for $\mathrm{Lip}_0(M)$, Lemma 0.1.6 yields that $m_{x,y}$ is strongly exposed by f.

(ii)⇒(iii). Assume that there are $\varepsilon > 0$ and a sequence $(u_n)_{n=1}^{\infty} \subset M \setminus (\{x\} \cup B(y, \varepsilon))$ such that

$$\lim_{n\to+\infty} \frac{(y,x)_{u_n}}{(u_n,y)_x} = 0.$$

We then clearly have

$$\lim_{n\to+\infty} \frac{(y,x)_{u_n}}{d(x,u_n)} = 0$$

since $(u_n,y)_x \le d(x,u_n)$. Let $f \in \mathrm{Lip}(M)$ be such that $\|f\|_L = 1$ and $\frac{f(x)-f(y)}{d(x,y)} = 1$. We may assume that $f(y) = 0$ and $f(x) = d(x,y)$. Consider b_n so that $\{x,y,u_n\}$ embeds isometrically into $\{x,y,u_n,b_n\}$. Notice that, if we denote f_n the unique 1-Lipschitz extension of $f\restriction_{\{x,y,u_n\}}$ to $\{x,y,u_n,b_n\}$, then $f_n(b_n) = (u_n,x)_y$ and therefore $|(u_n,x)_y - f(u_n)| \le (y,x)_{u_n}$. We have

$$\begin{aligned} f(x) - f(u_n) &= (f(x) - (u_n,x)_y) - ((u_n,x)_y - f(u_n)) \\ &= (d(x,y) - (u_n,x)_y) - ((u_n,x)_y - f(u_n)) \\ &\ge (u_n,y)_x - (y,x)_{u_n} \\ &= d(x,u_n) - 2(y,x)_{u_n}. \end{aligned}$$

It follows that

$$\lim_{n\to+\infty} \frac{f(x)-f(u_n)}{d(x,u_n)} = 1.$$

and so f is not peaking at (x,y) as $(u_n)_{n=1}^\infty$ does not converge to y.

(iii)$\Rightarrow$(iv). Assume that the pair (x,y) has the property (Z). Then for every $n\in\mathbb{N}$ there is $z_n \in M\setminus\{x,y\}$ such that $(x,y)_{z_n} \le \frac{1}{n}\min\{d(x,z_n),d(y,z_n)\}$. Passing to a subsequence and exchanging the roles of x and y we may assume that $d(x,z_n)\le d(y,z_n)$ for all $n\in\mathbb{N}$. We thus have $\dfrac{(x,y)_{z_n}}{d(x,z_n)} \to 0$ and $d(y,z_n) \ge \frac{1}{2}d(x,y)$. Therefore

$$\inf_{u\in M\setminus(\{x\}\cup B(y,\frac{1}{2}d(y,x)))} \frac{(y,x)_u}{d(x,u)} = 0.$$

Now notice that $d(x,u) = (u,y)_x + (x,y)_u \le 2\max\{(u,y)_x,(x,y)_u\}$. Thus,

$$\frac{(y,x)_u}{d(x,u)} \ge \frac{(y,x)_u}{2\max\{(u,y)_x,(x,y)_u\}} = \frac{(y,x)_u}{(u,y)_x}$$

where the equality above is true whenever the term on the left-hand side is less than $\frac{1}{2}$. It follows that

$$\inf_{u\in M\setminus(\{x\}\cup B(y,\frac{1}{2}d(y,x)))} \frac{(y,x)_u}{(y,u)_x} = 0,$$

a contradiction.

(iv)$\Rightarrow$(ii). By hypothesis, there is $\varepsilon_0 > 0$ such that

$$d(x,z) + d(z,y) > d(x,y) + \varepsilon_0 \min\{d(x,z),d(z,y)\}$$

for every $z \in M \setminus \{x, y\}$. We will show that (x, y) is a peak couple. To this end, fix $\varepsilon_1 > 0$ with $\frac{\varepsilon_1}{1-\varepsilon_1} < \frac{\varepsilon_0}{4}$ and let f be the Lipschitz function defined in [IKW1, Proposition 2.8], namely

$$f(z) := \begin{cases} \max\left\{\frac{d(x,y)}{2} - (1-\varepsilon_1)d(z,x), 0\right\} & \text{if } d(z,y) \geq d(z,x), \\ & \quad d(z,y) + (1-2\varepsilon_1)d(z,x) \geq d(x,y) \\ -\max\left\{\frac{d(x,y)}{2} - (1-\varepsilon_1)d(z,y), 0\right\} & \text{if } d(z,x) \geq d(z,y), \\ & \quad d(z,x) + (1-2\varepsilon_1)d(z,y) \geq d(x,y) \end{cases}$$

which is well defined and satisfies $\|f\|_L = 1$, $f(x) - f(y) = d(x, y)$, and

$$\frac{f(u) - f(v)}{d(u,v)} > 1 - \varepsilon_1 \text{ implies } \max\{d(x,u), d(y,v)\} < \frac{d(x,y)}{4}$$

for any $u, v \in M$, $u \neq v$ (see the proof of Proposition 2.8 in [IKW1]). Now, take $g = \frac{1}{2}(f + f_{x,y})$. We claim that g peaks at (x, y). Indeed, take sequences $(u_n)_{n=1}^\infty$ and $(v_n)_{n=1}^\infty$ in M with $\lim_{n\to\infty} \frac{g(u_n)-g(v_n)}{d(u_n,v_n)} = 1$. Fix $\varepsilon > 0$ and take $0 < \gamma < \varepsilon_1$ such that $\frac{\gamma}{1-\gamma} d(x, y) < \varepsilon_0 \varepsilon$. Now, take n_0 such that

$$\frac{g(u_n) - g(v_n)}{d(u_n, v_n)} > 1 - \frac{\gamma}{4} \tag{5.6}$$

for every $n \geq n_0$. We will show that $d(x, u_n), d(y, v_n) < \varepsilon$. First, note that (5.6) implies that

$$\frac{f(u_n) - f(v_n)}{d(u_n, v_n)} > 1 - \frac{\gamma}{2} > 1 - \varepsilon_1$$

and so $d(x, u_n), d(y, v_n) < \frac{d(x,y)}{4}$. Therefore $d(x, u_n) < d(y, u_n)$ and $d(y, v_n) < d(x, v_n)$. Moreover, it also follows from (5.6) that

$$\frac{f_{x,y}(u_n) - f_{x,y}(v_n)}{d(u_n, v_n)} > 1 - \frac{\gamma}{2} > 1 - \gamma$$

and so using Lemma 5.1.10 we get

$$(1 - \gamma) \max\{d(x, u_n) + d(y, u_n), d(x, v_n) + d(y, v_n)\} \leq d(x, y).$$

This and the hypothesis imposed on the pair (x, y) yield

$$d(x, y) + \varepsilon_0 d(x, u_n) < d(x, u_n) + d(u_n, y) \leq \frac{1}{1-\gamma} d(x, y).$$

Therefore,

$$d(x, u_n) \leq \frac{1}{\varepsilon_0} \left(\frac{1}{1-\gamma} - 1\right) d(x, y) < \varepsilon$$

for every $n \geq n_0$. Similarly, $d(y, v_n) < \varepsilon$. This shows that $(u_n)_{n=1}^\infty$ converges to x and $(v_n)_{n=1}^\infty$ converges to y. Thus, g peaks at (x, y) as desired. $\square$

Note that the proof of (ii)⇒(i) in Theorem 5.2.3 actually shows that the following holds:

Corollary 5.2.4. *Let $f \in \mathrm{Lip}_0(M)$ and $x, y \in M$, $x \neq y$. Then f peaks at the pair (x, y) if and only if f strongly exposes $m_{x,y}$ in $B_{\mathscr{F}(M)}$.*

In what follows we show that free spaces naturally strengthen their extremal structure. We will need the following lemma, which asserts that a net of molecules which converges to a molecule in the weak topology in fact converges in the norm topology.

Lemma 5.2.5. *Assume $(m_{x_\alpha,y_\alpha})_\alpha$ is a net in V_M which converges weakly to $m_{x,y}$. Then $\lim_\alpha d(x_\alpha, x) = 0$ and $\lim_\alpha d(y_\alpha, y) = 0$.*

Proof. Assume that $0 < \varepsilon < \min\{d(x, y), \limsup_\alpha d(x_\alpha, x)\}$. Consider the map f given by $f(t) = \max\{\varepsilon - d(x, t), 0\}$ and let $g = f - f(0) \in \mathrm{Lip}_0(M)$. Note that $\langle g, m_{x,y}\rangle = \frac{\varepsilon}{d(x,y)} > 0$. However,

$$\liminf_\alpha \langle g, m_{x_\alpha,y_\alpha}\rangle = \liminf_\alpha \frac{-f(y_\alpha)}{d(x_\alpha, y_\alpha)} \leq 0,$$

a contradiction. Therefore, $\lim_\alpha x_\alpha = x$. Analogously we get that $\lim_\alpha y_\alpha = y$. □

Proposition 5.2.6. *Let μ be weak-strongly exposed in $B_{\mathscr{F}(M)}$ by $f \in S_{\mathrm{Lip}_0(M)}$. Then μ is strongly exposed by f.*

Proof. First note that μ is a preserved extreme point of $B_{\mathscr{F}(M)}$ and so $\mu = m_{x,y}$ for some $x, y \in M$. Now take sequences $(u_n)_{n=1}^\infty, (v_n)_{n=1}^\infty$ in M such that $\langle f, m_{u_n,v_n}\rangle \to 1$. Since f weak-strongly exposes μ we have that $m_{u_n,v_n} \xrightarrow{w} \mu$. Now, Lemma 5.2.5 ensures that $\lim_n d(u_n, x) = \lim_n d(v_n, y) = 0$. Thus f peaks at μ and so μ is strongly exposed by f by Theorem 5.2.3. □

As a consequence of Proposition 5.2.6 we get the following corollary.

Corollary 5.2.7. *Let $f \in S_{\mathrm{Lip}_0(M)}$. If the norm of $\mathrm{Lip}_0(M)$ is Gâteaux differentiable at f, then it is also Fréchet differentiable at f (with the derivative $\mu \in \mathscr{F}(M)$ of the form $\mu = m_{x,y}$).*

Proof. Let us show that if $f \in \mathrm{Lip}_0(M)$ does not attain its norm on $B_{\mathscr{F}(M)}$, then f is not a point of Gâteaux differentiability of the norm $\|\ \|_L$. Indeed, let $(x_n)_{n=1}^\infty, (y_n)_{n=1}^\infty \subset M$ be such that $\langle f, m_{x_n,y_n}\rangle \to \|f\|_L$. It is enough to show that the functional $g \mapsto \lim_n \langle g, m_{x_n,y_n}\rangle$ defined on the linear span of f admits two different extensions on $\mathrm{Lip}_0(M)$.

