Compact convex sets that admit a strictly convex function

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September 15, 2016

Research partially supported by

UNIVERSIDAD DE MURCIA

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MINISTERIO DE ECONOMÍA Y COMPETITIVIDAD

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Fondo Europeo de Desarrollo Regional

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- The space of sequences ℓ_p, $0 < p < 1$, endowed with the metric $d((x_n), (y_n)) = \sum_n |x_n - y_n|^p$ is not a l.c.s.

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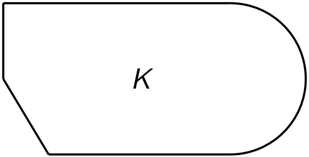

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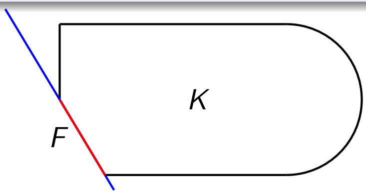
Let $K \subset X$ be compact and convex. We say that a closed subset $F \subset K$ is a **face** if there is a continuous affine function $w : K \to \mathbb{R}$ such that

$$F = \{x \in K : w(x) = \sup\{w, K\}\}.$$



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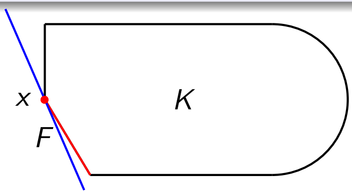

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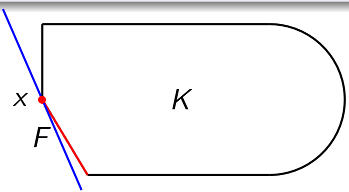

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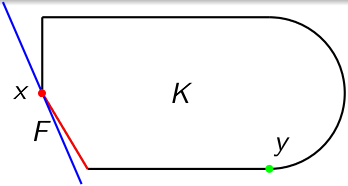

Each exposed point of K is an extreme point of K.

Lower semicontinuous functions

Definition

A function $f\colon (X,\tau)\to\mathbb{R}$ is said to be **continuous** if for every $x\in X$ and $\varepsilon>0$ there exists an open neighbourhood U of x such that $f(x)-\varepsilon<f(y)<f(x)+\varepsilon$ for every $y\in U$.

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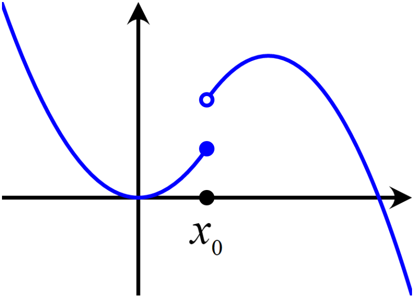

Points of continuity

Theorem (Raja, 2009)

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Let X be a locally convex topological vector space and let $f : X \to \mathbb{R}$ be lower semicontinuous, *strictly convex* and bounded on compact sets. Then for every $K \subset X$ compact and convex, the set of points in K which are both exposed and continuity points of $f|_K$ is dense in $\mathrm{ext}(K)$.

The class $\mathcal{SC}$

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Theorem (Hervé, 1961)

Let K be compact and convex. Then K is metrizable if, and only if, there exists a continuous strictly convex function $f : K \to \mathbb{R}$.

Embeddings into duals

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This result compares to these others

- K is *uniformly Eberlein* if and only if it embeds into a uniformly convex Banach space endowed with the weak topology.
- K is *Namioka-Phelps* if and only if it embeds into a dual Banach space with a LUR norm endowed with the weak* topology.
- K is *descriptive* if and only if it embeds into a dual Banach space with a weak*-LUR norm endowed with the weak* topology.

Ordinal indices

A function $\rho : K \times K \to \mathbb{R}$ is said to be a **symmetric** if satisfies $\rho(x, y) = \rho(y, x)$ and $\rho(x, y) = 0$ if and only if $x = y$.

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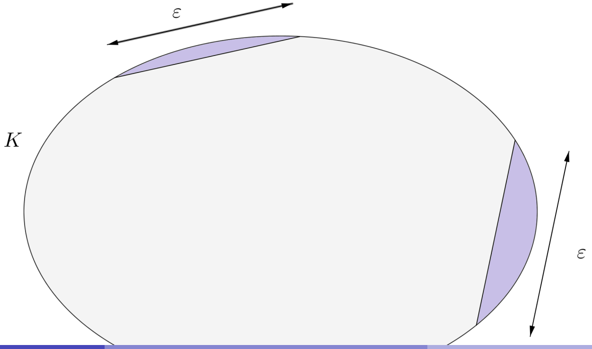
We consider the following set derivation:

$$[K]'_\varepsilon = \{x \in K : x \in S \text{ slice of } K \Rightarrow \operatorname{diam}(S) \geq \varepsilon\}, \ [K]^{n+1}_\varepsilon = [[K]^n_\varepsilon]'_\varepsilon$$



