Asymptotic uniform smoothness in spaces of compact operators

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Table of Contents

Table of Contents

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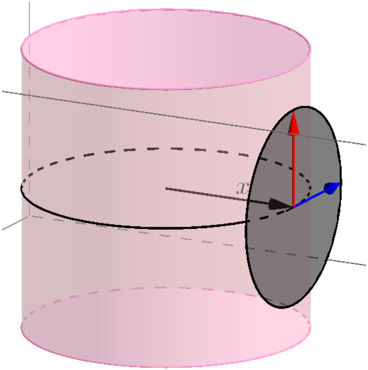

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The space X is said to be *asymptotically uniformly convex* (AUC for short) if

$$\overline{\delta}_X(t) > 0 \text{ for each } t > 0$$

and it is said to be *asymptotically uniformly smooth* (AUS for short) if

$$\lim_{t \to 0} t^{-1} \overline{\rho}_X(t) = 0$$

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Theorem (Causey, 2015)

$Sz(X \otimes_\varepsilon Y) = \max\{Sz(X), Sz(Y)\}$ for separable spaces X and Y.

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In particular, $X \otimes_\varepsilon Y$ admits an equivalent AUS norm if and only if X and Y do.

Table of Contents

Motivation

A sequence $(E_n)_n$ of finite dimensional subspaces of X is call a *finite dimensional decomposition* (FDD for short) if every $x \in X$ has a unique representation of the form $x = \sum_{n=1}^{\infty} x_n$, with $x_n \in E_n$ for every n.

Motivation

A sequence $(E_n)_n$ of finite dimensional subspaces of X is call a *finite dimensional decomposition* (FDD for short) if every $x \in X$ has a unique representation of the form $x = \sum_{n=1}^{\infty} x_n$, with $x_n \in E_n$ for every n. In addition, we shall denote $H_n = \bigoplus_{i=1}^{n} E_i$ and $H^n = \overline{\bigoplus_{i=n+1}^{\infty} E_i}$.

Assume that there is a shrinking FDD $(E_n)_n$ of X. For each $t > 0$ we have:

$$\overline{\delta}_X(t) = \inf_{n\in\mathbb{N}} \sup_{m\geq n} \inf\{||x + ty|| - 1 : x \in H_n \cap S_X, y \in H^m \cap S_X\},$$

$$\overline{\rho}_X(t) = \sup_{n\in\mathbb{N}} \inf_{m\geq n} \sup\{||x + ty|| - 1 : x \in H_n \cap S_X, y \in H^m \cap S_X\}.$$

Strongly AUC and strongly AUC spaces

Definition

Let X a Banach space with a monotone FDD (E_n). Denote $H_n = \bigoplus_{i=1}^{n} E_i$ and $H^n = \overline{\bigoplus_{i=n+1}^{\infty} E_i}$. X is said to be *strongly AUC* with respect to $(E_n)_n$ if the modulus defined by

$$\overline{s\delta}_{X,(E_n)}(t) = \inf\{||x+ty|| - 1 : x \in H_n, y \in H^n, ||x|| = ||y|| = 1, n \in \mathbb{N}\}$$

satisfies that $\overline{s\delta}_{X,(E_n)}(t) > 0$ for each $t > 0$.

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Some examples

a) If X is an ℓ_p-sum of finite dimensional spaces, $1 \leq p < \infty$, then $\overline{s\delta}_X(t) = \overline{s\rho}_X(t) = (1 + t^p)^{1/p} - 1$.

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c) The James space J with the norm

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e) Every uniformly smooth (resp. uniformly convex) space with a monotone FDD is strongly AUS (resp. strongly AUC).

Properties of strongly AUC and strongly AUS spaces

Let X be a Banach space with a monotone FDD $(E_n)_n$.

a) If X is strongly AUS w.r.t. $(E_n)_n$ then $(E_n)_n$ is shrinking.