First we claim that there is $g \in \mathrm{Lip}_0(M)$ such that $\lim_n \langle g, m_{x_n,y_n}\rangle$ does not exist. Indeed, assume that for every $g \in \mathrm{Lip}_0(M)$ the limit exists and denote it by $\varphi(g)$. Then $\varphi \in \mathrm{Lip}_0(M)^*$ by the uniform boundedness principle and $\|\varphi\| \leq 1$. Now, $(m_{x_n,y_n})_{k=1}^\infty$ is norm-Cauchy. Indeed, if this is not the case then there is $\varepsilon > 0$ and sequences $(p_k)_{k=1}^\infty$ and $(q_k)_{k=1}^\infty$ such that $\left\|m_{x_{p_k},y_{p_k}} - m_{x_{q_k},y_{q_k}}\right\| > \varepsilon$ for every k. So $(m_{x_{p_k},y_{p_k}} - m_{x_{q_k},y_{q_k}})_{k=1}^\infty$ is an ε-separated sequence with uniformly bounded cardinality of the supports. By Theorem 5.2

in [AK1] we get that $(m_{x_{p_k},y_{p_k}} - m_{x_{q_k},y_{q_k}})$ is not weakly-null, which is clearly a contradiction. Thus, the sequence (m_{x_n,y_n}) is norm-Cauchy and so it is also norm-convergent to φ. It follows that $\varphi \in B_{\mathscr{F}(M)}$ which is a contradiction that proves our claim.

Let now $(n_k)_{k=1}^{\infty}$ and $(m_k)_{k=1}^{\infty}$ be such that $\lim_k \left\langle g, m_{x_{n_k},y_{n_k}} \right\rangle = \limsup_n \langle g, m_{x_n,y_n} \rangle$ and $\lim_k \left\langle g, m_{x_{m_k} y_{m_k}} \right\rangle = \liminf_n \langle g, m_{x_n,y_n} \rangle$. It is clear that the Hahn-Banach extensions of these limits are different and they both extend the original limit. Thus $\| \|_L$ is not Gâteaux differentiable at the point f.

We now assume that the norm is Gâteaux differentiable at f. By the previous paragraph, the unique norming functional μ belongs to $\mathscr{F}(M)$. Then the version of the Šmulyan lemma 0.1.5 for Gâteaux differentiability yields that μ is weak-strongly exposed in $B_{\mathscr{F}(M)}$ by f. Now apply Proposition 5.2.6 and the version of Smulyan's lemma 0.1.5 for Fréchet differentiability. □

Let us end the section by giving the following characterisation under compactness assumptions, which improves [IKW1, Theorem 3.3].

Corollary 5.2.8. *Let M be a compact metric space. The following assertions are equivalent:*

(i) *M is geodesic.*
(ii) *For every $x, y \in M$ there is $z \in [x, y] \setminus \{x, y\}$.*
(iii) $\mathrm{Lip}_0(M)$ *has the Daugavet property.*
(iv) *The unit ball of $\mathscr{F}(M)$ does not have any preserved extreme point.*
(v) *The unit ball of $\mathscr{F}(M)$ does not have any strongly exposed point.*
(vi) *The norm of* $\mathrm{Lip}_0(M)$ *does not have any point of Gâteaux differentiability.*
(vii) *The norm of* $\mathrm{Lip}_0(M)$ *does not have any point of Fréchet differentiability.*

Proof. The equivalence between (i) and (iii) follows from Theorem 5.1.7 and the fact that compact length spaces are geodesic. Moreover, (i)⇒(ii) follows from Lemma 5.1.4. Now, if (ii) holds then every molecule $m_{x,y}$ can be written as a non-trivial convex combination as

$$m_{x,y} = \frac{d(x,z)}{d(x,y)} m_{x,z} + \frac{d(z,y)}{d(x,y)} m_{z,y}$$

and so it is not an extreme point of $B_{\mathscr{F}(M)}$. Since all the preserved extreme points are molecules [Wea2, Corollary 2.5.4], (iv) holds.

It is clear that (iv) implies (v). If (v) holds then by Theorem 5.2.3 we have that M has property (Z). Since M is compact then Proposition 2.8 in [IKW1] says that M is local, and so a length space by Proposition 5.1.5. This shows that (v) implies (i). Finally, the equivalence between (v), (vi) and (vii) follows from Corollary 5.2.7 (and holds even in the non-compact case). □

Remark 5.2.9. Note that the previous corollary means that, whenever M is a compact metric space, then either $\mathscr{F}(M)$ has the Daugavet property or its unit ball is dentable.

Such extreme behaviour related to the diameter of the slices of the unit ball does not hold for its dual $\mathrm{Lip}_0(M)$. Indeed, in [Iva] it is proved that every slice of $B_{\mathrm{Lip}_0(M)}$ has diameter two whenever M is unbounded or it is not uniformly discrete. Consequently $M = [0,1] \cup [2,3]$ is an example of a compact metric space such that every slice of $B_{\mathrm{Lip}_0(M)}$ has diameter two but $\mathrm{Lip}_0(M)$ fails the Daugavet property.

5.2.2 Preserved extreme points of the unit ball of $\mathscr{F}(M)$

Our next goal is to show that every preserved extreme point of $B_{\mathscr{F}(M)}$ is also a denting point. To this end, we need the following variation of Asplund–Bourgain–Namioka superlemma 2.1.3.

Lemma 5.2.10. *Let A, B be bounded closed convex subsets of a Banach space X and let $\varepsilon > 0$. Assume that $\mathrm{diam}(A) < \varepsilon$ and that there is $x_0 \in A \setminus B$ which is a preserved extreme point of $\overline{\mathrm{conv}}(A \cup B)$. Then there is a slice of $\overline{\mathrm{conv}}(A \cup B)$ containing x_0 which is of diameter less than ε.*

Proof. For each $r \in [0,1]$ let

$$C_r = \{x \in X : x = (1-\lambda)y + \lambda z, y \in A, z \in B, \lambda \in [r,1]\}.$$

The proof of the Superlemma says that there is r so that $\mathrm{diam}(\overline{\mathrm{conv}}(A \cup B) \setminus \overline{C_r}) < \varepsilon$. We will show that $x_0 \notin \overline{C_r}$. Thus, any slice separating x_0 from $\overline{C_r}$ will do the work. To this end, assume that there exist sequences $(y_n)_{n=1}^\infty \subset A$, $(z_n)_{n=1}^\infty \subset B$ and $(\lambda_n)_{n=1}^\infty \subset [r,1]$ such that $x_0 = \lim_n (1-\lambda_n)y_n + \lambda_n z_n$. By extracting a subsequence, we may assume that $(\lambda_n)_{n=1}^\infty$ converges to some $\lambda \in [r,1]$. Note that $x_0 = \lim_n (1-\lambda)y_n + \lambda z_n$. Since x_0 is a preserved extreme point, this implies that $(z_n)_{n=1}^\infty$ converges weakly to x_0 by Proposition 0.1.3. That is impossible since $x_0 \notin B$ and B is weakly closed as being convex and closed. □

Theorem 5.2.11. *Every preserved extreme point of $B_{\mathscr{F}(M)}$ is a denting point.*

Proof. Let μ be a preserved extreme point of $B_{\mathscr{F}(M)}$, which must be an element of V_M. Denote by $\mathscr{S}$ the set of weak-open slices of $B_{\mathscr{F}(M)}$ containing μ. Consider the order $S_1 \leq S_2$ if $S_2 \subset S_1$ for $S_1, S_2 \in \mathscr{S}$. Using (ii) of Proposition 0.1.3, every finite intersection of elements of $\mathscr{S}$ contains an element of $\mathscr{S}$ and so $(\mathscr{S}, \leq)$ is a directed set. Assume that μ is not a denting point. Then, there is $\varepsilon > 0$ so that $\mathrm{diam}(S) > 2\varepsilon$ for every $S \in \mathscr{S}$.

We distinguish two cases. Assume first that for every slice S of $B_{\mathscr{F}(M)}$ there is $\mu_S \in (V_M \cap S) \setminus B(\mu, \varepsilon/4)$. Then $(\mu_S)_{S \in \mathscr{S}}$ is a net in V_M which converges weakly to μ. By Lemma 5.2.5, it also converges in norm, which is impossible. Thus, there is a slice S of $B_{\mathscr{F}(M)}$ such that $\mathrm{diam}(V_M \cap S) \leq \varepsilon/2$. Note that

$$B_{\mathscr{F}(M)} = \overline{\mathrm{conv}}(V_M) = \overline{\mathrm{conv}}(\overline{\mathrm{conv}}(V_M \cap S) \cup \overline{\mathrm{conv}}(V_M \setminus S))$$

and so the hypotheses of Lemma 5.2.10 are satisfied for $A = \overline{\mathrm{conv}}(V_M \cap S)$, $B = \overline{\mathrm{conv}}(V_M \setminus S)$, and $\mu \in A \setminus B$. Then there is a slice of $B_{\mathscr{F}(M)}$ containing μ of diameter less than ε, a contradiction. □

Corollary 5.2.12. *Let M be a length space. Then $B_{\mathscr{F}(M)}$ does not have any preserved extreme point.*

Proof. The space $\mathscr{F}(M)$ has the Daugavet property whenever M is a length space [IKW1]. In particular, every slice of $B_{\mathscr{F}(M)}$ has diameter two. Thus, $B_{\mathscr{F}(M)}$ does not have any denting point. □

During the preparation of this work Aliaga and Guirao [AG] characterised metrically the preserved extreme points of free spaces. In the following pages we provide an alternative proof of their result which accidentally reproves our Theorem 5.2.11.

Theorem 5.2.13. *Let M be a metric space and $x, y \in M$. The following are equivalent:*

(i) *The molecule $m_{x,y}$ is a denting point of $B_{\mathscr{F}(M)}$.*
(ii) *For every $\varepsilon > 0$ there exists $\delta > 0$ such that every $z \in M$ satisfies*

$$(1-\delta)(d(x,z)+d(z,y)) < d(x,y) \Rightarrow \min\{d(x,z), d(y,z)\} < \varepsilon.$$

Proof of (i)⇒(ii). In fact we are going to show that negation of (ii) implies that $m_{x,y}$ is not a preserved extreme point. Since denting points are trivially preserved extreme points, this will show at once that $m_{x,y}$ is not denting.

So let us fix $\varepsilon > 0$ such that for every $n \in \mathbb{N}$ there exists $z_n \in M$ such that

$$\left(1-\frac{1}{n}\right)(d(x,z_n)+d(z_n,y)) < d(x,y)$$

but $\min\{d(x,z_n), d(y,z_n)\} \geq \varepsilon$. Let μ be a weak*-cluster point of $\{\delta(z_n)\}$ in $\mathrm{Lip}_0(M)^*$, which exists since $\{z_n\}$ is bounded. By lower semicontinuity of the norm we have

$$\|\delta(x)-\mu\| + \|\mu-\delta(y)\| = d(x,y).$$

If $\mu \in \{\delta(x), \delta(y)\}$, say $\mu = \delta(x)$ then by testing against the function $f(t) = d(t,x) - d(0,x)$ we get $\liminf_{n\to\infty} d(z_n, x) = 0$ which is a contradiction.

