# Compact convex sets that admit a strictly convex function


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# Outline

1. Introduction
2. The class $\mathcal{SC}$
3. Faces and exposed points
4. Ordinal indices

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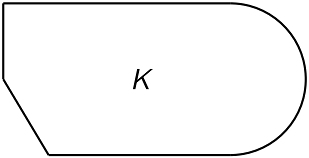

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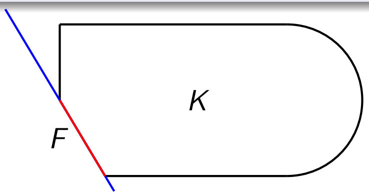

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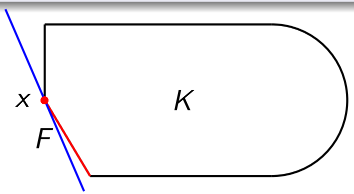

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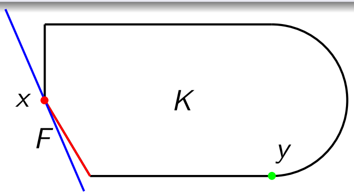


Each exposed point of $K$ is an extreme point of $K$.

# Motivation (I)

**Theorem (Raja, 2009)**

*Let $K$ be a convex compact subset of $X$ and let $f : K \to \mathbb{R}$ be a bounded convex lower semicontinuous function. Then $\operatorname{ext} K$ contains a dense subset of continuity points of $f$.*

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Is it possible to replace $\operatorname{ext} K$ by $\exp K$ in the above theorem?

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**Theorem (Hervé (1961))**

*If there exists $f : K \to \mathbb{R}$ continuous and strictly convex, then $K$ is metrizable.*

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- Metrizable convex compacts admits continuous strictly convex functions, so they are in the class.
- If $X$ is a Banach space endowed with its weak topology, then $\mathcal{SC}(X)$ is made up of all convex weakly compact subsets of $X$.

# Previous work about $\mathcal{SC}$

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**Theorem (Talagrand (1986))**

*If $K \in \mathcal{SC}$ then $[0, \omega_1]$ does not embed into $K$.*

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These operations allow us to improve the witnessing function.

- If $K \in \mathcal{SC}(X)$, then it is witnessed by a *bounded* strictly convex lower semicontinuous function.
- If $K \in \mathcal{SC}(X)$, then it is witnessed by the square of a lower semicontinuous rotund norm defined on $\operatorname{span}(K)$.

# Embeddings into duals

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This result compares to these others

- $K$ is *uniformly Eberlein* if and only if it embeds into a uniformly convex Banach space endowed with the weak topology.
- $K$ is *Namioka-Phelps* if and only if it embeds into a dual Banach space with a LUR norm endowed with the weak$^*$ topology.
- $K$ is *descriptive* if and only if it embeds into a dual Banach space with a weak$^*$-LUR norm endowed with the weak$^*$ topology.

# Relationship with $(*)$ property

**Theorem (Orihuela-Smith-Troyanski (2012))**

*Let $X^*$ be a dual Banach space. Then $(B_{X^*}, \omega^*)$ has $(*)$ with slices if and only if $X^*$ admits a dual rotund norm.*

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**Definition (Orihuela-Smith-Troyanski, 2012)**

Let $K$ be a compact subset of a locally convex topological space. $K$ is said to have $(*)$ if there exists a sequence $(\mathcal{U}_n)_{n=1}^{\infty}$ of families of open subsets of $K$ such that, given any $x, y \in K$, there exists $n \in \mathbb{N}$ such that:

i) $\{x, y\} \cap \bigcup \mathcal{U}_n$ is non-empty.

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Let $X$ be locally convex topological vector space and $K \subset X$ be compact and convex. Then $K \in \mathcal{SC}(X)$ if and only if $K$ has $(*)$ with slices.

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Let $f : X \to \mathbb{R}$ convex lower semicontinuous and bounded on compact subsets. Then for every compact convex subset $K \subset X$ and every open slice $S \subset K$, there is a face $F \subset S$ of $K$ such that $f|_K$ is constant and continuous on $F$.

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Let $f : X \to \mathbb{R}$ be lower semicontinuous, strictly convex and bounded on compact sets. Then for every $K \subset X$ compact and convex, the set of points in $K$ which are both exposed and continuity points of $f|_K$ is dense in ext$(K)$.