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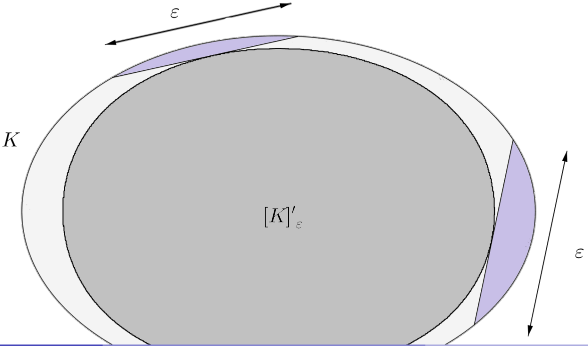

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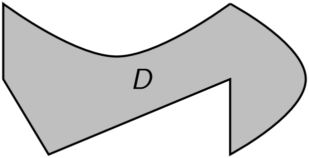
The following assertions are equivalent:

i) $K \in \mathcal{S}C$;

ii) there exists a symmetric ρ on K such that for every $\varepsilon > 0$ there is n so that $[K]^n_\varepsilon = \emptyset$.

Convex hull



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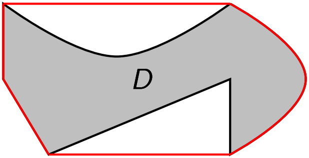

conv D

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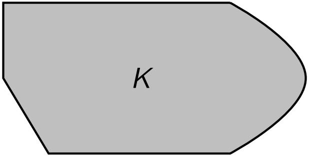

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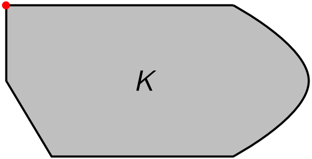

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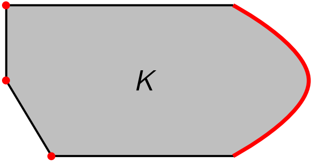

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Which conditions guarantee that $K = \operatorname{conv} \operatorname{ext} K$?

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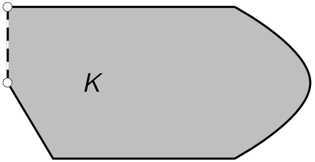

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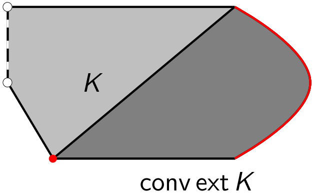

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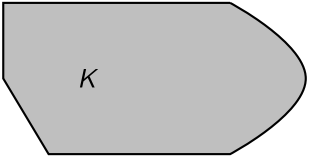

Theorem (Minkowski, 1911 – Caratheodory, 1930)

Let K be a compact convex subset of $\mathbb{R}^n$. Then $K = \operatorname{conv} \operatorname{ext} K$.

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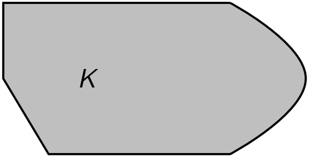

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This result is no longer true in infinite dimensions.

Krein–Milman theorem

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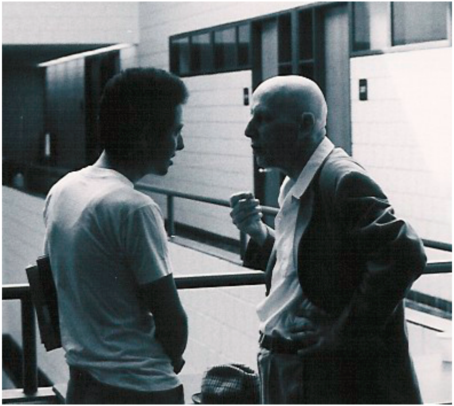

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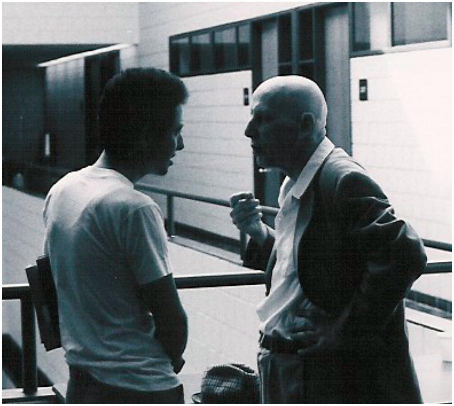

Assume that $K \in \mathcal{SC}$. Then $K = \overline{\operatorname{conv}}(\exp K)$

References

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L.C. García-Lirola, J. Orihuela, and M. Raja. "Compact convex sets that admit a lower semicontinuous strictly convex function". In: *J. of Convex Anal.* 24.3 (2017). to appear.

M. Raja. "Continuity at the extreme points". In: *J. Math. Anal. Appl.* 350.2 (2009), pp. 436–438.

Thank you for your attention