Thus $\mu \notin \{\delta(x), \delta(y)\}$. Then

$$\frac{\delta(x)-\delta(y)}{\|\delta(x)-\delta(y)\|} = \frac{\|\delta(x)-\mu\|}{\|\delta(x)-\delta(y)\|}\frac{\delta(x)-\mu}{\|\delta(x)-\mu\|} + \frac{\|\mu-\delta(y)\|}{\|\delta(x)-\delta(y)\|}\frac{\mu-\delta(y)}{\|\mu-\delta(y)\|}.$$

Thus $m_{x,y}$ is a non-trivial convex combination and so it is not preserved extreme. □

For the proof of the other implication, we need a couple of lemmata. The first of them shows that the diameter of the slices of the unit ball can be controlled by the diameter of the slices of a subset of the ball that is norming for the dual.

Lemma 5.2.14. *Let X be a Banach space and let $V \subset S_X$ be such that $B_X = \overline{\mathrm{conv}}(V)$. Let $f \in B_{X^*}$ and $0 < \alpha, \varepsilon < 1$. Then*

$$\mathrm{diam}(S(B_X, f, \varepsilon\alpha)) \leq 2\,\mathrm{diam}(S(B_X, f, \alpha) \cap V) + 4\varepsilon.$$

Proof. Fix a point $x_0 \in S(B_X, f, \alpha) \cap V$. It suffices to show that

$$\|x - x_0\| < \operatorname{diam}(S(B_X, f, \alpha) \cap V) + 2\varepsilon$$

for every $x \in S(B_X, f, \varepsilon\alpha) \cap \operatorname{conv}(V)$. To this end, let $x \in B_X$ be such that $f(x) > 1 - \varepsilon\alpha$, and $x = \sum_{i=1}^n \lambda_i x_i$, with $x_i \in V$, $\sum_{i=1}^n \lambda_i = 1$ and $\lambda_i > 0$ for all $1 \le i \le n$. Define

$$G = \{i \in \{1, \dots, n\} : f(x_i) > 1 - \alpha\}$$

and $B = \{1, \dots, n\} \setminus G$. We have

$$\begin{aligned} 1 - \varepsilon\alpha < f(x) &= \sum_{i\in G} \lambda_i f(x_i) + \sum_{i\in B} \lambda_i f(x_i) \\ &\le \sum_{i\in G} \lambda_i + (1-\alpha)\sum_{i\in B} \lambda_i = 1 - \alpha \sum_{i\in B} \lambda_i, \end{aligned}$$

which yields that $\sum_{i\in B} \lambda_i < \varepsilon$. Now,

$$\|x - x_0\| \le \sum_{i\in G} \lambda_i \|x_i - x_0\| + \sum_{i\in B} \lambda_i \|x_i - x_0\| \le \operatorname{diam}(S(B_X, f, \alpha) \cap V) + 2\varepsilon.$$

□

Lemma 5.2.15. *Let $x, y \in M$, $x \neq y$, such that $d(x, y) = 1$. For every $0 < \varepsilon < 1/4$ and $0 < \tau < 1$ there is a function $f \in \operatorname{Lip}_0(M)$ such that $\|f\|_L = 1$, $\langle f, m_{x,y}\rangle > 1 - 4\varepsilon\tau$ and satisfying that for every $u, v \in M$, $u \neq v$, if either $u, v \in B(x, \varepsilon)$ or $u, v \in B(y, \varepsilon)$, then $\langle f, m_{u,v}\rangle \le 1 - \tau$.*

Proof. Define $f \colon B(x, \varepsilon) \cup B(y, \varepsilon) \to \mathbb{R}$ by

$$f(t) = \begin{cases} \frac{1}{1+4\varepsilon\tau}(\tau + (1-\tau)d(y,t)) & \text{if } t \in B(x,\varepsilon), \\ \frac{1}{1+4\varepsilon\tau}(1-\tau)d(y,t) & \text{if } t \in B(y,\varepsilon). \end{cases}$$

Note that

$$\langle f, m_{x,y}\rangle = f(x) - f(y) = \frac{1}{1+4\varepsilon\tau} > 1 - 4\varepsilon\tau.$$

Moreover, note that if $u, v \in B(x, \varepsilon)$ or $u, v \in B(y, \varepsilon)$ then $\langle f, m_{u,v}\rangle \le \frac{1-\tau}{1+4\varepsilon\tau} \le 1 - \tau$, so the last condition in the statement is satisfied. Now we compute the Lipschitz norm of f. It remains to compute $\langle f, m_{u,v}\rangle$ with $u \in B(x, \varepsilon)$ and $v \in B(y, \varepsilon)$. In that case we have

$$\begin{aligned} |\langle f, m_{u,v}\rangle| &= \frac{|\tau + (1-\tau)(d(u,y) - d(v,y))|}{(1+4\varepsilon\tau)d(u,v)} \le \frac{\tau + (1-\tau)d(u,v)}{(1+4\varepsilon\tau)d(u,v)} \\ &\le \frac{1}{1+4\varepsilon\tau}\left(\frac{\tau}{1-2\varepsilon} + 1 - \tau\right) \le \frac{\tau(1+4\varepsilon) + 1 - \tau}{1+4\varepsilon\tau} = 1 \end{aligned}$$

where we are using that $(1-2\varepsilon)^{-1} \le 1 + 4\varepsilon$ since $\varepsilon < 1/4$. This shows that $\|f\|_L \le 1$. Next, find an extension of f with the same norm. Finally, replace f with the function $t \mapsto f(t) - f(0)$. □

Proof of (ii)⇒(i) of Theorem 5.2.13. We can assume that $d(x, y) = 1$. Fix $0 < \varepsilon < 1/4$. We will find a slice of $B_{\mathscr{F}(M)}$ containing $m_{x,y}$ of diameter smaller than 32ε. Let $\delta > 0$ be given by property (ii), clearly we may assume that $\delta < 1$. Let f be the function given by Lemma 5.2.15 with $\tau = \delta/2$. Consider

$$h(t) = \frac{f_{x,y}(t) + f(t)}{2}.$$

It is clear that $\|h\|_L \leq 1$. Moreover, note that

$$\langle h, m_{x,y} \rangle = \frac{\langle f_{x,y}, m_{x,y} \rangle + \langle f, m_{x,y} \rangle}{2} > 1 - 2\varepsilon\tau = 1 - \varepsilon\delta.$$

Now, take $\alpha = \delta/4$ and consider the slice $S = S(B_{\mathscr{F}(M)}, h, \alpha)$. Note that $m_{x,y} \in S(B_{\mathscr{F}(M)}, h, 4\varepsilon\alpha)$. We will show that $\operatorname{diam}(S \cap V_M) \leq 8\varepsilon$ and as a consequence of Lemma 5.2.14 we will get that $\operatorname{diam}(S(B_{\mathscr{F}(M)}, h, 4\varepsilon\alpha)) \leq 32\varepsilon$. So let $u, v \in M$ be such that $m_{u,v} \in S$. First, note that $\langle f_{x,y}, m_{u,v} \rangle > 1 - \delta$, since otherwise we would have

$$\langle h, m_{u,v} \rangle = \frac{1}{2}(\langle f_{x,y}, m_{u,v} \rangle + \langle f, m_{u,v} \rangle) \leq \frac{1}{2}(1 - \delta) + \frac{1}{2} = 1 - \frac{\delta}{2} < 1 - \alpha.$$

Thus, from the property (c) of the function $f_{x,y}$ in Lemma 5.1.10 and the hypothesis (ii) we have that

$$\min\{d(x, u), d(u, y)\} < \varepsilon \quad \text{and} \quad \min\{d(x, v), d(y, v)\} < \varepsilon.$$

On the other hand,

$$1 - \alpha < \langle h, m_{u,v} \rangle \leq \frac{1}{2} + \frac{1}{2}\langle f, m_{u,v} \rangle$$

and so $\langle f, m_{u,v} \rangle > 1 - 2\alpha = 1 - \frac{\delta}{2} = 1 - \tau$. Thus, we have that u and v do not belong simultaneously to neither $B(x, \varepsilon)$ nor $B(y, \varepsilon)$. If $d(x, v) < \varepsilon$ and $d(y, u) < \varepsilon$, then it is easy to check that $\langle f_{x,y}, m_{u,v} \rangle \leq 0$. So necessarily $d(x, u) < \varepsilon$ and $d(y, v) < \varepsilon$. Now, Lemma 4.1.3 yields that

$$\|m_{x,y} - m_{u,v}\| \leq 2\frac{d(x, u) + d(y, v)}{d(x, y)} \leq 4\varepsilon.$$

Therefore, $\operatorname{diam}(S \cap V_M) \leq 8\varepsilon$. $\square$

5.2.3 Extremal structure for spaces with additional properties

To the best of knowledge, two main questions in this domain remain open:

(a) If $\mu \in \operatorname{ext}(B_{\mathscr{F}(M)})$, is μ necessarily of the form $\mu = m_{x,y}$ for some $x \neq y \in M$?
(b) If the metric segment $[x, y]$ does not contain any other point of M than x and y, is $m_{x,y}$ an extreme point of $B_{\mathscr{F}(M)}$?

Our aim is to provide affirmative answers to both previous questions (a) and (b) in some particular cases. Namely, we are going to focus on uniformly discrete spaces, compact spaces, and metric spaces M such that $\mathscr{F}(M)$ admits a predual with some additional properties. To this end we consider the following notion.

Definition 5.2.16. Let M be a bounded metric space. We will say that a Banach space X is a *natural predual* of $\mathscr{F}(M)$ if $X^* = \mathscr{F}(M)$ isometrically and $\delta(M)$ is $\sigma(\mathscr{F}(M), X)$-closed.

It is clear that if M is a compact metric space then every isometric predual of $\mathscr{F}(M)$ is natural. Moreover, we showed in Theorem 4.2.7 that Kalton's predual is a natural one. However, we will show in Example 5.2.26 that there are isometric preduals of $\mathscr{F}(M)$ which are not natural.

We are going to focus now on the extreme points in the free spaces that admit a natural predual. Assuming moreover that the predual is made up of little-Lipschitz functions we get an affirmative answer the question of whether every extreme point of $B_{\mathscr{F}(M)}$ is a molecule. Note that this is an extension of Corollary 3.3.6 in [Wea2], where it is obtained the same result under the assumption that M is compact.

Proposition 5.2.17. *Let M be a bounded metric space. Assume that there is a subspace X of* $\mathrm{lip}_0(M)$ *which is a natural predual of $\mathscr{F}(M)$. Then* $\mathrm{ext}(B_{\mathscr{F}(M)}) \subset V_M$.

Proof. By the separation theorem we have that $B_{\mathscr{F}(M)} = \overline{\mathrm{conv}}^{w^*}(V_M)$. Thus, according to Milman theorem (see [FHH$^+$, Theorem 3.41]), we have $\mathrm{ext}(B_{\mathscr{F}(M)}) \subset \overline{V}^{w^*}$. So let us consider $\gamma \in \mathrm{ext}(B_{\mathscr{F}(M)})$. Take a net $(m_{x_\alpha,y_\alpha})_\alpha$ in V_M which w^*-converges to γ. By w^*-compactness of $\delta(M)$, we may assume (up to extracting subnets) that $(\delta(x_\alpha))_\alpha$ and $(\delta(y_\alpha))_\alpha$ converge to some $\delta(x)$ and $\delta(y)$ respectively.

