Extremal structure of Lipschitz free spaces

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Spaces of Lipschitz functions and Lipschitz free spaces

- Given a **complete** metric space (M, d) and a distinguished point $0 \in M$, the space

$$\mathrm{Lip}_0(M) := \{f\colon M \to \mathbb{R} : f \text{ is Lipschitz}, f(0) = 0\}$$

is a dual Banach space when equipped with the norm

$$\|f\|_L := \sup\left\{\frac{|f(x) - f(y)|}{d(x,y)} : x \neq y\right\}.$$

- The canonical predual of $\mathrm{Lip}_0(M)$ is the **Lipschitz free space** $\mathcal{F}(M) = \overline{\mathrm{span}}\{\delta_x : x \in M\}$, where $\langle f, \delta_x\rangle = f(x)$.

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- The elements of the form

$$\frac{\delta_x - \delta_y}{d(x, y)}, \; x, y \in M, x \neq y$$

are called **molecules**. Note that

$$B_{\mathcal{F}(M)} = \overline{\mathrm{conv}}\left\{\frac{\delta_x - \delta_y}{d(x, y)} : x, y \in M\right\}$$

Distinguished subsets of B_X

Let X be a Banach space and $x \in B_X$.

- x is an **extreme point** if $x = \frac{y+z}{2}$, $y, z \in B_X$, implies $x = y = z$.
- x is an **exposed point** if there is $f \in X^*$ such that

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- x is a **denting point** if the slices of B_X containing x are a neighbourhood basis for x in the norm topology.
- x is a **weak-strongly exposed point** if there is $f \in X^*$ providing slices that form a neighbourhood basis for x in the weak topology.
- x is a **strongly exposed point** if there is $f \in X^*$ providing slices that form a neighbourhood basis for x in the norm topology.

Extremal structure of $B_{\mathcal{F}(M)}$ and molecules

Theorem (Weaver, 1995)

Every preserved extreme point of $B_{\mathcal{F}(M)}$ is a molecule.

We do not know if every extreme point of $B_{\mathcal{F}(M)}$ is a molecule. We have shown that this is the case whenever $\mathcal{F}(M)$ has a predual with additional properties.

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Definition

Let $f \in S_{\mathrm{Lip}_0(M)}$. We say that f is **peaking at** (x, y) if

$$\frac{f(x)-f(y)}{d(x,y)} = 1 \text{ and } \lim_{n\to\infty} \frac{f(u_n)-f(v_n)}{d(u_n,v_n)} = 1 \Rightarrow u_n \to x, v_n \to y$$

Theorem (Weaver, 1999)

Assume that there is a Lipschitz function f peaking at (x, y). Then $\frac{\delta_x-\delta_y}{d(x,y)}$ is a preserved extreme point.

Strongly exposed points in $B_{\mathcal{F}(M)}$

Let $x, y \in M$, $x \neq y$. The following are equivalent.

(i) The molecule $\frac{\delta_x - \delta_y}{d(x,y)}$ is a strongly exposed point of $B_{\mathcal{F}(M)}$.

(ii) There is $f \in \mathrm{Lip}_0(M)$ peaking at (x, y).

(iii) There is $\varepsilon > 0$ such that for every $z \in M \setminus \{x, y\}$,

$$d(x,z) + d(y,z) > d(x,y) + \varepsilon \min\{d(x,z), d(y,z)\}$$

This result extends a characterisation of peaking functions in subsets of $\mathbb{R}$-trees due to by Dalet, Kaufmann and Procházka (2016).
The proof relies on ideas coming from Dalet–Kaufmann–Procházka characterisation and a paper by Ivakno, Kadets and Werner (2007).

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Corollary

Let M be a **compact** *metric space. Then* $\mathrm{Lip}_0(M)$ *has the Daugavet property if and only if* $B_{\mathcal{F}(M)}$ *does not have any strongly exposed point.*

Very recently, Aliaga and Guirao have characterised preserved extreme points of $B_{\mathcal{F}(M)}$. Their result says that $\frac{\delta_x - \delta_y}{d(x,y)}$ is a preserved extreme point if and only if for every $\varepsilon > 0$ there is $\delta > 0$ such that for every $z \in M \setminus \{x, y\}$,

$$(1-\delta)(d(x,z) + d(z,y)) < d(x,y) \Rightarrow \min\{d(x,z), d(y,z)\} < \varepsilon.$$

This solves a problem posed by Weaver and implies that if M is compact then every molecule which is an extreme point of $B_{\mathcal{F}(M)}$ is also a preserved extreme point.

Our next goal is to study the relationship between the different notions of extreme and exposed points for $B_{\mathcal{F}(M)}$. We need the following easy lemma.