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**Corollary (Asplund (1968) + Larman-Phelps (1979))**

*Let $X^*$ be a dual rotund Banach space. Then every convex $\omega^*$-compact is the closed convex hull of its $\omega^*$-exposed points.*

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Assume that $K \in \mathcal{SC}(X)$. Then $K$ is the closed convex hull of its exposed points.

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We shall mimic the Baire category arguments in RNP theory to find faces with $\rho$-diam$(F) = 0$.

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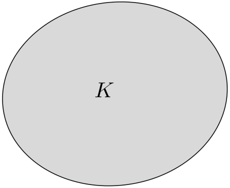

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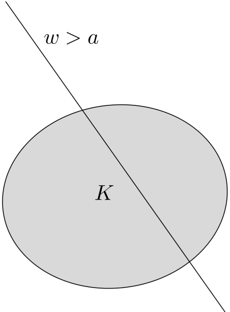


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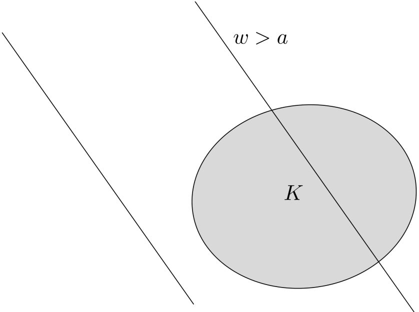


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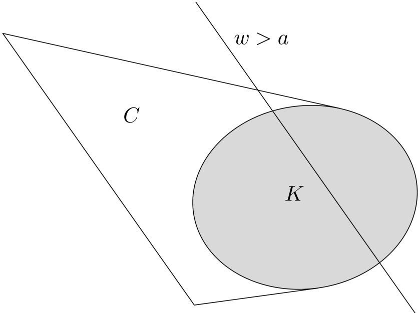


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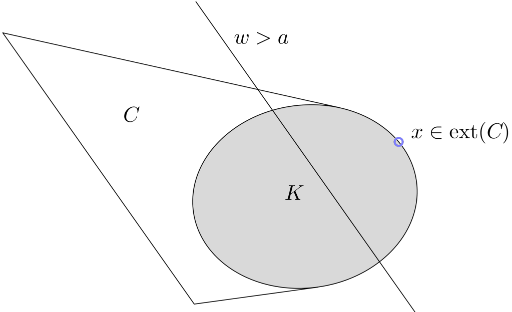


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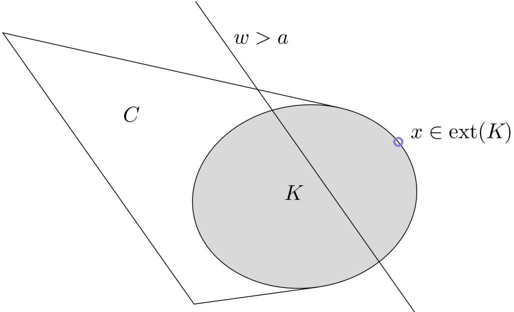


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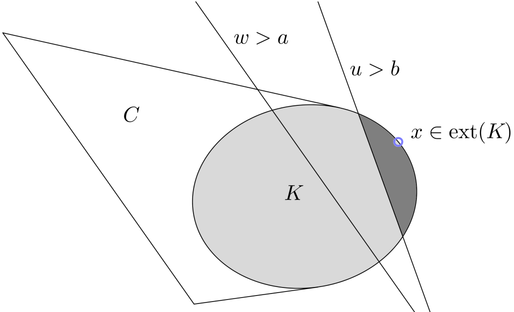


Take $w \in W$ and $\delta > 0$.
$\exists x \in \mathrm{ext}(C)$ where $f|_C$ is continuous.
By Choquet's lemma, $\exists u \in W$ and $b \in \mathbb{R}$ such that $\rho\text{-diam}(C \cap \{u > b\}) < \varepsilon$

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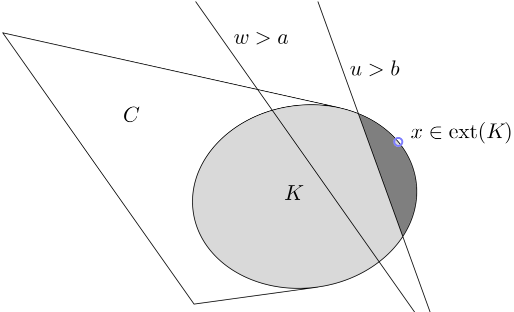
\begin{aligned}
G(K,\varepsilon) &= \{w \in W : \exists a < \sup\{w,K\}, \rho\text{-diam}(K \cap \{w > a\}) < \varepsilon\} \\
G(K) &= \bigcap_{n \geq 1} G(K, 1/n)
\end{aligned}
$$

Notice that if $w \in G(K)$ and $F$ is the face produced by $w$ then $f|_K$ is constant and continuous at each point of $F$.