Next, since M is bounded, we may assume up to extract a further subnet that $(d(x_\alpha, y_\alpha))_\alpha$ converges to $C \geq 0$. We claim that $C > 0$. Indeed, by assumption, there is $f \in X$ such that $\langle f, \gamma\rangle > \|\gamma\|/2 = 1/2$. Since $f \in \mathrm{lip}_0(M)$, there exists $\delta > 0$ such that whenever $z_1, z_2 \in M$ satisfy $d(z_1, z_2) \leq \delta$ then we have $|f(z_1) - f(z_2)| \leq \frac{1}{2} d(z_1, z_2)$. Since

$$\lim_\alpha \langle f, m_{x_\alpha,y_\alpha}\rangle = \langle f, \gamma\rangle > \frac{1}{2},$$

there is α_0 such that $\langle f, m_{x_\alpha,y_\alpha}\rangle > 1/2$ for every $\alpha > \alpha_0$. Thus $d(x_\alpha, y_\alpha) > \delta$ for $\alpha > \alpha_0$, which implies that $C \geq \delta > 0$. Summarising, we have a net $(m_{x_\alpha,y_\alpha})_\alpha$ which w^*-converges to $\frac{\delta(x)-\delta(y)}{C}$. So, by uniqueness of the limit, $\gamma = \frac{\delta(x)-\delta(y)}{C}$. Since $\gamma \in \mathrm{ext}(B_{\mathscr{F}(M)}) \subset S_{\mathscr{F}(M)}$, we get that $C = d(x, y)$ and so $\gamma = m_{x,y}$. □

A weaker version of the following proposition appears in the preprint [AG] for compact metric spaces. Our approach, which is independent of [AG], also yields a characterisation of exposed points of $B_{\mathscr{F}(M)}$.

Corollary 5.2.18. *Let M be a bounded* separable *metric space. Assume that there is a subspace X of* $\mathrm{lip}_0(M)$ *which is a natural predual of $\mathscr{F}(M)$. Then given $\mu \in B_{\mathscr{F}(M)}$ the following are equivalent:*

(i) $\mu \in \mathrm{ext}(B_{\mathscr{F}(M)})$.
(ii) $\mu \in \exp(B_{\mathscr{F}(M)})$.
(iii) *There are* $x, y \in M$, $x \neq y$, *such that* $[x, y] = \{x, y\}$ *and* $\mu = m_{x,y}$.

Proof. (i)⇒(iii) follows from Proposition 5.2.17. Moreover, (ii)⇒(i) is clear, so it only remains to show (iii)⇒(ii). To this end, let $x, y \in M$, $x \neq y$, be so that $[x, y] = \{x, y\}$. Consider

$$A = \{\mu \in B_{\mathscr{F}(M)} : \langle f_{x,y}, \mu \rangle = 1\}.$$

We will show that $A = \{m_{x,y}\}$ and so $m_{x,y}$ is exposed by $f_{x,y}$ in $B_{\mathscr{F}(M)}$. Let $\mu \in \mathrm{ext}(A)$. Since A is an extremal subset of $B_{\mathscr{F}(M)}$, μ is also an extreme point of $B_{\mathscr{F}(M)}$ and so $\mu \in V_M \cap A$. Recall that if $\langle f_{x,y}, m_{u,v} \rangle = 1$ then $u, v \in [x, y]$, therefore $V_M \cap A = \{m_{x,y}\}$. Thus $\mathrm{ext}(A) \subset \{m_{x,y}\}$. Finally note that A is a closed convex subset of $B_{\mathscr{F}(M)}$ and so $A = \overline{\mathrm{conv}}(\mathrm{ext}(A)) = \{m_{x,y}\}$ since the space $\mathscr{F}(M)$ has the RNP as being a separable dual. □

It is proved in Aliaga and Guirao's paper [AG] that if (M, d) is compact, then a molecule $m_{x,y}$ is extreme in $B_{\mathscr{F}(M)}$ if and only if it is preserved extreme if and only if $[x, y] = \{x, y\}$. Thus, if $\mathrm{lip}_0(M)$ separates points uniformly (and thus $\mathscr{F}(M) = \mathrm{lip}_0(M)^*$), Proposition 5.2.17 and Aliaga and Guirao's result provide a complete description of the extreme points: they are the molecules $m_{x,y}$ such that $[x, y] = \{x, y\}$. It is possible to obtain the same kind of complete descriptions in some different settings as it is proved in the following result.

Proposition 5.2.19. *Let* (M, d) *be a bounded metric space for which there is a compact Hausdorff topology* τ *such that* d *is* τ*-lower semicontinuous. Let* $0 < \alpha < 1$ *and let* (M, d^α) *be the* α*-snowflake of* M*. Then given* $\mu \in B_{\mathscr{F}(M,d^\alpha)}$ *the following are equivalent:*

(i) $\mu \in \mathrm{ext}(B_{\mathscr{F}(M,d^\alpha)})$.
(ii) $\mu \in \mathrm{strexp}(B_{\mathscr{F}(M,d^\alpha)})$.
(iii) *There are* $x, y \in M$, $x \neq y$, *such that* $\mu = m_{x,y}$.

Proof. Given $x \neq y$ in M, it is proved in [Wea2, Proposition 2.4.5] that there is a peaking function at (x, y). Thus $m_{x,y}$ is a strongly exposed point by Theorem 5.2.3 and (iii) implies (ii). The implication (ii)⇒(i) is obvious. To finish, notice that since $0 < \alpha < 1$ we have $[x, y] = \{x, y\}$ for every $x, y \in M$. Therefore the implication (i)⇒(iii) follows directly from Proposition 5.2.17 and the fact that $\mathrm{lip}_\tau(M, d^\alpha)$ separates points uniformly (this can be proved in the same way that Corollary 4.2.8 using that $\mathrm{Lip}_0(M, d) \subset \mathrm{lip}_0(M, d^\alpha)$). □

If we restrict our attention to uniformly discrete bounded metric spaces satisfying the hypotheses of the duality result, then all the families of distinguished points of $B_{\mathscr{F}(M)}$ that we have considered coincide. We will see in Example 5.2.26 that this fact does not hold for general uniformly discrete spaces.

Proposition 5.2.20. *Let* (M, d) *be a uniformly discrete bounded metric space such that* $\mathscr{F}(M)$ *admits a natural predual. Then for* $\mu \in B_{\mathscr{F}(M)}$ *it is equivalent:*

(i) $\mu \in \operatorname{ext}(B_{\mathscr{F}(M)})$.
(ii) $\mu \in \operatorname{strexp}(B_{\mathscr{F}(M)})$.
(iii) *There are* $x, y \in M$, $x \neq y$, *such that* $\mu = m_{x,y}$ *and* $[x, y] = \{x, y\}$.

Proof. (i) $\Rightarrow$ (iii) follows from Proposition 5.2.17. Moreover, (ii)$\Rightarrow$(i) trivially. Now, assume that $\mu = m_{x,y}$ with $[x, y] = \{x, y\}$. We will show that the pair (x, y) fails property (Z) and thus μ is a strongly exposed point. Assume, by contradiction, that there is a sequence $(z_n)_{n=1}^{\infty}$ in M such that

$$d(x, z_n) + d(y, z_n) \leq d(x, y) + \frac{1}{n} \min\{d(x, z_n), d(y, z_n)\}.$$

and so

$$(1 - 1/n)(d(x, z_n) + d(y, z_n)) \leq d(x, y).$$

The compactness with respect to the w^*-topology ensures the existence of a w^*-cluster point z of $(z_n)_{n=1}^{\infty}$ (M and $\delta(M) \subset \mathscr{F}(M)$ being naturally identified). Now, by the lower semicontinuity of the distance, we have

$$d(x, z) + d(y, z) \leq \liminf_{n\to\infty} (1 - 1/n)(d(x, z_n) + d(y, z_n)) \leq d(x, y).$$

Therefore, $z \in [x, y] = \{x, y\}$. Suppose $z = x$. Denote $\theta = \inf\{d(u, v) : u \neq v\} > 0$. The lower semicontinuity of d yields

$$\begin{aligned} \theta + d(x, y) &\leq \liminf_{n\to\infty} (1 - 1/n)(\theta + d(y, z_n)) \\ &\leq \liminf_{n\to\infty} (1 - 1/n)(d(x, z_n) + d(y, z_n)) \leq d(x, y), \end{aligned}$$

which is impossible. The case $z = y$ yields a similar contradiction. Thus the pair (x, y) does not have property (Z). $\square$

Next we show that the extremal structure of a free space has a significant impact on its isometric preduals. If a metric space M is countable and satisfies the assumptions of Proposition 5.2.17, then $\operatorname{ext}(B_{\mathscr{F}(M)})$ is also countable. Therefore, any isometric predual of $\mathscr{F}(M)$ is isomorphic to a polyhedral space by a theorem of Fonf [Fon], and so it is saturated with subspaces isomorphic to c_0. This applies for instance in the following cases.

Corollary 5.2.21. *Let M be a countable compact metric space. Then any isometric predual of $\mathscr{F}(M)$ (in particular* $\operatorname{lip}_0(M)$*) is isomorphic to a polyhedral space.*

Corollary 5.2.22. *Let M be a uniformly discrete bounded separable metric space such that $\mathscr{F}(M)$ admits a natural predual. Then any isometric predual of $\mathscr{F}(M)$ is isomorphic to a polyhedral space.*

It turns out that, for bounded uniformly discrete metric spaces, one of the questions stated above has a positive answer.

Proposition 5.2.23. *Let (M, d) be a bounded uniformly discrete metric space. Then a molecule $m_{x,y}$ is an extreme point of $B_{\mathscr{F}(M)}$ if and only if $[x, y] = \{x, y\}$.*

We will need the following observation, perhaps of independent interest: since a point $x \in B_X$ is extreme if and only if $x \in \operatorname{ext}(B_Y)$ for every 2-dimensional subspace Y of X, the extreme points of $B_{\mathscr{F}(M)}$ are separably determined. Let us be more precise.

