Compact convex sets that admit a strictly convex function

Luis Carlos García Lirola

Joint work with José Orihuela and Matías Raja

Universidad de Murcia

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1. Introduction
2. The class $\mathcal{SC}$
3. Faces and exposed points
4. Ordinal indices

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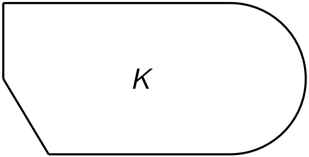

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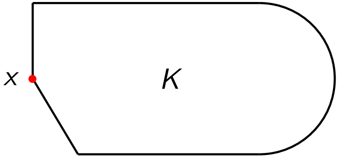

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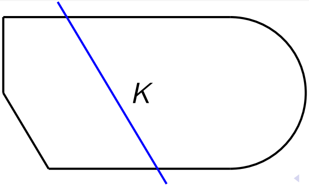
$$F = \{x \in K : w(x) = \sup\{w, K\}\}.$$



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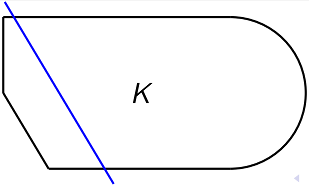

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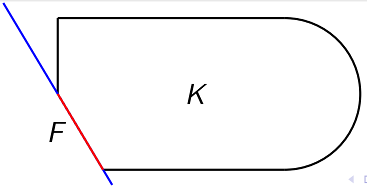

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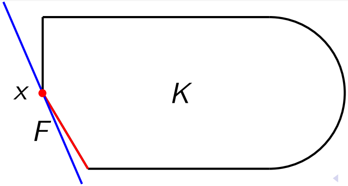

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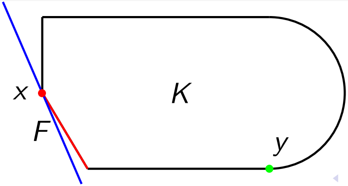

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A function $f : K \to \mathbb{R}$ is said to be *lower semicontinuous* if $\liminf f(x_\alpha) \geq f(x)$ whenever $x_\alpha \xrightarrow{\alpha} x$.

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Is it possible to replace $\operatorname{ext} K$ by $\exp K$ in the above theorem?

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Theorem (Hervé, 1961)

If there exists $f : K \to \mathbb{R}$ continuous and strictly convex, then K is metrizable.

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We denote by $\mathcal{SC}$ the class composed of all the families $\mathcal{SC}(X)$ for any locally convex space X.

Previous work about $\mathcal{SC}$

Theorem (Ribarska (1990))

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Theorem (Talagrand (1986))

If $K \in \mathcal{SC}$ then $[0, \omega_1]$ does not embed into K.

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- $\mathcal{SC}$ is stable by continuous linear images.

- If $K \in \mathcal{SC}(X)$, then it is witnessed by a *bounded* strictly convex lower semicontinuous function.
- If $K \in \mathcal{SC}(X)$, then it is witnessed by the square of a lower semicontinuous rotund norm defined on $\mathrm{span}(K)$.

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Let X be locally convex topological vector space and $K \subset X$ be compact and convex. Then $K \in \mathcal{SC}(X)$ if and only if K has $(*)$ with slices.

Faces and exposed points

Theorem (Raja, 2009)

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Let $f : X \to \mathbb{R}$ convex lower semicontinuous and bounded on compact subsets. Then for every compact convex subset $K \subset X$ and every open slice $S \subset K$, there is a face $F \subset S$ of K such that $f|_K$ is constant and continuous on F.

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Let $f : X \to \mathbb{R}$ be lower semicontinuous, strictly convex and bounded on compact sets. Then for every $K \subset X$ compact and convex, the set of points in K which are both exposed and continuity points of $f|_K$ is dense in ext(K).

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Let X^* be a dual rotund Banach space. Then every convex ω^*-compact is the closed convex hull of its ω^*-exposed points.

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Assume that $K \in \mathcal{SC}(X)$. Then K is the closed convex hull of its exposed points.

Some ideas

Consider the function

$$\rho(x,y) = \frac{f(x)^2 + f(y)^2}{2} - f\left(\frac{x+y}{2}\right)^2$$

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- Moreover, if f is strictly convex then ρ is a *symmetric*, that is,

$$\rho(x,y) = 0 \text{ if and only if } x = y$$

Some ideas

- We shall consider a notion of diameter with respect to ρ

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- We can use the existence of continuity extreme points of f to find slices of K with arbitrarily small ρ-diameter.
- Then Baire category arguments in RNP theory can be used to find faces where f remains constant.

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Let K be a convex and compact subset of a locally convex space

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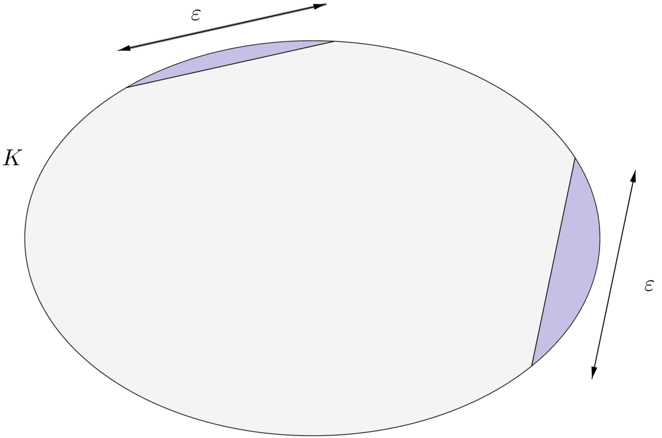

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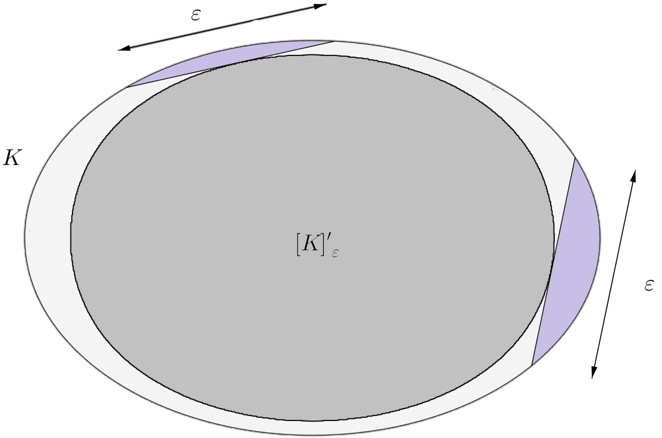

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- $Dz_\rho(K) = \sup_{\varepsilon>0} Dz_\rho(K,\varepsilon)$.

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- $Dz_\rho(K) = \sup_{\varepsilon>0} Dz_\rho(K, \varepsilon)$.
- $Sz_\rho(K)$ is defined by using open sets instead of slices.

Ordinal indices

The following assertions are equivalent:

i) $K \in \mathcal{S}C$;

ii) there exists a symmetric ρ on K such that $Dz_\rho(K) \leq \omega$;

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If there exists a metric d on K such that $Sz_d(K) \leq \omega_1$ then K is Gruenhage.

References

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