It suffices to show that $G(K,\varepsilon)$ is dense in $W$ for each $\varepsilon > 0$.




Take $w \in W$ and $\delta > 0$.
$\exists x \in \mathrm{ext}(C)$ where $f|_C$ is continuous.
By Choquet's lemma, $\exists u \in W$ and $b \in \mathbb{R}$ such that $\rho\text{-diam}(C \cap \{u > b\}) < \varepsilon$
$u \in G(K,\varepsilon)$ and $||\omega - u|| < \delta$.

Moreover, if $f$ is a strictly convex function then $\rho$ is a *symmetric*, that is,

$$\rho(x, y) = 0 \text{ if and only if } x = y$$

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Thus, $\rho$-diam$(F) = 0$ implies that $F$ is an exposed point.

1. Introduction
2. The class $\mathcal{SC}$
3. Faces and exposed points
4. Ordinal indices

# Ordinal indices

Let $K$ be a convex and compact subset of a normed space.

$$[K]'_\varepsilon = \{x \in K : x \in S \text{ slice of } K \Rightarrow \operatorname{diam}(S) \geq \varepsilon\}$$

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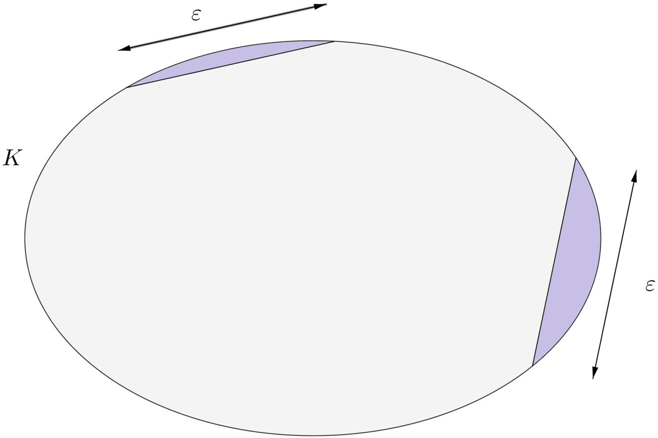

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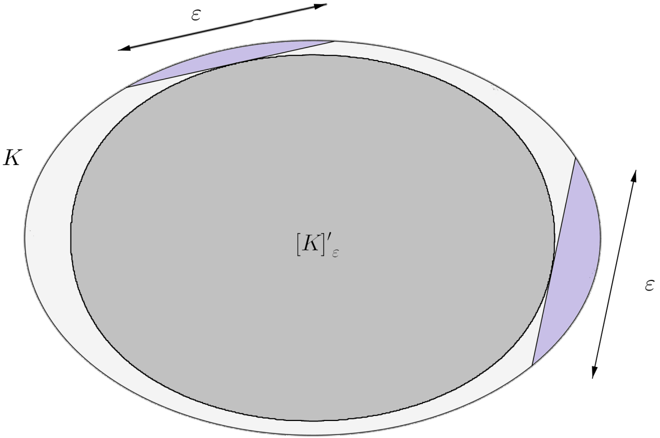

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Let $K$ be a convex and compact subset of a locally convex space **and** $\rho$ **a symmetric on** $K$.

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- $Sz_\rho(K)$ is defined by using open sets instead of slices.

# Ordinal indices

The following assertions are equivalent:

i) $K \in \mathcal{S}C$;

ii) there exists a symmetric $\rho$ on $K$ such that $Dz_\rho(K) \leq \omega$;

iii) there exists a symmetric $\rho$ on $K$ such that $Dz_\rho(K) \leq \omega_1$.

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If there exists a metric $d$ on $K$ such that $Sz_d(K) \leq \omega_1$ then $K$ is Gruenhage.

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Thank you for your attention