Lemma 5.2.24. *Let M be a metric space. Assume that $\mu_0 \in B_{\mathscr{F}(M)}$ is not an extreme point of $B_{\mathscr{F}(M)}$. Then there is a separable subset $N \subset M$ such that $\mu_0 \in \mathscr{F}(N)$ and $\mu_0 \notin \operatorname{ext}(B_{\mathscr{F}(N)})$.*

Proof. Write $\mu_0 = \frac{1}{2}(\mu_1 + \mu_2)$, with $\mu_1, \mu_2 \in B_{\mathscr{F}(M)}$. We can find sequences $(\nu_n^i)_{n=1}^\infty$ of finitely supported measures such that $\mu_i = \lim_{n\to\infty} \nu_n^i$ for $i = 0, 1, 2$. Let $N = \{0\} \cup \operatorname{supp}\{\nu_n^i\}$. Note that the canonical inclusion $\mathscr{F}(N) \hookrightarrow \mathscr{F}(M)$ is an isometry and $\nu_n^i \in \mathscr{F}(N)$ for each n, i. Since $\mathscr{F}(N)$ is complete, it is a closed subspace of $\mathscr{F}(M)$. Thus $\mu_0, \mu_1, \mu_2 \in \mathscr{F}(N)$ and so $\mu_0 \notin \operatorname{ext}(B_{\mathscr{F}(N)})$. $\square$

Proof of Proposition 5.2.23. Let $m_{x,y}$ be a molecule in M such that $[x, y] = \{x, y\}$ and assume that $m_{x,y} \notin \operatorname{ext}(B_{\mathscr{F}(M)})$. By Lemma 5.2.24, we may assume that M is countable. Write $M = \{x_n : n \geq 0\}$. Let $(e_n)_{n=1}^\infty$ be the unit vector basis of ℓ_1. It is well known that the map $\delta(x_n) \mapsto e_n$ for $n \geq 1$ defines an isomorphism from $\mathscr{F}(M)$ onto ℓ_1. Thus $(\delta(x_n))_{n=1}^\infty$ is a Schauder basis for $\mathscr{F}(M)$.

Assume that $m_{x,y} = \frac{1}{2}(\mu + \nu)$ for $\mu, \nu \in B_{\mathscr{F}(M)}$ and write $\mu = \sum_{n=1}^\infty a_n \delta(x_n)$. Fix $n \in \mathbb{N}$ such that $x_n \notin \{x, y\}$. Then, there is $\varepsilon_n > 0$ such that

$$d(x, y) \leq (1 - \varepsilon_n) \left(d(x, x_n) + d(x_n, y)\right).$$

Let $g_n = f_{x,y} + \varepsilon_n \mathbf{1}_{\{x_n\}}$, which is an element of $\operatorname{Lip}_0(M)$ since M is uniformly discrete. We will show that $\|g_n\|_L \leq 1$. To this end, take $u, v \in M$, $u \neq v$. Since $\|f_{x,y}\|_L \leq 1$, it is clear that $|\langle g_n, m_{u,v}\rangle| \leq 1$ if $u, v \neq x_n$. Thus we may assume $v = x_n$. Therefore (c) in Lemma 5.1.10 yields that $\langle f_{x,y}, m_{u,v}\rangle \leq 1 - \varepsilon_n$ and so $\langle g_n, m_{u,v}\rangle \leq 1$. Exchanging the roles of u and v, we get that $\|g_n\|_L \leq 1$. Moreover, note that

$$1 = \langle g_n, m_{x,y}\rangle = \frac{1}{2}(\langle g_n, \mu\rangle + \langle g_n, \nu\rangle) \leq 1$$

and so $\langle g_n, \mu\rangle = 1$. Analogously we show that $\langle f_{x,y}, \mu\rangle = 1$. Thus $a_n = \langle \mathbf{1}_{\{x_n\}}, \mu\rangle = 0$. Therefore $\mu = a\delta(x) + b\delta(y)$ for some $a, b \in \mathbb{R}$. Finally, let $f_1(t) := d(t, x) - d(0, x)$ and $f_2(t) := d(t, y) - d(0, y)$. Then $\|f_i\|_L = 1$ and $\langle f_i, m_{x,y}\rangle = 1$, so we also have $\langle f_i, \mu\rangle = 1$ for $i = 1, 2$. It follows from this that $a = -b = \frac{1}{d(x,y)}$, that is, $\mu = m_{x,y}$. This implies that $m_{x,y}$ is an extreme point of $B_{\mathscr{F}(M)}$. $\square$

Next we show that preserved extreme points are automatically strongly exposed for uniformly discrete metric spaces. Notice that, contrary to other results in this section, no boundedness assumption is needed.

Proposition 5.2.25. *Let M be a uniformly discrete metric space. Then every preserved extreme point of $B_{\mathscr{F}(M)}$ is also a strongly exposed point.*

Proof. Let $x, y \in M$ such that $m_{x,y}$ is a preserved extreme point of $B_{\mathscr{F}(M)}$. Assume that $m_{x,y}$ is not strongly exposed. By Theorem 5.2.3, the pair (x, y) enjoys property (Z). That is, for each $n \in \mathbb{N}$ we can find $z_n \in M \setminus \{x, y\}$ such that

$$d(x, z_n) + d(y, z_n) \leq d(x, y) + \frac{1}{n} \min\{d(x, z_n), d(y, z_n)\}.$$

Thus,

$$(1 - 1/n)(d(x, z_n) + d(y, z_n)) \leq d(x, y)$$

so it follows from condition (ii) in Theorem 5.2.13 that $\min\{d(x, z_n), d(y, z_n)\} \to 0$. Since M is uniformly discrete, this means that z_n is eventually equal to either x or y, a contradiction. $\square$

Let us include an example showing that in the non-compact case there are molecules which are extreme points of $B_{\mathscr{F}(M)}$ but not preserved extreme points. This example also provides a dual free space which does not admit a natural predual.

Example 5.2.26. Consider the sequence in c_0 given by $x_0 = 0, x_1 = 2e_1$, and $x_n = e_1 + (1 + 1/n)e_n$ for $n \geq 2$, where $\{e_n\}$ is the canonical basis. Let $M = \{0\} \cup \{x_n : n \in \mathbb{N}\}$.

(a) The molecule m_{0,x_1} is an extreme point of $B_{\mathscr{F}(M)}$ which is not preserved.
(b) $\mathscr{F}(M)$ does not admit any natural predual.
(c) The space $X = \{f \in \mathrm{Lip}_0(M) : \lim_n f(x_n) = f(x_1)/2\}$ is an isometric predual of $\mathscr{F}(M)$.

In [AG] it is proved that the molecule m_{0,x_1} is not a preserved extreme point of $B_{\mathscr{F}(M)}$ (this follows easily from Theorem 5.2.13). Proposition 5.2.23 implies that that $m_{0,x_1} \in \mathrm{ext}(B_{\mathscr{F}(M)})$. This fact and Proposition 5.2.20 yield that $\mathscr{F}(M)$ does not admit any natural predual.

Finally, for the proof of (c) we will employ the theorem of Petunīn and Plīčhko 4.2.4. The space X is clearly a separable closed subspace of $\mathscr{F}(M)^*$. Further, a simple case checking shows that for any $x \neq y \in M$, $y \neq 0$, the function $f(x) = 0, f(y) = d(x, y)$ can be extended as an element of X without increasing the Lipschitz norm. Since X is a lattice, Lemma 4.2.1 shows that X is separating. Finally, if $f \in X$ and

$$\lim_{k\to\infty} \frac{f(x_{n_k}) - f(x_{m_k})}{d(x_{n_k}, x_{m_k})} = \|f\|_L$$

then without loss of generality the sequence $(m_k)_{k=1}^{\infty}$ does not tend to infinity. Passing to a subsequence, we may assume that it is constant, say $m_k = m$ for all $k \in \mathbb{N}$. If $(n_k)_{k=1}^{\infty}$ does not tend to infinity, then $\frac{f(x_i)-f(x_m)}{d(x_i,x_m)} = \|f\|_L$ for some $i \neq m$. Otherwise, since $f \in X$, we have

$$\lim_{k\to\infty} \frac{f(x_{n_k}) - f(x_m)}{d(x_{n_k}, x_m)} = \frac{\frac{f(x_1)}{2} - f(m)}{d(x_1, x_m)}.$$

So in this case the norm is attained at $\frac{1}{d(x_1,x_m)}(\delta(x_1)/2-\delta(x_m)) \in B_{\mathscr{F}(M)}$. It follows that every $f \in X$ attains its norm. Thus by the theorem of Petunīn and Plīčhko, $X^* = \mathscr{F}(M)$.

The next example shows that there are uniformly discrete bounded metric spaces such that their free space does not admit any isometric predual at all.

Example 5.2.27. Let $M = \{0\} \cup \{1,2,3,\dots\} \cup \{a,b\}$ with the following distances:

$$\begin{aligned} d(0,n) &= d(a,n) = d(b,n) = 1+1/n, \\ d(a,b) &= d(0,a) = d(0,b) = 2, \text{ and} \\ d(n,m) &= 1 \end{aligned}$$

for $n,m \in \{1,2,3,\dots\}$. Then $\mathscr{F}(M)$ is not isometrically a dual space.

Indeed, let us assume that $\mathscr{F}(M) = X^*$. Then some subsequence of $(\delta(n))_{n=1}^{\infty}$ is weak*-convergent to some $\mu \in \mathscr{F}(M)$. By the weak*-lower semicontinuity of the norm, we have $\|\mu - \delta(a)\| \le 1$ and $\|\mu\| \le 1$. But Proposition 5.2.23 implies that $\delta(a)/2$ is an extreme point of $B_{\mathscr{F}(M)}$. This means that

$$\mu \in B_{\mathscr{F}(M)}(0,1) \cap B_{\mathscr{F}(M)}(\delta(a),1) = \{\delta(a)/2\}\,.$$

A similar argument shows that $\delta(b)/2 = \mu$. Hence $\delta(a) = \delta(b)$, which is a contradiction.

In the last part of the section, we focus on the case in which M is a compact metric space and $\mathscr{F}(M)$ is the dual of $\mathrm{lip}_0(M)$. Recall that in this case all extreme points of $B_{\mathscr{F}(M)}$ are molecules by Corollary 3.3.6 in [Wea2]. We will show that indeed $\mathscr{F}(M)$ is weak*-AUC, which implies in particular that the norm and the weak* topologies agree in $S_{\mathscr{F}(M)}$ and so every extreme point of the closed ball is also a denting point.

Recall that if X is a separable Banach space then the *modulus of weak*-asymptotic uniform convexity* of X^* can be computed as follows [BM] (see also Proposition 3.1.2):

$$\overline{\delta}^*_{X^*}(t) = \inf_{x^* \in B_{X^*}} \inf_{\substack{x_n^* \xrightarrow{w^*} 0 \\ \|x_n^*\| \ge t}} \liminf_{n\to\infty} \|x^* + x_n^*\| - 1.$$

Proposition 5.2.28. *Let M be a compact metric space. Assume that $\mathrm{lip}_0(M)$ separates points uniformly. Then $\mathscr{F}(M)$ is weak*-AUC.*

For the proof we need the following easy lemma.