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Let X be a Banach space with a monotone *shrinking* FDD and $0 < \sigma, \tau < 1$. Then

a) If $\overline{s\rho}_X(\sigma) < \sigma\tau$, then $\overline{s\delta}_{X^*}(3\tau) \geq \sigma\tau$.

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b) If $\overline{s\delta}_{X^*}(\tau) > \sigma\tau$, then $\overline{s\rho}_X(\sigma) \leq \sigma\tau$.

Thus, X is strongly AUS with power type p if and only if X^* is strongly AUC with power type p', the conjugate exponent of p.

AUS tensor products

Let X, Y be Banach spaces admitting monotone FDDs. Then

$$\overline{\rho}_{X\otimes_\varepsilon Y}(t) \leq (1+\overline{s\rho}_X(4t))(1+\overline{s\rho}_Y(4t))-1$$

for every $0<t<1/4$.

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- Assume that X is strongly AUC with power type p w.r.t. an shrinking FDD, and Y is strongly AUS with power type q. Then $\mathcal{K}(X, Y)$ is AUS with power type $\min\{p', q\}$.

Table of Contents

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$$h_M = \{(x_n) \in \mathbb{R}^{\mathbb{N}} : \sum_{n=1}^{\infty} M(|x_n|/\rho) < +\infty \text{ for some } \rho > 0\}$$

endowed with the *Luxemburg norm*

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The *Boyd indices* of an Orlicz function M are defined as follows:

$$\alpha_M = \sup\{q : \sup_{0<u,v\leq 1} \frac{M(uv)}{u^q M(v)} < +\infty\}$$

$$\beta_M = \inf\{q : \inf_{0<u,v\leq 1} \frac{M(uv)}{u^q M(v)} > 0\}$$

AUS and AUC Orlicz spaces

Theorem (Gonzalo–Jaramillo–Troyanski, 2007)

h_M is AUS if $\alpha_M > 1$. Moreover, α_M is the supremum of the numbers $\alpha > 1$ such that the modulus of asymptotic smoothness of h_M is of power type α.

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h_M is AUC if $\beta_M < \infty$, and β_M is the infimum of the numbers $\beta > 0$ such that its modulus of asymptotic convexity is of power type β.

Moreover, their proofs show that h_M is strongly AUS (resp. strongly AUC) whenever it is AUS (resp. AUC).

Compact operators on Orlicz spaces

Let M, N be Orlicz functions. The space $\mathcal{K}(h_M, h_N)$ is AUS if and only if $\alpha_M, \alpha_N > 1$ and $\beta_M < +\infty$.

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Let M, N be Orlicz functions such that $\alpha_M, \alpha_N > 1$ and $\beta_N < \infty$. Then $\mathcal{N}(h_M, h_N^*)$ has the weak* fixed point property.

References

Besbes, Mourad. "Points fixes dans les espaces des opérateurs nucléaires". In: *Bull. Austral. Math. Soc.* 46.2 (1992), pp. 287–294.

Borel-Mathurin, L. "Isomorphismes non linéaires entre espaces de Banach". PhD thesis. Université Paris 6, 2010.

Dilworth, S. J. et al. "Compactly uniformly convex spaces and property (β) of Rolewicz". In: *J. Math. Anal. Appl.* 402.1 (2013), pp. 297–307.

Dilworth, S. J. et al. "Equivalent norms with the property (β) of Rolewicz". In: *ArXiv e-prints* (June 2015). arXiv: `1506.07978 [math.FA]`.

Gonzalo, R., J. A. Jaramillo, and S. L. Troyanski. "High order smoothness and asymptotic structure in Banach spaces". In: *J. Convex Anal.* 14.2 (2007), pp. 249–269.

Lennard, Chris. "$\mathcal{C}_1$ is uniformly Kadec-Klee". In: *Proc. Amer. Math. Soc.* 109.1 (1990), pp. 71–77.

Thank you for your attention