Lemma

Assume $\frac{\delta_{x_\alpha}-\delta_{y_\alpha}}{d(x_\alpha,y_\alpha)}$ *converges weakly to* $\frac{\delta_x-\delta_y}{d(x,y)}$. *Then* $x_\alpha \to x$ *and* $y_\alpha \to y$. *Therefore* $\frac{\delta_{x_\alpha}-\delta_{y_\alpha}}{d(x_\alpha,y_\alpha)}$ *converges in norm to* $\frac{\delta_x-\delta_y}{d(x,y)}$.

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Proof.

Test the weak convergence against the function

$$f(t) = \max\{\varepsilon - d(x,t), 0\}$$

$\square$

Every preserved extreme point of $B_{\mathcal{F}(M)}$ is a denting point.

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Proof.

Denote V the set of molecules and let $\mu \in V$ be a preserved extreme point. Assume there is $\varepsilon > 0$ such that every slice of $B_{\mathcal{F}(M)}$ containing μ has diameter at least ε.

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There must be a slice S of $B_{\mathcal{F}(M)}$ such that $\mathrm{diam}(V \cap S) < \varepsilon/2$. Otherwise, there would be a net $\{\mu_\alpha\}$ of molecules that converges weakly to μ but not in norm, a contradiction.

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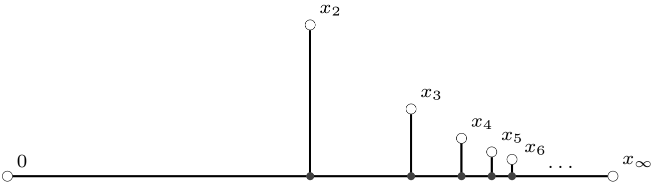
$$B_{\mathcal{F}(M)} = \overline{\operatorname{conv}}(V) = \overline{\operatorname{conv}}(\overline{\operatorname{conv}}(V \cap S) \cup \overline{\operatorname{conv}}(V \setminus S))$$

Now, a variation of Asplund–Bourgain–Namioka superlemma provides a slice of $B_{\mathcal{F}(M)}$ containing μ of diameter less than ε, a contradiction. $\square$

Example

There is a compact countable metric space M with a denting point of $B_{\mathcal{F}(M)}$ which is not strongly exposed.



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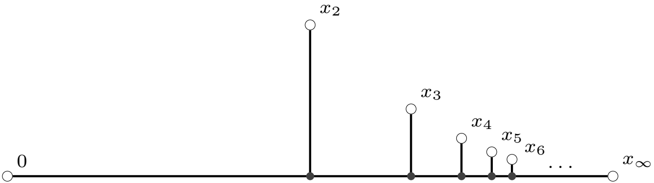

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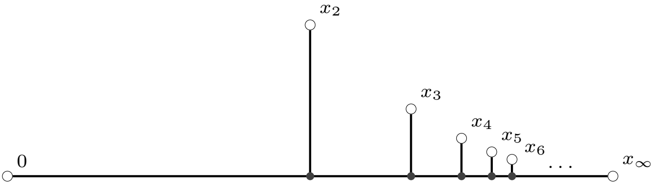

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Example

Consider the sequence in c_0 given by $x_0 = 0, x_1 = 2e_1$, and $x_n = e_1 + (1 + 1/n)e_n$ for $n \geq 2$. Let $M = \{0\} \cup \{x_n : n \in \mathbb{N}\}$. Aliaga and Guirao showed that the molecule $\frac{\delta(x_1)}{2}$ is not a preserved extreme point of $B_{\mathcal{F}(M)}$. However, we have shown that it is an extreme point.

Every weak-strongly exposed point of $B_{\mathcal{F}(M)}$ is a strongly exposed point.

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Corollary

The norm of $\mathrm{Lip}_0(M)$ *is Gâteaux differentiable at* f *if and only if it is Fréchet differentiable at* f.

Corollary

Let M be a **compact** *metric space. The following assertions are equivalent:*

(i) *M is geodesic, that is, for every pair of points in M there is a geodesic joining them.*

(ii) *For every $x, y \in M$ there is $z \in M \setminus \{x, y\}$ such that $d(x, y) = d(x, z) + d(z, y)$.*

(iii) $\mathrm{Lip}_0(M)$ *has the Daugavet property.*

(iv) *The unit ball of $\mathcal{F}(M)$ does not have any preserved extreme point.*

(v) *The unit ball of $\mathcal{F}(M)$ does not have any strongly exposed point.*

(vi) *The norm of* $\mathrm{Lip}_0(M)$ *does not have any point of Gâteaux differentiability.*

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Thank you for your attention