Lemma 5.2.29. *Let $(x_n^*)_{n=1}^{\infty} \subset X^*$ be a weak*-null sequence such that $\|x_n^*\| \ge 1$ and $F \subset X^*$ be a finite-dimensional subspace. Then $\liminf_{n\to\infty} d(x_n^*, F) \ge \frac{1}{2}$.*

Proof. Assume $\liminf_{n\to\infty} d(x_n^*, F) < \frac{1}{2} - \varepsilon$ for some $\varepsilon > 0$. By extracting a subsequence we may assume there is a sequence $(y_n^*)_{n=1}^{\infty} \subset F$ such that $\|y_n^* - x_n^*\| < \frac{1}{2} - \varepsilon$ for every

n. The sequence $(y_n^*)_{n=1}^\infty$ is bounded since $(x_n^*)_{n=1}^\infty$ is too. Thus, we may assume that $\lim_{n\to\infty} y_n^* = y^*$ for some $y^* \in F$. Note that

$$\|y^*\| \geq \lim_{n\to\infty} (\|x_n^*\| - \|x_n^* - y_n^*\|) \geq \frac{1}{2} + \varepsilon.$$

Take $x \in X$ such that $y^*(x) \geq \|y^*\| - \varepsilon/2$. We have

$$\frac{1}{2} - \varepsilon \geq \lim_{n\to\infty} \|x_n^* + y^*\| \geq \lim_{n\to\infty} (x_n^* + y^*)(x) \geq \|y^*\| - \frac{\varepsilon}{2} = \frac{1}{2} + \frac{\varepsilon}{2},$$

which is a contradiction. □

Proof of Proposition 5.2.28. We will use the same arguments as in the proof of Proposition 8 in [Pet]. Fix $t > 0$ and take $\gamma \in S_{\mathscr{F}(M)}$ and a weak*-null sequence $(\gamma_n)_{n=1}^\infty$ such that $\|\gamma_n\| \geq t$ for every $n \in \mathbb{N}$. We claim that

$$\liminf_{n\to\infty} \|\gamma + \gamma_n\| \geq 1 + \frac{t}{2}.$$

We may assume that γ is finitely supported. Pick $f \in \mathrm{lip}_0(M)$ with $\|f\|_L = 1$ and $\langle f, \gamma\rangle > 1 - \varepsilon$. Take $\theta > 0$ such that $\sup_{d(x,y)\leq\theta} |f(x) - f(y)| \leq \varepsilon d(x, y)$. Take $\delta < \frac{\varepsilon\theta}{2(1+\varepsilon)}$. By compactness, there exists a finite subset $E \subset M$ containing the support of γ and such that $\sup_{y\in M} d(y, E) < \delta$. We have $\liminf_{n\to\infty} d(\gamma_n/t, \mathscr{F}(E)) \geq \frac{1}{2}$ by Lemma 5.2.29. Now, by Hahn-Banach theorem, there exists a sequence $(f_n)_{n=1}^\infty \subset (1+\varepsilon)B_{\mathrm{Lip}_0(M)}$ such that $f_n\restriction_E = 0$ and $\liminf_{n\to\infty}\langle f_n, \gamma_n\rangle \geq \frac{t}{2}$. Consider $g_n = f + f_n$. By distinguishing the cases $d(x, y) < \theta$ and $d(x, y) > \theta$, one can show that $\|g_n\|_L \leq 1 + \varepsilon$. Now we have

$$\begin{aligned}\liminf_{n\to\infty} \|\gamma + \gamma_n\| &\geq \liminf_{n\to\infty} \frac{1}{1+\varepsilon}\langle g_n, \gamma + \gamma_n\rangle \\ &= \frac{1}{1+\varepsilon} \liminf_{n\to\infty}(\langle f, \gamma\rangle + \langle f, \gamma_n\rangle + \langle f_n, \gamma\rangle + \langle f_n, \gamma_n\rangle) \\ &\geq \frac{1}{1+\varepsilon}(1 - \varepsilon + \frac{t}{2} - \varepsilon)\end{aligned}$$

since $\gamma_n \xrightarrow{w^*} 0$ and $f \in \mathrm{lip}_0(M)$. Letting $\varepsilon \to 0$ proves the claim. It follows that $\overline{\delta}^*_{\mathscr{F}(M)}(t) \geq \frac{1}{2}t$ and so $\mathscr{F}(M)$ is weak*-AUC. □

It is well known and easy to show that if X^* is weak*-AUC then every point of the unit sphere has weak*-neighbourhoods of arbitrarily small diameter. This fact and the Choquet's lemma yield that if $x^* \in \mathrm{ext}(B_{X^*})$ then there are weak*-slices of B_{X^*} containing x^* of arbitrarily small diameter. That is, every extreme point of B_{X^*} is also a weak*-denting point.

Corollary 5.2.30. *Let M be a compact metric space. Assume that $\mathrm{lip}_0(M)$ separates points uniformly. Then every extreme point of $B_{\mathscr{F}(M)}$ is also a denting point.*

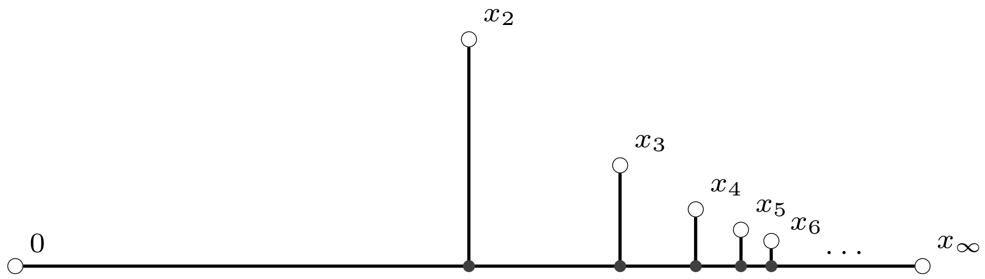

Figure 5.4: The metric space of Example 5.2.31

At this point one could be inclined to believe that the denting points and the strongly exposed points of $B_{\mathscr{F}(M)}$ coincide, at least when M is compact. The following example shows that this is not the case.

Example 5.2.31. Let (T, d) be the following set with its real-tree distance

$$[0,1] \times \{0\} \cup \bigcup_{n=2}^{\infty} \left\{1 - \frac{1}{n}\right\} \times \left[0, \frac{1}{n^2}\right].$$

We will consider (Ω, d) as the set

$$\{(0,0),(1,0)\} \cup \left\{\left(1 - \frac{1}{n}, \frac{1}{n^2}\right) : n \geq 2\right\}$$

together with the distance inherited from (T, d). Let us call for simplicity $0 := x_1 := (0,0)$, $x_\infty := (1,0)$ and $x_n := (1 - \frac{1}{n}, \frac{1}{n^2})$ if $n \geq 2$. Note that

$$(0, x_\infty)_{x_n} = \frac{1}{n^2} < \frac{1}{n}\left(\frac{1}{n^2} + \frac{1}{n}\right) = \frac{1}{n}\min\{d(0, x_n), d(x_\infty, x_n)\}$$

and so the couple $(x_\infty, 0)$ has property (Z). Therefore the characterisation of the points in $\text{strexp}(B_{\mathscr{F}(M)})$ given in Theorem 5.2.3 yields that $\delta(x_\infty)$ is not a strongly exposed point of $B_{\mathscr{F}(\Omega)}$. Aliaga and Guirao [AG] have proved that for a compact M, the condition $[x, y] = \{x, y\}$ implies that $\frac{\delta(x)-\delta(y)}{d(x,y)}$ is a preserved extreme point of $B_{\mathscr{F}(M)}$. In particular $\delta(x_\infty)$ is a preserved extreme point of $B_{\mathscr{F}(\Omega)}$.

5.2.4 Application to norm-attainment

Consider a metric space M and a Banach space Y. Recall that for every Lipschitz map $f \in \text{Lip}_0(M, Y)$ there is an unique linear operator $T_f : \mathscr{F}(M) \to Y$, given by $T_f(\delta(m)) =$

$f(m)$ for $m \in M$, such that $\|f\|_L = \|T_f\|$. Moreover, every operator $T \in \mathscr{L}(\mathscr{F}(M), Y)$ corresponds to a Lipschitz map, just consider $f(m) = T(\delta(m))$. It follows that the map $f \mapsto T_f$ defines a linear isometry from $\mathrm{Lip}_0(M, Y)$ onto $\mathscr{L}(\mathscr{F}(M), Y)$.

Therefore, one can consider two different notions of norm-attainment for a function $f \in \mathrm{Lip}_0(M, Y)$. First, we say that f *strongly attains its norm* if there are two different points $x, y \in M$ such that $\|f(x) - f(y)\| = \|f\|_L\, d(x, y)$. We denote by $\mathrm{SA}(M, Y)$ the set of all functions in $\mathrm{Lip}_0(M, Y)$ which strongly attain their norm. On the other hand, given Banach spaces X and Y, we denote $\mathrm{NA}(X, Y)$ the set of operators from X to Y which attain their norm. We say that f *attains its norm as an operator* if $T_f \in \mathrm{NA}(\mathscr{F}(M), Y)$.

It is clear that every Lipschitz function which strongly attains its norm also attains its norm as an operator. Kadets, Martín and Soloviova proved in [KMS] that if M is geodesic then $\mathrm{SA}(M, \mathbb{R})$ is not dense in $\mathrm{Lip}_0(M)$, and so $\mathrm{SA}(M, \mathbb{R}) \subsetneq \mathrm{NA}(\mathscr{F}(M), \mathbb{R})$ by Bishop–Phelps theorem. On the other hand, the following result was proved by Godefroy in [God3].

Proposition 5.2.32 (Godefroy). *Let M be a compact metric space such that $\mathrm{lip}_0(M)$ separates points uniformly. Let Y be a Banach space. Then $f \in \mathrm{SA}(M, Y)$ if and only if $T_f \in \mathrm{NA}(\mathscr{F}(M), Y)$. Moreover, if Y is finite-dimensional, then $\mathrm{NA}(\mathscr{F}(M), Y)$ is norm-dense in $\mathscr{L}(\mathscr{F}(M), Y)$. Equivalently, $\mathrm{SA}(M, Y)$ is norm-dense in $\mathrm{Lip}_0(M, Y)$.*

Our goal is to extend Godefroy's result to a more general context. First, recall that Bourgain proved in [Bou1] that if a Banach space X has the RNP then $\mathrm{NA}(X, Y)$ is dense in $\mathscr{L}(X, Y)$ for every Banach space Y. Thus, the last statement in Godefroy's result holds without assuming that the target space is finite-dimensional. Indeed, Bourgain's theorem says even more. An operator $T\colon X \to Y$ is said to be *strongly exposing* if there is $x \in S_X$ such that for every sequence $(x_n)_{n=1}^\infty \subset B_X$ such that $\lim_n \|Tx_n\|_Y = \|T\|$, there is a subsequence $(x_{n_k})_{k=1}^\infty$ which converges to either x or $-x$. Clearly every strongly exposing operator is norm-attaining. Bourgain proved that if X has the RNP then for every Banach space Y the set of strongly exposing operators from X to Y is dense in $\mathscr{L}(X, Y)$ (see [Bou1, Theorem 5]).

Proposition 5.2.33. *Let M be a complete metric space and Y be a Banach space. Assume that $\mathscr{F}(M)$ has the RNP. Then $\mathrm{SA}(M, Y)$ is dense in $\mathrm{Lip}_0(M, Y)$.*

For the proof we need the following lemma. Let us point out that a more general result holds, see Proposition 2.13 in [GLPPRZ].

