Asymptotic uniform smoothness in spaces of compact operators

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Asymptotic uniform smoothness

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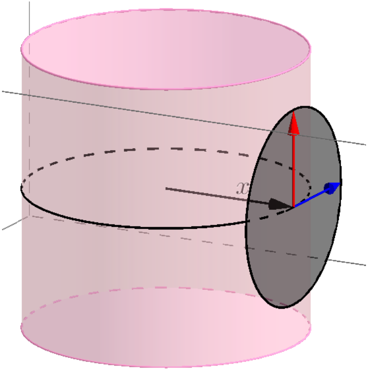

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$$\overline{\rho}_X(t) = \sup_{x \in S_X} \inf_{\dim(X/Y) < \infty} \sup_{y \in S_Y} ||x + ty|| - 1$$

The space X is said to be *asymptotically uniformly smooth* (AUS for short) if

$$\lim_{t \to 0} t^{-1} \overline{\rho}_X(t) = 0.$$

We say that X is *AUS with power type p* if $\overline{\rho}_X(t) \leq Ct^p$ for some $C > 0$.

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Recall that, under suitable assumptions, $\mathcal{K}(X, Y)$ is isometric to the injective tensor product $(X^* \widehat{\otimes}_\varepsilon Y, \| \|_\varepsilon)$

Strongly AUS spaces

A sequence $(E_n)_n$ of finite dimensional subspaces of X is call an *FDD* if every $x \in X$ has a unique representation of the form $x = \sum_{n=1}^{\infty} x_n$, with $x_n \in E_n$ for every n.

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Assume that there is a shrinking FDD $(E_n)_n$ of X. For each $t > 0$ we have:

$$\overline{\rho}_X(t) = \sup_{n\in\mathbb{N}} \inf_{m\geq n} \sup\{\|x + ty\| - 1 : x \in \oplus_{i=1}^{n} E_i \cap S_X, y \in \overline{\oplus_{i=m+1}^{\infty} E_i} \cap S_X\}.$$

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Definition

Let X a Banach space and let $\mathsf{E} = (E_n)_n$ be an FDD for X. The space X is said to be *strongly AUS with respect to* E if

$$\hat{\rho}_{\mathsf{E}}(t) = \sup_{n\in\mathbb{N}} \inf_{m\geq n} \sup\{\|x+ty\| - 1 : x \in \oplus_{i=1}^{m} E_i \cap S_X, y \in \overline{\oplus_{i=m+1}^{\infty} E_i} \cap S_X\}$$

satisfies $\lim_{t\to 0} t^{-1}\hat{\rho}_{\mathsf{E}}(t) = 0$. We say that X is *strongly AUS* if X is strongly AUS with respect to some FDD.

Some examples

a) Let $X = (\bigoplus_{n=1}^{\infty} E_n)_p$ be an ℓ_p-sum of finite dimensional spaces, $1 \leq p < \infty$, and consider $\mathsf{E} = (E_n)_{n=1}^{\infty}$. Then $\hat{\rho}_{\mathsf{E}}(t) = (1 + t^p)^{1/p} - 1$.

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b) Let $X = (\bigoplus_{n=1}^{\infty} E_n)_0$ be a c_0-sum of finite dimensional spaces, and $\mathsf{E} = (E_n)_{n=1}^{\infty}$. Then $\hat{\rho}_{\mathsf{E}}(t) = 0$ for each $t \in (0, 1]$.

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c) The James space J with the norm

$$||(x_n)_n||^2 = \sup_{1 \leq n_1 < \ldots < n_{2m+1}} \sum_{i=1}^{m} (x_{n_{2i-1}} - x_{n_{2i}})^2 + 2(x_{n_{2m+1}})^2$$

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e) Every uniformly smooth space with a monotone FDD is strongly AUS.

AUS tensor products

Let E, F be FDDs on Banach spaces X and Y, respectively. Then there exists a constant $K > 0$ such that

$$\overline{\rho}_{X\widehat{\otimes}_\varepsilon Y}(t) \leq (1 + \hat{\rho}_{\mathrm{E}}(Kt))(1 + \hat{\rho}_{\mathrm{F}}(Kt)) - 1$$

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In particular, if X is strongly AUS with power type p and Y is strongly AUS with power type q. Then $X\widehat{\otimes}_\varepsilon Y$ is AUS with power type $\min\{p, q\}$ and $\mathcal{N}(X, Y^*)$ is weak* AUC with power type $\max\{p', q'\}$ and has the weak* fixed point property.

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Moreover, if X^* is strongly AUS with power type p and Y is strongly AUS with power type q, then $\mathcal{K}(X, Y)$ is AUS with power type $\min\{p, q\}$.

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We have obtained the following:

Let X, Y be Banach spaces with dimension greater or equal than 2. Then $\mathcal{K}(X,Y)$ and $X\widehat{\otimes}_\varepsilon Y$ are not strictly convex.

Orlicz spaces

An *Orlicz function* is a continuous convex function M defined on $\mathbb{R}^+$ such that $M(0) = 0$ and $\lim_{t\to\infty} M(t) = +\infty$.
To any Orlicz function M we associate the space

$$h_M = \{(x_n) \in \mathbb{R}^{\mathbb{N}} : \sum_{n=1}^{\infty} M(|x_n|/\rho) < +\infty \text{ for some } \rho > 0\}$$

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The *Boyd indices* of an Orlicz function M are defined as follows:

$$\alpha_M = \sup\{q : \sup_{0<u,v\leq 1} \frac{M(uv)}{u^q M(v)} < +\infty\}$$

$$\beta_M = \inf\{q : \inf_{0<u,v\leq 1} \frac{M(uv)}{u^q M(v)} > 0\}$$

Compact operators on Orlicz spaces

Theorem (Gonzalo–Jaramillo–Troyanski, 2007)

h_M is AUS if $\alpha_M > 1$. Moreover, α_M is the supremum of the numbers $\alpha > 1$ such that the modulus of asymptotic smoothness of h_M is of power type α.

Borel-Mathurin (2010) proved an analogous statement relating β_M and the property of h_M being AUC. Moreover, their proofs show that h_M is strongly AUS (resp. strongly AUC) whenever it is AUS (resp. AUC).

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The space $\mathcal{K}(h_M, h_N)$ is AUS if and only if $\alpha_M, \alpha_N > 1$ and $\beta_M < +\infty$.

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Assume $\alpha_M, \alpha_N > 1$ and $\beta_N < \infty$. Then the space $\mathcal{N}(h_M, h_N)$ has the weak* fixed point property.

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Thank you for your attention