Lemma 5.2.34. *Let M be a complete metric space. Then V_M is norm-closed in $\mathscr{F}(M)$.*

Proof. Assume that $(m_{x_n,y_n})_{n=1}^\infty$ is a net of molecules in M which converges to $\mu \in \mathscr{F}(M)$. We will show that $(x_n)_{n=1}^\infty$ and $(y_n)_{n=1}^\infty$ are Cauchy sequences in M. To this end, take n_0 such that $\left\|m_{x_n,y_n} - m_{x_{n_0},y_{n_0}}\right\| < 1$ for $n \geq n_0$. By Lemma 4.1.3 we have

$$\max\{d(y_n, y_{n_0}), d(x_n, x_{n_0})\} < d(x_{n_0}, y_{n_0}) \text{ for } n \geq n_0,$$

and so $d(y_n, x_n) < 3d(x_{n_0}, y_{n_0})$ for $n \geq n_0$. Fix $\varepsilon > 0$ and take $n_\varepsilon \geq n_0$ such that $\|m_{x_n,y_n} - m_{x_{n_\varepsilon},y_{n_\varepsilon}}\| < \varepsilon/d(x_{n_0}, y_{n_0})$. Then, using again Lemma 4.1.3 we have

$$d(x_n, x_{n_\varepsilon}) < \varepsilon \frac{d(x_{n_\varepsilon}, y_{n_\varepsilon})}{d(x_{n_0}, y_{n_0})} < 3\varepsilon$$

whenever $n \geq n_\varepsilon$. This shows that $(x_n)_{n=1}^\infty$ is Cauchy, and the same holds for $(y_n)_{n=1}^\infty$, so they have limits in M, say x and y, respectively. Moreover, if $\varepsilon < d(x_{n_0}, y_{n_0})/2$ then $d(x_n, y_n) \geq \frac{1}{2} d(x_{n_0}, y_{n_0})$ for $n \geq n_\varepsilon$ and so $x \neq y$. One more application of Lemma 4.1.3 gives that $m_{x_n,y_n} \to m_{x,y}$ and so $\mu = m_{x,y}$. $\square$

Proof of Proposition 5.2.33. By Bourgain's theorem, it suffices to show that every strongly exposing operator $T \in \mathscr{L}(\mathscr{F}(M), Y)$ attains its norm at a molecule, and so $T \circ \delta \in \mathrm{SA}(M, Y)$. Let $T\colon \mathscr{F}(M) \to Y$ and $\mu \in \mathscr{F}(M)$ witnessing the definition of strongly exposing operator. Take a sequence $(y_n^*)_{n=1}^\infty \subset S_{Y^*}$ such that $\|T^* y_n^*\|_X > \|T\| - 1/n$ for every $n \in \mathbb{N}$. Since V_M is 1-norming, there is a sequence $(m_{x_n,y_n})_{n=1}^\infty \subset V_M$ such that $\langle T^* y_n^*, m_{x_n,y_n}\rangle > \|T\| - 1/n$ for every n. Note that

$$\langle T^* y_n^*, m_{x_n,y_n}\rangle = \langle y_n^*, T m_{x_n,y_n}\rangle \leq \|T m_{x_n,y_n}\|_Y$$

and so $\lim_n \|T m_{x_n,y_n}\|_Y = \|T\|$. Thus there is a subsequence $(m_{x_{n_k},y_{n_k}})_{k=1}^\infty$ which converges to either μ or $-\mu$. Since V_M is norm-closed we get that $\mu \in V_M$ as desired. $\square$

We can also replace the compactness hypothesis in Proposition 5.2.32 by a more general one.

Proposition 5.2.35. *Let Y be a Banach space and M be a separable bounded metric space such that $\mathscr{F}(M)$ admits a natural predual $X \subset \mathrm{lip}_0(M)$. Then $f \in \mathrm{SA}(M, Y)$ if and only if $T_f \in \mathrm{NA}(\mathscr{F}(M), Y)$. Moreover, $\mathrm{NA}(\mathscr{F}(M), Y)$ is norm-dense in $\mathscr{L}(\mathscr{F}(M), Y)$. Equivalently, $\mathrm{SA}(M, Y)$ is norm-dense in $\mathrm{Lip}_0(M, Y)$.*

Proof. Given $f \in \mathrm{Lip}_0(M, Y)$, assume that $T_f \in \mathrm{NA}(\mathscr{F}(M), Y)$. That is, there is $\gamma \in \mathscr{F}(M)$ with $\|\gamma\| \leq 1$ such that $\|T_f(\gamma)\|_Y = \|f\|_L$. By the Hahn-Banach theorem there is $y^* \in S_{Y^*}$ such that $\langle y^*, T_f(\gamma)\rangle = \|T_f(\gamma)\|_Y$. Now, we claim that $T_{y^* \circ f} = y^* \circ T_f$. Indeed, note that

$$T_{y^* \circ f}(\delta(m)) = (y^* \circ f)(m) = (y^* \circ T_f)(\delta(m))$$

for every $m \in M$. Thus $T_{y^* \circ f}$ coincides with $y^* \circ T_f$ on $\delta(M)$ and so they also coincide on $\mathscr{F}(M) = \overline{\mathrm{span}}\{\delta(M)\}$. This means that

$$T_{y^* \circ f}(\gamma) = \|T_f(\gamma)\|_Y = \|f\|_L \geq \|y^* \circ f\|_L.$$

Thus $T_{y^* \circ f}(\gamma) = \|f\|_L$ and the real-valued Lipschitz function $y^* \circ f$ attains its operator norm on γ. Since $\mathscr{F}(M)$ has the RNP as being a separable dual, the closed convex bounded set

$$C = \{\mu \in B_{\mathscr{F}(M)} : \langle T_{y^* \circ f}, \mu\rangle = \|f\|_L\}$$

has an extreme point, say μ. It is easily checked that μ is also an extreme point of $B_{\mathscr{F}(M)}$. Now, Proposition 5.2.17 yields that $\mathrm{ext}(B_{\mathscr{F}(M)}) \subset V_M$ and so there are $x, y \in M$, $x \neq y$, such that $\mu = m_{x,y}$. Thus,

$$\|f\|_L = \langle T_{y^* \circ f}, m_{x,y} \rangle = \frac{(y^* \circ f)(x) - (y^* \circ f)(y)}{d(x,y)} \leq \frac{\|f(x) - f(y)\|_Y}{d(x,y)}.$$

Therefore, f strongly attains its norm. This proves that $f \in \mathrm{SA}(M, Y)$ whenever $T_f = \mathrm{NA}(\mathscr{F}(M), Y)$. The converse is clear.

Finally, since $\mathscr{F}(M)$ has the RNP, Bourgain's theorem [Bou1] yields the denseness of $\mathrm{NA}(\mathscr{F}(M), Y)$ in $\mathscr{L}(\mathscr{F}(M), Y)$.

□

Let us note that any metric space satisfying the hypothesis of Theorem 4.2.7 satisfies the ones in Proposition 5.2.35. For instance, if M is compact and countable (see [Dal1]) or if M is a uniformly discrete metric space and there is a compact topology τ on M such that d is τ-lower semicontinuous (see Corollary 4.2.8).

5.3 Lipschitz free spaces over a compact convex set

In the last part of this chapter, we turn to a different topic. The aim of this section is to show that Pełczyński's universal space $\mathbb{P}$ is isomorphic to the Lipschitz free space over a compact convex set and that it is not isomorphic to the Lipschitz free space over c_0. Recall that $\mathbb{P}$ is a separable Banach space with a basis such that every Banach space with the BAP is isomorphic to a complemented subspace of $\mathbb{P}$. We refer the reader to [AK2] for the construction of this space.

Let us recall a property relating $\mathbb{P}$ and Lipschitz free spaces. A celebrated result of Godefroy and Kalton [GK] ensures that $\mathscr{F}(X)$ has the BAP provided X is a Banach space with the BAP. This result and the universal property of $\mathbb{P}$ yield that $\mathscr{F}(\mathbb{P})$ is isomorphic to a complemented subspace of $\mathbb{P}$. Moreover, the latter is isomorphic to a complemented subspace of $\mathscr{F}(\mathbb{P})$. Thus, the standard Pełczyński's decomposition method yields that $\mathbb{P}$ and $\mathscr{F}(\mathbb{P})$ are isomorphic.

We need a criterion for ensuring the BAP of the Lipschitz free space over a compact convex set. For that, we use a result by Pernecká and Smith [PS] which says that $\mathscr{F}(K)$ has the metric approximation property whenever K is a compact convex subset of $\mathbb{R}^n$. This result and a slight modification of [GK, Theorem 5.3] provide the following result.

Proposition 5.3.1. *Let X be a Banach space and $K \subset X$ be a compact convex subset containing 0. Assume that there exist $\lambda \geq 1$ and a sequence $(T_n)_n$ of finite-rank operators on X such that $\|T_n\| \leq \lambda$ and $T_n(K) \subset K$ for each n, and $(T_n)_n$ converges pointwise to the identity on K. Then $\mathscr{F}(K)$ has the λ-BAP.*

Proof. It is easy to check that a Banach space has the λ-BAP if and only if the identity can be uniformly approximated on finite sets by finite-rank operators whose norm is bounded by λ. By a density argument, in order to prove that $\mathscr{F}(K)$ has the λ-BAP it suffices to show that, given $\varepsilon > 0$ and $x_1, \dots x_k \in K$, there exists a finite-rank operator $T\colon \mathscr{F}(K) \to \mathscr{F}(K)$ such that $\|T\delta(x_i) - \delta(x_i)\| < \varepsilon$ for $1 \leq i \leq k$ and $\|T\| \leq \lambda$. Fix n such that $\|T_n(x_i) - x_i\| < \varepsilon/2$ for $1 \leq i \leq k$. Let $\hat{T}_n\colon \mathscr{F}(K) \to \mathscr{F}(T_n(K))$ be the induced linear map. Since $T_n(K)$ is a finite-dimensional compact convex set, $\mathscr{F}(T_n(K))$ has the MAP. Thus we can find a finite-rank operator $S\colon \mathscr{F}(T_n(K)) \to \mathscr{F}(T_n(K))$ so that $\|S\| = 1$ and $\left\|S\hat{T}_n\delta(x_i) - \hat{T}_n(\delta(x_i)\right\| < \varepsilon/2$. Then $T = S\hat{T}_n$ does the work. □

Theorem 5.3.2. *There exists a compact convex subset K of the Pełczyński space $\mathbb{P}$ such that $\mathbb{P}$ is isomorphic to $\mathscr{F}(K)$.*

Proof. Let $\{e_n : n \in \mathbb{N}\}$ be a normalized Schauder basis of $\mathbb{P}$ with associated projections $(P_n)_{n=1}^\infty$. Consider $K = \overline{\text{conv}}(\{e_n/n : n \in \mathbb{N}\})$, which clearly is a compact convex set satisfying $\mathbb{P} = \overline{\text{span}}(K)$. By [DL, Lemma 2.1] or [GO, Theorem 4] we get that $\mathbb{P}$ is isomorphic to a complemented subspace of $\mathscr{F}(K)$. Notice that $P_n(K) \subset K$ for each n. Thus, the space $\mathscr{F}(K)$ has the BAP by Proposition 5.3.1. Now, the universal property of $\mathbb{P}$ yields that $\mathscr{F}(K)$ is isomorphic to a complemented subspace of $\mathbb{P}$. Since $\mathbb{P}$ is isomorphic to its ℓ_1-sum, the conclusion follows by applying the standard Pełczyński's decomposition method. □

Kaufmann proved in [Kau] that the Lipschitz free space over a Banach space X and the Lipschitz free space over its unit ball are isomorphic. In particular, $\mathscr{F}(X)$ is isomorphic to a Lipschitz free space over a *bounded* convex set. We do not know if the free space over a separable Banach space is always isomorphic to the free space over a compact convex set.

Notice that the above question does not make sense for a nonseparable Banach space since the free space over a compact metric space is separable. We have shown that the answer to that question is affirmative for the space $\mathbb{P}$, although we do not know what happens for other spaces. The following fact could be useful for giving a general answer.

Proposition 5.3.3. *For every separable Banach space there exists a compact convex subset $K \subset \mathscr{F}(X)$ such that $\mathscr{F}(X)$ is isomorphic to a complemented subspace of $\mathscr{F}(K)$.*

Proof. Let $(\gamma_n)_{n=1}^\infty$ be a dense sequence in $S_{\mathscr{F}(X)}$ and take $K = \overline{\text{conv}}(\{\gamma_n/n : n \in \mathbb{N}\})$. Notice that $\mathscr{F}(X) = \overline{\text{span}}(K)$ and apply [DL, Lemma 2.1] or [GO, Theorem 4]. □

Now we focus on the relationship between $\mathbb{P}$ and other universal separable Banach spaces. By Aharoni theorem (see, e.g. [BL]) every separable metric space is Lipschitz embeddable into c_0. Therefore, if M contains a bi-Lipschitz copy of c_0 then $\mathscr{F}(M)$ is a universal separable Banach space. This is the case of $\mathscr{F}(c_0)$, the free space over the Gurariĭ space $\mathscr{F}(\mathbb{G})$, the Holmes space $\mathbb{H} = \mathscr{F}(\mathbb{U})$, where $\mathbb{U}$ denotes the Urysohn universal metric space, and the Pełczyński space $\mathbb{P} \cong \mathscr{F}(\mathbb{P})$. By [FW], the Holmes space is not isomorphic to $\mathbb{P}$. In [CDW, Question 2] it is asked whether $\mathbb{P}$ is isomorphic to $\mathscr{F}(c_0)$. We show next that the answer to this question is negative. Recall that a metric space is said to be an *absolute Lipschitz retract* if it is a Lipschitz retract of every metric space containing it. It is well known that c_0 is an absolute Lipschitz retract (see, e.g. Example 1.5 in [BL]). We also need the following fact: given a metric space M and $0 \in N \subset M$, if and $F\colon M \to N$ is a Lipschitz retraction then the associated operator $\hat{F}$ is a bounded projection from $\mathscr{F}(M)$ onto $\mathscr{F}(N)$.

Proposition 5.3.4. *Let M be a separable absolute Lipschitz retract. Then $\mathbb{P}$ is not isomorphic to a complemented subspace of $\mathscr{F}(M)$. In particular, $\mathbb{P}$ is not isomorphic to $\mathscr{F}(c_0)$.*

Proof. Since $\mathbb{U}$ contains an isometric copy of any separable metric space and M is an absolute Lipschitz retract, we have that M is a retract of $\mathbb{U}$. Thus, $\mathscr{F}(M)$ is isomorphic to a complemented subspace of the Holmes space $\mathbb{H} = \mathscr{F}(\mathbb{U})$. By [FW], $\mathbb{H}$ has the MAP and thus it is isomorphic to a complemented subspace of $\mathbb{P}$. Assume that $\mathbb{P}$ is isomorphic to a complemented subspace of $\mathscr{F}(M)$. It follows by applying the standard Pełczyński's decomposition method that $\mathbb{P}$ is isomorphic to $\mathbb{H}$, which is not true [FW, Theorem 4.2]. $\square$

Putting together some known results, it follows:

Proposition 5.3.5. *$\mathscr{F}(\mathbb{G})$ is isomorphic to a complemented subspace of $\mathbb{H}$.*

Proof. The Gurariĭ space $\mathbb{G}$ is an L_1-predual [Gur]. Thus, it is finitely hyperconvex [BL, Lemma 2.12.(ii) and Lemma 2.13] (i.e. every finite collection of mutually intersecting balls has a common point). By a theorem of Kubiś [Kub, Theorem 3.17], this implies that $\mathbb{G}$ is a non-expansive retract of $\mathbb{U}$. $\square$

To sum up, we know that the following relations hold:

$$\begin{tikzcd}
 & & & & \mathbb{H} \\
\mathscr{F}(c_0) \arrow[r, hook, "c"] & \mathscr{F}(\mathbb{G}) \arrow[r, hook, "c"] & \mathbb{H} \arrow[r, hook, "c"] & \mathbb{P} \arrow[ur, hook, "c", "/" marking] \arrow[dr, hook, "c", "/" marking] & \\
 & & & & \mathscr{F}(c_0)
\end{tikzcd}$$

We do not know if $\mathscr{F}(\mathbb{G})$ is isomorphic to $\mathscr{F}(c_0)$ or $\mathbb{H}$.

Corrigendum: It does not follow from the results in [FW] that the Holmes space is not isomorphic to $\mathbb{P}$. Therefore, Proposition 5.3.4 is not correct. However, the following holds: if the Holmes space is not isomorphic to $\mathbb{P}$ then $\mathbb{P}$ is not isomorphic to $\mathscr{F}(c_0)$.

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List of Symbols

$\|f\|_L$	Lipschitz norm of the map f, page 79
$\alpha(D)$	Kuratowski index of non-compactness of D, page 39
α_M	lower Boyd index of the Orlicz function M, page 73
$B_d(x,r)$	closed ball centred at x with radius r, page xxix
β_M	upper Boyd index of the Orlicz function M, page 73
B_X	closed unit ball of the Banach space X, page xxix
$\mathrm{conv}(A)$	convex hull of the set A, page xxix
$\mathscr{C}(T,\tau)$	space of continuous functions on the topological space (T,τ), page xxix
$\mathscr{D}(C,M)$	set of dentable maps from C to M, page 35
$\mathscr{D}_U(C,M)$	maps in $\mathscr{D}(C,M)$ which are uniformly continuous on bounded sets, page 35
$\hat{\delta}_{\mathsf{E}}$	modulus of strong asymptotic uniform convexity of E, page 63
$\overline{\delta}_X$	modulus of asymptotic uniform convexity of X, page 57
$\overline{\delta}^*_X$	modulus of weak* asymptotic uniform convexity of X, page 57
$\delta_X(t)$	modulus of uniform convexity, page 64
$\mathrm{Dz}(f)$	dentability index of the map f, page 40
$\mathrm{Dz}(X)$	dentability index of X, page 10
$\exp(A)$	exposed points of A, page 1
$\mathrm{ext}(A)$	extreme points of A, page 1
$\mathscr{F}(M)$	Lipschitz free space over M, page 80
$f_{x,y}$	Ivakhno–Kadets–Werner function, page 110
h_M	Orlicz space associated to the Orlicz function M, page 73
$\mathscr{K}_{\tau_1,\tau_2}(X,Y)$	space of τ_1-to-τ_2 continuous compact operators from X to Y, page xxix
$\mathscr{K}(X,Y)$	compact operators from X to Y, page xxix
$\mathrm{lip}_0(M)$	space of little-Lipschitz functions, page 84
$\mathrm{Lip}(M,N)$	space of Lipschitz functions from M to N, page 79
$\mathrm{lip}_0(M,X)$	space of X-valued little-Lipschitz functions, page 89
$\mathrm{lip}_{\omega,*}(B_{X^*},Y)$	space of Y-valued weak*-continuous functions on B_{X^*} which are little-Lipschitz with respect to the gauge ω, page 97
$\mathrm{lip}_\tau(M)$	space of little-Lipschitz functions on M which are τ-continuous, page 85
$\mathscr{L}_{\tau_1,\tau_2}(X,Y)$	space of τ_1-to-τ_2 continuous bounded operators from X to Y, page xxix
$\mathscr{L}(X,Y)$	bounded operators from X to Y, page xxix
$\mathrm{Mid}(x,y,\delta)$	δ-approximate midpoints of x and y, page 108

$\mu(X, Y)$	Mackey topology, page 87
$m_{x,y}$	molecule $\frac{\delta(x)-\delta(y)}{d(x,y)}$ in $\mathscr{F}(M)$, page 81
$\mathrm{NA}(X, Y)$	set of all operators from X to Y which attain their norm, page 136
$\mathscr{N}(X, Y)$	space of nuclear operators form X to Y, page 13
$\omega^{<\omega}$	set of finite sequences of natural numbers, page 39
$\mathbb{P}$	Pełczyński's universal space, page 138
$\hat{\rho}_{\mathsf{E}}$	modulus of strong asymptotic uniform smoothness of E, page 63
$\overline{\rho}_X$	modulus of asymptotic uniform smoothness of X, page 57
$\rho_X(t)$	modulus of uniform smoothness, page 64
$S_0(M)$	Dalet's space of Lipschitz functions, page 85
$S_0(M, X)$	Dalet's space of X-valued Lipschitz functions, page 89
$S(A, f, t)$	slice of the set A given by the functional f, page xxix
$\mathrm{SA}(M, Y)$	set of all functions in $\mathrm{Lip}_0(M, Y)$ which strongly attain their norm, page 136
$\mathscr{SC}$	class of compact convex subsets with admit a lower semicontinuous strictly convex function, page 15
$\underline{S}(C, h, \alpha)$	lower slice of the set C given by the function h, page 51
$\mathscr{SS}(f, A)$	set of all f-strongly slicing functionals on A, page 34
$s \frown t$	concatenation of s and t, page 39
$\mathrm{strexp}(A)$	strongly exposed points of A, page 1
$\mathrm{strext}(A)$	strongly extreme points of A, page 1
S_X	unit sphere of the Banach space X, page xxix
$\mathrm{Sz}_{[\eta,\mathscr{S}]}(\mathfrak{X})$	Szlenk index of $\mathfrak{X}$ associated to η and $\mathscr{S}$, page 9
$\mathrm{Sz}(X)$	Szlenk index of X, page 10
V_M	set of molecules in M, page 81
$x \otimes y$	operator given by $x^* \mapsto x^*(x)y$, page 12
$X \widehat{\otimes}_\varepsilon Y$	injective tensor product of X and Y, page 12
$X \widehat{\otimes}_\pi Y$	projective tensor product of X and Y, page 13
X^*	topological dual of X, page xxix
$[x, y]$	metric segment between x and y, page 110
$(x, y)_z$	Gromov product of x and y at z, page 118